Commun. Math. Phys. 163, 89–112 (1994)

Communications in
**Mathematical
Physics**

The Newtonian Limit for Asymptotically Flat Solutions of the Vlasov-Einstein System

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Received: 23 March 1993/in revised form: 4 October 1993

Abstract: It is shown that there exist families of asymptotically flat solutions of the Einstein equations coupled to the Vlasov equation describing a collisionless gas which have a Newtonian limit. These are sufficiently general to confirm that for this matter model as many families of this type exist as would be expected on the basis of physical intuition. A central role in the proof is played by energy estimates in unweighted Sobolev spaces for a wave equation satisfied by the second fundamental form of a maximal foliation.

1. Introduction

It is a well known empirical fact that in many situations a general relativistic description of the motion of self-gravitating matter can be replaced to a good approximation by a non-relativistic, Newtonian one. In the usual formulation of Newtonian gravity the interaction is described by a single scalar function, the Newtonian potential. The relation of this to general relativity, where the fundamental object is a Lorentz metric, is obscure. The basic idea required to understand this relation mathematically was provided by Cartan [4]. He showed that Newtonian theory can be formulated in such a way that the basic object is an affine connection whose non-zero components are components of the gradient of the Newtonian potential. The role of the potential itself is then merely that of providing a convenient representation of this connection in certain coordinate systems. It was realised by Friedrichs [12] that the natural way to connect the two theories is to require that the Levi-Civita connection of the spacetime metric go over in the limit as the speed of light c goes to infinity into the connection defined by Cartan. Since then many authors have extended this work on the relations between the equations of the two theories and the physical interpretations of their solutions. This knowledge has been systematised in the frame

Part of this work was done during a visit to the Department of Physics, Syracuse University, Syracuse, NY 13244-1130, USA

theory of Ehlers (see [11] and references therein). What has been achieved is to set up a precise definition of the Newtonian limit of general relativity which encodes that which is desirable on physical grounds. The major open question is whether this definition is compatible with the Einstein equations in the sense that there exists a sufficiently large class of solutions which satisfy all the axioms. The purpose of this paper is to answer this question in the affirmative.

The case of the Newtonian limit which is of most physical interest is that of an isolated system. This is expressed mathematically by restricting attention to asymptotically flat solutions of the Einstein equations. (The case of cosmological solutions, which is also of considerable interest, will not be treated here.) It is of prime importance to have results which do not only handle the vacuum Einstein equations since in that case only the trivial Newtonian solution (i.e. empty space) could be expected to arise as a limit of singularity free asymptotically flat spacetimes. It is at this point that the first serious difficulty is encountered. The most obvious type of matter to take would be one or more bodies made of fluid or an elastic solid. However only very limited results exist on the initial value problem for self-gravitating bodies of this type [20]. It is for this reason that the matter model chosen here is a collisionless gas described by the Vlasov equation. It is known that the local in time initial value problem for the Vlasov-Einstein system is well posed for a class of initial data which allows spatially localised matter distributions [5]. A large part of what follows does not crucially depend on any property of the particular matter model chosen beyond the fact that the local in time initial value problem is well posed. There is, however, one step where a special property of the Vlasov equation is used, namely in the last paragraph of Sect. 3. In order to generalise the results of this paper to a different matter model it would be essential to find a replacement for the argument of that paragraph or to modify the structure of the main proof significantly to avoid the need for that argument.

The next difficulty which hampers the development of rigorous theorems on the Newtonian limit is that this limit is singular in the sense that the Einstein equations, which are essentially hyperbolic, go over into the Poisson equation, which is elliptic. The hyperbolic nature of the Einstein equations is related to the propagation of gravitational waves. There is one special case, namely the case of spherical symmetry, where gravitational radiation is absent. This leads to a simplification of the problem and the Newtonian limit of spherically symmetric asymptotically flat solutions of the Vlasov-Einstein system was handled in [19].

When spherical symmetry is not assumed the problem of the singular limit has to be faced and to see which way to go it is useful to consider a simpler analogue of the Vlasov-Einstein system where a limiting situation occurs which is rather similar. This is the Vlasov-Maxwell system whose quasi-static limit has been considered in [1, 8, 22]. Of these papers the one which is of most relevance here is that of Degond [8]. He treats the limit using the fact that in the energy estimates for the Maxwell equations the terms in the equations which blow up as $c \to \infty$ make no contribution. The solutions discussed belong to a Sobolev space on each slice of constant time. For the Einstein equations this does not hold. An asymptotically flat metric falls off only as r^{-1} as $r \to \infty$ on a spacelike slice and the positive mass theorem implies that any attempt to impose faster fall-off excludes all but the trivial solution. Thus the metric does not belong to a Sobolev space. The usual way to get around this is to replace the ordinary Sobolev space by a weighted one. Unfortunately it is easily seen that such a replacement destroys the property used by Degond that singular terms drop out of the energy estimates. In the following this difficulty is circumvented with the help of a formulation of the Einstein equations used by Christodoulou and Klainerman [6]. There the only object which is determined by solving a non-trivial hyperbolic equation is the second fundamental form, which does lie in an ordinary Sobolev space. The r^{-1} part of the metric is generated by solving an elliptic equation.

Now that the strategy has been outlined, the main theorem will be stated. The notation is as follows: g_{ab} is the induced metric on the leaves of a maximal foliation of spacetime, k_{ab} is the second fundamental form of this foliation, ϕ is the lapse function, $\Gamma^{\alpha}_{\beta\gamma}$ are the Christoffel symbols of the spacetime metric and f is the phase space density of particles. The spacetime metric is of the form

$$ds^2 = -\phi^2 dt^2 + g_{ab} dx^a dx^b \,. \tag{1.1}$$

The notions of regular initial data and regular solutions appearing in the statement will be defined in Sect. 5. The exact interpretation of the order symbols used will be given at the end of Sect. 6. The parameter λ corresponds physically to c^{-2}.

Theorem 1.1. *Let $(g^0_{ab}(\lambda), k^0_{ab}(\lambda), f^0(\lambda))$ be a parameter-dependent initial data set for the Vlasov-Einstein system which is regular of order s for some $s \geq 6$ and satisfies the constraints and the maximal slicing condition. Suppose that as $\lambda \to 0$:*

(i) $g^0_{ab}(\lambda) = \lambda\delta_{ab} + O(\lambda^{3/2})$,
(ii) $k^0_{ab}(\lambda) = O(\lambda^{3/2})$,
(iii) $\partial_t k^0_{ab}(\lambda) = O(\lambda^{3/2})$,
(iv) $f^0(\lambda) = f^0_N + o(1)$,

for some f^0_N. Then a solution $(g_{ab}(\lambda), k_{ab}(\lambda), \phi(\lambda), f(\lambda))$ of the Vlasov-Einstein system, which is regular of order s and induces the given initial data on the hypersurface $t = 0$, exists on a λ-independent time interval $[0, T)$ and has the properties:

(i) $g_{ab}(\lambda) = \lambda\delta_{ab} + o(\lambda)$,
(ii) $k_{ab}(\lambda) = o(\lambda)$,
(iii) $\phi(\lambda) = 1 - U\lambda + o(\lambda)$ *for some* U,
(iv) $\Gamma^a_{00}(\lambda) = -\delta^{ab}\nabla_b U + o(1)$,
(v) *all other components* $\Gamma^{\alpha}_{\beta\gamma}$ *are* $o(1)$,
(vi) $f(\lambda) = f_N + o(1)$ *for some* f_N.

Moreover f_N and U solve the Vlasov-Poisson system with initial datum f^0_N.

In assumption (iii) in the hypotheses of this theorem the time derivative is to be calculated using the Einstein evolution equation (2.4) and the function ϕ in that equation is to be got by solving the lapse equation (2.12). This assumption may seem unnatural but it is essential. Its significance will be discussed further in Sect. 7.

It is appropriate at this point to mention some recent work related to the present paper. Fritelli and Reula [13] have suggested an interesting approach to proving convergence of solutions of the Einstein equations in the Newtonian limit on a spatially bounded region. Lottermoser [15] has proved the existence of rather general families of solutions of the Einstein constraint equations having a Newtonian limit. It was necessary to prove some new results on existence of families of solutions of the constraints in the present paper since Lottermoser's method is not suitable for producing solutions which satisfy a prescribed gauge condition (e.g. the maximal slicing condition used in the following). It is also of interest that the solutions whose existence is demonstrated in the present paper include ones which do not belong to the class produced in [15]. This is because the basic object Z_{ab} used there and supposed to behave regularly as $\lambda \to 0$ diverges in general in the Newtonian limit for the data constructed here.

The paper is organised as follows. In Sect. 2 the form of the Einstein equations used in [6] is discussed and it is shown how the parameter λ can conveniently be introduced into it. In the third and fourth sections estimates are derived for the Vlasov and Einstein equations respectively. In Sect. 5 there are used to prove the local existence of a solution on a λ-independent interval and the remainder of Theorem 1.1 is proved in Sect. 6. The existence of a large class of regular initial data is demonstrated in Sect. 7. Two appendices are concerned with some elliptic theory and estimates for modified Sobolev spaces which are needed in the body of the paper.

2. Derivation of the Reduced Equations

Consider first the $3+1$ form of the Einstein equations with zero shift. The constraints are

$$R - |k|^2 + (\mathrm{tr}\, k)^2 = 16\pi\phi^{-2}T_{00}\,, \tag{2.1}$$

$$\nabla^a k_{ab} - \nabla_b \,\mathrm{tr}\, k = -8\pi\phi^{-1}T_{0b}\,, \tag{2.2}$$

and the evolution equations are

$$\partial_t g_{ab} = -2\phi k_{ab}\,, \tag{2.3}$$

$$\begin{aligned}\partial_t k_{ab} = {}& -\nabla_a\nabla_b\phi + \phi(R_{ab} + \mathrm{tr}\, k k_{ab} - 2k_{ac}k^c_b \\ & - 8\pi T_{ab} - 4\pi\phi^{-2}T_{00}g_{ab} + 4\pi T g_{ab})\,.\end{aligned} \tag{2.4}$$

Here R_{ab} is the Ricci tensor of g_{ab}. The objects T_{00}, T_{0a} and T_{ab} are components of the matter tensor and T is obtained by taking the trace of T_{ab} with the metric g_{ab}. Define

$$A = \mathrm{tr}\, k\,, \tag{2.5}$$

$$B = R - |k|^2\,, \tag{2.6}$$

$$C_a = \nabla^b k_{ab} - \tfrac{1}{2}\nabla_a(\mathrm{tr}\, k)\,, \tag{2.7}$$

$$D_{ab} = \phi^{-1}\partial_t k_{ab} + \phi^{-1}\nabla_a\nabla_b\phi - R_{ab} + 2k_{ac}k^c_b\,. \tag{2.8}$$

It will also be useful to have the modified quantities

$$\tilde{B} = B - 16\pi\phi^{-2}T_{00}\,, \tag{2.9}$$

$$\tilde{C}_a = C_a + 8\pi\phi^{-1}T_{0a}\,, \tag{2.10}$$

$$\tilde{D}_{ab} = D_{ab} + 8\pi T_{ab} + 4\pi(\phi^{-2}T_{00} - T)g_{ab}\,. \tag{2.11}$$

