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An Explicit Description of the Fundamental Unitary for $SU(2)_q$

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Abstract: We give a concrete description of an isometry v from $\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})$ to $\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z})$ whose existence has recently been discovered by Woronowicz [11]. The isometry v gives the comultiplication δ on the C*-algebra A of the quantum group $SU(2)_q$ through the formula $\delta(x)=v(x\otimes 1)v^*(x\in A)$, where 1 is the identity operator on $\ell^2(\mathbb{Z}\times\mathbb{Z})$. The matrix entries of v are described in terms of little q-Jacobi polynomials. Using v, we give a concrete description of a unitary operator V on $H_\eta\otimes H_\eta$ such that $(\pi_\eta\otimes\pi_\eta)\delta(x)=V(\pi_\eta(x)\otimes 1)V^*$, where $H_\eta=\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{N})$ and π_η: $A\to L(H_\eta)$ is the GNS representation associated with the Haar state η on A. The operator V satisfies the pentagonal identity of Baaj and Skandalis [1].

1. Introduction

The C*-algebra A of the quantum group $SU(2)_q$, where $0<q<1$, is a unital C*-algebra with generators a, c and relations that make

$$\begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \tag{1.1}$$

a unitary element of $M_2(A)$, namely

$$a^*a+c^*c=aa^*+q^2c^*c=1\ ,$$

$$ac=qca,\quad ac^*=qc^*a,\quad cc^*=c^*c\ . \tag{1.2}$$

There is a natural representation of A on the Hilbert space $\ell^2(\mathbb{N}\times\mathbb{Z})$, which was described by Woronowicz [9] as follows. For any set I, denote the standard orthonormal basis of $\ell^2(I)$ by $\{\varepsilon_i: i\in I\}$; if J is another index set then we identify $\ell^2(I)\otimes\ell^2(J)$ with $\ell^2(I\times J)$ by the correspondence $\varepsilon_i\otimes\varepsilon_j\leftrightarrow\varepsilon_{(i,j)}$ $(i\in I, j\in J)$, and we abbreviate $\varepsilon_{(i,j)}$ to ε_{ij}. Define bounded linear operators a, c on $\ell^2(\mathbb{N}\times\mathbb{Z})$ by

$$a\varepsilon_{ki}=(1-q^{2k})^{1/2}\,\varepsilon_{k-1\,i},\quad c\varepsilon_{ki}=q^k\varepsilon_{k\,i-1}\ (k\in\mathbb{N}, i\in\mathbb{Z})\ . \tag{1.3}$$

Then it is easy to check that a, c satisfy the relations (1.2) and therefore define an action of A on $\ell^2(\mathbb{N}\times\mathbb{Z})$, which we call the *fundamental representation* of A (not to be confused with the "fundamental corepresentation" of A, which is the unitary element of $M_2(A)$ given by (1.1); the potential for confusion is compounded by the fact that many authors take the dual category to be more basic, and hence describe as "representations" what we call "corepresentations").

The algebra A becomes a bialgebra if we define a comultiplication δ on the generators by

$$\delta(a)=a\otimes a-qc^*\otimes c,\quad \delta(c)=c\otimes a+a^*\otimes c\,. \tag{1.4}$$

In general, a comultiplication on a unital C*-algebra is a *-homomorphism $\delta\colon A\to A\otimes A$ (where the tensor product is the minimal C*-algebraic tensor product) satisfying the coassociativity condition $(\iota\otimes\delta)\delta=(\delta\otimes\iota)\delta$. It is frequently convenient to describe a comultiplication by specifying its action on a set of generators, as in (1.4), and there is then a problem of showing that the specified mapping extends to a well-defined *-homomorphism on the algebra. For the $\mathrm{SU}(2)_q$ algebra, Woronowicz solved this problem in [9] by the following strategy. He observed that the elements $\delta(a)$, $\delta(c)$ given by (1.4) satisfy the same relations (1.2) as a and c. He then proved that the fundamental representation of A has a universal property: if a', c' are elements of a C*-algebra B which satisfy (1.2), and a, c are given by (1.3), then the mapping $a\mapsto a'$, $c\mapsto c'$ extends to a *-homomorphism from A to B.

Another strategy, which works well in the case of the comultiplication on a group C*-algebra (see [5]) is this: given a C*-algebra A acting on a Hilbert space H, look for a unitary operator v on $H\otimes H$ such that

$$\delta(x)=v(x\otimes 1_H)v^* \tag{1.5}$$

whenever x is a generator of A. The right-hand side of (1.5) evidently defines a *-homomorphism of A, so the extension problem becomes trivial.

In a recent paper, Woronowicz [11] has shown that a variant of this construction can be applied to the $\mathrm{SU}(2)_q$ C*-algebra. He constructs a unitary operator $v\colon H\otimes K\to H\otimes H$ such that

$$\delta(x)=v(x\otimes 1_K)v^*\quad (x\in A), \tag{1.6}$$

where $H=\ell^2(\mathbb{N}\times\mathbb{Z})$ (the Hilbert space for the fundamental representation of A) and $K=\ell^2(\mathbb{Z}\times\mathbb{Z})$. The construction of v is very ingenious and conceptually attractive, involving a contraction procedure from $\mathrm{SU}(2)_q$ onto $\mathrm{E}(2)_q$, but it is too indirect to be useful for computational purposes. It also has the methodological defect of using the comultiplication on A to define v, which rules out the possibility of using (1.6) to define δ. Our first aim in this paper is to present a self-contained and explicit construction of a unitary operator $v\colon H\otimes K\to H\otimes H$ which satisfies (1.6) for $x=a$ and $x=c$, and which therefore in principle enables (1.6) to be used to define the comultiplication on A.

The construction of v leads us to the consideration of a family of little q-Jacobi polynomials. It is not surprising that these polynomials should arise, since little q-Jacobi polynomials have previously been encountered in the analysis of the irreducible corepresentations of $\mathrm{SU}(2)_q$ by Masuda et al. [6], Vaksman and Soibelman [8] and Koornwinder [4]. We discuss these polynomials more fully in Sect. 3; but we want to indicate in this Introduction a difference between the series of little q-Jacobi polynomials that occur in our analysis of the fundamental representation of A and those that arise from the irreducible corepresentations of A. In both cases, one is concerned with sequences of polynomials $p_k(x;a,b;q^2)$ (see (3.3) for the definition). In the study of the (finite-dimensional) irreducible corepresentations, the parameter b is a positive power of q^2, but in the case of the fundamental representation, we shall encounter the (degenerate) case $b=0$.

