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On the Obstructions to Non-Cliffordian Pin Structures

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Abstract: We derive the topological obstructions to the existence of non-Cliffordian pin structures on four-dimensional spacetimes. We apply these obstructions to the study of non-Cliffordian pin-Lorentz cobordism. We note that our method of derivation applies equally well in any dimension and in any signature, and we present a general format for calculating obstructions in these situations. Finally, we interpret the breakdown of pin structure and discuss the relevance of this to aspects of physics.m

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I. Introduction

Suppose we are given a manifold, M, with tangent bundle τ_M which can be reduced to a bundle with structure group "O" say. Then one of the first things we might notice is that we generally have $\pi_1(O) \simeq G \not\simeq \{1\}$. What this means is that at a point $p \in M$ there exist paths O_1, $O_2 \in O$, which might act on the fibre $\tau_M|_p$ "equivalently" (in the sense that, for $x \in \tau_M|_p, O_1(x) = O_2(x)$), but with the property that

O_1 and O_2 (viewed as curves in O) are not homotopic, i.e., cannot be continuously deformed into each other. This might disturb us, and so we may be inclined to represent the information contained in the tangent bundle in a *simply connected* manner. What this amounts to locally (in a neighbourhood about p) is finding some bundle ς_M, with structure group $\bar{O}$ given by the exact sequence $1 \rightarrow \pi_1(O) \rightarrow \bar{O} \rightarrow O \rightarrow 1$. Then locally the bundle ς_M "encodes" all of the information that was contained in τ_M. However, we may not be able to find such a bundle *globally*, i.e., there are topological obstructions to globally "re-representing" the information of τ_M in a simply connected way.

In this paper, we are going to concentrate on *spacetimes*, M, which are not necessarily orientable. What this means is that the tangent bundle, τ_M, can at most be reduced to an $O(p, q)$ bundle. When the metric, g_{ab}, has signature $(- + + +)$ then the structure group will be $O(3, 1)$. When the metric has signature $(+ - - -)$ then the structure group will be $O(1, 3)$ (actually, $O(3, 1) \simeq O(1, 3)$, but as we shall see, it is necessary to keep the distinction when we pass to the double covers). Since $\pi_1(O_0(3, 1) \simeq \pi_1(O_0(1, 3)) \simeq \mathbb{Z}_2$, we are interested in finding all groups which are double covers of $O(3, 1)$ and $O(1, 3)$. However, there are *eight* distinct such double covers [2] of $O(p, q)$! Following Dabrowski, we will write these covers as

$$h^{a, b, c} : \mathrm{Pin}^{a, b, c}(p, q) \rightarrow O(p, q)$$

with $a, b, c \in \{+, -\}$. The signs of a, b, and c can be interpreted in the following way:

Recall, first, that $O(p, q)$ is not path connected; there are four components, given by the identity connected component, $O_0(p, q)$, and the three components corresponding to parity reversal P, time reversal T, and the combination of these two, PT(i.e., $O(p, q)$ decomposes into a semidirect product[1], $O(p, q) \simeq O_0(p, q) \odot (\mathbb{Z}_2 \times \mathbb{Z}_2)$). The signs of $a, b,$ and c then correspond to the signs of the squares of the elements in $\mathrm{Pin}^{a, b, c}(p, q)$ which cover space reflection, R_S, time reversal, R_T and a combination of the two respectively. (Recall that parity P is written $P = R_x R_y R_z$, the product of reflections about the three spacelike axes).

With this in mind we can, following Dabrowski [2], write out the explicit form of the groups $\mathrm{Pin}^{a, b, c}(p, q)$; they are given by the semidirect product

$$\mathrm{Pin}^{a, b, c}(p, q) \simeq \frac{(\mathrm{Spin}_0(p, q) \odot C^{a, b, c})}{\mathbb{Z}_2},$$

where the $C^{a, b, c}$ are the four double coverings of $\mathbb{Z}_2 \times \mathbb{Z}_2$; i.e., $C^{a, b, c}$ are the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (when $a = b = c = +$), D_4 (dihedral group, when there are two plusses and one minus in the triple a, b, c), $\mathbb{Z}_2 \times \mathbb{Z}_4$ (when there are two minuses and one plus in a, b, c), and Q_4 (quaternions, when $a = b = c = -$). Interestingly, the only groups which can be obtained from the Clifford algebras $Cl(p, q)$ (in the usual way) are

$$\mathrm{Pin}^{+, -, +}(p, q) \simeq \frac{(\mathrm{Spin}_0(p, q) \odot D_4)}{\mathbb{Z}_2}$$

and

[1] That is $O(p, q)$ is the disjoint union $O(p, q) = (O_0(p, q)) \cup P(O_0(p, q)) \cup T(O_0(p, q)) \cup PT(O_0(p, q))$, and the four element group $\{1, P, T, PT\}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\mathrm{Pin}^{-,+,+}(p,\,q) \simeq \frac{(\mathrm{Spin}_0(q,\,p) \odot D_4)}{\mathbb{Z}_2}\,.$$

These pin groups are therefore called "Cliffordian," and the obstruction theory for *Cliffordian* pin structure was worked out by Karoubi [3], see also [1]. We are concerned with the obstruction theory for the non-Cliffordian pin groups. To see how to approach this problem, let us first review the structures involved.

Recall, first of all, that $O(p,q)$ decomposes as a semidirect product $O(p,\,q) \simeq O_0(p,\,q) \odot (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Likewise, the pin groups decompose into semidirect products via $\mathrm{Pin}^{a,\,b,\,c}(p,\,q) \simeq \dfrac{(\mathrm{Spin}_0\, p,\,q \odot C^{a,\,b,\,c})}{\mathbb{Z}_2}$, where $\mathrm{Spin}_0(p,\,q)$ is the $2-1$ cover of $O_0(p,\,q) \simeq SO_0(p,\,q)$ and $C^{a,\,b,\,c}$ are $2-1$ covers of $\mathbb{Z}_2 \times \mathbb{Z}_2$. These semidirect products are naturally associated with the homomorphisms

$$\begin{cases} h_1 : \, C^{a,\,b,\,c} & \longrightarrow \quad \mathrm{Aut}\,(\mathrm{Spin}_0\,(p,\,q)) \\ h_2 : \, \mathbb{Z}_2 \times \mathbb{Z}_2 & \longrightarrow \quad \mathrm{Aut}\,(SO_0(p,\,q))\,, \end{cases}$$

i.e., for example, if e_1 represents time reflection in $\mathrm{Pin}^{a,b,c}(p,\,q)$, then $h(e_1)$ is equal to the map (automorphism) on $\mathrm{Spin}_0\,(p,\,q)$ given by conjugation:

$$\mathrm{Spin}_0\,(p,\,q) \ni v_1\,v_2 \ldots v_k \longrightarrow e_1 v_1\,v_2 \ldots v_k e_1^{-1}$$

and similarly for h_2. In other words, if $(\varsigma_1, c_1) \in \mathrm{Spin}_0\,(p,\,q) \odot C^{a,\,b,\,c}$ and $(\varsigma_2, c_2) \in \mathrm{Spin}_0(p,\,q) \odot C^{a,\,b,\,c}$, then multiplication of the two elements of the semidirect product is given by

$$(\varsigma_1,\, c_1)\,(\varsigma_2,\, c_1) = (\varsigma_1\, c_1 \varsigma_2\, c_1^{-1}, c_1\, c_2)$$

and so on.

What this means [4] is that we obtain exact sequences:

$$\begin{cases} 1 \longrightarrow \mathrm{Pin}_0^{a,\,b,\,c}\,(p,\,q) \longrightarrow \mathrm{Pin}^{a,\,b,\,c}\,(p,\,q) \longrightarrow C^{a,\,b,\,c} \longrightarrow 1 \\ 1 \longrightarrow O_0\,(p,\,q) \longrightarrow O\,(p,\,q) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1 \end{cases} \tag{1}$$

Furthermore, because the elements of the top sequence are $2-1$ covers of elements of the bottom sequence, we see that we must have the following (commutative) diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Pin}_0^{a,\,b,\,c}\,(p,\,q) & \longrightarrow & \mathrm{Pin}^{a,\,b,\,c}\,(p,\,q) & \longrightarrow & C^{a,\,b,\,c} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & O_0\,(p,\,q) & \longrightarrow & O\,(p,\,q) & \longrightarrow & \mathbb{Z}_2 \times \mathbb{Z}_2 & \longrightarrow & 1 \end{array} \tag{2}$$

Thus, diagram (2) "fixes" the structure of $\mathrm{Pin}^{a,b,c}(p,\,q)$, given $C^{a,b,c}$.

Including the short exact sequences which "express" the fact that $\mathrm{Pin}^{a,\,b,\,c}\,(p,\,q)$ and $\mathrm{Pin}_0^{a,\,b,\,c}\,(p,\,q)$ are $2-1$ covers of $O\,(p,\,q)$ and $O_0(p,\,q)$, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & | & & | & & | & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Pin}_0^{a,b,c}(p,q) & \longrightarrow & \mathrm{Pin}^{a,b,c}(p,q) & \longrightarrow & C^{a,b,c} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & O_0(p,q) & \longrightarrow & O(p,q) & \longrightarrow & \mathbb{Z}_2 \times \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & | & & | & & | & &
\end{array} \quad (3)$$

At first glance, the above diagram looks innocuous. However, as we shall see, when we view the algebraic structures in the diagram as *sheaves*, we will obtain a commutative diagram of sheaves, from which we will obtain a commutative diagram of sheaf cohomology groups, with which we will be able to derive our obstructions. Before we do this, however, it is useful to review sheaf cohomology.

