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Topological Entropy for Endomorphisms of Local C^*-Algebras

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Abstract: A notion of topological entropy for endomorphisms of local C^*-algebras is introduced as a generalisation of the topological entropy of classical dynamical systems. The basic properties are derived and a series of calculations are presented.

0. Introduction

The purpose with the following pages is to propose a definition of topological entropy for endomorphisms of C^*-algebras or, more generally, local C^*-algebras. In view of the significance of the topological entropy for the study of topological dynamical systems it is natural to try to extend this notion to non-commutative dynamical systems. In fact, several notions of entropy have already been introduced in the non-commutative setting, in particular by the work of Connes and Størmer [4], Connes [3] and of Connes, Narnhofer and Thirring [5]. See [14] for an overview. However, the classical model for these definitions is the measure theoretic entropy, and while this is natural for endomorphisms of von Neumann algebras, it seems that for C^*-dynamical systems it may be more appropriate to generalize the topological entropy rather than the measure theoretical. With the right definition it might even be possible to relate the non-commutative topological entropy to the entropy of Connes, as defined in [3], through a non-commutative version of the variational principle which relates the topological entropy to the measure theoretic in the commutative case.

Hudetz has proposed a definition of topological entropy for C^*-algebraic dynamical system in his thesis, [10, 11], and his work has been an inspiration for the work we present here. The definition we offer is even more elementary than the "pedestrian" approach of Hudetz and this may be the reason that the algebraic properties are better than with his. However, we share the problems with the continuity; good continuity properties of the entropy should compensate for the lack of something like a non-commutative Kolmogoroff-Sinai theorem as in case of the Connes-Størmer entropy or the notion of refinements in the case of the classical topological entropy. The entropy we define here does have some continuity properties, we derive these in Sect. 2 below, but they are far from satisfactory. Another parallel with Hudetz' approach is that we can almost imitate his calculations with our entropy (see Sect. 3 below) even though it may be impossible to relate our definitions directly.

The topological entropy $h(\phi)$ of a continuous selfmap $\phi: X \to X$ of a compact Hausdorff space X enjoys the following properties:

i) $h(\mathrm{id}_X) = 0$ and $h(\phi^{-1}) = h(\phi)$ when ϕ is a homeomorphism.

ii) When $\pi: X \to Y$ is a continuous surjection and $\psi: Y \to Y$ a continuous map such that $\psi \circ \pi = \pi \circ \phi$, then $h(\psi) \leq h(\phi)$.

iii) When $Y \subseteq X$ is a closed subset such that $\phi(Y) \subseteq Y$, then $h(\phi|_Y) \leq h(\phi)$.

iv) When $\psi: Y \to Y$ is a continuous map and $\pi: X \to Y$ a homeomorphism such that $\psi \circ \pi = \pi \circ \phi$, then $h(\psi) = h(\phi)$. (This follows from ii).)

v) $h(\phi^k) = kh(\phi)$, $k \in \mathbb{N}$.

These properties have been our guiding line in the quest for the right definition; in the next section we shall show that they can all be generalized to the non-commutative case.

1. Definition and Basic Properties

A *local C^*-algebra A* is a $*$-subalgebra of a C^*-algebra $\bar{A}$ which is closed under holomorphic functional calculus, cf. [2, 1]. We shall only consider unital local C^*-algebras in this paper and take the existence of a unit as part of the definition.

When $\alpha_1, \alpha_2, \ldots, \alpha_n$ are subsets of A we set

$$\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n = \bigcup_{\sigma \in \Sigma_n} \{a_1 a_2 \ldots a_n : a_i \in \alpha_{\sigma(i)}, i = 1, 2, \ldots, n\},$$

where we take the union over all elements of the symmetric group Σ_n. When $\alpha \subseteq A$ we set $\alpha\alpha^* = \{aa^* : a \in \alpha\}$ and $\alpha^*\alpha = \{a^*a : a \in \alpha\}$. When α is a finite set we let $\Sigma\alpha$ denote the sum of the elements in α. Recall that $\alpha \subseteq A$ is *selfadjoint* when $a \in \alpha \Rightarrow a^* \in \alpha$.

Definition 1.1. *A partition in A is a finite selfadjoint subset $\alpha \subseteq A$ such that $\Sigma\alpha^*\alpha > 0$.*

Note that a partition α automatically has $\Sigma\alpha\alpha^* > 0$. For any partition $\alpha \subseteq A$ set

$$N(\alpha) = \min\{\#\beta : \beta \subseteq \alpha, \Sigma\beta\beta^* > 0, \Sigma\beta^*\beta > 0\}$$

and

$$N_1(\alpha) = \min\{\#\beta : \beta \subseteq \alpha, \Sigma\beta^*\beta > 0\}.$$

Clearly,

$$N_1(\alpha) \leq N(\alpha) \leq 2N_1(\alpha), \tag{1.1}$$

Set $H_1(\alpha) = \log N_1(\alpha)$ and $H(\alpha) = \log N(\alpha)$. When $\alpha_1, \alpha_2, \ldots, \alpha_n$ are partitions then so is $\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$. Since $(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_k) \vee (\alpha_{k+1} \vee \alpha_{k+2} \vee \ldots \vee \alpha_n) \subseteq \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$ we have that

$$\begin{aligned} &H(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n) \\ &\quad \leq H((\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_k) \vee (\alpha_{k+1} \vee \alpha_{k+2} \vee \ldots \vee \alpha_n)). \end{aligned} \tag{1.2}$$

Since $N(\alpha \vee \beta) \leq N(\alpha)N(\beta)$, (1.2) yields that

$$\begin{aligned} &H(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n) \\ &\leq H(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_k) + H(\alpha_{k+1} \vee \alpha_{k+2} \vee \ldots \vee \alpha_n)). \end{aligned} \tag{1.3}$$

Finally, it is not difficult to see that

$$H(\alpha \vee \alpha \vee \ldots \vee \alpha) \leq H(\alpha), \tag{1.4}$$

regardless of how many times α is repeated in $\alpha \vee \alpha \vee \ldots \vee \alpha$.