In addition to its fundamental representation, the C*-algebra A has another very natural and important representation, namely the regular representation. It is shown in [9] that A has a distinguished state η, the Haar state, given by the formula

$$\eta(x)=(1-q^2)^{-1}\sum_{k=0}^{\infty} q^{2k}\langle\varepsilon_{k0}, x\varepsilon_{k0}\rangle \quad (x\in A)\,. \tag{1.7}$$

It is evident from (1.7) that the regular representation of A, which by definition is the GNS representation π_η associated with η, is (unitarily equivalent to) a direct sum of $\mathbb{N}$ copies of the fundamental representation, so that $\pi_\eta(x)$ can be identified with $x\otimes 1$ acting on the space $H_\eta=\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{N})$. We denote the elements of the standard orthonormal basis of H_η by

$$\varepsilon^{\alpha}_{ki}=\varepsilon_k\otimes\varepsilon_i\otimes\varepsilon_\alpha \quad (k,\alpha\in\mathbb{N}, i\in\mathbb{Z})$$

to emphasize that π_η acts like the fundamental representation on the first two (subscript) factors, whereas the third, superscript, factor just indicates multiplicity. It follows from (1.7) that the vector

$$(1-q^2)^{-\frac{1}{2}}\sum_{k=0}^{\infty} q^k\varepsilon^k_{k0} \tag{1.8}$$

in H_η has η as its associated vector state. But each term in the sum (1.8) is only determined up to a phase, and it will be convenient for us to take

$$\xi_\eta=(1-q^2)^{-\frac{1}{2}}\sum_{k=0}^{\infty}(-q)^k\varepsilon^k_{k0} \tag{1.9}$$

as the cyclic vector for the Haar state.

It follows from the general theory developed in [1] and [10] that the equation

$$V\cdot\pi_\eta(x)\xi_\eta\otimes\zeta=(\pi_\eta\otimes\pi_\eta)\delta(x)\cdot\xi_\eta\otimes\zeta \;(x\in A,\ \zeta\in H_\eta) \tag{1.10}$$

defines a unitary operator on $H_\eta\otimes H_\eta$. This unitary satisfies the pentagonal identity that Baaj and Skandalis use as the foundation for their duality theory for Hopf C*-algebras. They use the term "multiplicative" to denote a unitary that satisfies this identity, but most of the recent literature refers to V as the "fundamental" unitary (yet another use of the word) of its associated quantum group, in this case $SU(2)_q$. We shall give a formula connecting V with v, but first we need to introduce a bit more notation.

We shall use the now-standard "leg-numbering" notation for operators on tensor-product spaces (see [1] or [7] for an explanation of this notation). For example, if $x\in A$, then we could denote the element $\pi_\eta(x)$ of $L(H_\eta)$ by x_{12}, meaning that it acts like x on the first two factors of $\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{N})$ and like the identity on the third. Let K be the Hilbert space $\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{N})$. We denote the standard basis vectors of K by

$$\varepsilon^{\alpha\beta}_{rnij}=\varepsilon_r\otimes\varepsilon_n\otimes\varepsilon_\alpha\otimes\varepsilon_i\otimes\varepsilon_j\otimes\varepsilon_\beta \quad (r,\alpha,\beta\in\mathbb{N}, n,i,j\in\mathbb{Z})\,,$$

where (as in the case of $H_\eta \otimes H_\eta$) the third and sixth components appear as superscripts to indicate multiplicity. Then v_{1245} is the unitary operator from K to $H_\eta \otimes H_\eta$ which acts like v on the subscript indices and leaves the superscripts alone.

If $H = H_1 \otimes H_2 \otimes \cdots \otimes H_k$ is a tensor product of Hilbert spaces and σ is a permutation of $\{1, \ldots, k\}$, then we denote by Σ_σ the unitary operator from H to $H_{\sigma(1)} \otimes \cdots \otimes H_{\sigma(k)}$ obtained by permutating the factor spaces in the obvious way. The formula that we shall derive for V is

$$V = v_{1245} \Sigma_{(13)(452)} v^*_{1245} \Sigma_{(134)(52)} . \tag{1.11}$$

The proof that we give for this formula in Sect. 4 is just a computational verification, and provides very little insight into why the result should be true. It would be desirable to have a more conceptual proof of (1.11) which would help one to understand why this relation should hold.

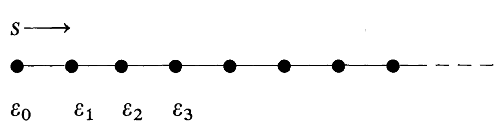
2. A Simple Example

In this section we describe a "quantum semigroup" which exhibits in a very simple form some aspects of the construction that we shall make in Sect. 4.

Let $C^*(s)$ be the C*-algebra generated by a non-unitary isometry s on a Hilbert space. The structure of $C^*(s)$ is well known [2], and it is independent of the multiplicity of s.

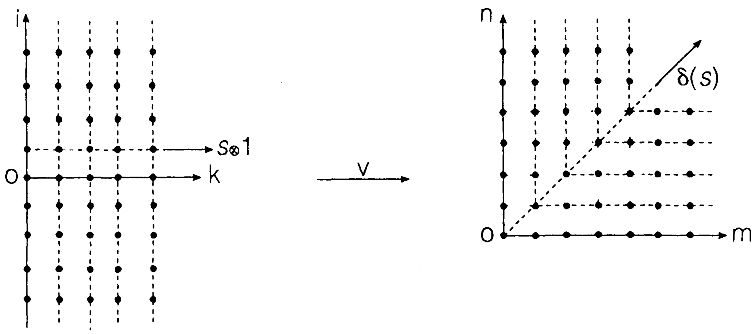
So if t is also an isometry then there is a unique *-isomorphism from $C^*(s)$ to $C^*(t)$ which maps s to t. In particular, $s \otimes s$ is an isometry, so there is a unique *-homomorphism $\delta: C^*(s) \to C^*(s) \otimes C^*(s)$ such that $\delta(s) = s \otimes s$. Since $(\iota \otimes \delta)\delta(s) = (\delta \otimes \iota)\delta(s) = s \otimes s \otimes s$, it is evident that δ is a comultiplication on $C^*(s)$. We call $(C^*(s), \delta)$ the quantum semigroup generated by an isometry. It does not have an antipode and cannot be made into a quantum group, but in some ways it serves as a prototype for the more complicated construction that we shall make later.

If we wish to obtain a concrete description of the comultiplication on $C^*(s)$ then we can proceed as follows. Take s to be the (forwards) unilateral shift on $\ell^2(\mathbb{N})$, whose action on the canonical orthonormal basis $\{\varepsilon_n : n \in \mathbb{N}\}$ is given by $s\varepsilon_n = \varepsilon_{n+1}$.



Then $s \otimes s$ is a shift of infinite multiplicity on $\ell^2(\mathbb{N} \times \mathbb{N})$. Define a bijective map $g: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{Z}$ by $g(m, n) = (m \wedge n, n - m)$, whose inverse is given by $g^{-1}(k, i) = (k - i \wedge 0, k + i \vee 0)$. (Here, $x \vee y$ means $\max\{x, y\}$ and $x \wedge y$ means $\min\{x, y\}$.)