II. Discussion of Sheaf[2] Theory

"Sheaf theory" is, broadly speaking, a mathematical technology that allows us to connect information which is local with information which is global. A *sheaf* is roughly something that tells us about localized information on M. To pass to global information, we need sheaf cohomology.

To make this more precise, let M be a topological space. Then *a presheaf* S over M is an assignment of a set $S(U)$ to every non-empty set $U \subset M$, such that for every pair of open sets $U_1 \subset U \subset M$ we have *restriction homomorphisms* $r^U_{U_1} : S(U) \longrightarrow S(U_1)$ which satisfy

(a) $r^U_U =$ "identity map on U,"
(b) For any open sets $U_2 \subset U_1 \subset U$, $r^U_{U_2} = r^{U_1}_{U_2} o\, r^U_{U_1}$.

Definition. *Let $\mathscr{A}$ and $\mathscr{B}$ be presheaves over M. Then we define a morphism of presheaves to be a set of mappings $f_u : \mathscr{A}(U) \longrightarrow \mathscr{B}(U)$, for each open set $U \subset M$, such that the diagram*

$$\begin{array}{ccc}
\mathscr{A}(U) & \longrightarrow & \mathscr{B}(U) \\
\downarrow r^U_{U_1} & & \downarrow r^U_{U_1} \\
\mathscr{A}(U_1) & \longrightarrow & \mathscr{B}(U_1)
\end{array}$$

is commutative, where $U_1 \subset U \subset M$, U_1 open. We write such a morphism as $f: \mathscr{A} \longrightarrow \mathscr{B}$.

Let $\{U_i\}$ be any collection of open subsets of M such that $U = \bigcup_i U_i$. A presheaf $\mathscr{A}$ is a *sheaf* iff it satisfies the following two "Sheaf Axioms":

Axiom 1. *If $a, b \in \mathscr{A}(U)$ and $\forall i, r^U_{U_i}(a) = r^U_{U_i}(b)$, then $a = b$.*

Axiom 2. *If for $a_i \in \mathscr{A}(U_i)$ and $U_i \cap U_k \neq \emptyset$ we have*

$$r^{U_i}_{U_i \cap U_k}(a_i) = r^{U_k}_{U_i \cap U_k}(a_k)$$

[2] This discussion is taken primarily from Wells [5], Chapter II.

for any i, then there exists $a \in \mathscr{A}(U)$ *such that* $r^U_{U_i}(a) = a_i, \forall\, i$.

Intuitively, Axiom 1 says that sheaves encode their information locally, whereas Axiom 2 says that we can "piece together" local information to get global information.

A mapping of sheaves, $\mathscr{A} \longrightarrow \mathscr{B}$, is a morphism of the underlying presheaves.

Now, there are many interesting examples of sheaves and their applications in geometry and mathematical physics, and we refer the reader to [5] and [6] for a thorough treatment. For our purposes, we shall be concerned with *constant* sheaves, i.e., sheaves which are simply the assignment $U \longrightarrow \mathscr{G}$ of some group $\mathscr{G}$ to any connected open set $U \subset M$.

Consider, now, the structure $\mathscr{A}_x$ obtained from a sheaf, $\mathscr{A}$ via

$$\mathscr{A}_x = \lim_{x \in U_i} \mathscr{A}(U_i)\,,$$

where "$\lim_{x\in U}$ refers to the direct limit of the restriction homomorphisms over nested neighbourhoods $U_1 \subset U_2 \subset \ldots \subset U_i \subset \ldots$ about x. The $\mathscr{A}_x$ is called the *stalk* of $\mathscr{A}$ at $x \in M$.

If $\mathscr{A}, \mathscr{B},$ and $\mathscr{C}$ are sheaves of groups on M, the sequence of morphisms

$$\mathscr{A} \xrightarrow{\mu} \mathscr{B} \xrightarrow{\nu} \mathscr{C}$$

is *exact* if the corresponding sequence on stalks

$$\mathscr{A}_x \xrightarrow{\mu_x} \mathscr{B}_x \xrightarrow{\nu_x} \mathscr{C}_x$$

is exact, $\forall x \in M$. A short exact sequence is a sequence of morphisms

$$1 \xrightarrow{f} \mathscr{A} \xrightarrow{g} \mathscr{B} \xrightarrow{h} \mathscr{C} \xrightarrow{j} 1 \tag{4}$$

with Im(f) =ker(g), Im(g) = Ker(h), Im(h) = ker(j). *Sheaf cohomology* is, roughly speaking, concerned with measuring "how exact" (4) is, i.e., to what extent Im(h)$\neq$ker(j). We now can develop sheaf cohomology theory [6] from the "Čech" point of view. The point now is that the coefficients for the cohomology will be sections of the sheaf, S, in question (i.e., sections are elements of $S(U)$). That is to say, we view (Čech) q-cochains as maps $C^q : U_0 \cap U_1 \cap \ldots \cap U_q \longrightarrow S(U_0 \cap U_1 \cap \ldots \cap U_q)$, where $U_0, U_1, \ldots U_q$ are $q+1$ open sets in M with non-empty intersection. We can define a coboundary operator, δ: $C^q \longrightarrow C^{q+1}$, in the usual way and so in an appropriate limit [6] we get the *sheaf cohomology groups of M with coefficients in S*:

$$H^*(M;\, S)\,.$$

Now, since our sheaves are all going to be constant, this cohomology will in fact reduce to the usual cohomology.

We now state the main result which we will need to calculate the obstructions in the next section:

Theorem [5]. *Let M be Hausdorff and paracompact*[3]*. Then*

(a) *For any sheaf* $\mathscr{A}$ *over M,*

[3] Recall that by Geroch [7] all spacetimes have these properties.

$$H^0(M;S) = \Gamma(M;S) = \text{“sections of } S \text{ over } M\text{”}.$$

(b) *For any sheaf morphism*

$$h\colon \mathscr{A} \longrightarrow \mathscr{B}$$

there is, for $q \geq 0$, *a group homomorphism*

$$h^q\colon H^q(M;\mathscr{A}) \longrightarrow H^q(M;\mathscr{B})$$

such that

(1) $h^0 = h_M\colon \mathscr{A}(M) \longrightarrow \mathscr{B}(M)$.
(2) $h^q =$ *identity if* $h =$ *identity*, $q \geqq 0$.
(3) $g^q \circ h^q = (g \circ h)^q, \forall q \geqq 0$, *if* $g : \mathscr{B} \longrightarrow \mathscr{C}$ *is another sheaf morphism.*

(c) *For each short exact sequence of sheaves*

$$1 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 1.$$

there is a group homomorphism $\delta^q : H^q(M;\mathscr{C}) \longrightarrow H^{q+1}(M;\mathscr{A}), \forall q \geqq 0$ *such that*

(1) *The induced sequence*

$$\begin{aligned}
1 &\longrightarrow H^0(M;\mathscr{A}) \longrightarrow H^0(M;\mathscr{B}) \\
&\longrightarrow H^0(M;\mathscr{C}) \xrightarrow{\delta^1} H^1(M;\mathscr{A}) \longrightarrow \ldots \\
&\longrightarrow H^q(M;\mathscr{A}) \longrightarrow H^q(M;\mathscr{B}) \longrightarrow H^q(M;\mathscr{C}) \\
&\xrightarrow{\delta^q} H^{q+1}(M;\mathscr{A}) \longrightarrow \ldots
\end{aligned}$$

is exact.

(2) *A commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathscr{A} & \longrightarrow & \mathscr{B} & \longrightarrow & \mathscr{C} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathscr{A}^1 & \longrightarrow & \mathscr{B}^1 & \longrightarrow & \mathscr{C}^1 & \longrightarrow & 1
\end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & H^0(M;\mathscr{A}) & \longrightarrow & H^0(M;\mathscr{B}) & \longrightarrow & H^0(M;\mathscr{C}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H^0(M;\mathscr{A}^1) & \longrightarrow & H^0(M;\mathscr{B}^1) & \longrightarrow & H^0(M;\mathscr{C}^1) \\
 & & & & & & \\
 & \longrightarrow & H^1(M;\mathscr{A}) & \longrightarrow & H^1(M;\mathscr{B}) & \longrightarrow & H^1(M;\mathscr{C}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & \longrightarrow & H^1(M;\mathscr{A}^1) & \longrightarrow & H^1(M;\mathscr{B}^1) & \longrightarrow & H^1(M;\mathscr{C}^1) \\
 & \longrightarrow & & & & & \\
 & \longrightarrow & \ldots & & & &
\end{array}$$

Proof. Wells [5], page 57.

The “connecting homomorphisms,” δ^q, are known as *Bockstein homomorphisms*, and will play a crucial role in our discussion in the next section.

