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Free Boson Realization of $U_q(\widehat{sl_N})$

Hidetoshi Awata[1,★], **Satoru Odake**[1,★,★★], **Jun'ichi Shiraishi**[2,★★★]

[1] Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-01, Japan
[2] Reasearch Institute for Mathematical Sciences, Kyoto University, Kyoto 606-01, Japan

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Abstract: We construct a realization of the quantum affine algebra $U_q(\widehat{sl_N})$ of an arbitrary level k in terms of free boson fields. In the $q \to 1$ limit this realization becomes the Wakimoto realization of $\widehat{sl_N}$. The screening currents and the vertex operators (primary fields) are also constructed; the former commutes with $U_q(\widehat{sl_N})$ modulo total difference, and the latter creates the $U_q(\widehat{sl_N})$ highest weight state from the vacuum state of the boson Fock space.

1. Introduction

Chiral algebras such as the Virasoro and current algebras play a central role in conformal field theory (CFT) in two dimensional space-time. This theory is a quantum field theory (QFT) of massless particles, in other words, a (massive) QFT at a critical point (renormalization-group fixed point) [1]. Perturbing CFT's suitably, we get integrable massive QFT's [2, 3, 4]. In these theories, the Virasoro algebra does not exist any longer. In many cases the quantum affine Lie algebra plays a crucial role instead of the Virasoro algebra [5]. This quantum algebra is, for a large part, at the origin of the integrability. Moreover it can almost determine the S-matrix of the theory, e.g. sine-Gordon model [5].

The Wess-Zumino-Novikov-Witten (WZNW) model is a fundamental example of CFT's; many CFT's can be realized through a coset construction of WZNW models. The WZNW model has been studied based on the representation theory of the affine Lie algebra. Correlation functions of this model, which are vacuum expectation values of vertex operators, satisfy certain holomorphic differential equations, what

★ Fellow of Soryushi Shogakukai
★★ Address after June 1: Department of Physics, Faculty of Liberal Arts, Shinshu University, Matsumoto 390, Japan
★★★ On leave from Department of Physics, University of Tokyo, Tokyo 113, Japan

is called, Knizhnik–Zamolodchikov (KZ) equations [6, 7]. We expect that the "q-WZNW model," which has a symmetry of the quantum affine algebra, is a certain massive deformation of the WZNW model. Correlation functions of the q-WZNW model satisfy q-difference equations (q-KZ equations) [8, 9]. Connection matrices of solutions for q-KZ equations are related to elliptic solutions of the Yang–Baxter equations of RSOS models [9]. An application of q-vertex operators based on $U_q(\widehat{sl_2})$ was performed in diagonalization of the XXZ spin chain [10].

Free field realizations of the Virasoro and affine Lie algebras were useful for studying representation theories [11] and calculating correlation functions [12, 13]. It is expected that this is also the case for the quantum affine algebras. In fact, the integral formula for correlation functions of the local operators of the XXZ spin chain was found by using the free boson realization of $U_q(\widehat{sl_2})$ and bosonized q-vertex operators [14, 15]. To study higher rank versions of the XXZ spin chain, sine-Gordon model, etc., we need free field realizations of the quantum affine algebras.

In this paper we construct a free boson realization of the quantum affine algebra $U_q(\widehat{sl_N})$ with an arbitrary level k. In the $q \to 1$ limit, it becomes the bosonized version of the Wakimoto realization of $\widehat{sl_N}$ [16, 17, 18]. Free field realizations of $U_q(\widehat{sl_N})$ with level 1 were constructed in [19]. Free field realizations of $U_q(\widehat{sl_2})$ with an arbitrary level were constructed by several authors [20, 21, 22, 23] and that of $U_q(\widehat{sl_3})$ was obtained by the present authors [24]. We construct a free boson realization of $U_q(\widehat{sl_N})$ by affinizing the Heisenberg realization (q-difference operator realization) of $U_q(sl_N)$ [25] and prove it by the OPE (operator product expansion) technique. The screening currents and the vertex operators (primary fields) are also constructed. They are necessary ingredients for calculating correlation functions. A certain integral of the screening current commutes with $U_q(\widehat{sl_N})$ and the vertex operator creates the highest weight state of $U_q(\widehat{sl_N})$ from the vacuum state of the boson Fock space.

This paper is organized as follows. In Sect. 2 we fix our notations and recall the definition of $U_q(\widehat{sl_N})$. We construct a free field realization of $U_q(\widehat{sl_N})$ in Sect. 3, and the screening currents and the vertex operators in Sect. 4. Section 5 is devoted to discussion. The grading operator is also bosonized. In Appendix A we present the Heisenberg realization of $U_q(sl_N)$. In Appendix B q-difference expressions of our free field realization are given. In Appendix C, D we give useful formulas and some details of calculations.

2. Notations

Throughout this paper, the complex numbers q and k are fixed. q is assumed to be a generic value with $|q| < 1$. We will use the standard symbol $[x]$,

$$[x] \stackrel{\text{def}}{=} \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{2.1}$$

and $\sum_{r=n}^{n-1} * \stackrel{\text{def}}{=} 0$, $\prod_{r=n}^{n-1} * \stackrel{\text{def}}{=} 1$. Let $\bar{\alpha}_i$, $\bar{\Lambda}_i$ $(1 \leqq i \leqq N-1)$, $(a_{ij})_{1 \leqq i,j \leqq N-1}$, be the simple roots, fundamental weights, the Cartan matrix of sl_N respectively. $(\cdot, \cdot)$ is the symmetric bilinear form; $(\bar{\alpha}_i, \bar{\alpha}_j) = a_{ij}$, $(\bar{\Lambda}_i, \bar{\alpha}_j) = \delta_{ij}$. g stands for the dual Coxeter number of sl_N, i.e., $g = N$.

The q-difference operator with a parameter α is defined by [20]

$$ {}_\alpha\partial_z f(z) \stackrel{\text{def}}{=} \frac{f(q^\alpha z)-f(q^{-\alpha}z)}{(q-q^{-1})z} . \tag{2.2}$$

The Jackson integral with parameters $p \in \mathbb{C}(|p|<1)$ and $s \in \mathbb{C}^\times$ is defined by

$$ \int_0^{s\infty} f(z)d_pz \stackrel{\text{def}}{=} s(1-p)\sum_{n\in\mathbb{Z}} f(sp^n)p^n . \tag{2.3}$$

These operations satisfy the following property:

$$ \int_0^{s\infty} {}_\alpha\partial_z f(z)d_pz = 0 \quad \text{for } p=q^{2\alpha} . \tag{2.4}$$

The deformed commutator with a parameter $p \in \mathbb{C}$ is

$$ [A,B]_p \stackrel{\text{def}}{=} AB - pBA . \tag{2.5}$$

The quantum affine algebra $U_q(\widehat{sl_N})$ is the associative algebra over $\mathbb{C}$ with Chevalley generators $e_i^\pm$, invertible t_i $(i=0,1,\dots,N-1)$, and the following relations [26][1]:

$$ [t_i,t_j]=0 , \tag{2.6}$$
$$ t_ie_j^\pm t_i^{-1} = q^{\pm a_{ij}^{\text{ext}}} e_j^\pm , \tag{2.7}$$
$$ [e_i^+,e_j^-] = \delta_{ij}\frac{t_i-t_i^{-1}}{q-q^{-1}} , \tag{2.8}$$

and

$$ \sum_{r=0}^{1-a_{ij}^{\text{ext}}} (-1)^r \begin{bmatrix} 1-a_{ij}^{\text{ext}} \\ r \end{bmatrix} (e_i^\pm)^{1-a_{ij}^{\text{ext}}-r} e_j^\pm (e_i^\pm)^r = 0 , \tag{2.9}$$

where $(a_{ij}^{\text{ext}})_{0\leqq i,j\leqq N-1}$ is the Cartan matrix of the extended Dynkin diagram of sl_N and $\left[{n \atop r}\right] \stackrel{\text{def}}{=} \frac{[n]!}{[r]![n-r]!}$, $[n]! \stackrel{\text{def}}{=} \prod_{r=1}^n [r]$.