Let now $\pi : A \to A$ be a unital $*$-endomorphism. π induces a map on the subsets of A in the obvious way and we have that

$$H(\pi(\alpha)) \leq H(\alpha) \tag{1.5}$$

for any partition α. Combining (1.2) with (1.5) it follows that the sequence $H(\alpha \vee \pi^k(\alpha) \vee \pi^{2k}(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha))$, $n \in \mathbb{N}$, is subadditive in n so that the limit

$$\hbar_k(\pi, \alpha) = \lim_{n\to\infty} \frac{1}{kn} H(\alpha \vee \pi_k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha))$$

exists and equals $\inf_n \frac{1}{kn} H(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha))$ for all $k \in \mathbb{N}$. Note that (1.1) implies that

$$\hbar_k(\pi, \alpha) = \lim_{n\to\infty} \frac{1}{kn} H_1(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha)).$$

Set

$$\hbar(\pi) = \sup_{\alpha,k} \hbar(\pi, \alpha),$$

where we take the supremum over all partitions α and all $k \in \mathbb{N}$.

Theorem 1.2. *$\hbar$ generalizes the topological entropy of dynamical systems;* i.e. *when X is a compact Hausdorff space and $\phi : X \to X$ a continuous map, then $\hbar(\pi_\phi) = h(\phi)$, where π_ϕ is the $*$-endomorphism of $C(X)$ induced by ϕ* (viz. $\pi_\phi(g) = g \circ \phi$, $g \in C(X)$).

Furthermore, we have the following:

i) $\hbar(\mathrm{id}_A) = 0$ *when* id_A *denotes the identity map of the local C^*-algebra A and $\hbar(\theta^{-1}) = \hbar(\theta)$, when θ is a $*$-automorphism of A.*

Let A be a local C^*-algebra and $\pi : A \to A$ a unital $*$-endomorphism.

ii) *When B is a local C^*-subalgebra of A such that $\pi(B) \subseteq B$, then $\hbar(\pi|_B) \leq \hbar(\pi)$.*

iii) *When B is a local C^*-algebra, $q : A \to B$ a surjective $*$-homomorphism and $\gamma : B \to B$ a unital $*$-endomorphism such that $\gamma \circ q = q \circ \pi$, then $\hbar(\gamma) \leq \hbar(\pi)$.*

iv) *When B is a local C^*-algebra, $\pi_1 : B \to B$ a unital $*$-endomorphism and $\theta : A \to B$ a $*$-isomorphism such that $\theta \circ \pi = \pi_1 \circ \theta$, then $\hbar(\pi) = \hbar(\pi_1)$.*

v) $\hbar(\pi^k) = k\hbar(\pi)$, $k \in \mathbb{N}$.

Proof. The proof of the first statement is almost straightforward. The essential points are the following: Every partition α of $C(X)$ defines a cover $\mathscr{U} = \{a^{-1}(\mathbb{C}\backslash\{0\}) : a \in \alpha\}$ of X. Then $N(\alpha)$ is the minimal number of elements in a subcover of $\mathscr{U}$. Conversely, every finite open cover of X gives rise to a partition in $C(X)$ by choosing a partition of unity subordinate to it. The cover defined by this partition is a shrinking of the given cover. We leave the details to the reader.

i) It follows immediately from (1.4) that $\hbar(\mathrm{id}_A) = 0$. To prove the identity $\hbar(\theta^{-1}) = \hbar(\theta)$, it suffices to note that

$$H(\alpha \vee \theta^{-k}(\alpha) \vee \ldots \vee \theta^{-kn+k}(\alpha)) = H(\alpha \vee \theta^{k}(\alpha) \vee \ldots \vee \theta^{kn-k}(\alpha))$$

for every partition α in A and all $n, k \in \mathbb{N}$.

ii) is trivial.

iii) Let β be a partition in B. It is easily seen that there is a finite selfadjoint set $\alpha_0 \subseteq A$ such that $q(\alpha_0) = \beta$. Since $q(\Sigma\alpha_0^*\alpha_0) \geq \Sigma\beta^*\beta > 0$, there is a $\delta > 0$ such that $q(\Sigma\alpha_0^*\alpha_0 - \delta 1) > 0$. Since B is a local C^*-algebra, $q(\Sigma\alpha_0^*\alpha_0 - \delta 1) = e^b$ for some $b = b^* \in B$. If $a = a^* \in A$ such that $q(a) = b$, then $y = e^a > 0$ in A and $x = \Sigma\alpha_0^*\alpha_0 - \delta 1 - y \in \ker q$. For $K > 0$ sufficiently large we have that $K^2x^2 + x + \delta 1 > 0$. If $x \neq 0$ we can choose K so large that $Kx \notin \alpha_0$. Set $\alpha = \alpha_0 \cup \{Kx\}$. Then $\Sigma\alpha^*\alpha = \Sigma\alpha_0^*\alpha_0 + K^2x^2 = \delta 1 + y + x + K^2x^2 > 0$, showing that α is a partition in A. Note that $q(\alpha) = \beta \cup \{0\}$. Thus $\hbar_k(\gamma, \beta) = \hbar_k(\gamma, q(\alpha))$ for all $k \in \mathbb{N}$. Since

$$\begin{aligned} &H(q(\alpha) \vee \gamma^k(q(\alpha)) \vee \ldots \vee \gamma^{kn-k}(q(\alpha))) \\ &\quad = H(q(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{kn-k}(\alpha))) \leq H(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{kn-k}(\alpha)) \end{aligned}$$

for all $k, n \in \mathbb{N}$, we see that $\hbar_k(\gamma, \beta) = \hbar_k(\gamma, q(\alpha)) \leq \hbar_k(\pi, \alpha) \leq \hbar(\pi)$ for all k. Since β was an arbitrary partition in B we see that $\hbar(\gamma) \leq \hbar(\pi)$.

iv) follows immediately from iii) but can of course also be shown directly.