III. Derivation of the Obstructions to Non-Cliffordian Pin Structures

First, let us adopt the shorthand $P = \mathrm{Pin}^{a,b,c}(p, q)$, $P_0 = \mathrm{Pin}_0^{a,b,c}(p, q)$, $O_0 = O_0(p, q), O = O(p, q), C = C^{a,b,c}$ in order to more efficiently describe the groups of Sect. I; associated to these groups are then constant sheaves $\mathscr{P}, \mathscr{P}_0, \mathscr{O}_0$, $\mathscr{O}$ and $\mathscr{C}$. Associated to diagram (3), then, is the following commutative diagram of sheaf morphisms:

$$\begin{array}{ccccccccc} & & | & & | & & | & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathscr{P}_0 & \longrightarrow & \mathscr{P} & \longrightarrow & \mathscr{C} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathscr{O}_0 & \longrightarrow & \mathscr{O} & \longrightarrow & \mathbb{Z}_2 \times \mathbb{Z}_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & | & & | & & | & & \end{array} \tag{5}$$

where the horizontal and vertical sequences are all exact. Combining diagram (5) with the above theorem, we obtain the following commutative diagram of sheaf cohomology groups:

$$\begin{array}{ccccccc} & & | & & | & & | \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^0(M;\,\mathbb{Z}_2) & & H^0(M;\,\mathbb{Z}_2) & & H^0(M;\,\mathbb{Z}_2) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \to & H^0(M;\,\mathscr{P}_0) & \to & H^0(M;\,\mathscr{P}) & \to & H^0(M;\,\mathscr{C}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \to & H^0(M;\,\mathscr{O}_0) & \to & H^0(M;\,\mathscr{O}) & \to & H^0(M;\,\mathbb{Z}_2 \times \mathbb{Z}_2) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^1(M;\,\mathbb{Z}_2) & & H^1(M;\,\mathbb{Z}_2) & & H^1(M;\,\mathbb{Z}_2) \\ & & \downarrow & & \downarrow & & \downarrow f \\ & \to & H^1(M;\,\mathscr{P}_0) & \to & H^1(M;\,\mathscr{P}) & \underset{\tilde{p}}{\to} & H^1(M;\,\mathscr{C}) \\ & & \downarrow & & \alpha \downarrow & & \downarrow \beta \\ & \to & H^1(M;\,\mathscr{O}_0) & \to & H^1(M;\,\mathscr{O}) & \underset{p}{\to} & H^1(M;\,\mathbb{Z}_2 \times \mathbb{Z}_2) \\ & & \downarrow & & \delta_0^2 \downarrow & & \delta_2 \downarrow \\ & & H^2(M;\,\mathbb{Z}_2) & & H^2(M;\,\mathbb{Z}_2) & & H^2(M;\,\mathbb{Z}_2) \end{array} \tag{6}$$

We are interested in the bottom part of this diagram (where we have labelled the maps between cohomology groups). Recalling that the vertical sequences in this diagram are exact, the derivation of the obstructions proceeds as follows.

Let $\xi \in H^1(M;\mathscr{O})$, i.e., ξ is a principal $O(p,q)$-bundle over M. We are concerned with the obstruction to the existence of a principal $\mathrm{Pin}(p, q)$ bundle, $\tilde{\xi} \in H^1(M;\mathscr{P})$, over ξ.

Thus, suppose that such a $\mathrm{Pin}(p, q)$ bundle, $\tilde{\xi}$, exists. Then $\alpha(\tilde{\xi}) \in H^1(M;\mathscr{O})$, and so by exactness

$$\delta_0^2(\alpha\,(\tilde{\xi}) = 0\,.$$

That is, if $\alpha(\tilde{\xi}) = \xi$, then we must have that $H^2(M;\mathbb{Z}_2) \ni \delta_0^2(\xi) = 0$. Likewise, if $\delta_0^2(\xi) = 0$, then such a $\tilde{\xi} \in H^1(M;\mathscr{P})$ exists, and so we see that the obstruction to the existence of a Pin(p, q) bundle $\tilde{\xi}$ is the vanishing of the class $\delta_0^2(\xi) = 0 \in H^2(M; \mathbb{Z}_2)$ (here we are regarding $\mathbb{Z}_2$ additively, i.e., $\mathbb{Z}_2 = \{0, 1\}$).

The point now is that we can "transfer" the above argument over to the vertical exact sequence on the far right in diagram (6). In other words, if $\tilde{\xi} \in H^1(M; \mathscr{P})$ exists over M, then by the commutativity of (6),

$$\beta(\tilde{p}(\tilde{\xi})) = p(\alpha(\tilde{\xi})) \in H^1(M; \mathbb{Z}_2 \times \mathbb{Z}_2)\,,$$

and so the obstruction is now

$$\delta_2(\beta(\tilde{p}(\tilde{\xi}))) = \delta_2(p(\alpha(\tilde{\xi}))) \in H^2(M; \mathbb{Z}_2)\,.$$

Now, by Milnor and Stasheff [8] the general form of this obstruction must be

$$H_2(M; \mathbb{Z}_2) \ni w_2(\tau_M) + w_1(\tau_M) \smile w_1(\tau_M)\,, \tag{7}$$

where $w_1(\tau_M)$ and $w_2(\tau_M)$ are the first and second Stiefel–Whitney classes of τ_M, respectively.

Decomposing the tangent bundle τ_M as

$$\tau_M \;\simeq\; \tau^+ \oplus \tau^-$$

(where the "plus" and "minus" signs of the subbundles refer to the behaviour of sections of these bundles with respect to the Lorentz metric) we obtain

$$w_1\,(\tau_M) = w_1(\tau^+ \oplus\, \tau^-) = w_1\,(\tau^+) + w_1\,(\tau^-) \tag{8}$$

and

$$w_2\,(\tau_M) = w_2(\tau^+) +\; w_2(\tau^-) + w_1\,(\tau^+) \smile w_1\,(\tau^-)\,. \tag{9}$$

Combining (8) and (9), and adopting the conventions $w_1(\tau^+) = w_1^+, w_1(\tau^-) = w_1^-$, $w_2(\tau^+) = w_2^+$, $w_2(\tau^-) = w_2^-$ we see that the obstruction must have the general form

$$aw_2^+ + bw_2^- + cw_1^- \smile w_1^- + dw_1^+ \smile w_1^- + ew_1^+ \smile w_1^+ = \delta_2(p(p(\xi)) \in H^2(M; \mathbb{Z}_2)\,, \tag{10}$$

where a, b, c, d, $e \in \mathbb{Z}_2$ are constants yet to be determined. Clearly, then, the determination of a, b, c, d, and e depends upon the nature of the double cover given by the exact sequence

$$1 \;\longrightarrow\; \mathbb{Z}_2 \;\longrightarrow\; C^{a,\,b,\,c} \;\longrightarrow\; \mathbb{Z}_2 \times \mathbb{Z}_2 \;\longrightarrow\; 1\,, \tag{11}$$

that is to say, the values of a, b, c, d, $e \in \mathbb{Z}_2$ depend upon the choice of $C^{a,\,b,\,c}$. We will treat each of these choices in turn. First, however, we need to understand the "Bockstein" homomorphism $\delta_2 : H^1(M; \mathbb{Z}_2 \times \mathbb{Z}_2) \longrightarrow H^2(M; \mathbb{Z}_2)$:

To begin, recall the interpretation of w_1^+ and w_1^-:

Once we have decomposed the tangent bundle τ_M via $\tau_M \simeq \tau^+ \oplus \tau^-$, we have the notions of *time-orientability* and *space-orientability* [9]. Then w_1^+ and w_1^- are cohomological data which tell us about the orientation of M. For example, if the signature is $(-+++)$ then (i) $w_1^- = 0 \Longleftrightarrow M$ is time-orientable, and (ii) $w_1^+ = 0 \Longleftrightarrow M$ is space-orientable.

More formally, what this means is that w_1^+ and w_1^- define a $\mathbb{Z}_2 \times \mathbb{Z}_2$ – valued Čech 1-cochain,

$$(w_1^+, w_1^-): \ U_a \cap U_b \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$$

(where U_a and U_b are two non-empty open sets in some arbitrary simple cover of M). In other words, $(w_1^+, w_1^-) \in H^1(M; \mathbb{Z}_2 \times \mathbb{Z}_2)$. We therefore expect the Bockstein homomorphism, δ_2, to relate the elements $(w_1^+, w_1^-) \in H^1\,(M; \mathbb{Z}_2 \times \mathbb{Z}_2)$ to the elements in $w_1^+ \smile w_1^+, w_1^- \smile w_1^-$, etc. in $H^2\,(M; \mathbb{Z}_2)$. To see how this occurs, recall the formal definition of δ_2 [5].

First, consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C^2(\mathbb{Z}_2) & \xrightarrow{f^1} & C^2(\mathscr{C}) & \xrightarrow{g^1} & C^2(\mathbb{Z}_2 \times \mathbb{Z}_2) & \longrightarrow 1 \\ & & \uparrow\alpha & & \uparrow\beta & & \uparrow\gamma & \\ 1 & \longrightarrow & C^1(\mathbb{Z}_2) & \xrightarrow{f} & C^1(\mathscr{C}) & \xrightarrow{g} & C^1(\mathbb{Z}_2 \times \mathbb{Z}_2) & \longrightarrow 1 \end{array} , \qquad (12)$$

where $C^n(\mathscr{A})$ is the set of n-cochains with coefficients in $\mathscr{A}$.