$U_q(\widehat{sl_N})$ is isomorphic to the associative algebra over $\mathbb{C}$ with Drinfeld generators $E_n^{\pm,i}$ $(n\in\mathbb{Z})$, H_n^i $(n\in\mathbb{Z}-\{0\})$, invertible K_i $(i=1,2,\dots,N-1)$, invertible γ, and the following relations [27]:

$$ \gamma : \text{central element} , \tag{2.10}$$
$$ [K_i,H_n^j]=0,\ K_iE_n^{\pm,j}K_i^{-1} = q^{\pm a_{ij}}E_n^{\pm,j} , \tag{2.11}$$
$$ [H_n^i,H_m^j] = \frac{1}{n}[a_{ij}n]\frac{\gamma^n-\gamma^{-n}}{q-q^{-1}}\delta_{n+m,0} , \tag{2.12}$$
$$ [H_n^i,E_m^{\pm,j}] = \pm\frac{1}{n}[a_{ij}n]\gamma^{\mp\frac{1}{2}|n|}E_{n+m}^{\pm,j} , \tag{2.13}$$
$$ [E_n^{+,i},E_m^{-,j}] = \frac{\delta^{ij}}{q-q^{-1}}\left(\gamma^{\frac{1}{2}(n-m)}\psi_{+,n+m}^i - \gamma^{-\frac{1}{2}(n-m)}\psi_{-,n+m}^i\right) , \tag{2.14}$$

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[1] For the grading operator d, see Sect 5

and

$$[E^{\pm,i}_{n+1}, E^{\pm,j}_{m}]_{q^{\pm a_{ij}}} + [E^{\pm,j}_{m+1}, E^{\pm,i}_{n}]_{q^{\pm a_{ij}}} = 0\,, \tag{2.15}$$

$$[E^{\pm,i}_{n}, E^{\pm,j}_{m}] = 0 \quad \text{for } a_{ij} = 0\,, \tag{2.16}$$

$$[E^{\pm,i}_{n}, [E^{\pm,i}_{m}, E^{\pm,j}_{\ell}]_{q^{\mp1}}]_{q^{\pm1}} + [E^{\pm,i}_{m}, [E^{\pm,i}_{n}, E^{\pm,j}_{\ell}]_{q^{\mp1}}]_{q^{\pm1}} = 0 \quad \text{for } a_{ij} = -1\,. \tag{2.17}$$

Here $\psi^i_{\pm,n}$ are defined by the following equation:

$$\sum_{n\in\mathbb{Z}} \psi^i_{\pm,n} z^{-n} \stackrel{\text{def}}{=} K_i^{\pm1} \exp\left(\pm(q-q^{-1}) \sum_{\pm n>0} H^i_n z^{-n}\right)\,. \tag{2.18}$$

Let H^i_0 be defined by

$$K_i \stackrel{\text{def}}{=} \exp\Big((q-q^{-1})\frac{1}{2}H^i_0\Big)\,, \tag{2.19}$$

then Eqs (2.11)–(2.13) hold for H^i_n $(n \in \mathbb{Z})$[2]. Equation (2.11) is derived from Eqs. (2.12), (2.13).

Defining the fields $H^i(z)$, $E^{\pm,i}(z)$ and $\psi^i_\pm(z)$ as

$$H^i(z) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} H^i_n z^{-n-1}, \quad E^{\pm,i}(z) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} E^{\pm,i}_n z^{-n-1},$$
$$\psi^i_\pm(z) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} \psi^i_{\pm,n} z^{-n}, \tag{2.20}$$

the above relations can be rewritten as formal power series equations:

$$[\psi^i_\pm(z), \psi^i_\pm(w)] = 0\,, \tag{2.21}$$

$$(z - q^{a_{ij}}\gamma^{-1}w)(z - q^{-a_{ij}}\gamma w)\psi^i_+(z)\psi^j_-(w)$$
$$= (z - q^{a_{ij}}\gamma w)(z - q^{-a_{ij}}\gamma^{-1}w)\psi^j_-(w)\psi^i_+(z)\,, \tag{2.22}$$

$$(z - q^{\pm a_{ij}}\gamma^{\mp\frac{1}{2}}w)\psi^i_+(z)E^{\pm,j}(w) = (q^{\pm a_{ij}}z - \gamma^{\mp\frac{1}{2}}w)E^{\pm,j}(w)\psi^i_+(z)\,, \tag{2.23}$$

$$(z - q^{\pm a_{ij}}\gamma^{\mp\frac{1}{2}}w)E^{\pm,j}(z)\psi^i_-(w) = (q^{\pm a_{ij}}z - \gamma^{\mp\frac{1}{2}}w)\psi^i_-(w)E^{\pm,j}(z)\,, \tag{2.24}$$

$$[E^{+,i}(z), E^{-,j}(w)] = \frac{\delta^{ij}}{(q-q^{-1})zw}(\delta(z^{-1}w\gamma)\psi^i_+(\gamma^{\frac{1}{2}}w)$$
$$-\delta(z^{-1}w\gamma^{-1})\psi^i_-(\gamma^{-\frac{1}{2}}w))\,, \tag{2.25}$$

and

[2] In the case of $n=0$, $\frac{1}{n}*$ should be understood as $\lim_{n\to0}\frac{1}{n}*$. For example, $\lim_{n\to0}\frac{1}{n}[n] = \frac{2\log q}{q-q^{-1}}$, $\lim_{n\to0}\frac{1}{n}[a_{ij}n]\frac{\gamma^n-\gamma^{-n}}{q-q^{-1}} = 0$, $\lim_{n\to0}\frac{1}{n}[a_{ij}n]\gamma^{\mp\frac{1}{2}|n|} = \frac{2\log q}{q-q^{-1}}a_{ij}$. In the following, this convention is assumed

$$(z - q^{\pm a_{ij}} w) E^{\pm,i}(z) E^{\pm,j}(w) = (q^{\pm a_{ij}} z - w) E^{\pm,j}(w) E^{\pm,i}(z) , \tag{2.26}$$
$$E^{\pm,i}(z) E^{\pm,j}(w) = E^{\pm,j}(w) E^{\pm,i}(z) \quad \text{for } a_{ij} = 0 , \tag{2.27}$$
$$\begin{aligned} &E^{\pm,i}(z_1) E^{\pm,i}(z_2) E^{\pm,j}(w) - (q+q^{-1}) E^{\pm,i}(z_1) E^{\pm,j}(w) E^{\pm,i}(z_2) \\ &\quad + E^{\pm,j}(w) E^{\pm,i}(z_1) E^{\pm,i}(z_2) \\ &\quad + (\text{replacement: } z_1 \leftrightarrow z_2) = 0 \quad \text{for } a_{ij} = -1 , \end{aligned} \tag{2.28}$$

where $\delta(x)$ is given by

$$\delta(x) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} x^n . \tag{2.29}$$

Correspondence between Chevalley generators and Drinfeld generators are [27]:

$$t_i \mapsto K_i \quad (i = 1, \ldots, N-1) , \tag{2.30}$$
$$e_i^{\pm} \mapsto E_0^{\pm,i} \quad (i = 1, \ldots, N-1) , \tag{2.31}$$
$$t_0 \mapsto \gamma K_1^{-1} \ldots K_{N-1}^{-1} , \tag{2.32}$$
$$\begin{aligned} e_0^+ \mapsto &[E_0^{-,N-1}, [E_0^{-,N-2}, [\ldots, [E_0^{-,2}, E_1^{-,1}]_{q^{-1}} \ldots]_{q^{-1}}]_{q^{-1}} \\ &\times K_1^{-1} \ldots, K_{N-1}^{-1} , \end{aligned} \tag{2.33}$$
$$\begin{aligned} e_0^- \mapsto &K_1 \ldots K_{N-1} \\ &\times [[\ldots [E_{-1}^{+,1}, E_0^{+,2}]_q, \ldots, \\ &E_0^{+,N-2}]_q, E_0^{+,N-1}]_q . \end{aligned} \tag{2.34}$$

$U_q(\widehat{sl_N})$ has the Hopf algebra structure. We take its coproduct Δ as

$$\Delta(t_i) = t_i \otimes t_i , \tag{2.35}$$
$$\Delta(e_i^+) = e_i^+ \otimes 1 + t_i \otimes e_i^+ , \tag{2.36}$$
$$\Delta(e_i^-) = e_i^- \otimes t_i^{-1} + 1 \otimes e_i^- , \tag{2.37}$$

and its antipode S is

$$S(t_i) = t_i^{-1}, \quad S(e_i^+) = -t_i^{-1} e_i^+, \quad S(e_i^-) = -e_i^- t_i . \tag{2.38}$$

An explicit coproduct formula for all the Drinfeld generators has not been obtained.