We define a unitary mapping $v: \ell^2(\mathbb{N} \times \mathbb{Z}) \to \ell^2(\mathbb{N} \times \mathbb{N})$ by $v \cdot \varepsilon_{ki} = \varepsilon_{mn}$ $(k \in \mathbb{N}, i \in \mathbb{Z})$, where $(m, n) = g^{-1}(k, i)$. It is easy to verify, and evident from the following diagram, that v converts $s \otimes 1$ to $\delta(s)$ in the sense that $v(s \otimes 1) = \delta(s)v$, where 1 is the identity map on $\ell^2(\mathbb{Z})$.



Hence $\delta(x)=v(x\otimes 1)v^*$ $(x\in C^*(s))$.

Since $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ are both separable Hillbert spaces they are isomorphic, and we could use a unitary mapping between these spaces to convert the domain space of v from $\ell^2(\mathbb{N}\times\mathbb{Z})$ to $\ell^2(\mathbb{N}\times\mathbb{N})$. In that way we could construct a unitary operator on $\ell^2(\mathbb{N}\times\mathbb{N})$ which would intertwine the actions of $C^*(s)\otimes 1$ and $\delta(C^*(s))$ in the same way that v does. But there is no canonical way of doing this, and it is clear from the above construction that the "natural" domain space for v is $\ell^2(\mathbb{N}\times\mathbb{Z})$ rather than $\ell^2(\mathbb{N}\times\mathbb{N})$. We shall find that an exactly similar situation occurs with the C*-algebra of $SU(2)_q$ and the unitary that describes its comultiplication.

3. A Family of Orthogonal Polynomials

We need to investigate some properties of a family of little q-Jacobi polynomials. We use the book of Gasper and Rahman [3] as a general reference for basic hypergeometric series, and we shall adopt their notation. The basic hypergeometric series ${}_{r+1}\phi_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r;q,z)$ is defined by

$${}_{r+1}\phi_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r;q,z)=\sum_{i=0}^{\infty}\frac{(a_1;q)_i(a_2;q)_i\cdots(a_{r+1};q)_i}{(q;q)_i(b_1;q)_i\cdots(b_r;q)_i}z^i,$$

where

$$(a;q)_i=\begin{cases}1 & \text{if } i=0,\\ (1-a)(1-aq)\cdots(1-aq^{i-1}) & \text{if } i\geqq 1.\end{cases}$$

We assume throughout that $0<q<1$. We also define

$$(a;q)_\infty=\lim_{i\to\infty}(a;q)_i\,.$$

We shall require Cauchy's q-binomial theorem in the form ([3, (1.3.2)])

$${}_1\phi_0(a;-;q,z)=\sum_{i=0}^{\infty}\frac{(a;q)_i}{(q;q)_i}z^i=\frac{(az;q)_\infty}{(z;q)_\infty}, \tag{3.1}$$

from which one deduces the finite form ([3, (1.3.14)])

$$\sum_{i=0}^{n}\frac{(q^{-n};q)_i}{(q;q)_i}z^i=(zq^{-n};q)_n\,. \tag{3.2}$$

The little q-Jacobi polynomials $p_k(x; a, b; q)$ are defined by

$$p_k(x; a, b; q) = {}_2\phi_1(q^{-k}, abq^{k+1}; aq; q, qx) . \tag{3.3}$$

We shall be concerned almost exclusively with the case $b=0$.

Proposition 3.1. For $0<a<q^{-1}$ *and* $b>0$ *we have*

$$\text{(i)}\ p_k(x; a, b; q) = \frac{(b^{-1}q^{-k}; q)_k}{(aq; q)_k} (abxq^{k+1})^k\ {}_3\phi_2(q^{-k}, a^{-1}q^{-k}, x^{-1}; bq, 0; q, q) , \tag{3.4}$$

$$\text{(ii)}\ p_k(x; a, b; q) = \frac{(x^{-1}b^{-1}q^{-k}; q)_k}{(aq; q)_k} (abxq^{k+1})^k\ {}_2\phi_1(q^k, x; bxq^k; q^{-1}, a^{-1}q^{-k-1}) . \tag{3.5}$$

Proof. (i) is a straightforward application of the transformation formula of Sears ([3, (1.5.6)]). To prove (ii), we use the technique of inversion ([3, Ex. 1.4(i)]) and then Jackson's transformation formula ([3, Ex. 1.15 (iii)]) to get

$$\begin{aligned} {}_2\phi_1(q^k, x; bxq^k; q^{-1}, a^{-1}q^{-k-1}) &= {}_2\phi_1(q^{-k}, x^{-1}; b^{-1}x^{-1}q^{-k}; q, a^{-1}b^{-1}q^{-k}) \\ &= \frac{(b^{-1}q^{-k}; q)_k}{(b^{-1}x^{-1}q^{-k}; q)_k}\ {}_3\phi_2(q^{-k}, x^{-1}, a^{-1}q^{-k}; bq, 0; q, q) , \end{aligned}$$

and the result follows from (i).

Corollary 3.2. *For* $0<a<q^{-1}$,

$$\text{(i)}\ p_k(x; a, 0; q) = \frac{(-ax)^k q^{\binom{k+1}{2}}}{(aq; q)_k}\ {}_3\phi_2(q^{-k}, x^{-1}, a^{-1}q^{-k}; 0, 0; q, q) , \tag{3.6}$$

$$\text{(ii)}\ p_k(x; a, 0; q) = \frac{(-a)^k q^{\binom{k+1}{2}}}{(aq; q)_k}\ {}_2\phi_1(q^k, x; 0; q^{-1}, a^{-1}q^{-k-1}) . \tag{3.7}$$

Proof. These follow from (3.4) and (3.5) as $b\to 0$. (The symbol $\binom{k+1}{2}$ denotes a binomial coefficient.)

We next state some orthogonality relations for the little q-Jacobi polynomials, for the case $b=0$.