Let $c \in \ker(\gamma)$. Then $c{=}g(c^1)$, for some $c^1 \in C^1(\mathscr{C})$, by exactness (c gives us a cohomology class in $C^1(\mathbb{Z}_2 \times \mathbb{Z}_2)$). By commutativity, we get $g^1(\beta(c^1)) = \gamma(g(c^1)) = 1$, and so $\beta(c^1) = f^1(a)$, for some $a \in C^2(\mathbb{Z}_2)$. We then get an induced mapping

$$\delta\colon \ H^1(M; \mathbb{Z}_2 \times \mathbb{Z}_2) \longrightarrow H^2(M; \mathbb{Z}_2)$$

from the map given above,

$$C^1(\mathbb{Z}_2 \times \mathbb{Z}_2) \ni c \longrightarrow (f^1)^{-1} \, o \, \beta(g^{-1}(c)) = a \in C^2(\mathbb{Z}_2)\,.$$

Since the homomorphism inducing g depends on the choice of $C^{a,b,c}$, we see that δ_2 depends upon our choice of $C^{a,b,c}$.

In fact, the above construction shows us how to calculate the images of $(w_1^+, 0)$, $(0, w_1^-) \in H^1\,(M; \mathbb{Z}_2 \times \mathbb{Z}_2)$ in $H^2\,(M; \mathbb{Z}_2)$ under δ_2. For example, if we take the signature to be $(-+++)$ then $(w_1^+, 0)$ and $(0, w_1^-)$ are related to the transformations $(R_S, 0)$and $(0, R_T)$ in the obvious way, i.e., $(w_1^+, 0)$ tells us whether or not we can continuously distinguish between systems under the operation $(R_S, 0)$, and likewise for time reversal. Now, the elements $(R_S, 0)$ and $(0, R_T)$ are double covered by elements $\pm\tilde{R}_S$ and $\pm\tilde{R}_T$ (respectively) in $C^{a,b,c}$. Corresponding to the way the elements $(R_S, 0)$, $(0, R_T) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ are covered by elements in $C^{a,b,c}$, there is also a "lifting" of the elements $(w_1^+, 0)$, $(0, w_1^-) \in H^1\,(M; \mathbb{Z}_2 \times \mathbb{Z}_2)$ to elements $\pm\tilde{w}_+$, $\pm\tilde{w}_- \in H^1\,(M; \mathscr{C})$ (corresponding to the map g^{-1} in (12) above). Next, we apply the Steenrod square operation Sq^1 (corresponding to the map β in (12)), i.e.,

$$Sq^1(\tilde{w}_1^\pm) = \tilde{w}_1^\pm \smile \tilde{w}_1^\pm \in C^2\,(\mathscr{C})\,.$$

Finally, we pull the elements $Sq^1(\tilde{w}\pm)$ to elements $w_1^\pm \smile w_1^\pm \in H^2\,(M; \mathbb{Z}_2)$ (corresponding to the map $(f^1)^{-1}$). The point is, when we pulled back $(w_1^+, 0)$ (say) to $\tilde{w}_1^+ \in C^1\,(\mathscr{C})$, we did so in a way compatible with the homomorphism $C^{a,b,c} \xrightarrow{f^*} \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e., if the 1-cycle, c_1, dual to $(w_1^+, 0)$ satisfies $\langle (w_1^+, 0), c_1 \rangle = a$, then the 1-cycle, c_1, dual to $\tilde{w}_1^+$ must satisfy $\langle \tilde{w}_1^+, c' \rangle = \tilde{a}$, where $\pm\tilde{a}$ covers a under the homomorphism f^*. When we then apply Sq^1 to $\tilde{w}_1^+$ we obtain $\tilde{w}_1^+ \smile \tilde{w}_1^+$, with the property that for some 2-cycle, c_2, dual to $\tilde{w}_1^+ \smile \tilde{w}_1^+$ we have $\langle \tilde{w}_1^+ \smile \tilde{w}_1^+, c_2\rangle = \langle \tilde{w}_1^+$, "front 1-face of c_2" $\rangle \cdot \langle \tilde{w}_1^+$, "back 1-face of c_2"$\rangle = \tilde{a}^2$. In other words, the pull back of $\tilde{w}_1^+ \smile \tilde{w}_1^+ \in C^2(\mathscr{C})$ to $\delta_2\,(w_1^+, 0) \in H^2\,(M;\, \mathbb{Z}_2)$ will depend upon whether or not $\tilde{a}^2 \in C^{a,b,c}$ pulls back to 0 or 1 in the group $\mathbb{Z}_2$ under the homomorphism f^*. If $\tilde{a}^2$ pulls back to 0, then $\delta_2(w_1^+)$=0. Otherwise, e=1.

Furthermore, we see from the above construction that the class $w_2(\tau_M) = w_2^+ + w_2^- + w_1^+ \smile w_1^-$ is unaffected by the choice of $C^{a,b,c}$, i.e., we always have $a = b = d = 1$.

We now proceed with a case by case analysis.

$\underline{C^{a,b,c} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2}$. Recall that taking $C^{a,b,c} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is equivalent (in Dabrowski's notation) to considering the groups $\mathrm{Pin}^{+,+,+}\,(p,q)$. We are then concerned with seeing how $(w_1^+,\, w_1^-) \in H^1\,(M;\, \mathbb{Z}_2 \times \mathbb{Z}_2)$ "pulls back" under the sequence of homomorphisms

$$H^1\,(M;\, \mathbb{Z}_2) \xrightarrow{f} H^1\,(M;\, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \xrightarrow{g} H^1\,(M;\, \mathbb{Z}_2 \times \mathbb{Z}_2)$$

induced by the exact sequence of homomorphisms $\mathbb{Z}_2 \xrightarrow{f*} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{g*} \mathbb{Z}_2 \times \mathbb{Z}_2$. Recall, however, that the homomorphisms $f*$ and $g*$ can be given explicitly as shown here in this example (signature $(-+++)$):

$$\begin{array}{ccccc}
\underline{\mathbb{Z}_2} & \xrightarrow{f*} & \underline{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2} & \xrightarrow{\beta*} & \underline{\mathbb{Z}_2 \times \mathbb{Z}_2} \\
\begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} \longrightarrow \\ \longrightarrow \end{matrix} & \left.\begin{matrix} (0,\,0,\,0) \\ (0,\,0,\,1) \end{matrix}\right\} & \longrightarrow & (0,\,0) \\
b=+ & & \left.\begin{matrix} (0,\,1,\,0) \\ (0,\,1,\,1) \end{matrix}\right\} & \longrightarrow & (0,\,R_T) \\
a=+ & & \left.\begin{matrix} (1,\,0,\,0) \\ (1,\,0,\,1) \end{matrix}\right\} & \longrightarrow & (R_S,\,0) \\
c=+ & & \left.\begin{matrix} (1,\,1,\,0) \\ (1,\,1,\,1) \end{matrix}\right\} & \longrightarrow & (R_S,\,R_T)\ .
\end{array}$$

Now, since the squares of all the elements covering $(R_S,\, 0)$, $(0,\, R_T)$, and $(R_S,\, R_T)$ are always $(0,\, 0,\, 0)$= "the identity in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$", we see that we can always pull back the elements $(w_1^+,\, 0)$, $(0,\, w_1^-) \in H^1\,(M;\, \mathbb{Z}_2 \times \mathbb{Z}_2)$ to elements $\tilde{w}_1^+,\, \tilde{w}_1^- \in H^1\,(M;\, \mathscr{C})$ with the property that $\tilde{w}_1^+ \smile \tilde{w}_1^+$, $\tilde{w}_1^- \smile \tilde{w}_1^- \in H^2\,(M;\, \mathscr{C})$ are both zero cocycles. Thus, pulling these cocycles back under f (induced by f^* given above) we get

$$\delta_2(w_1^+,\, 0) = \delta_2(0,\, w_1^-) = 0 \in H^2\,(M;\, \mathbb{Z}_2)\,.$$

In other words, $c = e$ =0, and so the information contained in $(w_1^+,\, 0)$ and $(0,\, w_1^-)$ is not relevant to the obstruction class in this situation. Thus, we have shown

Theorem 1. *Let M be a spacetime with tangent bundle τ_M an $O(p,q)$ bundle. Then M admits* $\mathrm{Pin}^{+,+,+}$ *(p,q) structure if and only if*

$$w_2^+ + w_2^- + w_1^- \smile w_1^+ = 0\,,$$

where $w_2^\pm$, $w_1^\pm$ are defined as above.

$\underline{C^{a,b,c} \simeq D_4}$. Recall that taking $C^{a,b\,c} \simeq D_4$ yields the *Cliffordian* pin groups $\mathrm{Pin}^{+,-,+}(p, q)$ and $\mathrm{Pin}^{-,++}(p, q)$. Although the obstructions to these structures have been worked out [3], we present our approach here for completeness.