In the following the maximal slicing condition $A = 0$ will be used. Under that assumption Eqs. (2.1), (2.2), and (2.4) are equivalent to $\tilde{B} = 0$, $\tilde{C}_a = 0$ and $\tilde{D}_{ab} = 0$ respectively. If $A = 0$ then the lapse equation

$$\Delta\phi = (|k|^2 + 4\pi\phi^{-2}T_{00} + 4\pi T)\phi \tag{2.12}$$

is satisfied. Following [6] it can be shown that (2.1)–(2.4) together with $A = 0$ imply a wave equation for k_{ab},

$$-(\phi^{-1}\partial_t)^2 k_{ab} + \Delta k_{ab} = N_{ab} + \tau_{ab}\,, \tag{2.13}$$

where

$$\tau_{ab} = 8\pi\big[\tfrac{1}{2}\,\phi^{-3}\dot{T}_{00}g_{ab} - \phi^{-1}(\nabla_a T_{0b} + \nabla_b T_{0a}) + \phi^{-1}\dot{T}_{ab} - \tfrac{1}{2}\,\phi^{-1}\dot{T}g_{ab} + \phi^{-2}(T_{0a}\nabla_b\phi + T_{0b}\nabla_a\phi) - (\phi^{-2}T_{00} - T)k_{ab} - \phi^{-4}\dot{\phi}T_{00}g_{ab}\big]\,, \tag{2.14}$$

$$N_{ab} = L_{ab} - H_{ab}\,, \tag{2.15}$$

$$\phi^2 L_{ab} = \nabla_a\nabla_b\dot{\phi} - \phi^{-1}\dot{\phi}\nabla_a\nabla_b\phi - \dot{\Gamma}^c_{ab}\nabla_c\phi + 2\phi(k^c_a\partial_t k_{bc} + k^c_b\partial_t k_{ac}) + 4\phi^2 k_{ac}k^{cd}k_{bd}\,, \tag{2.16}$$

$$\phi H_{ab} = \phi I_{ab} + \nabla^c\phi(2\nabla_c k_{ab} - \nabla_a k_{bc} - \nabla_b k_{ac}) - \nabla_a\phi\nabla^c k_{bc} - \nabla_b\phi\nabla^c k_{ac} - \nabla_c\nabla_b\phi k^c_a - \nabla_c\nabla_a\phi k^c_b + \Delta\phi k_{ab}\,, \tag{2.17}$$

$$I_{ab} = -3(R_{ac}k^c_b + R_{bc}k^c_a) + 2g_{ab}(k^{cd}R_{cd}) + k_{ab}R\,. \tag{2.18}$$

Here and in the following a dot is used to denote a derivative with respect to t whenever convenient. The reduced system of Einstein equations which will be used consists essentially of (2.3), (2.12), and (2.13). Unfortunately the occurrence of the Ricci tensor in I_{ab} causes trouble with the existence theory and so it will be replaced by using the relation

$$R_{ab} = \phi^{-1}\partial_t k_{ab} + \phi^{-1}\nabla_a\nabla_b\phi + 2k_{ac}k^c_b + 8\pi T_{ab} + 4\pi(\phi^{-2}T_{00} - T)g_{ab}\,, \tag{2.19}$$

which is equivalent to $\tilde{D}_{ab} = 0$. The following lemma was proved in the vacuum case in [6].

Lemma 2.1. *Let* (g_{ab}, k_{ab}, ϕ) *be a solution of* (2.3), (2.12), *and* (2.13) *(with* (2.19) *having been substituted into* (2.13) *to eliminate* R_{ab}*). Then if the data*

$$(g_{ab}(0), k_{ab}(0), \partial_t k_{ab}(0))$$

are such that A, $\tilde{B}$, $\tilde{C}_a$ *and* $\tilde{D}_{ab}$ *vanish for* $t = 0$ *and if* $\nabla_\alpha T^{\alpha\beta} = 0$ *then* A, $\tilde{B}$, $\tilde{C}_a$ *and* $\tilde{D}_{ab}$ *vanish everywhere so that* (g_{ab}, k_{ab}, ϕ) *defines a solution of the Einstein equations for which the hypersurfaces* $t =$ const *are maximal.*

Proof. This will only be sketched since it is very similar to the vacuum case. Equations (2.3), (2.12), and (2.13) imply (with $F = B + \operatorname{tr} D$):

$$\gamma^{-1}\partial_t A = \tilde{F} := F - 4\pi\phi^{-2}T_{00} - 4\pi T\,, \tag{2.20}$$

$$\phi^{-1}\partial_t F = \Delta A - 4\phi^{-1}\nabla^a\phi C_a + (\phi^{-1}\Delta\phi + R)A - 8\pi[-2\phi^{-1}\nabla^a T_{0a} + 2\phi^{-2}\nabla^a\phi T_{0a} - 1/2\phi^{-1}\dot{T} - 2k^{ab}T_{ab} - 3\phi^{-4}\dot{\phi}T_{00} + (-\phi^{-2}T_{00} + T)A + 3/2\phi^{-3}\dot{T}_{00}]\,, \tag{2.21}$$

$$\phi^{-1}\partial_t\tilde{C}_a = \nabla^b\tilde{D}_{ab} - 1/2\nabla_a(\operatorname{tr}\tilde{D}) + \phi^{-1}\nabla^b\phi\tilde{D}_{ab} - 1/2\phi^{-1}\nabla_a\phi\tilde{F} - \phi^{-1}A\nabla^b\phi k_{ab} - \nabla^b A k_{ab} + 8\pi\phi^{-1}T_{0a}A\,, \tag{2.22}$$

$$\phi^{-1}\partial_t\tilde{D}_{ab} = \nabla_a\tilde{C}_b + \nabla_b\tilde{C}_a\,. \tag{2.23}$$

In the derivation of (2.22) the vanishing of the 4-dimensional divergence of $T^{\alpha\beta}$ has been used. The latter can also be used to simplify the equation obtained from (2.21) by substituting for F and C_a in terms of $\tilde{F}$ and $\tilde{C}_a$. Differentiating Eqs. (2.20)–(2.23) and substituting these equations back in to eliminate unwanted derivatives gives wave equations of the form

$$-(\phi^{-1}\partial_t)^2 A + \Delta A = M\,, \tag{2.24}$$

$$-(\phi^{-1}\partial_t)^2 \tilde{F} + \Delta \tilde{F} = M'\,, \tag{2.25}$$

$$-(\phi^{-1}\partial_t)^2 \tilde{C}_a + \Delta \tilde{C}_a = M''_a\,, \tag{2.26}$$

where M, M' and M'' are homogeneous linear expressions in the quantities A, ∇A, $\nabla^2 A$, $\tilde{F}$, $\nabla\tilde{F}$, $\phi^{-1}\partial_t\tilde{F}$, $\tilde{C}$, $\nabla\tilde{C}$, $\phi^{-1}\partial_t\tilde{C}$, $\tilde{D}$, $\nabla\tilde{D}$. The coefficients in these expressions depend of course on the solution under consideration. Since a fixed solution is being examined, they can be treated as given functions. The proof of the lemma can now be completed by deriving an inequality of the form

$$\mathscr{E}(t) \le \int_0^t \mathscr{E}(s)\,ds\,, \tag{2.27}$$

where

$$\mathscr{E} = \int_{\mathbb{R}^3} (|A|^2 + |\nabla A|^2 + |\tilde{F}|^2 + |\nabla\tilde{F}|^2 + |\phi^{-1}\partial_t\tilde{F}|^2 + |\tilde{C}|^2 + |\nabla\tilde{C}|^2 + |\phi^{-1}\partial_t\tilde{C}|^2 + |\tilde{D}|^2)\,dV_g\,, \tag{2.28}$$

and dV_g is the volume element on $\mathbb{R}^3$ associated with the metric g_{ab}. This would be a straightforward consequence of the usual energy inequalities for wave equations if it were not for the occurrence of the quantities $\nabla\tilde{D}$ and $\nabla^2 A$ on the right-hand side of the equations. Note that these do not appear in the definition of $\mathscr{E}$. In fact $\nabla\tilde{D}$ only occurs in M'' and $\nabla^2 A$ only in M' and the problem can be overcome as follows. One of the terms which needs to be estimated is schematically of the form $\int_0^t \left[\int_{\mathbb{R}^3} (\partial_t\tilde{C}\nabla\tilde{D})(s)\right] ds$. Integrating by parts in time converts this into the sum of a spacetime integral and a boundary contribution on the hypersurface labelled by t. The spacetime integral can now be handled by a partial integration in space. A partial integration in space should also be applied to the boundary term. It is then schematically of the form $\int_{\mathbb{R}^3} (\nabla\tilde{C}\tilde{D})$. This can be estimated by an expression of the form

$$K\left(\eta \int_{\mathbb{R}^3} |\nabla\tilde{C}|^2 + \eta^{-1} \int_{\mathbb{R}^3} |\tilde{D}|^2\right),$$

where K is a constant and η may be chosen to be any positive real number. Choosing it so that $K\eta \le 1/2$ we can absorb the first term into $\mathscr{E}$. To handle the second term express it as the integral from 0 to t of its derivative and use Eq. (2.23). The term containing $\nabla^2 A$ can be estimated in an analogous way. Applying Gronwall's inequality to (2.27) now completes the proof. □

Lemma 2.1 shows that providing we are dealing with a matter model which guarantees that $\nabla_\alpha T^{\alpha\beta} = 0$ (and this is in particular true of matter described by the Vlasov equation) then solving the reduced system consisting of (2.3), (2.12), and

(2.13) suffices to solve the Einstein equations. In this paper the unknowns (g_{ab}, k_{ab}, ϕ) are time-dependent objects on $\mathbb{R}^3$. Normally, if asymptotically flat situations are to be studied, the boundary conditions $g_{ab} \to \delta_{ab}$ and $\phi \to 1$ as $|x| \to \infty$ will be imposed. In order to study the Newtonian limit the first of these will be replaced by $g_{ab} \to \lambda\delta_{ab}$, where the parameter λ corresponds to c^{-2}. The Newtonian limit then corresponds to the limit $\lambda \to 0$ and if this is to be regular it will be the case that $g_{ab} = O(\lambda)$ and $k_{ab} = O(\lambda)$ as $\lambda \to 0$. In the terminology of the frame theory (see [11]) g_{ab} is part of the temporal metric. In view of this dependence on λ it is convenient to use the variables $\gamma_{ab} = \lambda^{-1} g_{ab}$ and $\kappa_{ab} = \lambda^{-1} k_{ab}$. Then γ_{ab} satisfies the standard boundary condition that $\gamma_{ab} \to \delta_{ab}$ as $|x| \to \infty$. The basic equations are

$$\partial_t \gamma_{ab} = -2\phi\kappa_{ab}\,, \tag{2.29}$$

$$\Delta_\gamma \phi = \lambda[|\kappa|^2 + 4\pi\phi^{-2} T_{00} + 4\pi T]\phi\,, \tag{2.30}$$

$$-(\phi^{-1}\partial_t)^2 \kappa_{ab} + \lambda^{-1}\Delta_\gamma \kappa_{ab} = \lambda^{-1}(N_{ab} + \tau_{ab})\,. \tag{2.31}$$

In Eq. (2.30) the norm of κ_{ab} is defined by γ_{ab} and not by g_{ab}. Note that T has the same meaning as it had before, namely $g^{ab}T_{ab}$.