Proposition 3.3. For $0<a<q^{-1}$,

$$\text{(i)}\ \sum_{r=0}^{\infty} \frac{(aq)^r}{(q; q)_r} p_k(q^r; a, 0; q) p_\ell(q^r; a, 0; q) = \delta_{k\ell} \frac{(q; q)_k (aq)^k}{(aq; q)_k (aq; q)_\infty} , \tag{3.8}$$

$$\text{(ii)}\ \sum_{k=0}^{\infty} \frac{(aq; q)_k}{(q; q)_k (aq)^k} p_k(q^r; a, 0; q) p_k(q^s; a, 0; q) = \delta_{rs} \frac{(q; q)_r}{(aq; q)_\infty (aq)^r} . \tag{3.9}$$

Proof. The first of these is just a special case ($b=0$) of the standard orthogonality result for little q-Jacobi polynomials, proved in Sect. 7.3 of [3]. The proof of the dual formula (3.9) is a bit more elaborate, and to simplify the notation we shall adopt the convention of [3] that $(a_1, \ldots, a_n; q)_k$ means $(a_1; q)_k \cdots (a_n; q)_k$. In the following sequence of calculations, (3.10) uses (3.6), and (3.11) uses a partial inversion. We have

$$\sum_{k=0}^{\infty} \frac{(aq;q)_k}{(q;q)_k (aq)^k} p_k(q^r;a,0;q) p_k(q^s;a,0;q)$$

$$= \sum_{k=0}^{\infty} \frac{(aq^k)^k q^{k(r+s)}}{(q,aq;q)_k} {}_3\phi_2(q^{-r},q^{-k},a^{-1}q^{-k};0,0;q,q)$$
$$\times {}_3\phi_2(q^{-s},q^{-k},a^{-1}q^{-k};0,0;q,q) \tag{3.10}$$
$$= \sum_{k=0}^{\infty} \frac{(aq^k)^k q^{k(r+s)}}{(q,aq;q)_k} \sum_{i \geqq 0} \frac{(q^{-r};q)_i}{(q;q)_i} (q^k, aq^k; q^{-1})_i (aq^{2k})^{-i} q^{i^2}$$
$$\times {}_3\phi_2(q^{-s},q^{-k},a^{-1}q^{-k};0,0;q,q) \tag{3.11}$$
$$= \sum_{k=0}^{\infty} \sum_{i \geqq 0} \frac{(aq^{k-i})^{k-i} q^{k(r+s)}}{(q,aq;q)_{k-i}} \frac{(q^{-r};q)_i}{(q;q)_i} {}_3\phi_2(q^{-s},q^{-k},a^{-1}q^{-k};0,0;q,q)$$
$$= \sum_{k=0}^{\infty} \sum_{i=0}^{r} \frac{(aq^k)^k q^{(k+i)(r+s)}}{(q,aq;q)_k} \frac{(q^{-r};q)_i}{(q;q)_i} {}_3\phi_2(q^{-s},q^{-k-i},a^{-1}q^{-k-i};0,0;q,q). \tag{3.12}$$

If we can prove that

$$\sum_{i=0}^{r} q^{(k+i)(r+s)} \frac{(q^{-r};q)_i}{(q;q)_i} {}_3\phi_2(q^{-s},q^{-k-i},a^{-1}q^{-k-i};0,0;q,q) = \delta_{rs} \frac{(q;q)_r}{(aq)^r}, \tag{3.13}$$

then (3.9) will follow from (3.12) because of Cauchy's result that

$$\sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q,qz;q)_k} = \frac{1}{(qz;q)_\infty} \tag{3.14}$$

(cf. [3, (1.6.3)]). It will suffice to prove (3.13) under the assumption that $s \leqq r$. We then have, by (3.2),

$$\sum_{i=0}^{r} q^{i(r+s)} {}_3\phi_2(q^{-s},aq^{-i},bq^{-i};0,0;q,q)$$
$$= \sum_{i=0}^{r} \sum_{j=0}^{s} \frac{(q^{-r};q)_i}{(q;q)_i} \frac{(q^{-s};q)_j}{(q;q)_j} (aq^{-i},bq^{-i};q)_j q^{i(r+s)+j}$$
$$= \sum_{i,j} \frac{(q^{-r};q)_i (q^{-s};q)_j}{(q;q)_i (q;q)_j} \sum_{m=0}^{j} \frac{(q^{-j};q)_m}{(q;q)_m} (aq^{-i+j})^m \sum_{n=0}^{j} \frac{(q^{-j};q)_n}{(q;q)_n} (bq^{-i+j})^n q^{i(r+s)+j}$$
$$= \sum_{j,m,n} \frac{(q^{-s};q)_j (q^{-j};q)_m (q^{-j};q)_n}{(q;q)_j (q;q)_m (q;q)_n} a^m b^n q^{j(m+n+1)} \sum_{i=0}^{r} \frac{(q^{-r};q)_i}{(q;q)_i} q^{i(r+s-m-n)}$$
$$= \sum_{j,m,n} \frac{(q^{-s};q)_j (q^{-j};q)_m (q^{-j};q)_n}{(q;q)_j (q;q)_m (q;q)_n} a^m b^n q^{j(m+n+1)} (q^{s-m-n};q)_r. \tag{3.15}$$

In this last expression, fix m and n, and consider the sum over j. This should go from $j = \max\{m,n\}$ to $j = s$, but we shall make it go from $j = 0$ to $j = s$, since all the additional terms are zero. Using (3.2) again, we have

$$\sum_{j=0}^{s}\frac{(q^{-s};q)_j}{(q;q)_j}(q^{-j};q)_m(q^{-j};q)_n q^{j(m+n+1)}$$

$$=\sum_{j=0}^{s}\sum_{\alpha=0}^{m}\sum_{\beta=0}^{n}\frac{(q^{-s};q)_j}{(q;q)_j}\frac{(q^{-m};q)_\alpha}{(q;q)_\alpha}(q^{-j+m})^\alpha\frac{(q^{-n};q)_\beta}{(q;q)_\beta}(q^{-j+n})^\beta q^{j(m+n+1)}$$

$$=\sum_{\alpha,\beta}\frac{(q^{-m};q)_\alpha(q^{-n};q)_\beta}{(q;q)_\alpha(q;q)_\beta}q^{m\alpha+n\beta}\sum_{j=0}^{s}\frac{(q^{-s};q)_j}{(q;q)_j}q^{j(m-\alpha+n-\beta+1)}$$

$$=\sum_{\alpha,\beta}\frac{(q^{-m};q)_\alpha(q^{-n};q)_\beta}{(q;q)_\alpha(q;q)_\beta}q^{m\alpha+n\beta}(q^{-s+m-\alpha+n-\beta+1};q)_s\,. \tag{3.16}$$

The indices in the product represented by the final term in this expression cannot all be negative. So for the product to be nonzero we must have

$$-s+(m-\alpha)+(n-\beta)\geqq 0\,,$$

and hence $m+n\geqq s+\alpha+\beta\geqq s$. But then, by the same sort of reasoning, the final term in (3.15) must be zero, *unless* $s-m-n=-r$. Since $m\leqq j\leqq s$, this only happens when $m=n=j=s=r$. Thus for (3.15) to be nonzero we must have $s=r$, in which case the only nonzero term in the summation (3.15) is

$$\left(\frac{(q^{-r};q)_r}{(q;q)_r}\right)^3 a^r b^r q^{r(2r+1)}(q^{-r};q)_r=(q;q)_r\left(\frac{ab}{q}\right)^r.$$

Replacing a by q^{-k} and b by $a^{-1}q^{-k}$ and substituting into (3.13), we have the desired result.