Thus, recall that D_4 can be regarded as a semidirect product, $D_4 \simeq \mathbb{Z}_4 \odot \mathbb{Z}_2$, where $\mathbb{Z}_4 \subset D_4$ is a normal subgroup, i.e., elements (a_1, b_1), $(a_2, b_2) \in D_4 \simeq \mathbb{Z}_4 \odot \mathbb{Z}_2$ multiply according to $(a_1, b_1)\cdot(a_2, b_2) = (a_1 b_1 a_2 b_1^{-1}, b_1 b_2)$. If we regard a as the generator of the "$\mathbb{Z}_4$ part" ($a^4 = 0$) and b as the generator of the "$\mathbb{Z}_2$ part" ($b^2 = 0$), then what this means is that there are two different cases, corresponding to either the groups $\mathrm{Pin}^{+,-,+}(1, 3)$ and $\mathrm{Pin}^{-,+,+}(3, 1)$ or the groups $\mathrm{Pin}^{+,-,+}(3, 1)$ and $\mathrm{Pin}^{-,+,+}(1, 3)$. For the group $\mathrm{Pin}^{+,-,+}(1, 3)$ we get the sequence of homomorphisms

$$
\begin{array}{ccccc}
\underline{\mathbb{Z}_2} & \xrightarrow{f*} & \underline{D_4} & \xrightarrow{\beta*} & \underline{\mathbb{Z}_2 \times \mathbb{Z}_2} \\
\begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} \longrightarrow \\ \longrightarrow \end{matrix} & \left.\begin{matrix} (0, 0) \\ (a^2, 0) \end{matrix}\right\} & \longrightarrow & (0, 0) \\
a = + & & \left.\begin{matrix} (a, b) \\ (a^3, b) \end{matrix}\right\} & \longrightarrow & (0, R_S) \\
b = - & & \left.\begin{matrix} (a, 0) \\ (a^3, 0) \end{matrix}\right\} & \longrightarrow & (R_T, 0) \\
c = + & & \left.\begin{matrix} (0, b) \\ (a^2, b) \end{matrix}\right\} & \longrightarrow & (R_T, R_S) \, .
\end{array}
$$

Now, note the elements covering $(R_T, 0)$, $(a, 0)$ and $(a^3, 0)$, both satisfy $(a, 0)\cdot(a, 0) = a^2 = (a^3, 0)\cdot(a^3, 0)$, i.e., their squares are not equal to the identity element $(0, 0) \in D_4$. It follows that $(w_1^+, 0)$ pulls back to $w_1^+ \smile w_1^+$, i.e., $\delta_2(w_1^+, 0) = w_1^+ \smile w_1^+$ and so $e = 1$, $c = 0$. Thus, we have shown

Theorem 2. *Let M be a spacetime with tangent bundle τ_M either an $O(3, 1)$ bundle or an $O(1,3)$ bundle; then M admits either* $\mathrm{Pin}^{-,+,+}(3, 1)$ *or* $\mathrm{Pin}^{+,-,+}(1, 3)$ *structure (respectively) if and only if*

$$w_2^+ + w_2^- + w_1^+ \smile w_1^- + w_1^+ \smile w_1^+ = 0 \, .$$

When we consider the sequence of homomorphisms corresponding to the groups $\mathrm{Pin}^{+,-,+}(3, 1)$ and $\mathrm{Pin}^{-,+,+}(1, 3)$, we see that now it is $(0, w_1^-)$ that pulls back, and so $c = 1$, $e = 0$.

Theorem 3. *Let M be a spacetime with tangent bundle τ_M either an $O(3, 1)$ bundle or an $O(1,3)$ bundle; then M admits either* $\mathrm{Pin}^{+,-,+}(3, 1)$ *or* $\mathrm{Pin}^{-,+,+}(1, 3)$ *structure (respectively) if and only if*

$$w_2^+ + w_2^- + w_1^+ \smile w_1^- + w_1^- \smile w_1^- = 0 \, .$$

$\underline{C^{a,b,c} \simeq \mathbb{Z}_2 \times \mathbb{Z}_4}$. Recall that taking $C^{a,b,c} \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$ corresponds to considering the groups $\mathrm{Pin}^{a,b,c}(p, q)$, with two minuses and one plus occurring in the triple a, b, c. Now, we can as usual regard $\mathbb{Z}_2 \times \mathbb{Z}_4$ as the group given abstractly as

$$\mathbb{Z}_4 \times \mathbb{Z}_2 \simeq \{(a, b) \mid a^4 = b^2 = 1\} \, .$$

This means that the homomorphisms associated with the exact sequence (11) are given, for the group $\mathrm{Pin}^{-,-,+}(3, 1)$:

$$\begin{array}{cccccc}
\underline{\mathbb{Z}_2} & \xrightarrow{f*} & \underline{\mathbb{Z}_4 \times \mathbb{Z}_2} & \xrightarrow{\beta*} & \underline{\mathbb{Z}_2 \times \mathbb{Z}_2} \\
0 & \longrightarrow & \left.\begin{array}{c}(0,\,0)\\(a^2,\,0)\end{array}\right\} & \longrightarrow & (0,\,0) \\
1 & \longrightarrow & & & \\
a = - & & \left.\begin{array}{c}(a,\,b)\\(a^3,\,b)\end{array}\right\} & \longrightarrow & (R_S,\,0) \\
b = - & & \left.\begin{array}{c}(a,\,0)\\(a^3,\,0)\end{array}\right\} & \longrightarrow & (0,\,R_T) \\
c = + & & \left.\begin{array}{c}(0,\,b)\\(a^2,\,b)\end{array}\right\} & \longrightarrow & (R_S,\,R_T)\ .
\end{array}$$

It follows that both $(w_1^+,\,0)$ and $(0,\,w_1^-)$ pull back, and so $c = e = 1$. Furthermore, this same result clearly holds for the group $\mathrm{Pin}^{-,-,+}$ (1, 3). Thus, we have shown

Theorem 4. *Let M be a spacetime with tangent bundle τ_M an $O(p,q)$ bundle; then M admits* $\mathrm{Pin}^{-,-,+}$ (p,q) *if and only if*

$$w_2^+ + w_2^- + w_1^+ \smile w_1^- + w_1^+ \smile w_1^+ + w_1^- \smile w_1^- = 0\,.$$

For the groups $\mathrm{Pin}^{+,-,-}$ (3, 1) and $\mathrm{Pin}^{-,+,-}$ (1, 3), we see that only $(0,\,w_1^-)$ pulls back, hence

Theorem 5. *Let M be a spacetime with tangent bundle τ_M either an $O(3,\,1)$ bundle or an $O(1,3)$ bundle; then M admits either* $\mathrm{Pin}^{+,-,-}$ (3, 1) *or* $\mathrm{Pin}^{-,+,-}$ (1, 3) *structure (respectively) if and only if*

$$w_2^+ + w_2^- + w_1^+ \smile w_1^- + w_1^- \smile w_1^- = 0\,.$$

Finally, for the remaining cases we obtain

Theorem 6. *Let M be a spacetime with tangent bundle τ_M either an $O(3,\,1)$ bundle or an $O(1,3)$ bundle; then M admits either* $\mathrm{Pin}^{-,+,-}$ (3, 1) *or* $\mathrm{Pin}^{+,-,-}$ (1, 3) *structure (respectively) if and only if*

$$w_2^+ + w_2^- + w_1^+ \smile w_1^- + w_1^+ \smile w_1^+ = 0\,.$$

$\underline{C^{a,b,c} \simeq Q_4}$. Recall that taking $C^{a,b,c} \simeq Q_4$ is equivalent to considering the groups $\mathrm{Pin}^{-,-,-}$ $(p,\,q)$. Clearly then, both $(w_1^+,\,0)$ and $(0,\,w_1^-)$ always pull back. Thus,

Theorem 7. *Let M be a spacetime with tangent bundle τ_M an $O(p,q)$ bundle; then M admits* $\mathrm{Pin}^{-,-,-}$ (p,q) *structure if and only if*

$$w_1^- \smile w_1^- + w_1^+ \smile w_1^+ + w_2^+ + w_2^- + w_1^+ \smile w_1^- = 0\,.$$

IV. Applications of the Obstructions to Pin–Lorentz Cobordism

In this section, we use the obstructions developed above in Sect. III to derive the obstructions to pin-Lorentz cobordism. First, however, we review some elementary concepts from differential topology.

Now, recall that the existence of an everywhere non-singular Lorentz metric on M is equivalent to the existence of a global non-vanishing (smooth) *line field,*

$\{v, -v\}$, on M (when M is time-orientable, it suffices that M possess a global non-vanishing vector field v). The vectors $\pm v$ then have the usual interpretation as timelike vectors (see [9]).

Recall also the notion of *kink number*: Let $\Sigma \subset M$ be a three-dimensional, connected submanifold. Since $\dim(\Sigma) = 3$, we can always find a global framing $\{u_i\colon i = 1, 2, 3\}$ of Σ. Furthermore, even if M is not orientable we can always find a unit line field $\{n, -n\}$ which is normal to Σ (note that n has unit length with respect to the underlying Riemannian metric on M, g^R_{ab}, i.e., $g^R_{ab}u^a u^b = 1$). We can extend this tetrad framing (n, u_i) of Σ to a collar neighbourhood

$$N \cong \Sigma \times [0, 1]$$

(we extend to N to deal with the case $\Sigma \cong \partial M$). Let v be the timelike vector (line field) determined by g_{ab}. The v can be written as

$$v = v^0 n + v^i u_i ,$$

such that $\sum_i (v^i)^2 = 1$. Clearly, the v determines a map

$$K\colon \Sigma \longrightarrow \begin{cases} S^3, & \text{if M is time-orientable} \\ \mathbb{RP}^3, & \text{if } M \text{ } is \text{ } \underline{\text{not}} \text{ time-orientable} \end{cases}$$

by assigning to each point $p \in \Sigma$ the direction in T_pM (a point on the S^3 or $\mathbb{RP}^3$ determined by the tetrad (n, u_i)) that v_p points to. We then define the kink number of g_{ab} with respect to Σ by the formula

$$\operatorname{kink}(\Sigma; g_{ab}) = \deg(K) ,$$

where deg (K) is "the degree of the mapping K." If v is a timelike vector determined by g_{ab}, we shall often write

$$\operatorname{kink}(\Sigma; g_{ab}) = \operatorname{kink}(\Sigma; v) .$$

For our immediate purposes we shall be concerned with kinking with respect to ∂M, the boundary of our spacetime. In particular, we shall be concerned with the case M compact, with $\partial M \cong \Sigma_0 \cup \Sigma_1 \cup \ldots \cup \Sigma_n$, where the Σ_i's are closed, connected three-manifolds and "$\cup$" is the operation of disjoint union. We wish to define the quantity $\operatorname{kink}(\partial M; g_{ab}) = \operatorname{kink}(\Sigma_0 \cup \Sigma_1 \cup \ldots \cup \Sigma_n; g_{ab})$. On differential topological grounds (see [10]) we see that it makes sense to write

$$\operatorname{kink}(\partial M; g_{ab}) = \sum_i \operatorname{kink}(\Sigma_i; g_{ab}) .$$