3. Estimates for the Vlasov Equation

For a discussion of the definition of the Vlasov equation in general relativity see [18]. Recall that the phase space density, which is the unknown in this equation, is a real-valued function on the mass shell P, i.e. the submanifold of the tangent bundle of spacetime defined by the conditions $g_{\alpha\beta}p^\alpha p^\beta = -1$ and $p^0 > 0$. The manifold P can be coordinatized by the spacetime coordinates x^α together with the spatial components p^a of the momentum. In the situation considered in this paper it is identified in this way with $\mathbb{R}^6 \times [0, T)$. The Vlasov-Einstein system in a general spacetime takes the form:

$$\begin{aligned} &G_{\alpha\beta} = 8\pi T_{\alpha\beta}\,, \\ &x^\alpha \partial f/\partial x^\alpha - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \partial f/\partial p^\alpha = 0\,, \\ &T^{\alpha\beta} = -\int f p^\alpha p^\beta |g|^{1/2}/p_0 dp^1 dp^2 dp^3\,. \end{aligned} \tag{3.1}$$

Here units have been chosen where the speed of light takes the value one. In order to study the Newtonian limit the parameter λ needs to be introduced. This was already done for the Einstein equations in the previous section. It can be introduced into the Vlasov equation just as easily. All that needs to be done is to write the equation in $3+1$ form and then to introduce the rescaled variables γ_{ab} and κ_{ab}. The result is

$$\begin{aligned} &\frac{\partial f}{\partial t} + \phi(1+\lambda|p|^2)^{-1/2} p^a \frac{\partial f}{\partial x^a} \\ &\quad - [(1+\lambda|p|^2)^{1/2}\gamma^{ac}(\lambda^{-1}\nabla_c\phi) - 2\phi\kappa^a_c p^c \\ &\quad + \phi(1+\lambda|p|^2)^{-1/2}\Gamma^a_{bc}p^b p^c]\frac{\partial f}{\partial p^a} = 0\,, \end{aligned} \tag{3.2}$$

where $|p|^2 = \gamma_{ab}p^a p^b$. Attention will be confined to initial data for f which are compactly supported in $\mathbb{R}^6$. Also λ will be restricted to belong to the interval $(0, \lambda_0]$ for some $\lambda_0 > 0$. No loss of generality results since it is only the limiting behaviour as $\lambda \to 0$ which is of interest here. The introduction of λ into $T^{\alpha\beta}$ requires a little more thought. This object will be called the matter tensor since it will be normalised so that T^{00} becomes the mass density in the Newtonian limit. If we were to take the definition used in (3.1) directly and merely express it in terms of the rescaled variables then we would get the absurd result that the matter density vanishes like $\lambda^{3/2}$ for any matter distribution which is uniformly bounded in λ. This would be sufficient motivation for the following definition:

$$T^{\alpha\beta} = -\int f p^\alpha p^\beta \phi |\gamma|^{1/2} / p_0 dp^1 dp^2 dp^3 \,, \tag{3.3}$$

where γ denotes the determinant of γ_{ab}. In the following the definition (3.3) will *always* be used and never the third equation of (3.1). This has the pleasant effect that for distribution functions f which are bounded independently of λ the quantity T^{00} is neither forced to vanish nor blow up as λ tends to zero. To understand what this apparent redefinition means physically it is necessary to remember that, as already mentioned, g_{ab} is part of the temporal metric. In the definition of the matter tensor what is needed is a volume form and this should be formed using the spatial metric. In the Ehlers frame theory the relationship between these two metrics contains a power of λ [11]. In describing the Newtonian limit we can only take the equations of general relativity over exactly if the distinction between the temporal and spatial metrics is made from the start and this account for the apparent discrepancy.

In this section estimates for the solution of (3.2) will be obtained in a fixed background geometry. It will be supposed that the quantities Γ^a_{bc}, κ_{ab}, ϕ and $\lambda^{-1}\nabla_a \phi$ are continuous and bounded together with their first derivatives with respect to the spatial coordinates x^a. Let C_1 denote a common bound for these. It will furthermore be assumed that there exists a positive constant A such that $\gamma^{-1} < A$ and $A^{-1}\delta_{ab} \leq \gamma_{ab} \leq A\delta_{ab}$. It follows that a similar estimate holds for γ^{ab}. Equation (3.2) says that f is constant along characteristics (in the present context these are the lifts of geodesics to the mass shell) and the assumptions on the geometry are enough to guarantee the existence of these characteristics. Let $R(t)$ and $P(t)$ denote the maximum values of $|x|$ and $|p|$ respectively contained in the support of f at time t. Then (3.2) implies estimates of the form $P(t) \leq P(0)(1 + Ct)e^{Ct}$ and $R(t) \leq R(0) + \int_0^t P(s)ds$, where the constant C depends only on C_1 and A. Next the Sobolev norms of f will be estimated under the additional assumptions that γ_{ab} belongs to $L^\infty([0,T), K^s(\mathbb{R}^3))$ and that κ_{ab} and $\lambda^{-1}\nabla_a\phi$ belong to $L^\infty([0,T), H^s(\mathbb{R}^3))$. Here and in the following $H^s(\mathbb{R}^n)$ denotes the standard Sobolev space of order s and $\| \ \|_{H^s}$ is the norm on that space. The L^p norm is denoted by $\| \ \|_p$. The space $K^s(\mathbb{R}^3)$ is defined to consist of those functions f in $L^\infty(\mathbb{R}^3)$ with $\nabla f \in H^{s-1}(R^3)$ with norm $\|f\|_{K^s} = \|f\|_\infty + \|\nabla f\|_{H^{s-1}}$. Consider now the norm of $f(t)$ in $L^2(\mathbb{R}^6)$. Liouville's theorem implies that the L^2 norm of $f(t)$ with respect to the geometrically natural volume form defined by γ_{ab} is constant. On the other hand, under the assumptions already made on the geometry, this volume form defines an equivalent L^2 norm to that of the standard volume form on $\mathbb{R}^6$. Hence $\|f(t)\|_2 \leq C$ for a constant C only depending on A and C_1. To estimate $\|f(t)\|_{H^s}$ for $s > 0$ a method used in [8] will be adopted. If the Vlasov equation (3.2) is written schematically as $Xf = 0$ then the derivative $D^s f$ of order s satisfies an equation of the form $X(D^s f) = Q_s$ for a certain source term Q_s. At this stage it is necessary to use the assumptions that γ_{ab} belongs to $L^\infty([0,T), K^s(\mathbb{R}^3))$ and that κ_{ab} and $\lambda^{-1}\nabla_a\phi$ belong to $L^\infty([0,T), H^s(\mathbb{R}^3))$. Let

C_2 be a bound for their norms in these spaces. Note that if $s \geq 2$ the assumptions made up to now imply that $\gamma^{ab} \in L^\infty([0,T), K^s(\mathbb{R}^3))$ since $K^s(\mathbb{R}^3)$ is then a Banach algebra. To estimate Q_s the facts will be used that (see [14, 16 or 23]):

$$\|D^k(gh)\|_2 \leq C(\|g\|_\infty \|h\|_{H^s} + \|g\|_{H^s}\|h\|_\infty), \tag{3.4}$$

$$\|D^k(gh) - gD^k h\|_2 \leq C(\|Dg\|_\infty \|h\|_{H^{s-1}} + \|g\|_{H^s}\|h\|_\infty) \tag{3.5}$$

for $k \leq s$. These hold provided the norms on the right-hand side of the inequalities exist. They are valid for functions defined on the whole of $\mathbb{R}^n$ or on a bounded domain. In this section the latter case is the one which is relevant, the domain in question being an open subset of $\mathbb{R}^6$ which contains the intersection of the support of f with each hypersurface of constant time. There results the estimate

$$\|Q_{s-1}\|_2 \leq C\|f\|_{H^{s-1}} \quad \text{for} \quad s \geq 6\,. \tag{3.6}$$

Here C depends only on A, C_1 and C_2 and the Sobolev embedding theorem in $\mathbb{R}^6$ has been used. Combining (3.6), the equation $X(D^s f) = Q_s$ and the inequality

$$\|f\|_{H^s} \leq C(\|f\|_2 + \|D^s f\|_2) \tag{3.7}$$

shows that an integral inequality of the following form holds for $s \geq 6$,

$$\|f(t)\|^2_{H^{s-1}} \leq \|f(0)\|^2_{H^{s-1}} + C\int_0^t \|f(t')\|^2_{H^{s-1}} dt'\,, \tag{3.8}$$

where C only depends on A, C_1, and C_2. In fact C_1 can be estimated in terms of C_2 and $\|\phi\|_\infty$ using the Sobolev embedding theorem.

The H^{s-1} norm of the integrand in (3.3) can be estimated by a constant (depending only on A, C_1, and C_2) times the H^{s-1} norm of f. Also for all functions $F(x,p)$ whose supports are contained in a given compact set an inequality of the following form holds:

$$\|\bar{F}\|_{H^{s-1}} \leq C\|F\|_{H^{s-1}}\,, \tag{3.9}$$

where $\bar{F}(x) = \int F(x,p)\,dp$. Combining this information with (3.8) gives

$$\|T^{\alpha\beta}(t)\|^2_{H^{s-1}} \leq C\left(\|f(0)\|^2_{H^{s-1}} + \int_0^t \|f(t')\|^2_{H^{s-1}} dt'\right). \tag{3.10}$$

Suppose next that the time derivative of γ_{ab}, κ_{ab} and ϕ satisfy estimates of the same kind as already assumed for the quantities themselves. Then an argument analogous to that above leads to estimates of the form

$$\|\partial_t f(t)\|^2_{H^{s-2}} \leq \|\partial_t f(0)\|^2_{H^{s-2}} + C\int_0^t \|f(t')\|^2_{H^{s-1}} + \|\partial_t f(t')\|^2_{H^{s-2}} dt'\,, \tag{3.11}$$

$$\|\dot{T}^{\alpha\beta}(t)\|^2_{H^{s-2}} \leq C\left(\|\partial_t f(0)\|^2_{H^{s-2}} + \int_0^t \|f(t')\|^2_{H^{s-1}} + \|\partial_t f(t')\|^2_{H^{s-2}} dt'\right), \tag{3.12}$$

where the constant C now depends on A, C_1, C_2, $\|\dot{\phi}\|_\infty$, the norm of the time derivative of γ_{ab} in the space $L^\infty([0,T), H^{s-1}(\mathbb{R}^3))$ and the norms of the time derivatives of κ_{ab} and $\lambda^{-1}\nabla\phi$ in $L^\infty([0,T), H^s(\mathbb{R}^3))$.

The derivative $D^{s-1}(\partial_t f)$ still needs to be estimated. To do this, note that the Vlasov equation says that the solution is constant along characteristics. Thus the value of f at any point is equal to the value of f^0 at the point where the characteristic through that point intersects the initial hypersurface. Let $(X(s,t,x,p), V(s,t,x,p))$ be the characteristic satisfying $X(t,t,x,p)=(x,p)$ and $V(t,t,x,p)=(x,p)$. Then

$$f(t,x,p) = f^0(X(0,t,x,p), V(0,t,x,p)) \,. \tag{3.13}$$

Hence

$$\begin{aligned}\partial_t f(t,x,p) &= D_x f^0(X(0,t,x,p), V(0,t,x,p))\, dX/ds(t,x,p) \\ &\quad + D_p f^0(X(0,t,x,p), V(0,t,x,p))\, dV/ds(t,x,p) \,. \end{aligned}\tag{3.14}$$

Now the quantities dX/ds and dV/ds are bounded in $H^{s-1}(\mathbb{R}^6)$. Assuming that f^0 is in $H^s(\mathbb{R}^6)$ we see that in order to estimate the H^{s-1} norm of $\partial_t f$ it suffices to know the following two things. Firstly, the mapping taking (t,x,p) to $X(0,t,x,p)$ (which is a C^1 diffeomorpism) is bounded in H^{s-1}. Secondly, for s sufficiently large (in the present case $s>5$), the H^{s-1} norm of the composition $g\circ h$ of a mapping g of class H^{s-1} and a diffeomorphism h of class H^{s-1} can be estimated in terms of the H^{s-1} norms of g and h. These facts follow from the results of [10, 17]. Thus the H^{s-1} norm of $\partial_t f$ can be estimated by an expression of the form $C\|f^0\|_{H^s}$, where C depends only on the H^{s-1} norm of the coefficients in the Vlasov equation. An estimate for $\|\dot{T}^{\alpha\beta}\|_{H^{s-1}}$ follows.