Let $V(\lambda)$ be the Verma module over $U_q(\widehat{sl_N})$ generated by the highest weight state $|\lambda\rangle$, such that

$$H_n^i \mid \lambda\rangle = E_n^{\pm,i} \mid \lambda\rangle = 0 \ (n > 0) , \tag{2.39}$$
$$E_0^{+,i} \mid \lambda\rangle = 0 , \tag{2.40}$$
$$H_0^i \mid \lambda\rangle = \ell^i \mid \lambda\rangle , \tag{2.41}$$

where the classical part of the highest weight is $\bar{\lambda} = \sum_{i=1}^{N-1} \ell^i \bar{\Lambda}_i$.

Next we will introduce boson fields. For a set of bosonic oscillators a_n ($n \in \mathbb{Z}$), and zero modes $\hat{p}_a, \hat{q}_a$ whose commutation relations are

$$[a_n, a_m] = n \rho_a(n) \delta_{n+m,0}, \quad a_0 = \frac{2 \log q}{q - q^{-1}} \hat{p}_a , \tag{2.42}$$

$$[\hat{p}_a, \hat{q}_a] = \rho_a, \quad [a_n, \hat{q}_a] = 0 \ (n \neq 0) , \tag{2.43}$$

where ρ_a is a constant and $\rho_a(n)$ satisfies

$$\lim_{q\to 1}\rho_a(n) = \rho_a, \quad \lim_{n\to 0}\rho_a(n) = \left(\frac{2\log q}{q-q^{-1}}\right)^2 \rho_a\,, \tag{2.44}$$

we define free boson fields $a(z;\alpha)$ and $a_\pm(z)$ as follows:

$$a(z;\alpha) \stackrel{\mathrm{def}}{=} -\sum_{n\neq 0}\frac{a_n}{[n]}q^{-\alpha|n|}z^{-n} + \hat{q}_a + \hat{p}_a\log z\,, \tag{2.45}$$

$$a_\pm(z) \stackrel{\mathrm{def}}{=} \pm\left((q-q^{-1})\sum_{\pm n>0}a_n z^{-n} + \hat{p}_a\log q\right) \tag{2.46}$$

$$= \pm(q-q^{-1})\left(\sum_{\pm n>0}a_n z^{-n} + \frac{1}{2}a_0\right). \tag{2.47}$$

We abbreviate $a(z;0)$ as $a(z) \stackrel{\mathrm{def}}{=} a(z;0)$. In the $q\to 1$ limit $a(z;\alpha)$ becomes the free chiral boson field $\phi(z)$ used in the string theory and CFT (but the meaning of z is different). Correspondence between $a(z;\alpha)$ and $\phi(z) = \hat{x} - \sqrt{-1}\hat{p}\log z + \sqrt{-1}\sum_{n\neq 0}\frac{1}{n}\alpha_n z^{-n}$ is

$$a(z;\alpha) \to \sqrt{-1}\sqrt{\rho_a}\phi(z), \quad a_n \to \sqrt{\rho_a}\alpha_n, \quad \hat{p}_a \to \sqrt{\rho_a}\hat{p},$$
$$\hat{q}_a \to \sqrt{-1}\sqrt{\rho_a}\hat{x}\,. \tag{2.48}$$

Moreover let us define boson fields with parameters L, M as follows:

$$a(L_1,\ldots,L_r;M_1,\ldots,M_r \mid z;\alpha)$$
$$\stackrel{\mathrm{def}}{=} -\sum_{n\neq 0}\frac{[L_1 n]\ldots[L_r n]}{[M_1 n]\ldots[M_r n]}\frac{a_n}{[n]}q^{-\alpha|n|}z^{-n} + \frac{L_1\ldots L_r}{M_1\ldots M_r}(\hat{q}_a + \hat{p}_a\log z)\,, \tag{2.49}$$

$$a_\pm(L_1\ldots,L_r;M_1,\ldots,M_r \mid z)$$
$$\stackrel{\mathrm{def}}{=} \pm\left((q-q^{-1})\sum_{\pm n>0}\frac{[L_1 n]\ldots[L_r n]}{[M_1 n]\ldots[M_r n]}a_n z^{-n} + \frac{L_1\ldots L_r}{M_1\ldots M_r}\hat{p}_a\log q\right)$$
$$= \pm(q-q^{-1})\left(\sum_{\pm n>0}\frac{[L_1 n]\ldots[L_r n]}{[M_1 n]\ldots[M_r n]}a_n z^{-n} + \frac{L_1\ldots L_r}{M_1\ldots M_r}\frac{1}{2}a_0\right). \tag{2.50}$$

We abbreviate these as

$$\left(\frac{L_1}{M_1}\frac{L_2}{M_2}\cdots\frac{L_r}{M_r}a\right)(z;\alpha) \stackrel{\mathrm{def}}{=} a(L_1,L_2,\ldots,L_r;M_1,M_2,\ldots,M_r \mid z;\alpha)\,, \tag{2.51}$$

$$\left(\frac{L_1}{M_1}\frac{L_2}{M_2}\cdots\frac{L_r}{M_r}a_\pm\right)(z) \stackrel{\mathrm{def}}{=} a_\pm(L_1,L_2,\ldots,L_r;M_1,M_2,\ldots,M_r \mid z)\,. \tag{2.52}$$

Normal ordering prescription $:\ :$ is defined by

$$\begin{cases} \text{move } a_n\ (n>0) \text{ and } \hat{p}_a \text{ to right}\,, \\ \text{move } a_n\ (n<0) \text{ and } \hat{q}_a \text{ to left}\,. \end{cases} \tag{2.53}$$

For example,

$$: \exp(a(z;\alpha)) := \exp\left(-\sum_{n<0}\frac{a_n}{[n]}(q^{-\alpha}z)^{-n}\right)e^{\hat{q}_a}z^{\hat{p}_a}\exp\left(-\sum_{n>0}\frac{a_n}{[n]}(q^{\alpha}z)^{-n}\right). \quad (2.54)$$

For multicomponent a^i $(a^i_n, \hat{p}^i_a, \hat{q}^i_a)$, we treat them similarly; $[a^i_n, aj_m] = n\rho^{ij}_a(n)\delta_{n+m,0}$, etc. We can easily verify the following:

$$\left[\frac{1}{2}\sum_{i,j}\sum_{n\in\mathbb{Z}} : a^i_{-n}\rho_a^{-1,ij}(n)a^j_n :, a^\ell_m\right] = -ma^\ell_m , \quad (2.55)$$

where $\rho_a^{-1,ij}(n)$ is an inverse of $\rho^{ij}_a(n)$, i.e., $\sum_\ell \rho^{i\ell}_a(n)\rho_a^{-1,\ell j}(n) = \delta^{ij}$.