v) Let α be a partition. Then by definition and (1.2),

$$\begin{aligned} \sup_\beta \hbar_k(\pi^m, \beta) &\geq \hbar_k(\pi^m, \alpha \vee \pi^k(\alpha) \vee \pi^{2k}(\alpha) \vee \ldots \vee \pi^{km-k}(\alpha)) \\ &\geq \lim_{n\to\infty} \frac{1}{kn} H(\alpha \vee \pi^k(\alpha) \vee \pi^{2k}(\alpha) \vee \ldots \vee \pi^{nmk-k}(\alpha)) = m\hbar_k(\pi, \alpha) \end{aligned}$$

for all partitions α and all $k \in \mathbb{N}$. Hence $\hbar(\pi^m) \geq m\hbar(\pi)$. On the other hand we have also that $\hbar_k(\pi^m, \beta) = m\hbar_{km}(\pi, \beta)$ for all $k \in \mathbb{N}$ and all partitions β. Hence

$$\hbar(\pi^m) = \sup_{\beta,k} \hbar_k(\pi^m, \beta) = m \sup_{\beta,k} \hbar_{km}(\pi, \beta) \leq m \sup_{\beta,k} \hbar_k(\pi, \beta) = m\hbar(\pi). \qquad \square$$

We conclude this section by showing that $\hbar$ behaves in the expected way with respect to direct sums.

Proposition 1.3. *Let A_1 and A_2 be local C^*-algebras and $\pi_i : A_i \to A_i$ unital $*$-endomorphisms, $i = 1, 2$. Let $\pi_1 \oplus \pi_2 : A_1 \oplus A_2 \to A_1 \oplus A_2$ denote the direct sum endomorphism. Then*

$$\hbar(\pi_1 \oplus \pi_2) = \max\{\hbar(\pi_1), \hbar(\pi_2)\}.$$

Proof. It follows from Theorem 1.2 iii) that $\max\{\hbar(\pi_1), \hbar(\pi_2)\} \leq \hbar(\pi_1 \oplus \pi_2)$. To prove the reverse inequality, let α be a partition in $A_1 \oplus A_2$ and write $a = (a_1, a_2)$ for each element $a \in \alpha$. Then $\alpha_i = \{a_i : a \in \alpha\}$ is a partition in A_i, $i = 1, 2$, and it is easily seen that

$$\begin{aligned} &N(\alpha \vee (\pi_1 \oplus \pi_2)^k(\alpha) \vee (\pi_1 \oplus \pi_2)^{2k} \vee \ldots \vee (\pi_1 \otimes \pi_2)^{nk-k}(\alpha)) \\ &\quad \leq N(\alpha_1 \vee \pi_1^k(\alpha_1) \vee \ldots \vee \pi_1^{nk-k}(\alpha_1)) + N(\alpha_2 \vee \pi_2^k(\alpha_2) \vee \ldots \vee \pi_2^{nk-k}(\alpha_2)) \end{aligned}$$

for all $n, k \in \mathbb{N}$. Hence $\hbar_k(\pi_1 \oplus \pi_2, \alpha) \leq \max\{\hbar_k(\pi_1, \alpha_1), \hbar_k(\pi_2, \alpha_2)\} \leq \max\{\hbar(\pi_1), \hbar(\pi_2)\}$ for all $k \in \mathbb{N}$. Since α was an arbitrary partition in $A_1 \oplus A_2$, the proof is complete. $\square$

2. Continuity Properties

The purpose of this section is to show that the numbers $\hbar_k(\pi, \alpha)$ do depend continuously on α in a certain very weak sense. For this purpose we consider the partitions in A as a metric space with the Hausdorff distance D as metric:

$$D(\alpha, \beta) = \max \left\{ \sup_{a \in \alpha} \operatorname{dist}(a, \beta), \sup_{b \in \beta} \operatorname{dist}(b, \alpha) \right\}.$$

If α is a partition in A which contains at least one element ≥ 0 and we set $\alpha_\varepsilon = \{a + \varepsilon 1 : a \in \alpha\}$, then $\hbar_k(\pi, \alpha_\varepsilon) = 0$ since α_ε contains an invertible element. Since $D(\alpha, \alpha_\varepsilon) \leq \varepsilon$, this shows that a simple continuity statement like $\lim_{i \to \infty} D(\alpha_i, \alpha) = 0 \Rightarrow \lim_{i \to \infty} \hbar_k(\pi, \alpha_i) = \hbar_k(\pi, \alpha)$ can impossibly hold. In fact, even when we fix $n, k \in \mathbb{N}$, the number $H(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha))$ does not depend continuously on α in any straightforward way. We must therefore seek for a more subtle form for continuity.

Let α be a partition. For each $\varepsilon \geq 0$, let $M(\alpha, \varepsilon)$ be the least number of elements in any subset β of α with the property that for all states ω of A there is an element $b \in \beta$ such that $\omega(b^* b) > \varepsilon$; in symbols: $\forall \omega \in S_A \, \exists b \in \beta : \omega(b^* b) > \varepsilon$. We call then β for an *ε-subset*. We set $M(\alpha, \varepsilon) = \infty$ if no such subset exists. Then

$$M(\alpha, 0) = N_1(\alpha) \tag{2.1}$$

and

$$0 \leq \varepsilon \leq \delta \Rightarrow M(\alpha, \varepsilon) \leq M(\alpha, \delta). \tag{2.2}$$

Furthermore, we assert that

$$M(\alpha \vee \beta, \varepsilon\delta) \leq M(\alpha, \varepsilon) M(\beta, \varepsilon). \tag{2.3}$$

To prove this we may assume that both $M(\alpha, \varepsilon)$ and $M(\beta, \delta)$ are finite. Take an ε-subset $\alpha' \subseteq \alpha$ and a δ-subset $\beta' \subseteq \beta$. Then $\gamma = \{ba : a \in \alpha', b \in \beta'\}$ is a subset of $\alpha \vee \beta$. Since $\#\gamma \leq (\#\alpha')(\#\beta')$, it suffices to show that γ is an $\varepsilon\delta$-subset. So let $\omega \in S_A$. There is an $a \in \alpha'$ such that $\omega(a^* a) > \varepsilon$. Then $\omega(a^* a)^{-1} \omega(a^* \cdot a)$ defines a new state on A and hence there is a $b \in \beta'$ such that $\omega(a^* a)^{-1} \omega(a^* b^* b a) > \delta$. But then $ba \in \gamma$ and $\omega(a^* b^* b a) > \varepsilon\delta$.