4. Estimates for the Einstein Equations

First the form of N_{ab} will be examined when it is written in terms of γ_{ab} and κ_{ab}. The indices of κ_{ab} will be raised and lowered using γ_{ab} and its inverse,

$$\begin{aligned}\phi^2 L_{ab} &= \nabla_a\nabla_b\dot{\phi} - \phi^{-1}\dot{\phi}\nabla_a\nabla_b\phi - \dot{\Gamma}^c_{ab}\nabla_c\phi \\ &\quad + 2\lambda\phi(\kappa^c_a\partial_t\kappa_{bc} + \kappa^c_b\partial_t\kappa_{ac}) + 4\phi^2\lambda\kappa_{ac}\kappa^{cd}\kappa_{bd}\,, \end{aligned}\tag{4.1}$$

$$\begin{aligned}\phi H_{ab} &= \phi I_{ab} + \gamma^{cd}\nabla_c\phi(2\nabla_d\kappa_{ab} - \nabla_a\kappa_{bd} - \nabla_b\kappa_{ad}) \\ &\quad - \nabla_a\phi\gamma^{cd}\nabla_c\kappa_{bd} - \nabla_b\phi\gamma^{cd}\nabla_c\kappa_{ad} \\ &\quad - \nabla_c\nabla_a\phi\kappa^c_b - \nabla_c\nabla_b\phi\kappa^c_a + \Delta_\gamma\phi\kappa_{ab}\,, \end{aligned}\tag{4.2}$$

$$\phi I_{ab} = -3(R_{ac}\kappa^c_b + R_{bc}\kappa^c_a) + 2\gamma_{ab}(\kappa^{cd}R_{cd}) + \kappa_{ab}\gamma^{cd}R_{cd}\,. \tag{4.3}$$

In (4.3) it is still necessary to make the substitution

$$R_{ab} = \lambda\phi^{-1}\partial_t\kappa_{ab} + \phi^{-1}\nabla_a\nabla_b\phi + 2\lambda\kappa_{ac}\kappa^c_b + 8\pi T_{ab} + 4\pi\lambda(\phi^{-2}T_{00} - T)\gamma_{ab}\,. \tag{4.4}$$

Suppose that a collection of quantities $(\gamma_{ab}, \kappa_{ab}, \phi, T^{\alpha\beta})$ is given. These are not assumed to satisfy any equations. If $\bar{\gamma}_{ab}$ is a solution of

$$\partial_t\bar{\gamma}_{ab} = -2\phi\kappa_{ab}\,, \tag{4.5}$$

with initial datum $\gamma_{ab}(0)$ then an estimate of the form

$$\begin{aligned}\|\bar{\gamma}_{ab}(t) - \gamma_{ab}(0)\|^2_{H^s} \le C \int_0^t & \|\bar{\gamma}_{ab}(t') - \gamma_{ab}(0)\|^2_{H^s} \\ & + \|\nabla\phi(t')\|^2_{K^{s-1}}\|\kappa_{ab}(t')\|^2_{H^s}\,dt' \end{aligned}\tag{4.6}$$

holds for any $s \geq 2$ provided the norms appearing all exist. This can conveniently be proved using (A18). The time derivative of $\bar{\gamma}_{ab}$ can be estimated similarly in terms of ϕ, κ_{ab} and their time derivatives giving for any $s \geq 3$ the estimate

$$\|\partial_t \bar{\gamma}_{ab}(t)\|^2_{H^{s-1}} \leq \int_0^t \|\partial_t \gamma_{ab}(t')\|^2_{H^{s-1}} + \|\nabla\phi(t')\|^2_{K^{s-1}} \|\partial_t \kappa_{ab}(t')\|^2_{H^{s-1}} + \|\nabla\dot{\phi}(t')\|^2_{K^{s-1}} \|\kappa_{ab}(t')\|_{H^{s-1}} dt' . \tag{4.7}$$

Let $\bar{\Gamma}^a_{bc}$ denote the Christoffel symbols of $\bar{\gamma}_{ab}$. These satisfy

$$\partial_t \bar{\Gamma}^a_{bc} = -\nabla_b(\phi k^a_c) - \nabla_c(\phi k^a_b) - \nabla^a(\phi k_{bc}) . \tag{4.8}$$

In Sect. 5 this will be used to obtain a stronger estimate for $\partial_t \bar{\Gamma}^a_{bc}$ than could be derived from the estimate for $\partial_t \gamma_{ab}$. Next consider the following equation which is closely related to (2.30):

$$\Delta_f \bar{\phi} = -(\gamma^{ab} - \delta^{ab})\partial_a \partial_b \bar{\phi} - \gamma^{ab}\Gamma^c_{ab}\nabla_c \bar{\phi} + \lambda[|\kappa|^2 + 4\pi\phi^{-2}T_{00} + 4\pi T]\bar{\phi} . \tag{4.9}$$

It is shown in the appendix that if γ_{ab} is a metric such that $\|\gamma_{ab} - \delta_{ab}\|_{K^{s-1}} + \|\gamma_{ab} - \delta_{ab}\|_p$ is sufficiently small, if the contents of the square brackets are in $L^1(\mathbb{R}^3) \cap H^{s-1}(\mathbb{R}^3)$ and if λ is sufficiently small then (4.9) has a solution $\bar{\phi}$ tending to 1 at infinity. Moreover if bounds for $\|\gamma_{ab} - \delta_{ab}\|_{K^s}$, $\|\gamma_{ab} - \delta_{ab}\|_p$ and $\|\varrho\|_1 + \|\varrho\|_{H^{s-1}}$ are given then a bound for $\|\bar{\phi}\|_\infty + \lambda^{-1}\|\nabla\bar{\phi}\|_{H^s}$ is obtained. Here ϱ denotes the expression in square brackets in (4.9). Now the norms of ϱ appearing in this estimate can be estimated in terms of the H^s norm of κ_{ab}, the K^s norm of $\gamma_{ab} - \delta_{ab}$ and the H^{s-1} norm of $T^{\alpha\beta}$ together with a bound on the size of the support of $T^{\alpha\beta}$. If Eq. (4.9) is differentiated with respect to t then the resulting equation can be used to obtain bounds on $\partial_t \bar{\phi}$ and its spatial derivatives in a manner similar to the above.

The next equation for which estimates are needed is

$$-(\phi^{-1}\partial_t)^2 \bar{\kappa}_{ab} + \lambda^{-1}\Delta_\gamma \bar{\kappa}_{ab} = \lambda^{-1}(N_{ab} + \tau_{ab}) . \tag{4.10}$$

Suppose that a solution of (4.10) is given with initial data $(\kappa_{ab}(0), \partial_t \kappa_{ab}(0))$. The fundamental energy estimate for Eq. (4.10) is obtained by multiplying it by $\gamma^{ac}\gamma^{bd}\partial_t \bar{\kappa}_{cd}$ and integrating in space. Define

$$E = \int_{\mathbb{R}^3} |\phi^{-1}\partial_t \bar{\kappa}_{ab}|^2 + \lambda^{-1}|\nabla\bar{\kappa}_{ab}|^2 + |\bar{\kappa}_{ab}|^2 dV_\gamma . \tag{4.11}$$

Then the energy estimate takes the form

$$E(t) \leq E(0) + C\left(\int_0^t E(t')dt' + \int_0^t \left|\int_{\mathbb{R}^3} \lambda^{-1}[\gamma^{ac}\gamma^{bd}(N_{cd} + \tau_{cd})\partial_t \bar{\kappa}_{ab}](t')dV_\gamma\right| dt'\right), \tag{4.12}$$

where the constant C depends on the norm in the space $L^\infty([0,T), K^s(\mathbb{R}^3))$ of the quantity $\gamma_{ab} - \delta_{ab}$ for $s \geq 3$, the norm in the space $L^\infty([0,T), H^s(\mathbb{R}^3))$ of the quantities κ_{ab} and $\lambda^{-1}\nabla_a \phi$, the L^∞ norm of $\partial_t \gamma_{ab}$, the norm of $\lambda^{-1/2}\dot{\Gamma}^a_{bc}$ in the space $L^\infty([0,T), H^{s-1}(\mathbb{R}^3))$, the L^∞ norms of ϕ and $\dot{\phi}$ and a constant A such that $A^{-1}I \leq \gamma \leq AI$, where I is the identity matrix. This is proved in a way which is standard for quasilinear hyperbolic equations (cf. [16]). The important point is that the only possible λ dependence of the constant C in (4.12) is through the quantities $\lambda^{-1}\nabla\phi_n$ and $\lambda^{-1/2}\bar{\Gamma}^a_{bc}$ and it will be seen below that the latter quantities can be bounded in terms of quantities which do not depend on λ.

To estimate the second term on the right-hand side of (4.12) one could try to use the estimate

$$\left|\int_{\mathbb{R}^3} S^{ab}\partial_t\bar{\kappa}_{ab}\right| \leq C\int_{\mathbb{R}^3} |S|^2 + |\partial_t\kappa|^2 \tag{4.13}$$

for any tensor S_{ab}. This is sufficient for N_{ab} and for most of the terms in τ_{ab}. However it is not sufficient to estimate the terms involving spatial derivatives $T^{\alpha\beta}$ without resulting in a loss of differentiability in the iteration to be carried out in Sect. 5. This difficulty can be overcome by the device of doing a partial integration in time already used in the proof of Lemma 2.1. The term to be estimated is schematically of the form $\int_0^t \left[\int_{\mathbb{R}^3} (\partial_t\kappa\nabla T)(t')\right] dt'$. Now integrate by parts in time. The resulting spacetime integral can be handled by integration by parts in space. After partial integration in space the boundary term is of the form $\int_{\mathbb{R}^3} T\nabla\kappa$. Now estimate this by the sum of a small constant times the L^2 norm of $\nabla\kappa$ and a large constant times the L^2 norm of $T^{\alpha\beta}$. The former term can be absorbed into the energy and the latter can be estimatd by expressing it as the integral of its time derivative from 0 to t. If it is known that $\bar{\Gamma}^a_{bc}$ is $O(\lambda^{1/2})$ then the estimate for $\lambda^{-1}\int|\nabla_a\bar{\kappa}_{bc}|^2$ coming from (4.12) implies a similar one for $\lambda^{-1}\int|\partial_a\bar{\kappa}_{bc}|^2$.

Now consider the higher derivatives of κ_{ab}. Differentiate (4.10) up to $s-1$ times (using partial, not covariant, derivatives) and rearrange the result to give a hyperbolic equation for $D^m\kappa$, $m \leq s-1$. Let E_m be defined by replacing κ in the definition of E by $D^m\kappa$. An inequality similar to (4.12) can then be obtained with E replaced by $E + \sum_{i=1}^{s-1} E_i$ and the term involving N and τ being replaced by terms involving derivatives of N and τ up to order $s-1$. Here a bound for the H^{s-1} norm of $\lambda^{-1/2}\bar{\Gamma}^a_{bc}$ goes into the constant appearing for the following reason. Many terms of the form $\lambda^{-1}D^{m_1}\gamma D^{m_2}\bar{\kappa}$ with $m_1, m_2 \geq 1$ occur as source terms in the equation for $D^m\kappa$. Hence it is necessary to have bounds for spatial derivatives of $\lambda^{-1/2}\gamma$. These can be obtained from the equation

$$\partial_a\gamma_{bc} = \gamma_{bd}\Gamma^d_{ac} + \gamma_{cd}\Gamma^d_{ab}\,. \tag{4.14}$$

Note that the quantity $E + \sum_{i=1}^{s-1} E_i$ defines a norm which is stronger than $\|\kappa_{ab}\|_{H^s} + \|\partial_t\kappa_{ab}\|_{H^{s-1}}$ because of the fact that it contains λ^{-1}.

5. Existence on a Uniform Time Interval

In this section the existence statement of Theorem 1.1 will be proved. First the definition of regular initial data must be given.