We shall also require some q-contiguity results for the little q-Jacobi polynomials, as follows.

Proposition 3.4. *For* $0<a<q^{-1}$,

(i) $$(1-x)p_k(xq^{-1};a,0;q)=aq^{k+1}p_k(x;a,0;q)+(1-aq^{k+1})p_{k+1}(x;a,0;q)\,, \tag{3.17}$$

(ii) $$(1-a)p_k(x;aq^{-1},0;q)=(1-aq^k)p_k(x;a,0;q)-a(1-q^k)p_{k-1}(x;a,0;q)\,, \tag{3.18}$$

(iii) $$xp_k(x;aq,0;q)=(1-aq)[p_k(x;a,0;q)-p_{k+1}(x;a,0;q)]\,. \tag{3.19}$$

Proof. These are easily verified by comparing powers of x.

The orthogonality results from Proposition 3.3 enable us to construct a family of unitary operators on $\ell^2(\mathbb{N})$:

Proposition 3.5. *For* $0<q<1$ *and* $n\geqq 0$, *there is a unitary operator* u_n *on* $\ell^2(\mathbb{N})$ *whose action on the basis vectors is given by*

$$u_n\varepsilon_r=\sum_{k=0}^{\infty}\left\{\frac{(q^{2n+2};q^2)_\infty(q^{2n+2};q^2)_k}{(q^2;q^2)_r(q^2;q^2)_k}\right\}^{1/2}(-q^{n+1})^{r-k}p_k(q^{2r};q^{2n},0;q^2)\varepsilon_k\,. \tag{3.20}$$

Proof. Denote by $f(k,n,r)$ the coefficient of ε_k in (3.20). Then

$$\langle \varepsilon_k, u_n\varepsilon_r\rangle = f(k,n,r) .$$

The adjoint operator u_n^* must be given by $u_n^*\varepsilon_k = \sum_{r=0}^{\infty} f(k,n,r)\varepsilon_r$. The conditions for u_n to be unitary are

$$\langle u_n\varepsilon_r, u_n\varepsilon_s\rangle = \delta_{rs}, \quad \langle u_n^*\varepsilon_k, u_n^*\varepsilon_\ell\rangle = \delta_{k\ell} .$$

It is easy to see that these conditions are the same as (3.8) and (3.9), so Proposition 3.5 follows from Proposition 3.3.

4. Construction of v and V

Define u in $L(\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}))$ by

$$u\varepsilon_{rnij} = (u_{|n|}\varepsilon_r)\otimes\varepsilon_n\otimes\varepsilon_{i-r}\otimes\varepsilon_{j+r} \quad (r\in\mathbb{N}, n,i,j\in\mathbb{Z}) . \tag{4.1}$$

By Proposition 3.5, u is unitary. (There is no significance in the fact that the basis vectors are designated by multiple subscripts on one side of (4.1) and by a tensor product notation on the other side. It just seems more convenient to use multi-subscripts when the indices are single letters, and the tensor product notation otherwise.) Thus $(u\varepsilon_{rnij} = u_{|n|}\otimes 1\otimes s^r\otimes s^{-r})\varepsilon_{rnij}$,where s is the bilateral shift on $\ell^2(\mathbb{Z})$.

Also, define a unitary operator w: $\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})\to\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z})$ by

$$w\varepsilon_{knij} = \varepsilon_x\otimes\varepsilon_{i+y}\otimes\varepsilon_y\otimes\varepsilon_{j-x} \quad (r\in\mathbb{N}, n,i,j\in\mathbb{Z}) , \tag{4.2}$$

where $x=k-n\wedge 0$ and $y=k+n\vee 0$. Note that $(x,y)=g^{-1}(k,n)$, where g is the function defined in Sect. 2. Let $v=wu$. Using (3.20), (4.1) and (4.2), we can express v explicitly by

$$v\varepsilon_{rnij} = \sum_{k=0}^{\infty} f(k,|n|,r)\varepsilon_x\otimes\varepsilon_{i-r+y}\otimes\varepsilon_y\otimes\varepsilon_{j+r-x} \quad (r\in\mathbb{N}, n,i,j\in\mathbb{Z}) , \tag{4.3}$$

where $x=k-n\wedge 0$, $y=k+n\vee 0$ and

$$f(k,n,r) = \left\{\frac{(q^{2n+2};q^2)_\infty(q^{2n+2};q^2)_k}{(q^2;q^2)_k(q^2;q^2)_k}\right\}^{1/2}(-q^{n+1})^{r-k}p_k(q^{2r};q^{2n},0;q^2) , \tag{4.4}$$

as in the proof of Proposition 3.5. It is easy to check from (4.3) that the adjoint operator v^* is given by

$$v^*\varepsilon_{ki\ell j} = \sum_{r=0}^{\infty} f(k\wedge\ell, |\ell-k|, r)\,\varepsilon_r\otimes\varepsilon_{\ell-k}\otimes\varepsilon_{i+r-\ell}\otimes\varepsilon_{j-r+k} \quad (k,\ell\in\mathbb{N}, i,j\in\mathbb{Z}) . \tag{4.5}$$

It will be convenient for the proofs that follow to introduce some notation, as follows. For $k\in\mathbb{N}$ we write $(q)_k=(1-q^{2k})^{\frac{1}{2}}$, and

$$(q)_k! = \begin{cases} 1 & \text{if } k=0, \\ (q)_1(q)_2\cdots(q)_k & \text{if } k\geqq 1. \end{cases}$$

We also write $(q)_\infty!$ for $\lim_{k\to\infty}(q)_k!$. Thus $((q)_k!)^2=(q^2;q^2)_k$, and $(q^{2n+2};q^2)_k=((q)_{n+k}!/(q)_n!)^2$. With this notation, it follows from (4.4) and (3.6) that

$$\begin{aligned} f(k,n,r) &= \frac{(q)_\infty!((q^{2n+2};q^2)_k)^{\frac{1}{2}}}{(q)_n!(q)_r!(q)_k!}(-q^{n+1})^{r-k}p_k(q^{2r};q^{2n},0;q^2) \\ &= \frac{(q)_\infty!(-q^{n+1})^{r-k}(-q^{2n+2r})^k q^{2\binom{k+1}{2}}}{(q)_n!(q)_r!((q^{2n+2};q^2)_k)^{1/2}}\,{}_3\phi_2(q^{-2r},q^{-2k},q^{-2(n+k)};0,0;q^2,q^2) \\ &= \frac{(q)_\infty!(-q)^r q^{k(n+k)}q^{rk}q^{r(n+k)}}{(q)_r!(q)_k!(q)_{n+k}!}\,{}_3\phi_2(q^{-2r},q^{-2k},q^{-2(n+k)};0,0;q^2,q^2). \end{aligned} \tag{4.6}$$

Notice that this last expression is symmetric in k and $n+k$.