3. Free Boson Realization of $U_q(\widehat{sl_N})$

To construct the Drinfeld $U_q(\widehat{sl_N})$ generators of level k in terms of free boson fields, we need $N^2 - 1$ free boson fields a^i $(1 \leqq i \leqq N-1)$, b^{ij} and c^{ij} $(1 \leqq i < j \leqq N)$. Their commutation relations are

$$[a^i_n, a^j_m] = \frac{1}{n}[(k+g)n][a_{ij}n]\delta_{n+m,0}, \quad [\hat{p}^i_a, \hat{q}^j_a] = (k+g)a_{ij} , \quad (3.1)$$

$$[b^{ij}_n, b^{i'j'}_m] = -\frac{1}{n}[n]^2\delta^{ii'}\delta^{jj'}\delta_{n+m,0}, \quad [\hat{p}^{ij}_b, \hat{q}^{i'j'}_b] = -\delta^{ii'}\delta^{jj'} , \quad (3.2)$$

$$[c^{ij}_n, c^{i'j'}_m] = \frac{1}{n}[n]^2\delta^{ii'}\delta^{jj'}\delta_{n+m,0}, \quad [\hat{p}^{ij}_c, \hat{q}^{i'j'}_c] = \delta^{ii'}\delta^{jj'} , \quad (3.3)$$

and the remaining commutators vanish.

Let us define fields $H^i(z)$, $\psi^i_\pm(z)$ and $E^{\pm,i}(z)$ $(1 \leqq i \leqq N-1)$ as follows[3]:

$$\begin{aligned}
H^i(z) &\stackrel{\text{def}}{=} \frac{1}{(q-q^{-1})z} \\
&\times \Bigg(\sum_{j=1}^{i}(b^{j,i+1}_+(q^{\frac{k}{2}+j-1}z) - b^{j,i}_+(q^{\frac{k}{2}+j}z)) \\
&\quad + a^i_+(q^{\frac{g}{2}}z) + \sum_{j=i+1}^{N}(b^{i,j}_+(q^{\frac{k}{2}+j}z) - b^{i+1,j}_+(q^{\frac{k}{2}+j-1}z))\Bigg) \\
&\quad - (\text{replacement}: \ x_+(q^\alpha z) \mapsto x_-(q^{-\alpha}z) \text{for } x = a, b) , \quad (3.4)
\end{aligned}$$

[3] These operators are well-defined on the boson Fock space that will be defined in the next section

$$\begin{aligned}\psi_\pm^i(q^{\pm\frac{k}{2}}z) \stackrel{\text{def}}{=} :\exp\Bigg(&\sum_{j=1}^{i}(b_\pm^{j,i+1}(q^{\pm(k+j-1)}z) - b_\pm^{j,i}(q^{\pm(k+j)}z))\\&+ a_\pm^i(q^{\pm\frac{k+g}{2}}z)\\&+\sum_{j=i+1}^{N}(b_\pm^{i,j}(q^{\pm(k+j)}z) - b_\pm^{i+1,j}(q^{\pm(k+j-1)}z))\Bigg):,\end{aligned}\tag{3.5}$$

$$\begin{aligned}E^{+,i}(z) \stackrel{\text{def}}{=} \frac{-1}{(q-q^{-1})z}&\sum_{j=1}^{i}:\exp((b+c)^{j,i}(q^{j-1}z))\\&\times\big(\exp(b_+^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^jz))\\&-\exp(b_-^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^{j-2}z))\big)\\&\times\exp\Bigg(\sum_{\ell=1}^{j-1}(b_+^{\ell,i+1}(q^{\ell-1}z) - b_+^{\ell,i}(q^\ell z))\Bigg):,\end{aligned}\tag{3.6}$$

$$\begin{aligned}E^{-,i}(z) \stackrel{\text{def}}{=} &\frac{-1}{(q-q^{-1})z}\\&\times\Bigg(\sum_{j=1}^{i-1}:\exp\big((b+c)^{j,i+1}(q^{-(k+j)}z)\big)\\&\qquad\times\big(\exp\big(-b_-^{j,i}(q^{-(k+j)}z) - (b+c)^{j,i}(q^{-(k+j-1)}z)\big)\\&\qquad-\exp\big(-b_+^{j,i}(q^{-(k+j)}z) - (b+c)^{j,i}(q^{-(k+j+1)}z)\big)\big)\\&\qquad\times\exp\Bigg(\sum_{\ell=j+1}^{i}\big(b_-^{\ell,i+1}(q^{-(k+\ell-1)}z) - b_-^{\ell,i}(q^{-(k+\ell)}z)\big)\\&\qquad+a_-^i\big(q^{-\frac{k+g}{2}}z\big) + \sum_{\ell=i+1}^{N}\big(b_-^{i,\ell}(q^{-(k+\ell)}z) - b_-^{i+1,\ell}(q^{-(k+\ell-1)}z)\big)\Bigg):\\&+:\exp\big((b+c)^{i,i+1}(q^{-(k+i)}z)\big)\\&\quad\times\exp\Bigg(a_-^i\big(q^{-\frac{k+g}{2}}z\big) + \sum_{\ell=i+1}^{N}\big(b_-^{i,\ell}(q^{-(k+\ell)}z) - b_-^{i+1,\ell}(q^{-(k+\ell-1)}z)\big)\Bigg):\\&-:\exp\big((b+c)^{i,i+1}(q^{k+i}z)\big)\\&\quad\times\exp\Bigg(a_+^i\big(q^{\frac{k+g}{2}}z\big) + \sum_{\ell=i+1}^{N}\big(b_+^{i,\ell}(q^{k+\ell}z) - b_+^{i+1,\ell}(q^{k+\ell-1}z)\big)\Bigg):\\&-\sum_{j=i+2}^{N}:\exp\big((b+c)^{i,j}(q^{k+j-1}z)\big)\\&\qquad\times\big(\exp\big(b_+^{i+1,j}(q^{k+j-1}z) - (b+c)^{i+1,j}(q^{k+j}z)\big)\\&\qquad-\exp\big(b_-^{i+1,j}(q^{k+j-1}z) - (b+c)^{i+1,j}(q^{k+j-2}z)\big)\big)\\&\qquad\times\exp\Bigg(a_+^i\big(q^{\frac{k+g}{2}}z\big)\\&\qquad+\sum_{\ell=j}^{N}\big(b_+^{i,\ell}(q^{k+\ell}z) - b_+^{i+1,\ell}(q^{k+\ell-1}z)\big)\Bigg):\Bigg),\end{aligned}\tag{3.7}$$

where $b^{ii} \stackrel{\text{def}}{=} 0$, $c^{ii} \stackrel{\text{def}}{=} 0$ and $(b+c)^{ij} \stackrel{\text{def}}{=} b^{ij} + c^{ij}$. These expressions are guessed from free boson realizations of $U_q(\widehat{sl_2})$ [20], $U_q(\widehat{sl_3})$ [24] and the Heisenberg realization of $U_q(sl_N)$ [25] (Appendix A). q-difference expressions of these fields are given in Appendix B. In the $q \to 1$ limit, Eqs. (3.4), (3.6) and (3.7) become the bosonized version of the Wakimoto realization of $\widehat{sl_N}$ with level k [16, 17, 18].