It is clear that

$$M(\pi(\alpha), \varepsilon) \leq M(\alpha, \varepsilon). \tag{2.4}$$

We set $H_1(\alpha, \varepsilon) = \log M(\alpha, \varepsilon)$. Now (2.3) and (2.4) show that the limit

$$\hbar_k(\pi, \alpha, \varepsilon) = \lim_{n \to \infty} \frac{1}{kn} H_1(\alpha \vee \pi^k(\alpha) \vee \pi^{2k}(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \varepsilon^n)$$

exists in $[0, \infty]$ and equals

$$\inf_n \frac{1}{kn} H_1(\alpha \vee \pi^k(\alpha) \vee \pi^{2k}(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \varepsilon^n)$$

for all $k \in \mathbb{N}$ and all $\varepsilon \in [0, \infty[$. Note that $H_1(\pi, \alpha, 0) = H_1(\pi, \alpha)$ and $\hbar_k(\pi, \alpha, 0) = \hbar_k(\pi, \alpha)$.

Lemma 2.1. *The function $\varepsilon \to \hbar_k(\pi, \alpha, \varepsilon)$ is non-decreasing and upper semi-continuous on $[0, \infty[$. In particular, $\lim_{\varepsilon \downarrow 0} \hbar_k(\pi, \alpha, \varepsilon) = \hbar_k(\pi, \alpha)$.*

Proof. It follows from (2.2) that $\hbar_k(\pi, \alpha, \varepsilon)$ is non-decreasing in ε. So we only have to show that $\lim_{\varepsilon \downarrow \varepsilon_0} \hbar_k(\pi, \alpha, \varepsilon) = \hbar_k(\pi, \alpha, \varepsilon_0)$ for all $\varepsilon_0 \in [0, \infty[$. Let $\delta > 0$ and choose $n \in \mathbb{N}$ such that $\frac{1}{kn} H_1(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \varepsilon_0^n) \leq \hbar_k(\pi, \alpha, \varepsilon_0) + \delta$. Then $H_1(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \varepsilon^n) = H_1(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \varepsilon_0^n)$ for all $\varepsilon \geq \varepsilon_0$ sufficiently close to ε_0. Thus

$$\hbar_k(\pi, \alpha, \varepsilon) \leq \frac{1}{kn} H_1(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \varepsilon^n) \leq \hbar_k(\pi, \alpha, \varepsilon_0) + \delta$$

for all $\varepsilon \geq \varepsilon_0$ sufficiently close to ε_0. □

Proposition 2.2. *Let α a partition in A and let α_n, $n = 1, 2, \ldots$, be a sequence of partitions in A such that $\lim_{n\to\infty} \frac{1}{n} \log D(\alpha, \alpha_n) = -\infty$. For each $k \in \mathbb{N}$ we have*

a)
$$\begin{aligned}\sup_{\delta<\varepsilon} \hbar_k(\pi, \alpha, \delta) &\leq \liminf_n \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \varepsilon^n) \\ &\leq \limsup_n \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \varepsilon^n) \\ &\leq \hbar_k(\pi, \alpha, \varepsilon)\end{aligned}$$

for all $\varepsilon \in]0, \infty[$. In particular,

$$\hbar_k(\pi, \alpha) = \lim_{\varepsilon\downarrow 0} \left(\limsup_n \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \varepsilon^n) \right).$$

b) *If $\delta_n \in]0, \infty[$ is a sequence such that $\lim_{n\to\infty} \delta_n = 0$ and*

$$\lim_{n\to\infty} \left(\log \delta_n - \frac{1}{n} \log D(\alpha, \alpha_n) \right) = \infty,$$

then

$$\hbar_k(\pi, \alpha) = \lim_{n\to\infty} \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \delta_n^n).$$

Proof. a) Since $\lim_{n\to\infty} D(\alpha, \alpha_n) = 0$, there is a $M \geq 0$ such that $\|a\| \leq M$ for all $a \in \alpha \cup \bigcup_{n=1}^{\infty} \alpha_n$. Then

$$\begin{aligned}&D(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n)) \\ &\quad \leq nM^{n-1} D(\alpha, \alpha_n).\end{aligned}$$

Let $0 < \delta_1 < \varepsilon$ and $\delta_2 > \varepsilon$. By assumption $D(\alpha_n, \alpha) \leq \frac{\delta_1^n}{2nM^{2n-1}}$ for all sufficiently large n. It follows that if ω is a state of A and a an element of $\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n)$ such that $\omega(a^*a) > \varepsilon^n$, then there is an element b of $\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha)$ such that $\omega(b^*b) > \varepsilon^n - \delta_1^n > \delta_1^n$ for all sufficiently large n. Hence

$$\begin{aligned}&\frac{1}{kn} H_1(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \delta_1^n) \\ &\quad \leq \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \varepsilon^n)\end{aligned}$$

for all sufficiently large n. In a similar way we conclude that

$$\frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \varepsilon^n) \leq \frac{1}{kn} H_1(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \delta_2^n)$$

for all sufficiently large n. Hence

$$\begin{aligned} \hbar_k(\pi, \alpha, \delta_1) &\leq \liminf_n \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \varepsilon^n) \\ &\leq \limsup_n \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \varepsilon^n) \\ &\leq \hbar_k(\pi, \alpha, \delta_2). \end{aligned}$$

Since $\delta_1 < \varepsilon$ was arbitrary and $\lim_{\delta_2 \downarrow \varepsilon} \hbar_k(\pi, \alpha, \delta_2) = \hbar_k(\pi, \alpha, \varepsilon)$ by Lemma 2.1, we get the desired inequalities.