Definition. *An initial data set* $(g^0_{ab}(\lambda), k^0_{ab}(\lambda), f^0(\lambda))$ *for the Vlasov-Einstein system depending on a parameter* $\lambda \in (0, \lambda_0]$ *is called regular of order* s *if for some* $p > 6$,

(i) $g^0_{ab}(\lambda) - \lambda\delta_{ab}$ *belongs to the space* $L^p(\mathbb{R}^3) \cap K^{s+2}(\mathbb{R}^3)$ *for each fixed* λ,

(ii) $k^0_{ab}(\lambda)$ *belongs to the space* $H^s(\mathbb{R}^3)$ *for each* λ,

(iii) $f^0(\lambda)$ *belongs to* $H^s(\mathbb{R}^6)$ *and has compact support for each* λ.

It is of course assumed that $g^0_{ab}(\lambda)$ is a Riemannian metric for each λ.

If a regular initial data set satisfies the conditions (i) and (ii) occurring in the hypotheses of Theorem 1.1 then the quantities $\gamma^0_{ab} = \lambda^{-1}g^0_{ab}$ and $\kappa^0_{ab} = \lambda^{-1}k^0_{ab}$ satisfy the conditions

$$\begin{aligned} \gamma^0_{ab} &= \delta_{ab} + O(\lambda^{1/2}), \\ \kappa^0_{ab} &= O(\lambda^{1/2}). \end{aligned} \tag{5.1}$$

The O-symbol is to be understood in the sense of the function spaces occurring in the above definition, both in (5.1) and in the hypotheses of Theorem 1.1. Consider now Eq. (2.30) on the initial hypersurface. Using the definition (3.3) the expression in the square bracket in (2.30) can be brought into a form which does not contain ϕ. The results of the appendix show that for λ sufficiently small this equation has a solution ϕ which tends to 1 at infinity and that $\nabla\phi$ is $O(\lambda)$ as $\lambda \to 0$ in the space $H^s(\mathbb{R}^3)$. It follows that if $\partial_t\kappa_{ab}$ is defined with the help of (2.4) then it belongs to $H^s(\mathbb{R}^3)$ for each λ. Assumption (iii) of Theorem 1.1 implies that it is $O(\lambda^{1/2})$ in that space as $\lambda \to 0$. In this way data for the reduced system (2.29)–(2.31) can be constructed from regular initial data for the Vlasov-Einstein system.

To prove the existence theorem an iteration will be set up. First define γ_0, κ_0, ϕ_0, f_0 by extending the initial data γ^0, f^0 and the function ϕ^0 in a time independent manner and defining $\kappa_0(t) = \kappa^0 + (\partial_t\kappa)^0 t$. These functions do not satisfy Eqs. (2.29)–(2.31) but do satisfy the desired initial conditions. If now γ_n, κ_n, ϕ_n and f_n have been defined the next iterate is obtained as follows. First solve (4.5) with ϕ and κ replaced by ϕ_n and κ_n and γ^0 as initial datum. Define γ_{n+1} to be equal to the solution $\bar{\gamma}$. This should only be done on an interval $[0, T_n)$ short enough so that γ_{n+1} is a Riemannian metric. Next solve (4.9) with γ, κ, ϕ and f replaced by the corresponding quantities with a subscript n and define ϕ_{n+1} to be equal to the solution $\bar{\phi}$. To ensure the existence of the solution it may be necessary to reduce the size of T_n. The value of λ_0 must also be restricted in a way described below. Now solve (4.10) with γ replaced by γ_n and the quantities occurring in the definitions of N_{ab} and τ_{ab} replaced by the corresponding quantities with subscript n. As initial data use κ^0 and $(\partial_t\kappa)^0$. Let κ_{n+1} be equal to the solution $\bar{\kappa}$ obtained. Finally, in order to obtain f_{n+1} solve (3.2) with γ, κ and ϕ and f replaced by γ_n, κ_n, ϕ_n and f_{n+1} respectively and initial datum f^0.

The next step is to show that this iteration is bounded in certain function spaces and that the T_n can be chosen so that $T_n \geq T$ for all n, where T is a positive constant. Define

$$\begin{aligned} a_n(t) = \max_{0\leq m\leq n} \sup_\lambda \{ &\|\gamma_m(t) - \gamma_m(0)\|^2_{H^s} + \|\partial_t\gamma_m(t)\|^2_{H^{s-1}} + \|\kappa_m(t)\|^2_{H^s} \\ &+ \|\partial_t\kappa_m(t)\|^2_{H^{s-1}} + \|\lambda^{-1/2}\nabla\kappa_m(t)\|^2_{H^{s-1}} \\ &+ \|f_m(t)\|^2_{H^{s-1}} + \|\partial_t f_m(t)\|^2_{H^{s-2}} \} \end{aligned} \tag{5.2}$$

$$\begin{aligned} b_n(t) = \max_{0\leq m\leq n} \sup_\lambda \{ &\|\phi_m(t)\|^2_\infty + \|\lambda^{-1}\nabla\phi_m(t)\|^2_{H^s} + \|\dot{\phi}_m(t)\|^2_\infty \\ &+ \|\lambda^{-1}\nabla\dot{\phi}_m(t)\|^2_{H^s} + \|\lambda^{-1/2}\Gamma_m(t)\|^2_{H^{s-1}} + \|D^{s-1}(\partial_t f_m)(t)\|^2_2 \} \end{aligned} \tag{5.3}$$

Lemma 5.1. *Let $a_n(t)$ and $b_n(t)$ be the functions on $[0, T_n)$ defined as* (5.2) *and* (5.3). *Let λ_0 be a positive constant. Define*

$$\varrho_n = |\kappa_n|^2 + 4\pi\phi_n^{-2}(T_{00})_n + 4\pi T_n\,. \tag{5.4}$$

Suppose that for some $p < 6$ the inequalities $\|\gamma_{ab} - \delta_{ab}\|_{K^{s-1}} + \|\gamma_{ab} - \delta_{ab}\|_p \le C_1$ and $\lambda_0(\|\varrho_n(t)\|_\infty + \|\varrho_n(t)\|_{H^{s-1}}) \le C_2$ hold on the whole interval $[0, T_n)$, where the constants C_1 and C_2 are chosen so that Lemma A1 is applicable. Then there exists a constant C, only depending on the initial data, and positive real-valued functions D and D' on $\mathbb{R}^2$ and $\mathbb{R}$ respectively which are bounded on bounded subsets such that for all $n > 1$:

$$a_{n+1}(t) \le C + D(\sup a_n, \sup b_n)\int_0^t a_{n+1}(t') + b_n(t')dt'\,, \tag{5.5}$$

$$b_{n+1}(t) \le D'(\sup a_n)\,. \tag{5.6}$$

Proof. This is an application of the various estimates derived for the Vlasov and Einstein equations in Sects. 3 and 4. Note first that the required estimates for $\|\gamma_n(t) - \gamma_n(0)\|_{H^s}$ and $\|\partial_t\gamma_n(t)\|_{H^{s-1}}$ follow from (4.6) and (4.7) respectively. The estimates for $\|\kappa_n\|_{H^s}$, $\partial_t\kappa_n\|_{H^{s-1}}$ and $\|\lambda^{-1/2}\nabla\kappa_n\|_{H^{s-1}}$ are obtained from (4.12) and the analogous estimate for $E + \sum_{i=1}^{s-1} E_i$ as described towards the end of Sect. 4. The inequalities (3.8) and (3.11) provide the desired estimates for $\|f_n\|_{H^{s-1}}$ and $\|\partial_t f_n\|_{H^{s-2}}$. This completes the proof of (5.5). The estimation of ϕ_{n+1} and its spatial derivatives is accomplished by applying the information on the solution (4.9) given in Sect. 4. The quantity $\|\lambda^{-1/2}\dot{\Gamma}_n(t)\|_{H^{s-1}}$ and its spatial derivatives can then be estimated using (4.8). Next apply the argument of the last paragraph of Sect. 3 to estimate the last term in (5.3). Finally, the equation for $\dot{\phi}_m$ implies an estimate for it in terms of a_n. □

I claim that the inequalities (5.5) and (5.6) imply the boundedness of a_n and b_n uniformly in n on an appropriate time interval. To see this, first let K_1 be a constant which is greater than C and the supremum of $a_0(t) + b_0(t)$. Next choose λ_0 and T so that if λ and t are restricted to lie in the intervals $(0, \lambda_0]$ and $[0, T)$ respectively the quantity $\|\gamma_{ab} - \delta_{ab}\|_{K^{s-1}} + \|\gamma_{ab} - \delta_{ab}\|_p$ is small enough whenever a_n is less than K_1 so that the equation for ϕ_{n+1} can be solved and (5.5) and (5.6) hold. Let K_2 be a bound for D' under the condition that $\sup a_n \le K_1$. Let K_3 be a bound for D under the conditions that $\sup a_n \le K_1$ and $\sup b_n \le K_2$. Reduce the size of T if necessary so that

$$(C + K_2K_3T)e^{K_3T} \le K_1\,. \tag{5.7}$$

By induction $a_n(t) \le K$ and $b_n(t) \le K_2$ for all $t \in [0, T)$ and all n. Thus the iteration is bounded as claimed. This bounded iteration is what is needed to show the existence of a regular solution of the equations, as will now be shown.

Definition. *A solution $(g_{ab}(\lambda), k_{ab}(\lambda), \phi(\lambda), f(\lambda))$ of the Vlasov-Einstein system is called regular of order s if for each λ,*

(i) *$g_{ab}(\lambda) - \lambda\delta_{ab}$ and its time derivative belong to $L^\infty([0, T), K^s(\mathbb{R}^3))$,*

(ii) *$k_{ab}(\lambda)$ belongs to $L^\infty([0, T), H^s(\mathbb{R}^3))$ and $\partial_t k_{ab}(\lambda)$ belongs to $L^\infty([0, T), H^{s-1}(\mathbb{R}^3))$,*

(iii) $\phi(\lambda)$ *and its time derivative belong to* $L^\infty([0,T), K^{s+1}(\mathbb{R}^3))$,
(iv) $f(\lambda)$ *and its time derivative belong to* $L^\infty([0,T), H^{s-1}(\mathbb{R}^6))$.

The boundedness of the iteration implies on the one hand that the iterates converge weakly to a limit in the spaces introduced in this definition and on the other hand that they converge strongly to a solution of the Vlasov-Einstein system in the analogous spaces of functions with $\mathbb{R}^3$ replaced by any ball $B_R \subset \mathbb{R}^3$ and s replaced by $s-1$. The latter fact follows from the Ascoli theorem for vector-valued functions [9].