Now suppose v is a smooth vector field on M which vanishes on some discrete set of points $p_1, p_2, \ldots p_n \in M$. Associated to each of these vanishing points p_i is the *index of v at p_i*, which is precisely the degree of mapping given by $\frac{v(x)}{\|v(x)\|}$, which takes a little sphere $s(p_i)$ about p_i into the unit sphere. We write "$\sum i_v$" to mean "the sum of the indices of v." We then have the following formula [10]:

$$\sum i_v = e(M) + \operatorname{kink}(\partial M; v) ,$$

where $e(M)$ is the Euler number of M and $\operatorname{kink}(\partial M;\ v)$ is as above. In particular, if M is a spacetime then the timelike line field $\{v, -v\}$ is non-vanishing and so $\sum i_v = 0$, hence,

$$e(M) = -\operatorname{kink}(\partial M;\ g_{ab})\,. \tag{13}$$

Now, a direct application of Wu's formula ([1] or [8]) shows the following identity: For any $x_2 \in H^2(M;\ \mathbb{Z}_2)$,

$$w_2(M) \smile x_2 - (w_1(M) \smile w_1(M)) \smile x_2 = x_2 \smile x_2\,. \tag{14}$$

Writing the intersection pairing as $h: H_2(M;\ \mathbb{Z}_2) \times H_2(M;\ \mathbb{Z}_2) \to \mathbb{Z}_2$ (defined explicitly via $h(x, y) = x \cdot y = (x_2 \smile y_2) \frown w$, where $x_2, y_2 \in H^2(M;\ \mathbb{Z}_2)$ satisfy $x_2 \frown w = x$ and $y_2 \frown w = y$, where $w \in H_4(M;\ \mathbb{Z}_2)$ is the fundamental homology class) we recall the important

Lemma (Milnor and Kervaire, [11], page 517). *Let M be a smooth manifold of dimension 4. Let $u(\partial M)$ (the* mod 2 *Kervaire semicharacteristic) be given by*

$$u(\partial M) = \dim_{\mathbb{Z}_2}(H_0(\partial M;\ \mathbb{Z}_2) \oplus H_1(\partial M;\ \mathbb{Z}_2)) \bmod 2\,.$$

Then the rank of the intersection pairing, h, satisfies

$$\operatorname{rank}(h) = (u(\partial M) + e(M)) \bmod 2\,.$$

Note. Actually, our version of the above lemma differs slightly from that in [11] in that we allow M to be non-orientable. However, the lemma is still true since Poincaré–Lefshetz duality still holds in $\mathbb{Z}_2$ coefficients for non-orientable M.

From the definition of h and Eq. (14) it follows immediately that $\operatorname{rank}(h)=0$ if and only if $w_2 + w_1 \smile w_1 = 0$. If M is a spacetime, then the lemma together with Eq. (13) then give us

Lemma 1. *Let M be a spacetime with tangent bundle τ_M. Then*

$$w_2(\tau_M) + w_1(\tau_M) \smile w_1(\tau_M) = 0 \Longleftrightarrow$$

$$(u(\partial M) + \operatorname{kink}(\partial M;\ g_{ab})) \bmod 2 = 0\,.$$

Combining Lemma 1 with Eqs. (8) and (9) and the above set of theorems, we obtain the following:

Definition. *Let $\Sigma_1, \Sigma_2 \ldots \Sigma_n$ be a collection of closed three-manifolds. Then we say that there exists a* $\mathrm{Pin}^{a,b,c}$ *(p, q) cobordism for $\{\Sigma_i : i = 1, \ldots n\}$ if and only if there exists a spacetime M admitting* $\mathrm{Pin}^{a,b,c}$ *(p, q) structure and satisfying*

$$\partial M \cong \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_n\,.$$

In the below Corollaries, $\{\Sigma_i : i = 1, \ldots n\}$ always denotes some collection of closed three-manifolds.

Corollary 1. *There exists a* $\mathrm{Pin}^{+,+,+}$ *(p, q) cobordism, M, for $\{\Sigma_i : i = 1, \ldots n\}$ if and only if the following holds:*

$$(u(\partial M) + \operatorname{kink}(\partial M;\, g_{ab})) \bmod 2 = 0 \Longleftrightarrow$$

$$w_1^+ \smile w_1^+ + w_1^- \smile w_1^- = 0$$

Corollary 2. *There exists either a* $\mathrm{Pin}^{-,+,+}$ (3, 1) *or a* $\mathrm{Pin}^{+,-,+}$ (1, 3) *cobordism* M *for* $\{\Sigma_i : i = 1, \dots n\}$ *if and only if the following holds*:

$$(u(\partial M) + \operatorname{kink}(\partial M;\, g_{ab})) \bmod 2 = 0 \Longleftrightarrow w_1^- \smile w_1^- = 0\,.$$

Corollary 3. *There exists either a* $\mathrm{Pin}^{+,-,+}$ (3, 1) *or a* $\mathrm{Pin}^{-,+,+}$ (1, 3) *cobordism* M *for* $\{\Sigma_i$: i=1, ... $n\}$ *if and only if the following holds*:

$$(u(\partial M) + \operatorname{kink}(\partial M; g_{ab})) \bmod 2 = 0 \Longleftrightarrow w_1^+ \smile w_1^+ = 0\,.$$

Corollary 4. *There exists a* $\mathrm{Pin}^{-,-,+}$ (p,q) *cobordism* M *for* $\{\Sigma_i$: i=1, ... $n\}$ *if and only if*

$$(u(\partial M) + \operatorname{kink}(\partial M; g_{ab})) \bmod 2 = 0\,.$$

Corollary 5. *There exists either a* $\mathrm{Pin}^{+,-,-}$ (3, 1) *or a* $\mathrm{Pin}^{-,+,-}$ (1, 3) *cobordism* M *for* $\{\Sigma_i$: i=1, ... $n\}$ *if and only if the following holds*:

$$(u(\partial M) + \operatorname{kink}(\partial M; g_{ab})) \bmod 2 = 0 \Longleftrightarrow w_1^+ \smile w_1^+ = 0\,.$$

Corollary 6. *There exists either a* $\mathrm{Pin}^{-,+,-}$ (3, 1) *or a* $\mathrm{Pin}^{+,-,-}$ (1,3) *cobordism* M *for* $\{\Sigma_i$: i=1, ... $n\}$ *if and only if the following holds*:

$$(u(\partial M) + \operatorname{kink}(\partial M; g_{ab})) \bmod 2 = 0 \Longleftrightarrow w_1^- \smile w_1^- = 0\,.$$

Corollary 7. *There exists a* $\mathrm{Pin}^{-,-,-}$ (p,q) *cobordism* M *for* $\{\Sigma_i$: i=1, ... $n\}$ *if and only if*

$$(u(\partial M) + \operatorname{kink}(\partial M; g_{ab})) \bmod 2 = 0\,.$$

Thus, we see that the topological obstructions to $\mathrm{Pin}^{a,b,c}$ (p,q) cobordism depend only upon boundary data (i.e., kink number), the values of a, b, $c \in \{\pm\}$, the choice of signature, and the behaviour of the 1-cocycles $w_1^{\pm}$ under the cup product operation.

V. Interpreting the Breakdown of Pin Structure

We now interpret the breakdown of pin structure on M in two different ways: First, by examining the behaviour of pinor fields as we parallel propagate them around closed loops in M and secondly, by examining the behaviour of the determinant of the world line Dirac operator (the fermion effective action which arises in the quantization of a point particle possessing world line supersymmetry) in these situations.

Now, first recall that since we are generically dealing with non-orientable spacetimes M in this paper, we automatically have $\pi_1(M) \neq 0$, i.e., M cannot be simply connected. This means that there exist loops (closed curves), γ, in M with the property that when we parallel propagate some tetrad e^α around γ we will reverse the orientation of e^α.

Explicitly, suppose that we are given an "initial" tetrad $e^\alpha_{(i)}$ at some point $p \in \gamma$, and that after we parallel propagate around γ we are left with a "final tetrad" $e^\beta_{(f)}$.

The two tetrads will then be related by the equation $e^{\beta}_{(f)} = e^{\alpha}_{(i)} L^{\beta}_{\alpha}$, where $L^{\beta}_{\alpha} \in O(p,q)$ is some general Lorentz transformation (note that L^{β}_{α} cannot lie in the identity connected component, $O_0(p,q)$, since the final tetrad will generically have a different orientation than the initial one). For example, if $e^{\beta}_{(f)}$ has a different *spacelike* orientation than $e^{\alpha}_{(i)}$, then L^{β}_{α} must lie in $P(O_0(p,q))$ (the component of $O(p,q)$ containing parity reversal), and so on.