b) Let $\varepsilon > 0$. By Lemma 2.1 there is a $\delta > 0$ such that $\hbar_k(\pi, \alpha, \delta) \leq \hbar_k(\pi, \alpha) + \varepsilon$. If a is an element of $\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n)$ and ω a state of A such that $\omega(a^*a) > \delta_n^n$, then there is an element $b \in \alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha)$ such that $\omega(b^*b) > \delta_n^n - 2nM^{2n-1}D(\alpha, \alpha_n) > 0$ for all sufficiently large n. If instead $a \in \alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha)$ and $\omega(a^*a) > (2\delta_n)^n$, then there is an element $b \in \alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n)$ such that $\omega(b^*b) > (2\delta_n)^n - 2nM^{2n-1}D(\alpha, \alpha_n) > \delta_n^n + \delta_n^n - 2kM^{2n-1}D(\alpha, \alpha_n) > \delta_n^n$ for all sufficiently large n. Thus

$$\begin{aligned} H_1(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), 0) &\leq H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \delta_n^n) \\ &\leq H_1(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), (2\delta_n)^n) \\ &\leq H_1(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \delta^n) \end{aligned}$$

for all sufficiently large n. Consequently

$$\begin{aligned} \hbar_k(\pi, \alpha) &\leq \liminf_n \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \delta_n^n) \\ &\leq \limsup_n \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \delta_n^n) \\ &\leq \lim_{n\to\infty} \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha), \delta^n) \\ &= \hbar_k(\pi, \alpha, \delta) \leq \hbar_k(\pi, \alpha) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary the conclusion follows. □

In general it is not true that $\lim_{n\to\infty} \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n)) = \hbar_k(\pi, \alpha)$, no matter how rapidly α_n approaches α. But we have the following conclusion:

Corollary 2.3. *Under the assumption of Proposition 2.2 it follows that*

$$\hbar_k(\pi, \alpha) \geq \limsup_n \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n)).$$

If, furthermore,

$$\lim_{n\to\infty} \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \varepsilon^n) = \inf_n \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \varepsilon^n),$$

for all sufficiently small $\varepsilon \geq 0$, *then we have that*

$$\hbar_k(\pi, \alpha) = \lim_{n\to\infty} \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n)).$$

Proof. By a) of Proposition 2.2,

$$\hbar_k(\pi, \alpha, \varepsilon) \geq \limsup_n \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n))$$

for all $\varepsilon > 0$. Since $\lim_{\varepsilon\downarrow 0} \hbar_k(\pi, \alpha, \varepsilon) = \hbar_k(\pi, \alpha)$ by Lemma 2.1, we get the stated inequality. To prove the equality, note that under the given assumption we can prove, as in Lemma 2,1, that

$$\lim_{\varepsilon\downarrow 0}\left(\lim_{n\to\infty} \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n), \varepsilon^n)\right) = \lim_{n\to\infty} \frac{1}{kn} H_1(\alpha_n \vee \pi^k(\alpha_n) \vee \ldots \vee \pi^{k(n-1)}(\alpha_n)).$$

The equality therefore follows by letting $\varepsilon \to 0$ in a) of Proposition 2.2. □

3. Calculations

In this section we calculate the topological entropy of a series of $*$-endomorphisms of local algebras that are generated by a sequence of finite dimensional C^*-algebras. Fundamental to our calculations is the following simple lemma whose proof we leave to the reader. We use the notation $D(B)$ for the dimension of a maximal abelian C^*-subalgebra of a finite dimensional C^*-algebra B.

Lemma 3.1. *Let A be a local C^*-algebra and α a partition in A. Assume that B is a finite-dimensional unital C^*-subalgebra of A and that $\alpha \subseteq B$. Then $N(\alpha) \leq D(B)$. (In other words, if $B \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$, then $N(\alpha) \leq n_1 + n_2 + \ldots + n_k$.)* □

We shall also need the following lemma which is the analogue of 3.2.9 in [10].

Lemma 3.2. *Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ be an increasing sequence of finite dimensional C^*-algebras with the same unit and set $A = \bigcup_{i\in\mathbb{N}} A_i$. Then A is a local C^*-algebra. Let $\pi: A \to A$ be a unital injective $*$-endomorphism satisfying the following conditions:*

i) *For each $j, m \in \mathbb{N}$, $D(C^*(A_j, \pi(A_j), \ldots, \pi^{m-1}(A_j))) \leq D(A_{j+m})$.*

ii) *For each $j \in \mathbb{N}$, there is an integer $n_j \in \mathbb{N}$ such that A_j commutes with $\pi^{kn_j}(A_j)$ for all $k \in \mathbb{N}$ and the natural $*$-homomorphism*

$$A_j \otimes \pi^{n_j}(A_j) \otimes \pi^{2n_j}(A_j) \otimes \ldots \otimes \pi^{kn_j}(A_j) \to C^*(A_j, \pi^{n_j}(A_j), \pi^{2n_j}(A_j), \ldots, \pi^{kn_j}(A_j))$$

is an isomorphism for all $k \in \mathbb{N}$.

iii)
$$\lim_{j\to\infty} \frac{n_j - j}{j} = 0 .$$

Then
$$\hbar(\pi) = \limsup_j \frac{\log D(A_j)}{j} .$$

Proof. Let α be a partition in A. Then $\alpha \subseteq A_j$ for some $j \in \mathbb{N}$. By use of (1.3), (1.5), condition i) and Lemma 3.1, we find that

$$\begin{aligned}
\hbar_k(\pi,\alpha) &= \lim_{n\to\infty} \frac{1}{kn} H(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha)) \\
&\leq \lim_{n\to\infty} \frac{1}{kn} (H(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(n-j)}(\alpha)) \\
&\quad + H(\pi^{k(n-j+1)}(\alpha) \vee \ldots \vee \pi^{k(n-1)}(\alpha))) \\
&\leq \limsup_n \frac{1}{kn} (\log D(A_{j+k(n-1)+1}) + H(\alpha \vee \pi^k(\alpha) \vee \ldots \vee \pi^{k(j-2)}(\alpha))) \\
&\leq \limsup_j \frac{\log D(A_j)}{j}
\end{aligned}$$

for all $k \in \mathbb{N}$. This shows that $\hbar(\pi) \leq \limsup_j \frac{\log D(A_j)}{j}$. On the other hand, if we now let α_j denote the partition in A_j consisting of the minimal non-zero projections in a maximal abelian C^*-subalgebra of A_j, the assumption ii) and the injectivity of π gives the following estimate:

$$\begin{aligned}
\hbar(\pi) \geq \hbar_{n_j}(\pi,\alpha_j) &= \lim_{n\to\infty} \frac{1}{n_j n} H(\alpha_j \vee \pi^{n_j}(\alpha_j) \vee \ldots \vee \pi^{n_j(n-1)}(\alpha_j)) \\
&= \lim_{n\to\infty} \frac{1}{n_j n} n \log D(A_j) = \frac{\log D(A_j)}{n_j}
\end{aligned}$$

for all $j \in \mathbb{N}$. Hence $\hbar(\pi) \geq \limsup_j \frac{\log D(A_j)}{n_j}$. Since $\limsup_j \frac{\log D(A_j)}{n_j} = \limsup_j \frac{\log D(A_j)}{j}$ by iii), the proof is complete. □

Thus we have the same conclusion as in Theorem 3.2.9 in [10], except that we have not shown that the sequence $\frac{\log D(A_j)}{j}$ is actually convergent. This follows from the proof in [10].