6. Convergence to the Newtonian Limit

First it is necessary to get hold of the functions f_N and U which occur in the statement of Theorem 1.1. Now $f(\lambda)$ is bounded in $C^1([0,T), H^{s-1}(\mathbb{R}^6))$. Choose a sequence $\{\lambda_n\}$ converging to zero. Let $f_n = f(\lambda_n)$. (This should not be confused with the sequence of iterates used in the last section.) The sequence $\{f_n\}$ is bounded in $C^1([0,T), H^{s-1}(\mathbb{R}^6))$ and hence, the Ascoli theorem for vector-valued functions, has a subsequence which converges strongly in $C^0([0,T), H^{s-2}(\mathbb{R}^6))$. Call the limit f_N. Next consider the sequence $\{\lambda_n^{-1}(1-\phi_n)\}$, where $\phi_n = \phi(\lambda_n)$. This is bounded in $C^1([0,T) \times \mathbb{R}^3))$ and the sequence $\{\lambda_n^{-1}\nabla\phi_n\}$ is bounded in $C^1([0,T), H^s(\mathbb{R}^3))$. By the type of argument used in the previous section it can be seen that, after passing to a subsequence, $\{\lambda_n^{-1}(1-\phi_n)\}$ converges uniformly on compact subsets to a limit, which will be denoted by U, and that for any $R > 0$ the restriction of $\lambda_n^{-1}\nabla\phi_n$ converges in $C^0([0,T), H^{s-1}(B_R))$ to the restriction of ∇U. Moreover, using the weak* convergence argument, U is bounded and ∇U is in $L^\infty([0,T), H^s(\mathbb{R}^3))$. In the same way it can be concluded that there is a sequence $\{\lambda_n\}$ converging to zero such that $\gamma_n = \gamma(\lambda_n)$ and $\kappa_n = \kappa(\lambda_n)$ converge in suitable function spaces as $n \to \infty$. From Eq. (4.4) it can be seen that the Ricci tensor of γ_n tends to zero as $n \to \infty$. Hence the limiting metric is flat. Consider now the time derivative of the connection. This is given by Eq. (4.8). We know already that $\nabla\phi$ is $O(\lambda)$ and that $\nabla\kappa$ is $O(\lambda^{1/2})$ in certain spaces. Hence Eq. (4.8) implies that $\dot{\Gamma}^a_{bc}$ is $O(\lambda^{1/2})$. Integrating in time shows that the Christoffel symbols of γ_{ab} are $O(\lambda^{1/2})$. Now the partial derivatives of γ_{ab} with respect to the spatial coordinates can be written in terms of these Christoffel symbols and γ_{ab} itself as in (4.14) and it follows that the partial derivatives of γ_{ab} must be $O(\lambda^{1/2})$. In particular the partial derivatives of the limit of the γ_n vanish and so this limiting metric must in fact be given by δ_{ab}. As a consequence the limit of the κ_n must also vanish. The characteristics of the Vlasov equation converge uniformly to those of the non-relativistic Vlasov equation with force term ∇U along the sequence λ_n. Hence f_N must coincide with the unique solution of the latter equation with the C^1 initial datum f_N^0. Passing to the limit in (2.30) and (3.2) then shows that f_N and U satisfy the Vlasov-Poisson system with initial datum f_N^0. Unfortunately the convergence statements derived up to now are not enough to ensure that the function U satisfies the standard boundary condition that $U \to 0$ as $r \to \infty$. On the other hand it is known that U is bounded and that it solves the Poisson equation with the compactly supported source $T_{00}(0)$. If U' is the unique solution of the Poisson equation with this source for which $U' \to 0$ at infinity then $U - U'$ is a bounded solution of the Laplace equation and hence constant. Thus U can be made to satisfy the desired boundary condition by subtracting from it this constant. It will be supposed from now on that this alteration has been made so that U vanishes at infinity. Since the solution of the Vlasov-Poisson system with a given C^1 initial datum is unique, it follows that for every sequence $\{\lambda_n\}$ the sequence $\{\lambda^{-1}\nabla\phi_n, f_n\}$ has a subsequence tending to the same limit, namely $(\nabla U, f_N)$. Hence the restriction of $\lambda^{-1}\nabla\phi$ to any ball of radius R converges to ∇U in H^{s-1} and f_n converges to f in H^{s-2}. It is also possible to show weak* convergence globally in $\mathbb{R}^3$.

The convergence statements obtained up to now can be improved by estimating the difference between the Newtonian and relativistic solutions,

$$\Delta_f(\lambda^{-1}(\phi-1)+U) = -(\gamma^{ab}-\delta^{ab})\partial_a\partial_b(\lambda^{-1}\phi) - \gamma^{ab}\Gamma^c_{ab}(\lambda^{-1}\partial_c\phi) + [|\kappa|^2 + 4\pi\phi^{-2}T_{00} - 4\pi T_{00}(0) + 4\pi T]\phi\,. \tag{6.1}$$

Hence

$$\|\lambda^{-1}(\phi-1)+U\|_{K^s} \le C(\|\gamma_{ab}-\delta_{ab}\|_{K^{s-1}} + \|\kappa_{ab}\|_{H^{s-2}} + \|T_{00}-T_{00}(0)\|_{H^{s-2}} + \lambda)\,. \tag{6.2}$$

Differentiating (6.1) gives the estimate

$$\|\lambda^{-1}\dot\phi + \dot U\|_{K^s} \le C(\|\gamma_{ab}-\delta_{ab}\|_{K^{s-1}} + \|\partial_t\gamma_{ab}\|_{H^{s-1}} + \|\kappa_{ab}\|_{H^{s-2}} + \|\partial_t\kappa_{ab}\|_{H^{s-2}} + \|\dot T_{00} - \dot T_{00}\|_{H^{s-2}} + \lambda)\,. \tag{6.3}$$

Using the definition of T_{00} it can be shown straightforwardly that the terms involving T_{00} in (6.2) and (6.3) can be replaced in these inequalities by expressions involving $f - f_N$,

$$\|\lambda^{-1}(\phi-1)+U\|_{K^s} \le C(\|\gamma_{ab}-\delta_{ab}\|_{K^{s-1}} + \|\kappa_{ab}\|_{H^{s-2}} + \|f-f_N\|_{H^{s-2}} + \lambda)\,. \tag{6.4}$$

The inequality (6.3) can be modified similarly. Subtracting the Vlasov equation for f_N from that for f gives the estimate

$$\|f(t)-f_N(t)\|^2_{H^{s-2}} \le \|f(0)-f_N(0)\|^2_{H^{s-2}} + C\int_0^t (\|f(t')-f_N(t')\|^2_{H^{s-2}} + \|\gamma_{ab}(t')-\delta_{ab}\|^2_{K^{s-2}} + \|\lambda^{-1}\nabla_c\phi(t') + \nabla_c U(t')\|^2_{H^{s-1}} + \lambda^2)dt' \tag{6.5}$$

and an analogous estimate for $\|\dot f(t) - \dot f_N(t)\|_{H^{s-2}}$. Equation (2.29) implies an estimate of the form

$$\|\gamma_{ab}(t)-\gamma_{ab}(0)\|^2_{H^{s-1}} \le C\int_0^t \|\gamma_{ab}(t')-\gamma_{ab}(0)\|^2_{H^{s-1}} + \|\kappa_{ab}(t')\|^2_{H^{s-1}}dt'\,. \tag{6.6}$$

To close the argument and obtain a useful differential inequality, it remains to estimate κ_{ab}. This can be done by examining carefully the third term in (4.12). This is mostly routine but there are two expressions which require particular care and these will be handled explicitly here. The first, which arises from N_{ab} is:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \lambda^{-1}\gamma^{ac}\gamma^{bd}\phi^{-2}\nabla_c\nabla_d\dot\phi\,\partial_t\kappa_{ab}dV_\gamma\right| \\
&\quad\le \left|\int_{\mathbb{R}^3} \lambda^{-1}\gamma^{ac}\gamma^{bd}\phi^{-2}\nabla_d\dot\phi\nabla_c(\partial_t\kappa_{ab})dV_\gamma\right| + \dots \\
&\quad\le C\lambda^{1/2}(\|\lambda^{-1}\nabla_d\dot\phi\|_2^2 + \|\lambda^{-1/2}\nabla_c(\partial_t\kappa_{ab})\|_2^2) + \dots\,.
\end{aligned}$$

The other, which arises from τ_{ab}, is $\left|\int_{\mathbb{R}^3} \gamma^{ac}\gamma^{bd}\phi^{-3}\dot{T}_{00}\gamma_{cd}\partial_t\kappa_{ab}dV_\gamma\right|$. To estimate this, first note that $\dot{T}^{00} - \dot{T}^{00}(0)$ can be controlled straigthforwardly so that $\dot{T}^{00}$ can be replaced without loss of generality by $\dot{T}^{00}(0)$. Now the fact can be used that $\dot{T}^{00}(0) = -\partial_a T^{0a}(0)$. After this substitution has been made it suffices to integrate by parts in space.

The quantity $\partial_t\gamma_{ab}$ satisfies $\|\partial_t\gamma_{ab}\|_{H^{s-1}} \le \|\kappa_{ab}\|_{H^{s-1}}$. Furthermore:

$$\|\gamma_{ab}(t) - \delta_{ab}\|_{K^{s-1}} \le \|\gamma_{ab}(t) - \gamma_{ab}(0)\|_{H^{s-1}} + \|\gamma_{ab}(0) - \delta_{ab}\|_{H^{s-1}_\delta} . \tag{6.7}$$

Define

$$\begin{aligned} a(t) = \sup_\lambda\{ &\|\gamma_{ab}(t) - \gamma_{ab}(0)\|^2_{H^{s-1}} + \|\kappa_{ab}(t)\|^2_{H^{s-1}} \\ &+ \|\partial_t\kappa_{ab}(t)\|^2_{H^{s-2}} + \|\lambda^{-1/2}\nabla\kappa_{ab}(t)\|^2_{H^{s-2}} \\ &+ \|f(t) - f_N(t)\|^2_{H^{s-2}} + \|\partial_t f(t) - \partial_t f_N(t)\|^2_{H^{s-2}}\} . \end{aligned} \tag{6.8}$$

Then the above estimates show that

$$a(t) \le C\left(\lambda^{1/2} + \int_0^t a(t')dt'\right) . \tag{6.9}$$

It follows using Gronwall's inequality that $a(t) = O(\lambda^{1/2})$.

The meaning of the order symbols in the statement of Theorem 1 can now be explained. They refer to the function spaces obtained from those occurring in the definition of a regular solution of the Vlasov-Einstein system by replacing s by $s-1$. In other words, convergence is obtained in a space involving one less derivative than that where the existence of the solution has been obtained. Given this definition, the conclusions (i)–(vi) of Theorem 1.1 follow from (6.9), (6.7), and (6.4).

7. Solution of the Constraints

In terms of γ_{ab} and κ_{ab} these take the form

$$R_\gamma - \lambda|\kappa|^2 = 16\pi\lambda\phi^{-2}T_{00} , \tag{7.1}$$

$$\gamma^{ab}\nabla_a\kappa_{bc} = -8\pi\phi^{-1}T_{0c} \tag{7.2}$$

for a maximal hypersurface. Let $\mu = \phi^{-2}T_{00}$ and $J_a = \phi^{-1}T_{0a}$. Note that when μ and J_a are expressed in terms of f and γ_{ab} using (3.3) then it is seen that they do not explicitly depend on ϕ. To solve these equations we start with the following λ-dependent objects:

(i) a metric $\tilde{\gamma}_{ab}$ satisfying $\tilde{\gamma}_{ab} = \delta_{ab} + O(\lambda^{3/2})$,

(ii) a non-negative real-valued function $\tilde{f}$ of compact support on the mass shell defined by $\tilde{\gamma}_{ab}$ having the form $\tilde{f} = \tilde{f}(0) + O(\lambda^{1/2})$.