We now make a simple but useful observation: if $k,\ell\in\mathbb{N}$ then the unordered sets $\{k,\ell\}$ and $\{k\wedge\ell, k\wedge\ell+|\ell-k|\}$ are the same. It follows from the symmetry in (4.6) that

$$f(k\wedge\ell,|\ell-k|,r)=\frac{(q)_\infty!(-q)^r q^{k\ell}q^{r(k+\ell)}}{(q)_r!(q)_k!(q)_\ell!}\,{}_3\phi_2(q^{-2r},q^{-2k},q^{-2\ell};0,0;q^2,q^2)\,. \tag{4.7}$$

Putting (4.7) into (4.5) greatly simplifies the formula for v^*.

We now prove that v has the property claimed in the Introduction. The proof of Theorem 4.1 consists of an uninformative verification, but this is followed by an indication of where the formula for v actually comes from.

Theorem 4.1. *For all x in A, $\delta(x)=v\cdot x\otimes 1\cdot v^*$, where 1 denotes the identity operator on $\ell^2(\mathbb{Z}\times\mathbb{Z})$.*

Proof. It will be sufficient to verify that $\delta(x)v=v\cdot x\otimes 1$ for the cases $x=a$ and $x=c$.

When $x=a$, we have

$$\begin{aligned} v\cdot a\otimes 1\cdot\varepsilon_{rnij} &= (q)_r v\varepsilon_{r-1\,nij} \\ &= \sum_{k=0}^{\infty}(q)_r f(k,|n|,r-1)\varepsilon_x\otimes\varepsilon_{i-r+1+y}\otimes\varepsilon_y\otimes\varepsilon_{j+r-1-x}\,, \end{aligned} \tag{4.8}$$

where $x=k-n\wedge 0$ and $y=k+n\vee 0$ as before; whereas, from (1.3) and (1.4),

$$\begin{aligned} \delta(a)v\cdot\varepsilon_{rnij} &= \sum_{k=0}^{\infty} f(k,|n|,r)\delta(a)\varepsilon_x\otimes\varepsilon_{i-r+y}\otimes\varepsilon_y\otimes\varepsilon_{j+r-x} \\ &= \sum_{k=0}^{\infty} f(k,|n|,r)[(q)_x(q)_y\varepsilon_{x-1}\otimes\varepsilon_{i-r+y}\otimes\varepsilon_{y-1}\otimes\varepsilon_{j+r-x} \\ &\quad -q^{x+y+1}\varepsilon_x\otimes\varepsilon_{i-r+y+1}\otimes\varepsilon_y\otimes\varepsilon_{j+r-x-1}]\,. \end{aligned}$$

In the first term of this last expression, we replace k by $k+1$. Noting that this changes x to $x+1$ and y to $y+1$, and observing that the unordered sets $\{k,k+|n|\}$ and $\{x,y\}$ coincide, we see that

$$\begin{aligned} \delta(a)v\cdot\varepsilon_{rnij} &= \sum_{k=0}^{\infty}[(q)_{k+1}(q)_{|n|+k+1}f(k+1,|n|,r)-q^{|n|+2k+1}f(k,|n|,r)] \\ &\quad\times\varepsilon_x\otimes\varepsilon_{i-r+y+1}\otimes\varepsilon_y\otimes\varepsilon_{j+r-x-1}\,. \end{aligned} \tag{4.9}$$

Comparing coefficients in (4.8) and (4.9), we see that we need to check that

$$(q)_r f(k,|n|,r-1)=(q)_{k+1}(q)_{|n|+k+1}f(k+1,|n|,r)-q^{|n|+2k+1}f(k,|n|,r)\,.$$

But this follows in a routine way from (4.4) and (3.17).

The verification that $v\cdot c\otimes 1=\delta(c)v$ is similar, and we omit the details: it is necessary to consider separately the cases $n>0$ and $n\leqq 0$ and to use (3.18) and (3.19) for the respective cases.

How might one set about determining v without prior knowledge of the result? As a preliminary to this, consider the problem of reconstructing the canonical orthonormal basis $\{\varepsilon_{ki}\colon k\in\mathbb{N},\, i\in\mathbb{Z}\}$ of $\ell^2(\mathbb{N}\times\mathbb{Z})$ from a knowledge of the action of A on this space. It would be sufficient to know ε_{00}: for then we can obtain ε_{ki} from the fact that

$$a^{*k}c^{*i}\varepsilon_{00}=(q)_k!\,\varepsilon_{ki}\,. \tag{4.10}$$

But ε_{00} is in the kernel of a, which is also the eigenspace of c^*c corresponding to the eigenvector 1. Since 1 is an isolated point in the spectrum of c^*c, the projection p_0 onto this space is an element of A. Take any unit vector ε'_{00} in the range of p_0, and let $\varepsilon'_{ki}=((q)_k!)^{-1}a^{*k}c^{*i}\varepsilon'_{00}$. This reconstructs $\{\varepsilon_{ki}\}$, as required, modulo a unitary in the commutant of A (which is the von Neumann algebra generated by a bilateral shift on the second coordinate space).

If a unitary operator v satisfying (1.6) exists then it must map $\{\varepsilon_{0nij}\colon n,i,j\in\mathbb{Z}\}$ to an orthonormal basis for the kernel of $\delta(a)$. So it is natural to look for elements of $\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z})$ in the kernel of $\delta(a)$. It does not take much experimentation to discover that the vectors ζ_{nij} $(n,i,j\in\mathbb{Z})$ satisfy $\delta(a)\zeta_{nij}=0$, where

$$\zeta_{nij}=\begin{cases}\displaystyle\sum_{k=0}^{\infty}\frac{q^{k(k+n)}}{(q)_k!(q)_{k+n}!}\,\varepsilon_k\otimes\varepsilon_{i+k}\otimes\varepsilon_{n+k}\otimes\varepsilon_{j-k} & (n\geqq 0)\,,\\[2ex] \displaystyle\sum_{k=0}^{\infty}\frac{q^{k(k-n)}}{(q)_k!(q)_{k-n}!}\,\varepsilon_{k-n}\otimes\varepsilon_{i+k}\otimes\varepsilon_{k}\otimes\varepsilon_{j-k} & (n<0)\,.\end{cases} \tag{4.11}$$