Example 3.3. Let B be a unital C^*-algebra and let $\bar{A} = \bigotimes_{n\in\mathbb{N}} B$ be the infinite tensor product C^*-algebra of a countable number of copies of B. For each $k \in \mathbb{N}$ the simple tensors of the form $b_1 \otimes b_2 \otimes b_3 \otimes \ldots \otimes b_k \otimes 1 \otimes 1 \otimes \ldots$ generate a unital C^*-subalgebra A_k of $\bar{A}$ which is $*$-isomorphic to the tensor product $B \otimes B \otimes \ldots \otimes B$ of k copies of B. Thus $A = \bigcup_k A_k$ is a local C^*-algebra such that $\bar{A}$ is the closure of A. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be an injective map. For each $b \in B$, let $b(i)$ denote the element $1 \otimes 1 \otimes \ldots \otimes 1 \otimes b \otimes 1 \otimes 1 \otimes \ldots$, where b occurs as the i^{th} tensor factor, $i \in \mathbb{N}$.

There is then a unique unital $*$-endomorphism π_σ of A given by the condition that $\pi_\sigma(b(i)) = b(\sigma(i))$, $b \in B$, $i \in \mathbb{N}$. Then

i) $\hbar(\pi_\sigma) = 0$ unless σ has an infinite orbit in $\mathbb{N}$,

ii) $\hbar(\pi_\sigma) = \infty$ if σ has an infinite orbit in $\mathbb{N}$ and B is infinite dimensional, and

iii) $\hbar(\pi_\sigma) = r \log D(B)$ when B is finite dimensional, where r is the number of infinite orbits of σ in $\mathbb{N}$.

Proof. i) Let α be a partition in A. Then $\alpha \subseteq A_m$ for some $m \in \mathbb{N}$. If σ has no infinite orbit in $\mathbb{N}$ there is an integer N such that π_σ^N is the identity on A_m. In particular $\pi_\sigma^{kN}(\alpha) = \alpha$ for all $k \in \mathbb{N}$. It follows that for each $k \in \mathbb{N}$ and all sufficiently large n, $N(\alpha \vee \pi_\sigma^k(\alpha) \vee \ldots \vee \pi_\sigma^{k(n-1)})$ can not exceed $N(\alpha \vee \pi_\sigma^k(\alpha) \vee \ldots \vee \pi_\sigma^{kN}(\alpha))$ and therefore $\hbar_k(\pi_\sigma, \alpha) = 0$.

ii) Let $N \in \mathbb{N}$ be arbitrary. If B is infinite dimensional there is an infinite dimensional unital abelian C^*-subalgebra D of B. By using this it is easily seen that B contains a partition $\beta = \{b_1, b_2, \ldots, b_N\}$ in D with the property that for every $i \in \{1, 2, \ldots, N\}$ there is a state ω_i of B such that $\omega_i(b_i) = 1$ while $\omega_i(b_j) = 0$, $i \neq j$. Now let $m \in \mathbb{N}$ be an integer such that $\{\sigma^n(m) : n \in \mathbb{N}\}$ is infinite. Then $\alpha = \{b_1(m), b_2(m), \ldots, b_N(m)\}$ is a partition in A such that $\hbar_k(\pi_\sigma, \alpha) \geq \log N$ for all $k \in \mathbb{N}$. Since $N \in \mathbb{N}$ was arbitrary we conclude that $\hbar(\pi_\sigma) = \infty$.

iii) We first derive the following expression for the number r of infinite orbits:

$$r = \lim_{n \to \infty} \left(\inf_{k \in \mathbb{N}} \frac{1}{k} \# F_n \cup \sigma(F_n) \cup \sigma^2(F_n) \cup \ldots \cup \sigma^k(F_n) \right), \tag{3.1}$$

where $F_n = \{1, 2, \ldots, n\}$. To prove (3.1) note first that by subadditivity we have that

$$\begin{aligned} &\lim_{k \to \infty} \frac{1}{k} \# F \cup \sigma(F) \cup \ldots \cup \sigma^k(F) \\ &= \inf_{k \in \mathbb{N}} \frac{1}{k} \# F \cup \sigma^2(F) \cup \ldots \cup \sigma^2(F) \cup \ldots \cup \sigma^k(F) \end{aligned}$$

for any finite set F in $\mathbb{N}$. If now F is such a subset we can write $F = F_1 \cup F_2$, where F_1 is the elements of F whose orbit under σ is finite and F_2 is the compliment. Then

$$\begin{aligned} &\# F \cup \sigma(F) \cup \sigma^2(F) \cup \ldots \cup \sigma^k(F) \\ &\leq \# F_1 \cup \sigma(F_1) \cup \ldots \cup \sigma^k(F_1) \\ &\quad + \# F_2 \cup \sigma(F_2) \cup \sigma^2(F_2) \cup \ldots \cup \sigma^k(F_2), \end{aligned}$$

and since there is a k-independent upper bound on $\# F_1 \cup \sigma(F_1) \cup \ldots \cup \sigma^k(F_1)$ we conclude that

$$\begin{aligned} &\lim_{k \to \infty} \frac{1}{k} \# F \cup \sigma(F) \cup \sigma^2(F) \cup \ldots \cup \sigma^k(F) \\ &= \lim_{k \to \infty} \frac{1}{k} \# F_2 \cup \sigma(F_2) \cup \ldots \cup \sigma^k(F_2). \end{aligned}$$