It is assumed that $\tilde{\gamma}_{ab} - \delta_{ab} \in H^s_\delta(\mathbb{R}^3)$ for some δ in the interval $(-1, -1/2)$, that $\tilde{f} \in H^s(\mathbb{R}^6)$ and that these objects depend continuously on λ in the given spaces. The O-symbols here also refer to those spaces. Define $\tilde{\mu}$ and $\tilde{J}_a$ in terms of $\tilde{f}$ and $\tilde{\gamma}_{ab}$ in the same way as $\mu = \phi^{-2}T_{00}$ and $J_a = \phi^{-1}T_{0a}$ are defined in terms of f and γ_{ab} (i.e. by (3.3)). Then $\tilde{\mu}$ and $\tilde{J}_a$ belong to $H^s(\mathbb{R}^3)$ and depend continuously on λ in this space. Now let $\tilde{\kappa}_{ab}$ be a traceless symmetric tensor satisfying $\tilde{\kappa}_{ab} = O(\lambda^{1/2})$ and $\tilde{\gamma}^{ab}\tilde{\nabla}_a\tilde{\kappa}_{bc} = -8\pi\tilde{J}_c$. Tensors $\tilde{\kappa}_{ab}$ of this type can be constructed in a standard way using the York decomposition. From these objects it is possible to determine initial data for the Einstein equations by solving the equation [3]

$$\Delta_{\tilde{\gamma}}\psi + (1/8)(-R_{\tilde{\gamma}}\psi + \lambda|\tilde{\kappa}|^2\psi^{-7}) + 2\pi\lambda\tilde{\mu}\psi^{-3} = 0 \tag{7.3}$$

and defining $\gamma = \psi^4\tilde{\gamma}$, $\kappa = \psi^{-2}\tilde{\kappa}$ and $f(p^a) = \psi^{-8}\tilde{f}(\psi^2 p^a)$. Note that f is a function on the mass shell defined by γ_{ab}. A computation shows that $J_a = \psi^{-6}\tilde{J}_a$ and $\mu = \psi^{-8}\tilde{\mu}$. It is well known that Eq. (7.3) can be solved for a unique ψ which tends to 1 at infinity and which has the property that $\psi - 1$ belongs to a weighted Sobolev space provided that $\tilde{\gamma}_{ab}$ is close to δ_{ab} in a weighted Sobolev space. Because of assumption (i) above this will be satisfied when λ is close to zero. Note that the condition $\delta > -1$ implies that if $s \geq 2$ the space $H^s_\delta(\mathbb{R}^3)$ is continuously embedded in $L^p(\mathbb{R}^3)$ for some $p < 6$. Let ψ_1 be the solution of the equation $\Delta_f\psi_1 = -2\pi\tilde{\mu}(0)$ which tends to zero at infinity. Then since U satisfies the Poisson equation and ψ is identically one when $\lambda = 0$ it follows that $\psi_1 = (1/2)U$. Now a comparison with (7.3) shows that

$$\Delta_f(\psi - \lambda\psi_1) = O(\lambda^{3/2}). \tag{7.4}$$

Hence $\psi = 1 + \lambda\psi_1 + O(\lambda^{3/2})$. This can be used to calculate the Ricci tensor of γ_{ij} to order λ using the formula for conformal transformations. The result is that the contributions of order λ on the right-hand side of (2.4) cancel so that assumption (iii) of Theorem 1.1 is satisfied by the given initial data.

Now the significance of this assumption will be discussed. It is used only once in the proof, namely to start the iteration. If it did not hold then it would be impossible to choose functions $(\gamma_0, \kappa_0, f_0)$ inducing the correct initial data for the reduced system in such a way that the quantity a_0 defined by Eq. (5.2) was finite. This consideration makes it clear that the condition is essential for the above proof but leaves open the possibility that it might not be necessary for the theorem. It also gives no clue as to why the form of $\tilde{\gamma}_{ab}$ used in this section leads to solutions of the constraints where the condition is satisfied. An answer to these questions is suggested by the results of [21] where it was shown that a necessary condition for the existence of a sufficiently regular Newtonian limit is that the coefficient of λ^2 in the expansion of the spatial part of the metric (i.e. the coefficient of λ in the expansion of γ_{ab}), considered as a linearised metric, has vanishing linearised Bach tensor. This means that it satisfies the linearised version of the condition of conformal flatness. Another way of expressing this would be to say that γ_{ab} is conformally flat to first order in λ. The technical assumptions in [21] are not easy to compare with those used here and so instead the relevant computations will be done directly in the present set up. Suppose that $(g_{ab}(\lambda), k_{ab}(\lambda), \phi(\lambda), f(\lambda))$ is a solution of th Vlasov-Einstein system which is regular of order s and has all the properties contained in the conclusion of Theorem 1.1 except that (ii) is replaced by $k_{ab}(\lambda) = O(\lambda)$. Then passing to the limit in (2.3) implies that $k_{ab}(\lambda) = o(\lambda)$. Now look at (2.4). In terms of the rescaled variables it reads:

$$\begin{aligned}\partial_t\kappa_{ab} = &-\lambda^{-1}\nabla_a\nabla_b\phi + \phi\lambda^{-1}R_{ab} + \operatorname{tr}\kappa\kappa_{ab} - 2\kappa_{ac}\kappa^c_b \\ &- 8\pi\lambda^{-1}T_{ab} - 4\pi\phi^{-2}T_{00}\gamma_{ab} + 4\pi T\gamma_{ab}\,.\end{aligned} \tag{7.5}$$

Under the given assumptions the third, fourth, fifth and seventh terms converge to zero as $\lambda \to 0$. It follows that

$$\partial_t\kappa_{ab} - \lambda^{-1}R_{ab} = \nabla_a\nabla_b U + \Delta U\delta_{ab} + o(\lambda)\,. \tag{7.6}$$

In principle it might happen that each term on the left-hand side of (7.6) fails to converge as $\lambda \to 0$ although their sum converges. For this reason it is difficult to get a clean statement. To proceed further, suppose that the solution has enough regularity properties so that $\partial_t \kappa_{ab}$ has a limit as $\lambda \to 0$ in some sense which is strong enough to allow this limit and the time derivative to be interchanged. Then $\partial_t \kappa_{ab}$ must vanish in the limit. So it is not reasonable to expect that Theorem 1.1 would continue to hold if $O(\lambda^{3/2})$ in condition (iii) were replaced by $O(\lambda)$. Whether it could be replaced by $o(\lambda)$ remains an open question. With the assumption on the convergence of $\partial_t \kappa_{ab}$ it can be concluded that the limit of $\lambda^{-1} R_{ab}$ exists. Its value is determined by (7.6). Now consider the restriction which this implies on initial data, supposed to be differentiable enough in λ so that on the initial hypersurface we can write

$$\gamma_{ab} = \delta_{ab} + \lambda h_{ab} + o(\lambda) \tag{7.7}$$

for some h_{ab} which does not depend on λ. The result is:

$$\tfrac{1}{2}\, \delta^{ab}(\partial_c \partial_d h_{ab} + \partial_a \partial_b h_{cd} - \partial_b \partial_d h_{ac} - \partial_a \partial_c h_{bd}) - \nabla_c \nabla_d U + \Delta U \delta_{cd} = 0\,. \tag{7.8}$$

Let K_{cd} denote the first of the three terms in this equation. If h_{ab} is thought of as a linearised metric then K_{ab} is its linearised Ricci tensor. Let $K = \delta^{ab} K_{ab}$. The linearised Bach tensor of h_{ab} is defined to be

$$B_{abc} = \nabla_c K_{ab} - \nabla_b K_{ac} + \tfrac{1}{4}\,(\delta_{ab} \nabla_c K - \delta_{ac} \nabla_b K)\,. \tag{7.9}$$

Lemma 3 of [21] says that if h_{ab} belongs to a suitable weighted Sobolev space and the corresponding B_{abc} vanishes then there exist a function F and a covector X_a, both belonging to weighted Sobolev spaces, such that

$$h_{ab} = F \delta_{ab} + \nabla_a X_b + \nabla_b X_a\,. \tag{7.10}$$

Equation (7.8) implies that the linearised Bach tensor of h_{ab} vanishes and so h_{ab} must be of the form (7.10). Substituting this back into (7.8) gives $F = 2U$. Thus

$$\gamma_{ab} = \delta_{ab}(1 + 2\lambda U) + \lambda(\nabla_a X_b + \nabla_b X_a) + o(\lambda)\,. \tag{7.11}$$

In the special case $X_a = 0$ this means that γ_{ab} is conformal to a metric $\tilde{\gamma}_{ab}$ of the form $\delta_{ab} + o(\lambda)$, which is essentially the ansatz used above to construct solutions of the constraints.

It remains to discuss the meaning of the quantity X_a. This quantity represents gauge freedom in the sense that it can be transformed away using a λ-dependent change of coordinates. Unfortunately, for a general solution of the Vlasov-Einstein system the coordinates which would be required are not consistent with the condition $g_{0a} = 0$ used in this paper. This fact is relevant to the discussion of the importance of condition (iii) of Theorem 1.1 for the following reason. To get clear-cut results it would be very useful to have (7.7) holding for the whole solution and not just on the initial hypersurface. The discussion in [21] suggests that in order to do this it would be necessary to analyse the post-Newtonian equations, i.e. to extend the asymptotic expansions in the conclusions of Theorem 1.1 by an additional power of λ. The post-Newtonian equations are very complicated and the only thing which makes them amenable to analysis at all is the use of a coordinate condition which sets X_a to zero. As stated above, this cannot be done in the coordinates used in the proof of Theorem 1.1. Thus it seems likely that a full clarification of the meaning of condition (iii) would require a generalisation of the results of this paper which allows coordinate conditions better adapted to the post-Newtonian equations.

Appendix 1. Some Elliptic Theory

The discussion here will be limited to results which are not easily extracted from the literature. For background material on the Poisson integral see [7]. In this appendix Δ always denotes the Laplacian of the standard flat metric on $\mathbb{R}^3$. For any function ϕ on $\mathbb{R}^3$ and a positive real number ε let $\phi_\varepsilon(x) = \phi(\varepsilon x)$. If f is a continuous function belonging to $L^p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for some $p < 3/2$ then it possesses a Newtonian potential u which is C^1. This is a solution of the Poisson equation $\Delta u = f$ in the sense of distributions and can be obtained as the limit for $\varepsilon \to 0$ of the Newtonian potentials of the compactly supported functions $\phi_\varepsilon f$, where ϕ is any C^∞ function of compact support which takes the value 1 in a neighbourhood of the origin. The Newtonian potentials of the functions $\phi_\varepsilon f$ can be represented by the familiar Poisson integral. They tend to zero at infinity. Pick a sequence ε_n tending to zero and let $f_n = \phi_{\varepsilon_n} f$. Then the f_n are bounded in $L^\infty(\mathbb{R}^3)$ and converge to f in $L^p(\mathbb{R}^3)$. Let u_n denote the Newtonian potential of f_n. A useful estimate for the Poisson integral will now be derived (cf. [2]). For any $R > 0$:

$$\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\,dy = \int_{|x-y|<R} \frac{f(y)}{|x-y|}\,dy + \int_{|x-y|>R} \frac{f(y)}{|x-y|}\,dy$$
$$\leq C(\|f\|_\infty R^2 + \|f\|_p R^{(3-q)/q}).$$

Here it has been assumed that the conjugate exponent q of p is greater than 3, which implies that $p < 3/2$. Putting $R = (\|f\|_p/\|f\|_\infty)^{p/3}$ gives the estimate

$$\|u_n\|_\infty \leq C\|f_n\|_p^{2p/3}\|f_n\|_\infty^{1-2p/3}, \quad p < 3/2. \tag{A1}$$

The derivatives of u_n can be estimated similarly.

$$\|\nabla u_n\|_\infty \leq C\|f_n\|_p^{p/3}\|f_n\|_\infty^{1-p/3}, \quad p < 3. \tag{A2}$$