From Eqs. (3.4) and (2.19), H_n^i and K_i are

$$\begin{aligned} H_n^i &= \sum_{j=1}^{i} \left(b_n^{j,i+1} q^{-(\frac{k}{2}+j-1)|n|} - b_n^{j,i} q^{-(\frac{k}{2}+j)|n|} \right) \\ &\quad + a_n^i q^{-\frac{g}{2}|n|} + \sum_{j=i+1}^{N} \left(b_n^{i,j} q^{-(\frac{k}{2}+j)|n|} - b_n^{i+1,j} q^{-(\frac{k}{2}+j-1)|n|} \right), \end{aligned} \tag{3.8}$$

$$K_i = q \sum_{j=1}^{i} \left(\hat{p}_b^{j,i+1} - \hat{p}_b^{j,i} \right) + \hat{p}_a^i + \sum_{j=i+1}^{N} \left(\hat{p}_b^{i,j} - \hat{p}_b^{i+1,j} \right). \tag{3.9}$$

We obtain the following proposition:

Proposition 1. *$H^i, \psi_\pm^i, E^{\pm,i}$ in Eqs.* (3.4)–(3.7) *satisfy the relations Eqs.* (2.10)–(2.13) *with $\gamma = q^k$, Eq.* (2.28), *and the following relations:*

$$\begin{aligned} &E^{+,i}(z) E^{-,j}(w) \simeq E^{-,j}(w) E^{+,i}(z) \\ &\sim \text{reg.} + \frac{\delta^{ij}}{(q-q^{-1})w} \left(\frac{1}{z-q^k w} \psi_+^i (q^{\frac{k}{2}} w) - \frac{1}{z-q^{-k} w} \psi_-^i (q^{-\frac{k}{2}} w) \right), \end{aligned} \tag{3.10}$$

$$(z - q^{\pm a_{ij}} w) E^{\pm,i}(z) E^{\pm,j}(w) \simeq (q^{\pm a_{ij}} z - w) E^{\pm,j}(w) E^{\pm,i}(z) \sim \text{reg.}, \tag{3.11}$$

$$E^{\pm,i}(z) E^{\pm,j}(w) \simeq E^{\pm,j}(w) E^{\pm,i}(z) \sim \text{reg} \quad \textit{for } a_{ij} = 0, \tag{3.12}$$

where the symbol $\simeq$ and $\sim$ mean equality in the OPE *sense* (*in other words analytic continuation sense*), *and $\sim$ means equality modulo regular parts.*

Proof. A straightforward but tedious OPE calculation shows this proposition. We give the useful formulas in Appendix C and how the poles cancel each other in Appendix D. For Eq. (2.28) some explanation is needed. Let us denote OPE of each term of $E^{\pm,i}(z)$ as follows (see Appendix D for notation):

$$E^{\pm,i(A)}(z) E^{\pm,j(B)}(w) \simeq f_\pm^{ijAB}(z,w) \colon E^{\pm,i(A)}(z) E^{\pm,j(B)}(w) \colon . \tag{3.13}$$

For $i = j$ there are three cases:

$$f_\pm^{iiAB}(z,w) = q^\ell \frac{z-w}{z-q^{\pm 2} w} \quad \text{and} \quad f_\pm^{iiBA}(w,z) = q^\ell \frac{w-z}{w-q^{\pm 2} z}, \tag{3.14}$$

$$f_\pm^{iiAB}(z,w) = q^\ell \frac{q^{\pm 2} z - w}{z - q^{\pm 2} w} \quad \text{and} \quad f_\pm^{iiBA}(w,z) = q^\ell, \tag{3.15}$$

$$f_\pm^{iiAB}(z,w) = q^\ell \quad \text{and} \quad f_\pm^{iiBA}(w,z) = q^\ell \frac{q^{\pm 2} w - z}{w - q^{\pm 2} z}, \tag{3.16}$$

where $\ell \in \mathbb{Z}$ depends on $i,\ j,\ A,\ B,\ \pm$. For $a_{ij}=-1$ there are two cases[4]:

$$f_{\pm}^{ijAB}(z,w)=q^{m}\frac{q^{\mp 1}z-w}{z-q^{\mp 1}w} \quad \text{and} \quad f_{\pm}^{jiBA}(w,z)=q^{m}\ , \tag{3.17}$$

$$f_{\pm}^{ijAB}(z,w)=q^{m} \quad \text{and} \quad f_{\pm}^{jiBA}(w,z)=q^{m}\frac{q^{\mp 1}w-z}{w-q^{\mp 1}z}\ , \tag{3.18}$$

where $m \in \mathbb{Z}$ depends on $i,\ j,\ A,\ B,\ \pm$. These OPE equations can be translated to formal power series equations:

$$E^{\pm,i(A)}(z)E^{\pm,j(B)}(w)=g_{\pm}^{ijAB}(z,w):E^{\pm,i(A)}(z)E^{\pm,j(B)}(w):\ . \tag{3.19}$$

Equations (3.14)–(3.18) are translated to

$$\begin{aligned} g_{\pm}^{iiAB}(z,w) &= q^{\ell}(z-w)\frac{1}{z}\sum_{n\geqq 0}\left(q^{\pm 2}\frac{w}{z}\right)^{n} \\ &\quad \text{and} \quad g_{\pm}^{iiBA}(w,z)=q^{\ell}(w-z)\frac{1}{w}\sum_{n\geqq 0}\left(q^{\pm 2}\frac{z}{w}\right)^{n}\ , \end{aligned} \tag{3.20}$$

$$g_{\pm}^{iiAB}(z,w)=q^{\ell}(q^{\pm 2}z-w)\frac{1}{z}\sum_{n\geqq 0}\left(q^{\pm 2}\frac{w}{z}\right)^{n} \quad \text{and} \quad g_{\pm}^{iiBA}(w,z)=q^{\ell}\ , \tag{3.21}$$

$$g_{\pm}^{iiAB}(z,w)=q^{\ell} \quad \text{and} \quad g_{\pm}^{iiBA}(w,z)=q^{\ell}(q^{\pm 2}w-z)\frac{1}{w}\sum_{n\geqq 0}\left(q^{\pm 2}\frac{z}{w}\right)^{n}\ , \tag{3.22}$$

$$g_{\pm}^{ijAB}(z,w)=q^{m}(q^{\mp 1}z-w)\frac{1}{z}\sum_{n\geqq 0}\left(q^{\mp 1}\frac{w}{z}\right)^{n} \quad \text{and} \quad g_{\pm}^{jiBA}(w,z)=q^{m}\ , \tag{3.23}$$

$$g_{\pm}^{ijAB}(z,w)=q^{m} \quad \text{and} \quad g_{\pm}^{jiBA}(w,z)=q^{m}(q^{\mp 1}w-z)\frac{1}{w}\sum_{n\geqq 0}\left(q^{\mp 1}\frac{z}{w}\right)^{n}\ , \tag{3.24}$$

respectively. A product of three E's can be expressed as

$$\begin{aligned} &E^{\pm,i_1(A_1)}(z_1)E^{\pm,i_2(A_2)}(z_2)E^{\pm,i_3(A_3)}(z_3) \\ &= g_{\pm}^{i_1i_2A_1A_2}(z_1,z_2)g_{\pm}^{i_1i_3A_1A_3}(z_1,z_3)g_{\pm}^{i_2i_3A_2A_3}(z_2,z_3) \\ &\quad \times :E^{\pm,i_1(A_1)}(z_1)E^{\pm,i_2(A_2)}(z_2)E^{\pm,i_3(A_3)}(z_3):\ . \end{aligned} \tag{3.25}$$

We remark that this is a consequence of the bosonic realization. Using this fact, we obtain

$$\begin{aligned} &E^{\pm,i(A_1)}(z_1)E^{\pm,i(A_2)}(z_2)E^{\pm,j(B)}(w)-(q+q^{-1})E^{\pm,i(A_1)}(z_1)E^{\pm,j(B)}(w)E^{\pm,i(A_2)}(z_2) \\ &\quad +E^{\pm,j(B)}(w)E^{\pm,i(A_1)}(z_1)E^{\pm,i(A_2)}(z_2) \\ &= g_{\pm}^{iiA_1A_2}(z_1,z_2)(g_{\pm}^{ijA_1B}(z_1,w)g_{\pm}^{ijA_2B}(z_2,w) \\ &\quad -(q+q^{-1})g_{\pm}^{ijA_1B}(z_1,w)g_{\pm}^{jiBA_2}(w,z_2) \\ &\quad +g_{\pm}^{jiBA_1}(w,z_1)g_{\pm}^{jiBA_2}(w,z_2)) \\ &\quad \times :E^{\pm,i(A_1)}(z_1)E^{\pm,i(A_2)}(z_2)E^{\pm,j(B)}(w):\ . \end{aligned} \tag{3.26}$$

In each case, this coefficient is antisymmetric with respect to z_1 and z_2. Therefore Eq. (2.28) holds. □

[4] For E^{-}, there are extra poles. However, we can discard them because they cancel each other

We remark that Eqs. (3.10), (3.11), (3.12) imply Eqs. (2.25), (2.26), (2.27) respectively. Therefore we obtain our main statement:

Corollary 2. H^i, $\psi_\pm^i$, $E^{\pm,i}$ *in Eqs.* (3.4)–(3.7) *realize the quantum affine algebra* $U_q(\widehat{sl_N})$ *in the Drinfeld realization with* $\gamma = q^k$.