Since the latter limit is $\leq \# F_2 \leq r$ and the right-hand side of (3.1) equals

$$\sup_F \left(\lim_{k \to \infty} \frac{1}{k} (\# F \cup \sigma(F) \cup \sigma^2(F) \cup \ldots \cup \sigma^k(F)) \right),$$

where we take the supremum over all finite subsets of $\mathbb{N}$, we have proved one of the inequalities in (3.1). Let next $i_1, i_2, \dots, i_m$ be elements of $\mathbb{N}$ with disjoint infinite orbits under σ and set $F = \{i_1 i_2, \dots, i_m\}$. Then

$$\lim_{k\to\infty} \frac{1}{k} \# F \cup \sigma(F) \cup \dots \cup \sigma^k(F) = \# F = m\,,$$

proving that the right-hand side of (3.1) is $\geq m$. We have now established (3.1) and proceed to the calculation of $\hbar(\pi_\sigma)$. If F is a finite subset of $\mathbb{N}$ we let A_F denote the $*$-algebra generated by the elements of the form $b(i)$, $b \in B$, $i \in F$. Then A_F is a finite dimensional C^*-algebra and $D(A_F) = D(B)^{\# F}$. Note that $A_{F_n} = A_n$, $n \in \mathbb{N}$. Let α be a partition in A. Then $\alpha \subseteq A_m$ for some $m \in \mathbb{N}$ and

$$\alpha \vee \pi_\sigma^k(\sigma) \vee \pi_\sigma^{2k}(\alpha) \vee \dots \vee \pi_\sigma^{k(n-1)}(\alpha) \in A_G\,,$$

where $G = F_n \cup \sigma^k(F_n) \cup \sigma^{2k}(F_n) \cup \dots \cup \sigma^{k(n-1)}(F_n)$, $k, n \in \mathbb{N}$. By using Lemma 3.1 and (3.1) we find that

$$\begin{aligned}
&(\log D(B))^{-1} \hbar_k(\pi_\sigma, \alpha) \\
&= \lim_{n\to\infty} \frac{1}{kn} (\# F_n \cup \sigma^k(F_n) \cup \dots \cup \sigma^{k(n-1)}(F_n)) \\
&\leq \lim_{n\to\infty} \frac{1}{kn} (\# F_n \cup \sigma(F_n) \cup \sigma^2(F_n) \cup \dots \cup \sigma^{kn}(F_n)) \leq r\,.
\end{aligned}$$

It follows that $\hbar(\pi_\sigma) \leq r \log D(B)$. To prove the reverse inequality, take $n \in \mathbb{N}$ and let $\beta = \{e_1, e_2, \dots, e_d\}$ be a partition in B consisting of the minimal non-zero projections in a maximal abelian C^*-subalgebra of B. For each $m \in \mathbb{N}$, let α_m be the partition in A_{F_m} given by the simple tensors

$$e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_m} \otimes 1 \otimes 1 \otimes \dots\,,$$

$i_1, i_2, \dots, i_m \in \{1, 2, \dots, d\}$. Thus α_m consists of $D(B)^m$ mutually orthogonal projections. It is easy to see that

$$\begin{aligned}
&(\log D(B))^{-1} H(\alpha_m \vee \pi_\sigma(\alpha_m) \vee \pi_\sigma^2(\alpha_m) \vee \dots \vee \pi_\sigma^n(\alpha_m)) \\
&\quad = \# F_m \cup \sigma(F_m) \cup \sigma^2(F_m) \cup \dots \cup \sigma^n(F_m)
\end{aligned}$$

for all $n \in \mathbb{N}$, so that

$$(\log D(B))^{-1} \hbar_1(\pi_\sigma, \alpha_m) = \inf_{k\in\mathbb{N}} \frac{1}{k} \# F_m \cup \sigma(F_m) \cup \dots \cup \sigma^k(F_m)\,.$$

Since this holds for all $m \in \mathbb{N}$, we conclude from (3.1) that $\hbar(\pi_\sigma) \geq r \log D(B)$. □

Example 3.4. Let $0 < \tau \leq 1$. As shown by Jones in [12], the hyperfinite II_1 factor R is generated by the unit 1 and a sequence of projections $e_0, e_1, e_2, \dots$ such that

(a) $e_i e_{i\pm 1} e_i = \tau e_i$,

(b) $e_i e_j = e_j e_i$ for $|i - j| \geq 2$, and

(c) $\mathrm{tr}(w e_i) = \tau\, \mathrm{tr}(w)$ when w is a word in $1, e_0, e_1, \dots, e_{n-1}$,

exactly when $\tau \in \left(0, \frac{1}{4}\right] \cup \left\{ \left(4\cos^2 \frac{\pi}{m}\right)^{-1} : m \in \mathbb{N}, m \geq 3 \right\}$. There is a tr-preserving unital $*$-endomorphism θ_τ such that $\theta_\tau(e_i) = e_{i+1}$, $i = 0, 1, 2, \dots$. Let A_n be the

C^*-subalgebra of R generated by $1, e_0, e_1, e_2, \ldots, e_n$ and set $A = \bigcup_n A_n$. Then A is a local C^*-algebra and $\theta_\tau(A) \subseteq A$. We have that

$$\hbar(\theta_\tau|_A) = \begin{cases} -\frac{1}{2}\log\tau, & \text{when } \tau \in \left\{\left(4\cos^2\frac{\pi}{m}\right)^{-1} : m \in \mathbb{N}, m \geq 3\right\} \\ \log 2, & \text{when } \tau \in \left(0, \frac{1}{4}\right]. \end{cases}$$