These estimates show that u_n and its first derivatives converge uniformly to u and its first derivatives. In particular u tends to zero as $|x| \to \infty$. Now all the functions u_n have the property that $u_n(x) = O(|x|^{-1})$ and $\nabla u_n(x) = O(|x|^{-2})$ as $|x| \to \infty$. Hence a partial integration shows that

$$\|\nabla u_n\|_2^2 \leq \|u_n\|_\infty \|f_n\|_1. \tag{A3}$$

If $f \in L^1(R^3)$ it follows that ∇u_n is a bounded sequence in $L^2(R^3)$. Since this sequence also converges pointwise it follows from Fatou's lemma that ∇u is in $L^2(R^3)$ and that it satisfies the equivalent of (A3). Putting this together with (A1) gives

$$\|\nabla u\|_2 \leq C\|f\|_\infty^{1/6}\|f\|_1^{5/6}. \tag{A4}$$

Another estimate which is satisfied by u when $f \in H^s(\mathbb{R}^3)$ is

$$\|\partial_a \partial_b u\|_{H^s} \leq \|f\|_{H^s}. \tag{A5}$$

It has now been shown that if $f \in L^1(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$ then its Newtonian potential u belongs to $L^\infty(\mathbb{R}^3)$ and $\partial_a u$ belongs to $H^{s+1}(\mathbb{R}^3)$. In the following another related result will be required. Let H^{ab} be a tensor on $\mathbb{R}^3$ whose components belong to $L^p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for some $p < 6$ and which has the property that $\partial_c H^{ab}$ belongs to $H^{s-1}(\mathbb{R}^3)$. Let h be a function which is bounded and whose first derivatives are in $H^s(\mathbb{R}^3)$. Consider the equation $\Delta u = H^{ab}\partial_a\partial_b h$. By Hölder's inequality the expression on the right-hand side of this equation belongs to $L^q(\mathbb{R}^3)$ for some $q < 3/2$. Thus it can be concluded from the above discussion that u is in $L^\infty(\mathbb{R}^3)$ and an estimate for its norm in that space follows from (A1). Now let $H_n^{ab} = \phi_{\varepsilon_n} H^{ab}$ and let u_n be the solution of $\Delta u_n = H_n^{ab}\partial_a\partial_b h$,

$$\|\nabla u_n\|_2^2 = -\int_{\mathbb{R}^3} u_n H_n^{ab}\partial_a\partial_b h$$
$$= -\int_{\mathbb{R}^3} (\partial_a u_n H_n^{ab}\partial_b h + u_n\partial_a H_n^{ab}\partial_b h)\,.$$

Thus the L^2 norm of ∇u_n is bounded and it is possible to argue as above that ∇u is in $L^2(\mathbb{R}^3)$ and that its norm can be estimated in terms of $\|H^{ab}\|_\infty$, $\|\partial_a H^{ab}\|_2$ and $\|\partial_b h\|_2$.

Lemma A1. *Let γ_{ab} be a Riemannian metric such that $\gamma_{ab}-\delta_{ab} \in L^p(\mathbb{R}^3)\cap K^s(\mathbb{R}^3)$ for some $p < 6$ and some $s \geq 3$ and let Γ^a be its contracted Christoffel symbols. Let ϱ be a function belonging to $L^1(\mathbb{R}^3)\cap H^{s-1}(\mathbb{R}^3)$ and λ a non-negative real number. Then there exist positive constants C_1 and C_2 such that if $\|\gamma_{ab} - \delta_{ab}\|_p + \|\gamma_{ab} - \delta_{ab}\|_{K^{s-1}} \leq C_1$ and $(\|\varrho\|_1 + \|\varrho\|_{H^{s-1}})\lambda \leq C_2$ then the equation*

$$\Delta\phi = (\delta^{ab} - \gamma^{ab})\partial_a\partial_b\phi + \Gamma^a\partial_a\phi + \lambda\varrho\phi \tag{A6}$$

has a unique solution with the property that $\phi \to 1$ as $|x| \to \infty$. Moreover for any $k \leq s$ the solution satisfies an estimate of the form

$$\|\lambda^{-1}(\phi - 1)\|_\infty + \|\lambda^{-1}\nabla\phi\|_{H^k} \leq C \tag{A7}$$

for a constant C only depending on C_1, C_2, $\|\gamma_{ab} - \delta_{ab}\|_{K^s}$ and $\|\varrho\|_1 + \|\varrho\|_{H^{s-1}}$.

Proof. Define an iteration by solving

$$\Delta\phi_{n+1} = (\delta^{ab} - \gamma^{ab})\partial_a\partial_b\phi_n + \Gamma^a\partial_a\phi_n + \lambda\varrho\phi_n\,, \tag{A8}$$

with $\phi_{n+1} \to 1$ as $|x| \to \infty$ and $\phi_0 = 1$. Let $q_n = \Gamma^a\partial_a\phi + \lambda\varrho\phi$. If $\phi_n \in L^\infty(\mathbb{R}^3)$ and $\nabla\phi_n \in H^s(\mathbb{R}^3)$ then $q_n \in L^1(\mathbb{R}^3) \cap H^{s-1}(\mathbb{R}^3)$, and so by the above remarks concerning the Poisson equation the solution ϕ_{n+1} exists. Furthermore

$$\begin{aligned}\|q_n\|_1 + \|q_n\|_{H^{k-1}} \leq\ & C(\|\gamma_{ab} - \delta_{ab}\|_{K^k} + \lambda(\|\varrho\|_1 + \|\varrho\|_{H^{k-1}}))\\ & \times (\|\phi_n\|_\infty + \|\nabla\phi_n\|_{H^k})\end{aligned} \tag{A9}$$

for any k with $3 \leq k \leq s$. Thus there exist constants C_1 and C_2 such that if the inequalities $\|\gamma_{ab} - \delta_{ab}\|_{L^p} + \|\gamma_{ab} - \delta_{ab}\|_{K^3} \leq C_1$ and $(\|\varrho\|_1 + \|\varrho\|_{H^{s-1}})\lambda \leq C_2$ hold then

$$\begin{aligned}&\|\phi_{n+1} - \phi_n\|_\infty + \|\nabla\phi_{n+1} - \nabla\phi_n\|_{H^3}\\ &\quad\leq K(\|\phi_n - \phi_{n-1}\|_\infty + \|\nabla\phi_n - \nabla\phi_{n-1}\|_{H^3})\end{aligned} \tag{A10}$$

for some $K < 1$. It follows that $\{\phi_n\}$ and $\{\nabla\phi_n\}$ are Cauchy sequences in $L^\infty(\mathbb{R}^3)$ and $H^3(\mathbb{R}^3)$ respectively and hence ϕ_n converges to a solution of (A6) with $\phi \to 1$ as $|x| \to \infty$. This solution satisfies

$$\|\phi\|_\infty + \|\nabla\phi\|_{H^3} \leq C \tag{A11}$$

for some C depending only on C_1 and C_2. Dividing (A6) by λ gives

$$\Delta(\lambda^{-1}\phi) = (\delta^{ab} - \gamma^{ab})\partial_a\partial_b(\lambda^{-1}\phi) + \Gamma^a\partial_a(\lambda^{-1}\phi) + \varrho\phi . \tag{A12}$$

Estimating the quantity $\lambda^{-1}\phi_n$ in the same way as ϕ_n was estimated above shows that $\{\lambda^{-1}(\phi_n - 1)\}$ and $\{\lambda^{-1}\nabla\phi_n\}$ are Cauchy sequences in $L^\infty(\mathbb{R}^3)$ and $H^3(\mathbb{R}^3)$ respectively and that (A7) holds for $k = 3$. To see that it is true for any $k \leq s$, use the information we already have in the right-hand side of (A12). Using the estimates for the Poisson equation stated at the beginning of this appendix then shows that if the hypotheses of the lemma hold and if $s \geq 4$ we obtain (A7) for $k = 4$ after possibly reducing the size of C_1. This process can be repeated until $k = s$. □

Appendix 2. Estimates for Modified Sobolev Spaces

This appendix is concerned with proving some useful estimates for the modified Sobolev spaces $K^s(\mathbb{R}^3)$ introduced in Sect. 3. Recall that

$$\|f\|_{K^s} = \|f\|_\infty + \|\nabla f\|_{H^{s-1}} . \tag{A13}$$

The results to be proved are analogues of the results (3.4) and (3.5) for functions belonging to ordinary Sobolev spaces and are proved in a similar way. Estimates of this type are discussed in [14, 16 and 23]. First analogues of (3.4) will be discussed. Supposed that f, g belong to $K^s(\mathbb{R}^3)$ for some $s \geq 2$. Note first the obvious fact that $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$. If α is a multi-index with $1 \leq |\alpha| \leq s$ then

$$D^\alpha(fg) = D^\alpha f g + f D^\alpha g + \sum \frac{\alpha!}{\beta!\gamma!} D^\beta f D^\gamma g . \tag{A14}$$

The sum is taken over all multi-indices β and γ with $|\beta| + |\gamma| = |\alpha|$ and $\max(|\beta|, |\gamma|) < |\alpha|$. The first estimate we wish to prove is that

$$\|D^\alpha(fg)\|_2 \leq C\|f\|_{K^s}\|g\|_{K^s} \tag{A15}$$

for some constant C. It is elementary to estimate the first two terms in (A14) by the right-hand side of (A15) and so we can concentrate on the third term. Consider one of the summands there. Suppose first that either β or γ is less than $s - 3/2$. Without loss of generality we can assume that it is β. Then

$$\|D^\beta f D^\gamma g\|_2 \leq \|D^\beta f\|_\infty \|D^\gamma g\|_2 \leq C\|D^\beta f\|_{H^2}\|D^\gamma g\|_2 . \tag{A16}$$

If on the other hand both β and γ are greater than $s - 3/2$ then it can be concluded that $s < 3$. The only estimate which remains to be done to establish (A15) is

$$\|DfDg\|_2 \leq \|Df\|_4\|Dg\|_4 \leq \|Df\|_{H^1}\|Dg\|_{H^1} , \tag{A17}$$

where the Hölder and Sobolev inequalities have been used. The inequality (A15) shows that multiplication is a continuous mapping from $K^s(\mathbb{R}^3)$ to itself. This is a statement of a weaker type than (3.4). A stronger result could presumably be obtained using the Gagliardo-Nirenberg inequality but that will not be attempted here since (A15) is sufficient for the applications in this paper. In a similar way it can be shown that if $f \in H^s(\mathbb{R}^3)$ and $g \in K^s(\mathbb{R}^3)$ for $s \geq 2$, then $fg \in H^s(\mathbb{R}^3)$ and

$$\|fg\|_{H^s} \leq \|f\|_{H^s}\|g\|_{K^s} . \tag{A18}$$

An estimate related to (3.5) can also be obtained if $s \geq 3$, namely

$$\|D^\alpha(fg) - fD^\alpha g\|_2 \leq C\|f\|_{K^s}\|g\|_{K^{s-1}}, \quad 1 \leq |\alpha| \leq s. \tag{A19}$$

The final estimate which is required concerns composition with a C^r function. Suppose then that U is an open interval in $\mathbb{R}$ and $F: U \to \mathbb{R}$ a C^r function whose derivatives up to order r are bounded on U. Let f be a function in $K^s(\mathbb{R}^3)$, where $2 \leq s \leq r$, whose range is contained in U. Now

$$D^\alpha(F(f)) = \sum C_{r\alpha_1\ldots\alpha_l} \frac{d^k F}{df^k} D^{a_1} f \ldots D^{\alpha_l} f, \tag{A20}$$

where $1 \leq k \leq |\alpha|$, $|\alpha_1| + \ldots + |\alpha_l| = |\alpha|$ and $|\alpha_i| \geq 1$ for all i. By an argument similar to those given above it is seen that the terms on the right-hand side of (A20) can be estimated straightforwardly unless $s = 2$. In that exceptional case the embedding $H^1(\mathbb{R}^3) \to L^4(R^3)$ can be used. There results the estimate

$$\|F(f)\|_{K^s} \leq C\|F\|_{C^r}\|f\|^s_{K^s}. \tag{A21}$$

In this paper (A21) is only needed to estimate ϕ^{-1} and the inverse metric γ^{ab}. In both cases it is applied to the function $F(f) = 1/f$ with $U = (c, \infty)$ for some $c > 0$.

Acknowledgements. I thank D. Christodoulou for showing me the argument involving integration by parts in time used in the proof of Lemma 2.1 and again in Sect. 4. I am also grateful to G. Rein for helpful comments.

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Communicated by S.-T. Yau