4. Screening Currents and Vertex Operators

To calculate correlation functions and investigate the irreducible representation, we need screening operators, which commute with $U_q(\widehat{sl_N})$. Let us define the screening currents $S^i(z)$ $(i = 1, \ldots, N-1)$ as follows:

$$S^i(z) \stackrel{\text{def}}{=} :\exp\left(-\left(\frac{1}{k+g}a^i\right)\left(z;\frac{k+g}{2}\right)\right):\tilde{S}^i(z)\,, \tag{4.1}$$

$$\begin{aligned}\tilde{S}^i(z) \stackrel{\text{def}}{=} & \frac{-1}{(q-q^{-1})z} \\ & \times \sum_{j=i+1}^{N} :\exp\left((b+c)^{i+1,j}(q^{N-j}z)\right) \\ & \qquad \times \left(\exp\left(-b_-^{i,j}(q^{N-j}z) - (b+c)^{i,j}(q^{N-j+1}z)\right)\right. \\ & \qquad \left. - \exp\left(-b_+^{i,j}(q^{N-j}z) - (b+c)^{i,j}(q^{N-j-1}z)\right)\right) \\ & \qquad \times \exp\left(\sum_{\ell=j+1}^{N}\left(b_-^{i+1,\ell}(q^{N-\ell+1}z) - b_-^{i,\ell}(q^{N-\ell}z)\right)\right):\,.\end{aligned} \tag{4.2}$$

We remark that $\tilde{S}^i(z)$ is nothing else but $E^{+,N-i}(z)$ with replacement $b_\pm^{i,j} \mapsto -b_\mp^{N+1-j,N+1-i}$, $(b+c)^{i,j} \mapsto (b+c)^{N+1-j,N+1-i}$. These screening currents have the following properties.

Proposition 3. S^i, $\tilde{S}^i$ *in Eqs.* (4.1), (4.2) *and* H^i, $E^{\pm,i}$ *in Eqs.* (3.4)–(3.7) *satisfy the following relations:*

$$[H_n^i, S^j(z)] = 0\,, \tag{4.3}$$

$$E^{+,i}(z)S^j(w) \simeq S^j(w)E^{+,i}(z) \sim \text{reg.}, \tag{4.4}$$

$$\begin{aligned}E^{-,i}(z)S^j(w) &\simeq S^j(w)E^{-,i}(z) \\ &\sim \text{reg.} + \delta^{ij}{}_{k+g}\partial_w \\ &\quad \times \left(\frac{1}{z-w}:\exp\left(-\left(\frac{1}{k+g}a^j\right)\left(w;-\frac{k+g}{2}\right)\right):\right),\end{aligned} \tag{4.5}$$

and

$$(z-q^{-a_{ij}}w)\tilde{S}^i(z)\tilde{S}^j(w) \simeq (q^{-a_{ij}}z-w)\tilde{S}^j(w)\tilde{S}^i(z) \sim \text{reg.}\,, \tag{4.6}$$

$$\tilde{S}^i(z)\tilde{S}^j(w) \simeq \tilde{S}^j(w)\tilde{S}^i(z) \sim \text{reg.} \quad \text{for } a_{ij} = 0\,. \tag{4.7}$$

Proof. Straightforward (see Appendices C, D). □

Equations (4.3)–(4.5) can be expressed in the commutator form.

Corollary 4.

$$[H_n^i, S^j(z)] = 0\,, \tag{4.8}$$

$$[E_n^{+,i}, S^j(z)] = 0\,, \tag{4.9}$$

$$[E_n^{-,i}, S^j(z)] = \delta^{ij}{}_{k+g}\partial_z\left(z^n\colon \exp\left(-\left(\frac{1}{k+g}a^j\right)\left(z;-\frac{k+g}{2}\right)\right)\colon\right)\,. \tag{4.10}$$

From this we get the desired property of the screening charges.

Corollary 5. *If the Jackson integrals of the screening currents Eq.* (4.1),

$$\int_0^{s\infty} S^i(z)d_pz, \quad p = q^{2(k+g)}\,, \tag{4.11}$$

are convergent, they commute with $U_q(\widehat{sl_N})$ *generated by Eqs.* (3.4)–(3.7).

Next we will construct the vertex operators (primary fields), which create the $U_q(\widehat{sl_N})$ highest weight states from the vacuum state of the boson Fock space. The vacuum state of the boson Fock space, $|\mathbf{0}\rangle$, is defined by

$$a_n^i|\mathbf{0}\rangle = b_n^{ij}|\mathbf{0}\rangle = c_n^{ij}|\mathbf{0}\rangle = 0 \ (n \geqq 0)\,. \tag{4.12}$$

Let $|p_a,p_b,p_c\rangle$ be

$$|p_a,p_b,p_c\rangle \stackrel{\mathrm{def}}{=} \exp\left(\sum_{i,j=1}^{N-1} p_a^i \frac{a_{ij}^{-1}}{k+g}\hat{q}_a^j + \sum_{1\leqq i<j\leqq N} p_b^{ij}(-1)\hat{q}_b^{ij} + \sum_{1\leqq i<j\leqq N} p_c^{ij}\hat{q}_c^{ij}\right)|\mathbf{0}\rangle\,, \tag{4.13}$$

then $|p_a,p_b,p_c\rangle$ is the highest weight state of the boson Fock space, i.e.,

$$a_n^i|p_a,p_b,p_c\rangle = b_n^{ij}|p_a,p_b,p_c\rangle = c_n^{ij}|p_a,p_b,p_c\rangle = 0 \ (n > 0)\,, \tag{4.14}$$

$$\hat{p}_a^i|p_a,p_b,p_c\rangle = p_a^i|p_a,p_b,p_c\rangle,\ \hat{p}_x^{ij}|p_a,p_b,p_c\rangle = p_x^{ij}|p_a,p_b,p_c\rangle \quad (x = b,c)\,. \tag{4.15}$$

The boson Fock space $F(p_a,p_b,p_c)$ is generated by oscillators of negative mode on the highest weight state $|p_a,p_b,p_c\rangle$. $E_n^{\pm,i}$ change $p_b - p_c$ only, S_n^i changes p_a and $p_b - p_c$, H_n^i does not change p_a, p_b, p_c. $|p_a,0,0\rangle$ has the following property:

Proposition 6. H^i, $E^{\pm,i}$ *in Eqs.* (3.4)–(3.7) *act on* $|p_a,0,0\rangle$ *as follows:*

$$X_n|p_a,0,0\rangle = 0 \quad (n > 0; X = H^i, E^{\pm,i})\,, \tag{4.16}$$

$$E_0^{+,i}|p_a,0,0\rangle = 0\,, \tag{4.17}$$

$$H_0^i|p_a,0,0\rangle = p_a^i|p_a,0,0\rangle\,. \tag{4.18}$$

Proof. Straightforward. X_n $(n > 0)$ annihilate $|p_a,p_b,p_c\rangle$ with $p_b + p_c = 0$, and $E_0^{+,i}$ annihilate $|p_a,0,0\rangle$. □

This property is just the highest weight state condition of $U_q(\widehat{sl_N})$.