Furthermore, it is easy to check that

$$\delta(c)\zeta_{nij}=\begin{cases}\zeta_{n-1\,i-1\,j} & \text{if } n>0\,,\\ \zeta_{n-1\,i\,j-1} & \text{if } n\leqq 0\,;\end{cases} \tag{4.12}$$

and an application of (3.14) shows that $\|\zeta_{nij}\|=1/(q)_\infty!$. So $\{(q)_\infty!\zeta_{nij}\colon n,i,j\in\mathbb{Z}\}$ is an orthonormal basis for the kernel of $\delta(a)$ and (apart from some shifts in the i and j coordinates) we can identify it with the image under v of $\{\varepsilon_{0nij}\colon n,i,j\in\mathbb{Z}\}$. To find $v\varepsilon_{rnij}$, it is necessary (by (4.10)) to compute $\delta(a^*)^r\zeta_{nij}$, and this is done as follows. Since (by (1.2))

$$(c\otimes c^*)(a^*\otimes a^*)=q^2(a^*\otimes a^*)(c\otimes c^*)\,,$$

it follows from the q-binomial formula (Exercise 1.35 in [3]) that

$$\delta(a^*)^r=(a^*\otimes a^*-qc\otimes c^*)^r=\sum_{\alpha=0}^{r}\begin{bmatrix}r\\ \alpha\end{bmatrix}_{q^2}(-q)^{r-\alpha}(a^*\otimes a^*)^\alpha(c\otimes c^*)^{r-\alpha}\,,$$

where the symbol $\begin{bmatrix} r \\ \alpha \end{bmatrix}_{q^2}$ denotes the q-binomial coefficient, given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = (-q^n)^k q^{-\binom{k}{2}} \frac{(q^{-n};q)_k}{(q;q)_k} \tag{4.13}$$

(see Appendix I in [3]). Therefore

$$\delta(a^*)^r \varepsilon_{ki\ell j} = \sum_{\alpha=0}^{r} \begin{bmatrix} r \\ \alpha \end{bmatrix}_{q^2} (-q)^{r-\alpha} \frac{(q)_{k+\alpha}!(q)_{\ell+\alpha}!}{(q)_k!(q)_\ell!} q^{(k+\ell)(r-\alpha)} \varepsilon_{k+\alpha} \otimes \varepsilon_{i-r+\alpha} \otimes \varepsilon_{\ell+\alpha} \otimes \varepsilon_{j+r-\alpha} . \tag{4.14}$$

We now combine (4.14) and (4.11). For simplicity, we treat only the case $n \geqq 0$, and we suppress the indices on the second and fourth coordinate vectors (replacing them with a bullet symbol). We obtain

$$\begin{aligned} \delta(a^*)^r \zeta_{nij} &= \sum_{k=0}^{\infty} \sum_{\alpha \geqq 0} \frac{q^{k(k+n)}}{(q)_{k+\alpha}!(q)_{k+n+\alpha}!} \begin{bmatrix} r \\ \alpha \end{bmatrix}_{q^2} (-q)^{r-\alpha} (q^{2(k+\alpha)}, q^{2(k+n+\alpha)}; q^{-2})_\alpha \\ &\quad \times q^{(n+2k)(r-\alpha)} \varepsilon_{k+\alpha} \otimes \varepsilon_\bullet \otimes \varepsilon_{k+n+\alpha} \otimes \varepsilon_\bullet \\ &= \sum_{k=0}^{\infty} \sum_{\alpha \geqq 0} \frac{q^{(k-\alpha)(k+n-\alpha)}}{(q)_k!(q)_{k+n}!} \begin{bmatrix} r \\ \alpha \end{bmatrix}_{q^2} (-q)^{r-\alpha} (q^{2k}, q^{2k+2n}; q^{-2})_\alpha q^{(n+2k-2\alpha)(r-\alpha)} \\ &\quad \times \varepsilon_k \otimes \varepsilon_\bullet \otimes \varepsilon_{k+n} \otimes \varepsilon_\bullet . \end{aligned}$$

After some straightforward calculations using (4.13) and an inversion (cf. proof of Proposition 3.1), the coefficient of $\varepsilon_{k\bullet\, k+n\bullet}$ is seen to be

$$\frac{(-q)^r q^{k(n+k)} q^{r(n+2k)}}{(q)_k!(q)_{k+n}!} \, {}_3\phi_2(q^{-2r}, q^{-2k}, q^{-2(k+n)}; 0, 0; q^2, q^2) . \tag{4.15}$$

This leads us to the expression for v given in (4.6).

The formula for v is not unique (and indeed may not coincide with that given by Woronowicz [11]), since v is only determined up to premultiplication by a unitary in the commutant of $A \otimes 1$. We now move on to consider the operator V, whose structure is much more tightly controlled than that of v, by the formula (1.10). In effect, the nonuniqueness in v is "cancelled out" by the presence of v^* as well as v in the formula (1.11). We shall prove (1.11) by the following strategy. It follows from (1.9) that

$$\pi_\eta(p_0)\pi_\eta(a)^\alpha \xi_\eta = (1-q^2)^{-\frac{1}{2}} (-q)^\alpha (q)_\alpha! \varepsilon_{0\,0}^\alpha ,$$

where p_0 denotes, as before, the projection onto the kernel of a (so that $\pi_\eta(p_0)$ is the projection onto the subspace spanned by $\{\varepsilon_{0i}^\lambda : i \in \mathbb{Z}, \lambda \in \mathbb{N}\}$). Therefore

$$\varepsilon_{ki}^\alpha = \frac{(1-q^2)^{\frac{1}{2}}}{(q)_\alpha!(q)_k!(-q)^\alpha} \pi_\eta(a^*)^k \pi_\eta(c^*)^i \pi_\eta(p_0) \pi_\eta(a)^\alpha \xi_\eta ,$$

and by (1.10)

$$V \varepsilon_{ki\ell j}^{\alpha\beta} = \frac{(1-q^2)^{\frac{1}{2}}}{(q)_\alpha!(q)_k!(-q)^\alpha} (\pi_\eta \otimes \pi_\eta) \delta(a^{*k} c^{*i} p_0 a^\alpha) \cdot \xi_\eta \otimes \varepsilon_{\ell j}^\beta . \tag{4.16}$$

In order to compute the right-hand side of (4.16), we need the following result about $\delta(p_0)$.

Proposition 4.2. *If* p_0 *is the projection onto the kernel of* a *then*

$$\delta(p_0)\varepsilon_{ki\ell j}=f(k\wedge\ell,|\ell-k|,0)v\cdot\varepsilon_0\otimes\varepsilon_{\ell-k}\otimes\varepsilon_{i-\ell}\otimes\varepsilon_{j+k}\quad(k,\ell\in\mathbb{N},i,j\in\mathbb{Z}), \tag{4.17}$$

where f *is given by* (4.4) *or* (4.7).

Proof. Since $\{\varepsilon_{0n\alpha\beta}: n,\alpha,\beta\in\mathbb{Z}\}$ is an orthonormal basis for the range of $p_0\otimes 1$, it follows from Theorem 4.1 that $\{v\varepsilon_{0n\alpha\beta}\}$ is an orthonormal basis for the range of $\delta(p_0)$. Therefore

$$\delta(p_0)\varepsilon_{ki\ell j}=\sum_{n,\alpha,\beta}\langle v\varepsilon_{0n\alpha\beta},\varepsilon_{ki\ell j}\rangle v\varepsilon_{0n\alpha\beta}\,. \tag{4.18}$$

From (4.3) it follows that the inner product on the right-hand side of (4.18) is zero unless $n=\ell-k$, $\alpha=i-\ell$ and $\beta=j+k$, in which case it is $f(k\wedge\ell,|\ell-k|,0)$.