Now, we wish to view γ as the initial (and final) curve in a continuous family of curves, $\{\gamma(\upsilon)|\upsilon \in [0,1]\}$, which begins and ends at γ i.e., $\gamma(0) = \gamma(1) = \gamma$. This family of curves sweeps out a smooth 2-cycle T. Thus, for each $\upsilon \in [0,1]$ we have a curve $\gamma(\upsilon)$ and for each $\gamma(\upsilon)$ we parallel propagate some tetrad $e^{\alpha}_{(i)}(\upsilon)$ around $\gamma(\upsilon)$ to obtain a new tetrad $e^{\beta}_{(f)}(\upsilon)$, related to the old one by $e^{\beta}_{(f)}(\upsilon) = L^{\beta}_{\alpha}(\upsilon) e^{\alpha}_{(i)}(\upsilon)$, where $L^{\beta}_{\alpha}(\upsilon) \in O(p,q)$ for each value of υ.

Clearly then, since $\gamma(0) = \gamma(1) = \gamma$ we must have $L^{\beta}_{\alpha}(0) = L^{\beta}_{\alpha}(1) = I^{\beta}_{\alpha}$.

Now, consider the elements of some "pin bundle" (covering the bundle of frames) which "represent" the tetrads $e^{\alpha}_{(i)}(\upsilon)$ (which, we note for completeness, constitute a smooth field of tetrads on T as we vary υ), and write these elements as $\psi^{\alpha}_{i}(\upsilon)$. Then we can consider the problem of parallel propagating these initial "pinor fields" around each $\gamma(\upsilon)$ to obtain final pinor fields (ψ_f) which are related to the initial ones (on each curve $\gamma(\upsilon)$) by some transformation $\psi^{\beta}_{f}(\upsilon) = \tilde{L}^{\beta}_{\alpha}(\upsilon)\psi^{\alpha}_{i}(\upsilon)$, where $\pm\tilde{L}^{\beta}_{\alpha}(\upsilon) \in \mathrm{Pin}^{a,b,c}(p,q)$ are the elements of the pin group $\mathrm{Pin}^{a,b,c}\,(p,q)$ covering the corresponding Lorentz transformations $L^{\beta}_{\alpha}(\upsilon) \in O(p,q)$. The point is, again since we have $\gamma(0) = \gamma(1) = \gamma$, we expect to have $\tilde{L}^{\beta}_{\alpha}(0) = \tilde{I}^{\beta}_{\alpha}$ and $\tilde{L}^{\beta}_{\alpha}(1) = \tilde{I}^{\beta}_{\alpha}$; however, if there is a breakdown of pin structure we will have $\tilde{L}^{\beta}_{\alpha}(0) = +\tilde{I}^{\beta}_{\alpha}$ but $\tilde{L}^{\beta}_{\alpha}(1) = -\tilde{I}^{\beta}_{\alpha}$. Now we saw above (in Sect. III) that such an anomaly occurs depending upon the value of a certain obstruction class, which in turn depends upon the choice of signature and the values of $a, b, c, \in \{+,-\}$ (the symmetries of the pinor fields). Because of this, it is useful to consider an explicit example in order to have a clear picture of what is going on.

Thus, let M be a spacetime, with signature $(-+++)$, which is neither space nor time-orientable ($w_1^+ = w_1^- \neq 0$), and consider the problem of putting a *Cliffordian* pin structure on M, i.e., let the pin group be $\mathrm{Pin}^{+,-,+}$ (3, 1). Then from Sect. III (Theorem 3) above we know that the obstruction to putting this sort of pin structure on M is that the following hold:

$$w_2^+ + w_2^- + w_1^+ \smile w_1^- + w_1^- \smile w_1^- = 0\,.$$

A natural question is, why does $w_1^- \smile w_1^-$ contribute to the possibility of an anomaly but not $w_1^+ \smile w_1^+$? To see how to answer this question, assume that there exist 2-cycles, T and T', such that $w_1^- \smile w_1^-[T] \neq 0$ and $w_1^+ \smile w_1^+\,[T'] \neq 0$ (here we are regarding $\mathbb{Z}_2$ additively). It follows that there are closed curves, γ and γ', embedded in T and T' respectively, with the property that when we parallel propagate a tetrad $e_{(i)}$ around γ the final tetrad has opposite time-orientation (i.e., assume for simplicity that $e_{(f)}$ can be written $e_{(f)} = R_T e_{(i)}$, where R_T is time-reversal); also, it follows that when we propagate some tetrad $e'_{(i)}$ around γ' the final tetrad is related to the initial one by some reflection, R_L, about a spacelike axis L, i.e.,

$e'_{(f)} = R_L e'_{(i)}$. In terms of the pinors ψ, ψ' representing e, e' (respectively) we then have (using now gamma matrix notation since our pin group is Cliffordian)

$$\psi_f = \gamma_0 \psi_i \quad (\text{on}\, \gamma)\,.$$

$$\psi'_f = \gamma_L \psi'_i \quad (\text{on}\, \gamma')\,. \tag{13}$$

where of course γ_0 represents time reflection and γ_L represents reflection about axis L. Now, since we have chosen $a = +, b = -$ recall that we have

$$\gamma_0^2 = -\text{Identity} = -I$$

and

$$\gamma_L^2 = +\text{Identity} = I\,.$$

We now wish to view γ and γ' as the initial and final curves in the two families of curves $\{(\gamma(\upsilon)|\upsilon \in [0,1]\}$ and $\{\gamma'(\upsilon)|\upsilon \in [0,1]\}$ which sweep out T and T' respectively. We are then concerned with the following question: To what extent is an anomaly on T or T' determined simply by insisting that $w_1^- \smile w_1^-[T] \neq 0$ or $w_1^+ \smile w_1^+\,[T']\, \neq 0$ and that $a = +, b = -$?

First consider the curves $\gamma'(\upsilon)$ sweeping out T'. Suppose (for the purpose of contradiction) that there was an anomaly. Then we would have

$$\psi'_f(0) = \gamma_L \psi'_i(0)\,, \tag{14}$$

$$\psi'_f(1) = -\gamma_L \psi'_i(1)\,. \tag{15}$$

Furthermore, because $w_1^+ \smile w_1^+\,[T']\, \neq 0$ it follows that there is a curve $c'(\upsilon)$ in T' (generated by the parameter υ) with the property that propagating tetrads around c' also reverses spacelike orientation, i.e., we have

$$\gamma_L \psi'_f(1) = \psi'_f(0)\,, \tag{16}$$

$$\psi'_i(1) = \gamma_L \psi'_i(0)\,. \tag{17}$$

However, combining Eqs. (15) and (17) we obtain

$$\psi'_f(1) = -\gamma_L \gamma_L \psi'_i(0) = -\gamma_L \psi'_f(0)\,,$$

but this contradicts (16)! Thus, we see that $w_1^+ \smile w_1^+\,[T']\, \neq 0$ together with $\gamma_L \gamma_L = +I$ imply that we cannot have an anomaly (arising from the "parity reversal" part of some arbitrary pin transform). But this is exactly why $w_1^+ \smile w_1^+$ is not relevant to the obstruction class. If $w_1^+ = 0$, then γ_L does not even arise in our considerations, and so the question of an anomaly in γ_L becomes moot.

On the other hand, consider the curve $\gamma(\upsilon)$ sweeping out T. *Now* suppose (for the purpose of contradiction) that there is *no* anomaly. Then we have

$$\psi_f(0) = \gamma_0 \psi_i(0)\,, \tag{18}$$

$$\psi_f(1) = \gamma_0 \psi_i(1)\,. \tag{19}$$

Furthermore, the assumption that $w_1^- \smile w_1^-[T] \neq 0$ again implies

$$\gamma_0 \psi_f(1) = \psi_f(0)\,, \tag{20}$$

$$\psi_i(1) = \gamma_0 \psi_i(0) \,. \tag{21}$$

However, combining Eqs. (19) and (21) now gives us

$$\psi_f(1) = \gamma_0 \gamma_0 \psi_i(0) = -\psi_i(0) = \gamma_0 \psi_f(0) \,.$$

However, Eq. (20) is equivalent to

$$\psi_f(1) = -\gamma_0 \psi_f(0) \,,$$

and so again we have a contradiction. But this means that $w_1^- \smile w_1^- [T] \neq 0$ together with $\gamma_0^2 = -I$ imply that we *must* have an anomaly on T! But this is exactly why $w_1^- \smile w_1^-$ *is* relevant to the obstruction class. In other words, we have shown the following

Fact. Suppose there is a two-cycle, T, in M such that $w_1^d \smile w_1^d [T] \neq 0$ (where d can be $+$ or $-$). Then the values of $a, b \in \{\pm\}$ *alone* can affect the anomalous behaviour of pinor fields on T (and hence on M).

Indeed, we now see that the above constructions can be used to rederive the results of Sect. III.

Now, however, let us briefly go one step further and analyse the breakdown of pin structure by generalising a construction of Witten [14] (see also [15] and [16] for related reading).

Recall that Witten (in [14]) interprets the breakdown of spin structure in terms of anomalies in the fermion effective action which arises when one quantizes a point particle with world line supersymmetry.