Corollary 7. *Using the highest weight state* $|p_a,0,0\rangle = |(\ell^1,\dots,\ell^{N-1}),0,0\rangle$, *we get the highest weight left module of* $U_q(\widehat{sl_N})$, $V(\lambda)$,

$$V(\lambda) \hookrightarrow \bigoplus_{r\in\mathbb{Z}^{N(N-1)/2}} F\big((\ell^1,\dots,\ell^{N-1}),r,-r\big)\,, \tag{4.19}$$

where the classical part of the highest weight is $\bar{\lambda} = \ell^1\bar{\Lambda}_1 + \cdots + \ell^{N-1}\bar{\Lambda}_{N-1} = (\ell^1,\dots,\ell^{N-1})$.

As is well known in CFT, this module is reducible.

Let us define the vertex operator with a weight $\bar{\lambda} = (\ell^1,\dots,\ell^{N-1})$ and a parameter α, $\phi^{\bar{\lambda}}(z;\alpha)$, as follows:

$$\phi^{\bar{\lambda}}(z;\alpha) \stackrel{\text{def}}{=} \exp\left(\sum_{i,j=1}^{N-1}\left(\frac{\ell^i}{k+g}\frac{\min(i,j)}{N}\frac{N-\max(i,j)}{1}a^j\right)(z;\alpha)\right): \,. \tag{4.20}$$

The highest weight state of $U_q(\widehat{sl_N})$, $|(\ell^1,\dots,\ell^{N-1}),0,0\rangle$, is created from the vacuum $|\mathbf{0}\rangle$ by this operator with any parameters α and β,

$$|(\ell^1,\dots,\ell^{N-1}),0,0\rangle = \lim_{z\to 0}\phi^{\bar{\lambda}}(q^\beta z;\alpha)|\mathbf{0}\rangle\,. \tag{4.21}$$

Moreover this vertex operator has the following properties.

Proposition 8. $\phi^{\bar{\lambda}}$ *in Eq.* (4.20) *and* $H^i, E^{\pm,i}$ *in Eqs.* (3.4) – (3.7) *satisfy the following relations:*

$$[H^i_n,\phi^{\bar{\lambda}}(z;\alpha)] = \frac{1}{n}[\ell^i n]q^{-(\alpha+\frac{g}{2})|n|}z^n\phi^{\bar{\lambda}}(z;\alpha)\,, \tag{4.22}$$

$$[E^{+,i}_n,\phi^{\bar{\lambda}}(z;\alpha)] = 0\,, \tag{4.23}$$

and

$$(z-q^{\ell^i}w)E^{-,i}(z)\phi^{\bar{\lambda}}\left(w;-\frac{k+g}{2}\right) \simeq (q^{\ell^i}z-w)\phi^{\bar{\lambda}}\left(w;-\frac{k+g}{2}\right)E^{-,i}(z) \sim \text{reg.} \tag{4.24}$$

Proof. Straightforward. We use the q-analogue of the inverse of the Cartan matrix:

$$\sum_{r=1}^{N-1}\frac{[a_{ir}n]}{[n]}\frac{[\min(r,j)n][(N-\max(r,j))n]}{[Nn][n]} = \delta_{ij}\,. \tag{4.25}$$

We remark that Eq. (4.24) can be rewritten as

$$\left[E^{-,i}_n,\phi^{\bar{\lambda}}\left(z;-\frac{k+g}{2}\right)\right]_{q^{\ell^i}} = -z\left[\phi^{\bar{\lambda}}\left(z;-\frac{k+g}{2}\right),E^{-,i}_{n-1}\right]_{q^{\ell^i}}\,. \tag{4.26}$$

From $\phi^{\bar{\lambda}}(q^\beta z;\alpha)$ with appropriate α and β, we can construct the q-vertex operator $\Phi(z)$ [9], which has an intertwining property. We will discuss this problem in the next section.

5. Discussion

In this paper we have constructed a free boson realization of $U_q(\widehat{sl_N})$. We can also bosonize the grading operator d. d is defined by the property for the Chevalley generators,

$$[d,t_i]=0, \quad \left[d,e_i^{\pm}\right]=\pm\delta_{i0}e_i^{\pm}\,, \tag{5.1}$$

or equivalently, for the Drinfeld generators,

$$\left[d,H_n^i\right]=nH_n^i, \quad \left[d,E_n^{\pm,i}\right]=nE_n^{\pm,i}\,. \tag{5.2}$$

Using Eqs. (2.55) and (4.25), let us define the q-analogue of the Virasoro L_0 operator [17, 18] as follows:

$$\begin{aligned} L_0 \stackrel{\mathrm{def}}{=} & \frac{1}{2}\sum_{i,j=1}^{N-1}\sum_{n\in\mathbb{Z}} : a_{-n}^i \frac{n^2}{[n][(k+g)n]}\frac{[\min(i,j)n][(N-\max(i,j))n]}{[Nn][n]}a_n^j : \\ & + \sum_{i,j=1}^{N-1}\bar{\rho}^i\frac{a_{ij}^{-1}}{k+g}\hat{p}_a^j \\ & + \frac{1}{2}\sum_{1\leqq i<j\leqq N}\sum_{n\in\mathbb{Z}} : b_{-n}^{ij}(-1)\frac{n^2}{[n]^2}b_n^{ij} : + \frac{1}{2}\sum_{1\leqq i<j\leqq N}\hat{p}_b^{ij} \\ & + \frac{1}{2}\sum_{1\leqq i<j\leqq N}\sum_{n\in\mathbb{Z}} : c_{-n}^{ij}\frac{n^2}{[n]^2}c_n^{ij} : + \frac{1}{2}\sum_{1\leqq i<j\leqq N}\hat{p}_c^{ij}\,, \end{aligned} \tag{5.3}$$

where $\bar{\rho}^i=1$, i.e., $\bar{\rho}=(1,1,\ldots,1)=\sum_{i=1}^{N-1}\bar{\Lambda}_i$ is the half sum of positive roots of sl_N. Then $d=-L_0$ satisfies Eq. (5.2) on the representation space given in Corollary. 7. The L_0 eigenvalue of $|(\ell^1,\cdots,\ell^{N-1}),0,0\rangle$ is $\frac{1}{2(k+g)}\ell^i a_{ij}^{-1}(\ell^j+2\bar{\rho}^j)=\frac{1}{2(k+g)}(\bar{\lambda},\bar{\lambda}+2\bar{\rho})$.

We have also constructed the screening currents and the vertex operators. Using these, we can start the representation theory and calculation of correlation functions. Like $\widehat{sl_N}$ [28, 29], it is expected that the projection from the boson Fock space to the irreducible $U_q(\widehat{sl_N})$ representation space can be done by BRST cohomology technique. In fact, recently, this procedure has been worked out for $U_q(\widehat{sl_2})$ [30]. The BRST operator is constructed by using the screening current.

To calculate the Jackson integral formulas for the correlation functions, which are solutions of the q-KZ equation, we must first prepare the q-vertex operators Φ. We will restrict ourselves to the type I [10] vertex operator $\Phi_{V(\mu)}^{V(\nu)V_\lambda}(z): V(\mu)\to V(\nu)\otimes V_{\lambda z}$. $\Phi_{V(\mu)}^{V(\nu)V_\lambda}(z)$ can be constructed from $\phi^{\bar{\lambda}}(q^\beta z;\alpha)$ with appropriate α,β. From Eq. (4.24), we choose $\alpha=-\frac{k+g}{2}$. This choice agrees with refs. [31] ($U_q(\widehat{sl_2})$ with an arbitrary level k) and [32] (vector representation of $U_q(\widehat{sl_N})$ with $k=1$). Starting from $\phi^{\bar{\lambda}}(z)\stackrel{\mathrm{def}}{=}\phi^{\bar{\lambda}}\left(z;-\frac{k+g}{2}\right)$, we define $\phi_{i_1,\quad,i_n}^{\bar{\lambda}}(z)$ as follows:

$$\phi_{i_1,\quad,i_n}^{\bar{\lambda}}(z)\stackrel{\mathrm{def}}{=}\left[\phi_{i_1,\quad,i_{n-1}}^{\bar{\lambda}}(z),E_0^{-,i_n}\right]_{q^x}, \quad x=\left(\bar{\lambda}-\sum_{j=1}^{n-1}\bar{\alpha}_{i_j},\bar{\alpha}_{i_n}\right)\,. \tag{5.4}$$

To determine β, we must specify the finite dimensional representation of $U_q(\widehat{sl_N})$. Results of refs. [31, 32] suggest $\beta = k + g$. Once the finite dimensional representation is obtained and β is determined, we can construct the q-vertex operator $\Phi_{V(\mu)}^{V(\nu)V_\lambda}(z)$ from our $\phi^{\bar{\lambda}}_{i_1,\ \ ,i_n}(q^\beta z)$. Then, we can calculate correlation functions of the q-vertex operators in the standard way. These problems are now under investigation.