Theorem 4.3. *The operator* V *defined by* (1.10) *satisfies*

$$V=v_{1245}\Sigma_{(13)(452)}v^*_{1245}\Sigma_{(134)(52)}=v_{1245}\Sigma_{(354)}v^*_{4532}\,,$$

where the notation is as explained in the Introduction.

Proof. We compute the right-hand side of (4.16). First, by arguing as in (4.14)–(4.15) we obtain

$$\begin{aligned}(\pi_\eta\otimes\pi_\eta)\delta(a)^\alpha\cdot\xi_\eta\otimes\varepsilon^\beta_{\ell j}&=\sum_{\gamma,\delta\geq 0}\begin{bmatrix}\alpha\\ \gamma\end{bmatrix}_{q^2}(-q)^{\alpha-\gamma}(c^*\otimes c)^{\alpha-\gamma}(a\otimes a)^\gamma\frac{(-q)^\delta}{(1-q^2)^{\frac12}}\varepsilon^{\delta\beta}_{\delta 0\ell j}\\ &=(1-q^2)^{-1/2}\sum_{\gamma,\delta}\begin{bmatrix}\alpha\\ \gamma\end{bmatrix}_{q^2}(-q)^{\alpha-\gamma+\delta}\frac{(q)_\delta!(q)_\ell!}{(q)_{\delta-\gamma}!(q)_{\ell-\gamma}!}\\ &\quad\times q^{(\alpha-\gamma)(\delta+\ell-2\gamma)}\varepsilon^\delta_{\delta-\gamma\,\alpha-\gamma\,\ell-\gamma}{}^\beta_{j-\alpha+\gamma}\,.\end{aligned}$$

Then, by Proposition 4.2 together with (4.7),

$$\begin{aligned}&(\pi_\eta\otimes\pi_\eta)\delta(p_0a^\alpha)\cdot\xi_\eta\otimes\varepsilon^\beta_{\ell j}\\ &=(1-q^2)^{-1/2}\sum_{\gamma,\delta}\begin{bmatrix}\alpha\\ \gamma\end{bmatrix}_{q^2}(-q)^{\alpha-\gamma+\delta}(q^{2\delta},q^{2\ell};q^{-2})_\gamma q^{(\alpha-\gamma)(\delta+\ell-2\gamma)}\\ &\quad\times\frac{(q)_\infty!\,q^{(\delta-\gamma)(\ell-\gamma)}}{(q)_\delta!(q)_\ell!}v'\varepsilon^\delta_{0\,\ell-\delta\,\alpha-\ell}{}^\beta_{j-\alpha+\delta}\,,\end{aligned} \tag{4.19}$$

where we have written v' for v_{1245}.

Next, we apply $(\pi_\eta\otimes\pi_\eta)\delta(a^{*k}c^{*i})$ to both sides of (4.19). Since

$$(\pi_\eta\otimes\pi_\eta)\delta(a^{*k}c^{*i})v'=v'\cdot\pi_\eta(a^{*k}c^{*i})\otimes 1\,,$$

we see that $(\pi_\eta\otimes\pi_\eta)\delta(a^{*k}c^{*i}p_0a^\alpha)\cdot\xi_\eta\otimes\varepsilon^\beta_{\ell j}$ is equal to an expression which looks just like the right-hand side of (4.19) except that the basis vector is replaced by

$$(q)_k!\,\varepsilon^\delta_{k\,i+\ell-\delta\,\alpha-\ell}{}^\beta_{j-\alpha+\delta}\,.$$

We write this latter expression as $(q)_k!\varepsilon^{\bullet\;\bullet}_{\bullet\bullet\bullet\bullet}$ since none of the suffixes will change during the remainder of the calculation. It follows from (4.16) that

$$V\varepsilon^{\alpha\beta}_{ki\ell j}=\sum_{\gamma,\delta}\frac{(q)_\infty!(-q)^{\delta-\gamma}}{(q)_\alpha!(q)_\delta!(q)_\ell!}\begin{bmatrix}\alpha\\ \gamma\end{bmatrix}_{q^2}(q^{2\delta},q^{2\ell};q^{-2})_\gamma q^{(\delta-\gamma)(\ell-\gamma)}q^{(\alpha-\gamma)(\delta+\ell-2\gamma)}v'\varepsilon^{\bullet\;\bullet}_{\bullet\bullet\bullet\bullet}\,.$$

As in (4.14)–(4.15), this expression simplifies to

$$\sum_{\delta=0}^{\infty}\frac{(q)_\infty!(-q)^\delta q^{\alpha\ell}q^{\delta(\alpha+\ell)}}{(q)_\delta!(q)_\alpha!(q)_\ell!}\,{}_3\phi_2(q^{-2\delta},q^{-2\alpha},q^{-2\ell};0,0;q^2,q^2)v'\varepsilon^{\bullet\;\bullet}_{\bullet\bullet\bullet\bullet}$$

$$=v'\left[\sum_{\delta=0}^{\infty}f(\alpha\wedge\ell,|\ell-\alpha|,\delta)\varepsilon^{\delta\qquad\quad\beta}_{k\,i+\ell-\delta\,\alpha-\ell\,j-\alpha+\delta}\right]. \tag{4.20}$$

But by (4.5) we have

$$v^*\varepsilon_{\ell j\alpha i}=\sum_{\delta=0}^{\infty}f(\alpha\wedge\ell,|\ell-\alpha|,\delta)\varepsilon_\delta\otimes\varepsilon_{\alpha-\ell}\otimes\varepsilon_{j-\alpha+\delta}\otimes\varepsilon_{i+\ell-\delta}\,.$$

Thus the expression in square brackets in (4.20) is the same as would be obtained by making v^* act on coordinates 4, 5, 3 and 2 of $\varepsilon^{\alpha\,\beta}_{ki\ell j}$ and carrying them to coordinates 3, 4, 5 and 2 (leaving coordinates 1 and 6 unaffected). This is the same as the action of $\Sigma_{(354)}v^*_{4532}$ or $\Sigma_{(13)(452)}v^*_{1245}\Sigma_{(134)(52)}$, and so the proof of the theorem is completed.

This paper has been entirely devoted to the spatial analysis of the single quantum group $SU(2)_q$. But this object has played the role of a prototype in the study of quantum groups, and it seems important to have a thorough understanding of it. As is apparent from the results of this paper, there are aspects of the C^*-algebraic structure of $SU(2)_q$ that still await investigation.

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