Explicitly, Witten takes the world line of the particle to be a closed curve γ in the spacetime. He then constructs the fermion effective action, $\sqrt{\det(D)}(\gamma)$, where D is the "world line Dirac operator,"

$$D = i\left(\frac{d}{dt}\delta^i_j + \frac{dx^l}{dt} S^i_{lj}\right) \,,$$

where S_{ilj} is the spin connection. Thus, to define $\sqrt{\det(D)}$ we need only know the eigenvalues of D. Now, let us (following Witten) just consider the relationship between anomalies in $\sqrt{\det(D)}$ and the breakdown of Pin(4) structure; that is, we do not decompose the tangent bundle τ_M into "spacelike" and "timelike" parts determined by some Lorentz structure on M (i.e., we are concentrating here simply on lifting the $O(4)$ structure of the tangent bundle). Then the first thing we must recall is that there are *two* types of Pin(4)-structure, which we write $\mathrm{Pin}^+(4)$ and $\mathrm{Pin}^-(4)$. $\mathrm{Pin}^+(4)$ is the $2-1$ cover of $O(4)$ with the property that the element γ_+, which generates the non-identity connected component of $\mathrm{Pin}^+(4)$, satisfies $\gamma_+^2 = \mathrm{Id}$. The obstruction to $\mathrm{Pin}^+(4)$ structure on M can then be calculated using the above constructions, and we can see that the obstruction is that the following hold:

$$w_2(\tau_M) = 0$$

(see also [17] for another derivation).

On the other hand, $\mathrm{Pin}^-(4)$ is the $2-1$ cover of $O(4)$ with the property that the element γ_-, which generates the non-identity connected component of $\mathrm{Pin}^-(4)$, satisfies $\gamma_-^2 = -\mathrm{Id}$. Thus, the obstruction to $\mathrm{Pin}^-(4)$ structure is that the following hold:

$$w_2(\tau_M) + w_1(\tau_M) \smile w_1(\tau_M) = 0 .$$

Now, as Witten notes, with the choice of $O(4)$ for tangent bundle structure group we have that $A^i_j = \frac{dx^l}{dt} S^i_{lj}$ is on $O(4)$-invariant gauge field on γ. We are concerned with how choosing our pin group (i.e., boundary conditions) affects this gauge field and hence the eigenvalues of D (and thus the value of $\det(D)$). To see how this happens, let us again consider an explicit example.

First, let M be a manifold with $w_2(\tau_M) = w_1(\tau_M) \neq$ and let there be a 2-cycle T in M such that $w_1 \smile w_1[T] \neq 0$. Then M does admit $\mathrm{Pin}^-(4)$ structure but does not admit $\mathrm{Pin}^+(4)$ structure. This means we must differentiate between the determinants used in the two situations; thus, let $\det^{\pm}(D)$ denote the determinants obtained using the groups $\mathrm{Pin}^{\pm}(4)$, respectively.

Now, for $\det^+(D)$ we see that we get the same boundary condition as the one that Witten considers (i.e., he takes his gamma matrix to have square equal to *plus* the identity). It follows that A can be gauge transformed into the form

$$A = \frac{1}{2\pi} \begin{pmatrix} 0 & \theta_1 & & \\ -\theta_1 & 0 & & \\ & & 0 & \theta_2 \\ & & -\theta_2 & 0 \end{pmatrix}$$

regardless of the fact that $w_1 \smile w_1\ [T]\ \neq 0$ (here we are again regarding T as being swept out by a continuous family of world lines $\gamma(v)$). Thus, using [14] we obtain

$$\sqrt{\det{}^+(\mathrm{D})} = \prod_{i=1}^{2} \sin\left(\frac{\theta_i}{2}\right) . \tag{22}$$

The relevance of the fact that $w_1 \smile w_1[T] \neq 0$ becomes clear when we realise that the form of $\sqrt{\det^-(D)}$ is exactly the same as expression (22), but the boundary conditions satisfied by the angles θ_1, θ_2 are different. Explicitly, the total amount that the angles change in $\sqrt{\det^-(D)}$ (as we interpolate from $\gamma(0)$ to $\gamma(1)$) must differ from the amount they change in $\sqrt{\det^+(D)}$ by π.

In our example, we are assuming there is an anomaly in $\sqrt{\det^+(D)}$ (i.e., there is no $\mathrm{Pin}^+(4)$ structure). It follows that one of the angles must change by 2π while the other stays fixed, that is, we must have something like

$$\theta_1(0) = \theta_1(1) ,$$
$$\theta_2(0) = \theta_2(1) + 2\pi .$$

However, the angles appearing in $\sqrt{\det^-(D)}$ (1) have an extra π added in. But this means that the total change in *both* angles appearing in $\sqrt{\det^-(D)}$ is essentially π, and so there is no anomaly in the expression $\prod_{i=1}^{2} \sin\left(\frac{\theta_i}{2}\right)$. Thus, we see that as expected there is an anomaly in $\sqrt{\det^+(D)}$ but not $\sqrt{\det^-(D)}$.

When $w_1 \smile w_1 = 0$ then the boundary conditions are the same for both pin structures (which is what we expect since the obstruction classes are identical when $w_1 \smile w_1 = 0$).

VI. Format for Solving the General Problem

In the general situation, we will be given a manifold M with tangent bundle τ_M an "O"-bundle satisfying $\pi_1(O) \simeq G \not\simeq \{1\}$. We are then concerned with globally lifting τ_M to another bundle with structure group $\bar{O}$ satisfying

$$1 \longrightarrow \pi_1(O) \longrightarrow \bar{O} \longrightarrow O \longrightarrow 1 .$$

Using the theorem from Sect. II, we will again obtain a commutative diagram of sheaf cohomology groups.

If δ_2; $H^1(M;\mathcal{O}) \to H^2(M;\pi_1(O))$ is the Bockstein homomorphism for the vertical sequence in the diagram (as above) and $\xi \in H^1(M;\mathcal{O})$ denotes a choice of principal O-bundle, then we see that the obstruction to lifting to a principal $\bar{O}$-bundle is now the element $\delta_2(\xi) \in H^2(M;\pi_1(O))$.

For example, if we take a four-manifold with "Kleinian" metric g_{ab} (signature $(+\,+\,-\,-)$) then τ_M has structure group $O(2, 2)$ satisfying $\pi_1(O(2,2)) \simeq \mathbb{Z} \times \mathbb{Z}$. Thus, the obstruction to representing (globally) the information in τ_M in a simply connected way is, in this case, an element of $H^2(M;\mathbb{Z} \times \mathbb{Z})$. The point is, we could again write out the general form of this obstruction, and then use the commutative diagram of cohomology groups (analogous to (6)) to calculate the explicit form the obstruction takes in the various cases corresponding to how the "discrete" part of $O(2, 2)$ is covered by the discrete part of the cover, $\bar{O}(2, 2)$.

Finally, some readers may be worried about how far we have extended the vertical sequences in (6). However, it is shown ([6]), p. 207) that we can always extend as far as we need to (i.e., to $H^2(M;\pi_1(O))$) as long as $\pi_1(O)$ is abelian (regardless of whether or not $C^{a,b,c}$ is abelian).

VII. Conclusion

Finally, we mention one further application of these results, namely, the calculation of amplitudes in the Hartle–Hawking approach to treating gravitation in a quantum-mechanical way ([12, 13]).

Recall that in this approach the basic idea is to take Feynman's "sum over histories" philosophy to its logical conclusion, in other words, we sum over manifolds as well as metrics. We allow the topology of the universe to fluctuate. More explicitly, suppose that $\{\Sigma^i_j,\ \psi^i_j,\ h^i_j)|j = 1,\dots n\}$ is a collection of three-manifolds $\Sigma^i_1,\ \Sigma^i_2,\ \dots\ \Sigma^i_n$ with matter fields ψ^i_j and three metrics h_j (representing an "initial" configuration) and $\{(\Sigma^f_k,\ \psi^f_k,\ h^f_k)|k = 1,\ \dots m\}$ is a collection of three-manifolds $\Sigma^f_1,\ \Sigma^f_2,\ \dots \Sigma^f_m$ with matter fields ψ^f_k and three metrics h_k (representing a "final" configuration). Then the amplitude to go from the initial state to the final state is given, in this picture, by

$$\langle(\Sigma^f,\ \psi^f,\ h^f)|(\Sigma^i,\ \psi^i,\ h^i)\rangle = \textstyle\sum_M \upsilon(M) \int_C \delta\phi\delta g e^{(-I\{\phi,g,M\})} .$$

where the sum is over all manifolds M with boundary

$$\partial M \cong \Sigma_1^f \cup \Sigma_2^f \cup \ \ldots \cup \Sigma_m^f \cup \Sigma_1^f \cup \Sigma_2^i \cup \ldots \cup \Sigma_n^i ,$$

weighted by $v(M)$, with I the Euclidean action for matter fields ϕ and metrics g on M inducing the given configurations on the boundary. The point is, we might want to use the "selection rules" derived above (Sect. IV) to assign "weight zero" ($v(M) = 0$) to those manifolds which do not admit some pin or spin structure, i.e., which are not $\mathrm{Pin}^{a,b,c}(p,q)$ (or spin) cobordisms for the boundary three-manifolds. If we demand the three surfaces Σ^f, Σ^i be everywhere spacelike, then $\mathrm{kink}(\partial M; g_{ab}) = 0$, and so we see that such restrictions would be perhaps non-trivial. The precise effect such a procedure would have on the class of manifolds appearing in the path integral is, however, at present unclear.

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