Proof. By using iii) it is seen that the conditions of Lemma 3.2 are satisfied (with $n_j = j + 2$). Thus $\hbar(\theta_\tau|_A) = \limsup_j \frac{\log D(A_j)}{j}$. This number is easily found from the literature: If $\tau > \frac{1}{4}$ the inclusion pattern for $A_1 \subseteq A_2 \subseteq \ldots$ is periodic (of period 2) in the sense of [8] and hence, by [10], Lemma 3.2.7, $\limsup_j \frac{\log D(A_j)}{j} = \frac{1}{2}\log\beta$, where β is the Perron-Frobenius eigenvalue for the inclusion matrix of $A_j \subseteq A_{j+2}$ for all sufficiently large j. It is well-known that $\beta = \tau^{-1}$. If instead $\tau \leq \frac{1}{4}$, the Bratteli diagram of $A_1 \subseteq A_2 \subseteq \ldots$ was described by Jones in Sect. 5 of [12]. In particular, one finds that

$$D(A_n) = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \frac{n+2-2k}{n+2} \frac{(n+2)!}{k!(n-k+2)!},$$

and it follows easily that $\lim_{j\to\infty} \frac{D(A_j)}{j} = \log 2$. □

The Connes-Størmer entropy of θ_τ with respect to the trace state of R was calculated by Choda in [8], Example 2, and it interesting to compare with the result above. For $\tau \geq \frac{1}{4}$ the results agree, but for $\tau < \frac{1}{4}$ Choda gets the value $-t\log t - (1-t)\log(1-t)$, where $\tau = t(1-t)$, while $\hbar(\theta_\tau|_A) = \log 2$ is τ-independent. Note that $-t\log -(1-t)\log t(1-t) \leq \log 2$ for all $t \in]0, 1[$.

There is also an automorphic version of θ_τ obtained from a twosided sequence $\{e_i : i \in \mathbb{Z}\}$ of generating projections in R satisfying almost the same relations as above, see [13, 8]. Again there is a canonical local C^*-subalgebra A generated by an increasing sequence of finite dimensional subalgebras of R such that $\theta_\tau(A) = A$. By the same arguments as above, using [7] in place of [12], we get exactly the same values for $\hbar(\theta_\tau|_A)$ as before.

Example 3.5. Let S be a finite subset in $\mathbb{N}$ and $n \in \mathbb{N}$. There is then a sequence $\{u_i : i = 0, 1, 2, \ldots\}$ of unitaries, which generates the hyperfinite II_1 factor R, such that $u_i^n = 1$ for all i, $u_i u_j = \exp(2\pi i/n) u_j u_i$ when $|i - j| \in S$ and $u_i u_j = u_j u_i$ when $|i - j| \notin S$. Furthermore, there is a unital $*$-endomorphism θ of R such that $\theta(u_i) = u_{i+1}$, $i = 0, 1, 2, \ldots$. See [6] for all of this. The C^*-algebra A_n generated by $u_0, u_1, u_2, \ldots, u_n$ is finite dimensional, and hence $A = \bigcup_n A_n$ is a local C^*-algebra such that $\theta(A) \subseteq A$. Lemma 3.2 applies to $\theta : A \to A$ and by the same arguments as in 3.4 (for the case $\tau > \frac{1}{4}$) we get that $\hbar(\theta|_A) = \frac{1}{2}\log n$; the same value as Choda gets in [8] for the Connes-Størmer entropy with respect to the trace. Again there is also an automorphic version, see Example 3 of [8], giving the same value. □

Example 3.6. There is a canonical way to associate an AF-algebra to a given subshift of finite type (also called a topological Markov chain), containing the subshift as a canonical diagonal D, in such a way that the shift can be extended from the diagonal to a $*$-automorphism θ of the AF-algebra, see [9] and [10]. The union of the finite dimensional C^*-algebras defining the AF-algebra is a local C^*-algebra A left globally invariant by θ. Then $\hbar(\theta|_{A\cap D}) = \hbar(\theta|_A) = \log\lambda$, where λ is the spectral radius of the matrix defining the subshift. It follows that $\hbar(\theta|_B) = \log\lambda$ for every local C^*-algebra such that $A \cap D \subseteq B \subseteq A$. □

Concluding Comment 3.7. In all the examples considered above the $*$-endomorphisms are restrictions of $*$-endomorphisms of an enveloping C^*-algebra, in fact most of them extend to a canonical enveloping von Neumann algebra. It is an interesting question to decide if the values of the $*$-endomorphisms on the topological entropy of the C^*-algebra level (and maybe even on the von Neumann algebra level) remains the same as the ones obtained here. The continuity results of Sect. 2 show at least that the entropy on the C^*-level depends only on how the $*$-endomorphism acts on partitions in the finite-dimensional subalgebras. However, the results of Sect. 2 are not strong enough (or else the author is not strong enough) to decide if the values are no larger on the C^*-level. □

References

1. Blackadar, B.: K-theory for operator algebras. MSRI Publication Series 5. Berlin, Heidelberg, New York: Springer 1986
2. Connes, A.: Non commutative differential geometry Publ. Math. I.H.E.S. **62**, 257–360 (1986)
3. Connes, A.: Entropie de Kolmogroff Sinai et mechanique statistique quantique. C.R. Acad. Sci. Paris **301**, 1–6 (1985)
4. Connes, A., Størmer, E.: Entropy of automorphisms of II_1 von Neumann algebras. Acta Math. **134**, 289–306 (1975)
5. Connes, A., Narnhofer, H., Thirring, W.: Dynamical entropy of C^*-algebras and von Neumann algebras. Commun. Math. Phys. **112**, 691–719 (1987)
6. Choda, M.: Shifts on the hyperfinite II_1 factor. J. Operator Theory **17**, 223–235 (1987)
7. Choda, M.: Index for factors generated by Jones' two sided sequence of projections. Pacific J. Math. **139**, 1–16 (1989)
8. Choda, M.: Entropy for $*$-endomorphisms and relative entropy for subalgebras. J. Operator Theory **25**, 125–140 (1991)
9. Evans, D.: Entropy of automorphisms of AF algebras. Publ. RIMS, Kyoto Univ. **18**, 1045–1051 (1982)
10. Hudetz, T.: Quantenmechanische topologische Entropie. Thesis. Univ. Wien, 1992
11. Hudetz, T.: Quantum topological entropy: first steps of a "Pedestrian" approach. In: Quantum Probability VIII. Accardi, L., von Waldenfels, W. (eds.). Singapore: World Scientific (1992)
12. Jones, V.: Index for subfactors. Invent. Math. **72**, 1–25 (1983)
13. Pimsner, M., Popa, S.: Entropy and index for subfactors. Ann. Sci. École Norm. Sup. (4), **19**, 57–106 (1986)
14. Størmer, E.: Entropy in operator algebras. Department of Mathematics, University of Oslo, 1992

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