To extend our results to arbitrary quantum affine Lie algebras, it may be important to consider the geometrical interpretation of the free boson realization. For $q = 1$ case, the β–γ system is suitable for the geometrical interpretation [17]. For $q \neq 1$ case, we define the quantum β–γ fields, $\beta^{ij}_{\alpha,\pm}(z)$ $(\alpha = \pm 1), \gamma^{ij}(z)$, as follows:

$$\beta_{1,\pm}(z) \overset{\text{def}}{=} \frac{-1}{(q-q^{-1})z} : \exp(b_\pm(z) - (b+c)(q^{\pm 1}z)) : , \tag{5.5}$$

$$\beta_{-1,\pm}(z) \overset{\text{def}}{=} \frac{-1}{(q-q^{-1})z} : \exp(-b_\mp(z) - (b+c)(q^{\pm 1}z)) : , \tag{5.6}$$

$$\gamma(z) \overset{\text{def}}{=} : \exp((b+c)(z)) : , \tag{5.7}$$

where we suppress the superscript ij. They are not free fields any longer. They satisfy

$$(z - q^{\alpha+\alpha'}w)\beta_{\alpha,\varepsilon}(z)\beta_{\alpha',\varepsilon'}(w) = \left(q^{\alpha+\alpha'}z - w\right)\beta_{\alpha',\varepsilon'}(w)\beta_{\alpha,\varepsilon}(z) \quad (\varepsilon, \varepsilon' = \pm)\,, \tag{5.8}$$

$$\beta_{\pm 1,\pm}(z)\beta_{\mp 1,\pm}(w) = \beta_{\mp 1,\pm}(w)\beta_{\pm 1,\pm}(z)\,, \tag{5.9}$$

$$(z - q^{\mp 1}w)\beta_{\pm 1,\pm}(z)\gamma(w) = (q^{\mp 1}z - w)\gamma(w)\beta_{\pm 1,\pm}(z)\,, \tag{5.10}$$

$$(z - q^{\mp 1}w)\gamma(z)\beta_{\pm 1,\mp}(w) = (q^{\mp 1}z - w)\beta_{\pm 1,\mp}(w)\gamma(z)\,, \tag{5.11}$$

$$\gamma(z)\gamma(w) = \gamma(w)\gamma(z)\,. \tag{5.12}$$

Our free boson realization of $U_q(\widehat{sl_N})$ is reexpressed by these quantum β–γ fields. In the $q \to 1$ limit, $\beta_{\alpha,+}(z) - \beta_{\alpha,-}(z)$ and $\gamma(z)$ become usual $\beta(z)$ and $\gamma(z)$ respectively. These $\beta_{\alpha,\pm}$, γ fields are the affinization of q-oscillator $(aa^\dagger - q^{\pm 1}a^\dagger a = q^{\mp\mathcal{N}}); a \to \gamma$, $a^\dagger \to \beta_{\alpha,+} - \beta_{\alpha,-}$ (see Appendix A). We expect that our realization in terms of the quantum β–γ system acts on the q-deformed semi-infinite flag manifold [17].

Our free boson realization may also be useful to analyze the q-analogue of the Virasoro and W algebras by the Hamiltonian reduction, and the representation at the critical level $k = -g$.

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Appendix A

For the reader's convenience, we give the result of [25], the Heisenberg realization of $U_q(sl_N)$ with the weight $\lambda_i \in \mathbb{C}$. Let us consider variables x_{ij} and derivatives $\frac{\partial}{\partial x_{ij}}$ $(1 \leqq i < j \leqq N)$. Their commutation relations are

$$\left[\frac{\partial}{\partial x_{ij}}, x_{i'j'}\right] = \delta_{ii'}\delta_{jj'}\,, \quad \text{others} = 0\,. \tag{A.1}$$

Standard Chevalley generators of $U_q(sl_N), e_i^{\pm}, t_i = q^{h_i} (i = 1, \ldots, N-1)$, are realized as follows:

$$h_i \stackrel{\text{def}}{=} -\sum_{j=1}^{i} \left(\vartheta_{j,i+1} - \vartheta_{j,i}\right) + \lambda_i - \sum_{j=i+1}^{N} \left(\vartheta_{i,j} - \vartheta_{i+1,j}\right), \tag{A.2}$$

$$e_i^+ \stackrel{\text{def}}{=} \sum_{j=1}^{i} x_{j,i} \frac{1}{x_{j,i+1}} \left[\vartheta_{j,i+1}\right] q^{-\sum_{\ell=1}^{j-1}(\vartheta_{\ell,i+1}-\vartheta_{\ell,i})}, \tag{A.3}$$

$$\begin{aligned} e_i^- \stackrel{\text{def}}{=} & \sum_{j=1}^{i-1} x_{j,i+1} \frac{1}{x_{j,i}} \left[\vartheta_{j,i}\right] q^{\sum_{\ell=j+1}^{i}(\vartheta_{\ell,i+1}-\vartheta_{\ell,i})-\lambda_i+\sum_{\ell=i+1}^{N}(\vartheta_{i,\ell}-\vartheta_{i+1,\ell})} \\ & + x_{i,i+1} \left[\lambda_i - \sum_{\ell=i+1}^{N} \left(\vartheta_{i,\ell} - \vartheta_{i+1,\ell}\right)\right] \\ & - \sum_{j=i+2}^{N} x_{i,j} \frac{1}{x_{i+1,j}} \left[\vartheta_{i+1,j}\right] q^{\lambda_i - \sum_{\ell=j}^{N}(\vartheta_{i,\ell}-\vartheta_{i+1,\ell})}, \end{aligned} \tag{A.4}$$

where $\vartheta_{ij} \stackrel{\text{def}}{=} x_{ij}\frac{\partial}{\partial x_{ij}}, x_{ii} \stackrel{\text{def}}{=} 1, \vartheta_{ii} \stackrel{\text{def}}{=} 0$.

Our free field realization of $U_q(\widehat{sl_N})$ is obtained by the following replacement with suitable argument:

$$x \mapsto e^{(b+c)(z)}, \tag{A.5}$$

$$-\vartheta \mapsto \pm b_{\pm}(z), \tag{A.6}$$

$$\lambda \mapsto \pm a_{\pm}(z), \tag{A.7}$$

$$[A(z)] \mapsto \frac{e^{A_+(z)} - e^{A_-(z)}}{(q-q^{-1})z}. \tag{A.8}$$

Appendix B

In this appendix, we reexpress Eqs. (3.4), (3.6), (3.7) and (4.1) by using the q-difference operator. These expressions are not unique and we give one of them.

Using the following formulas:

$$\frac{1}{(q-q^{-1})z}\left(a_+(q^\alpha z) - a_-(q^{-\alpha} z)\right) = {}_1\partial_z a(z;\alpha) = \sum_{n\in\mathbb{Z}} a_n q^{-\alpha|n|} z^{-n-1}, \tag{B.1}$$

$$\begin{aligned} &\frac{1}{(q-q^{-1})z} : \left(\exp(\pm b_{\pm}(z) - (b+c)(qz)) - \exp(\pm b_{\mp}(z) - (b+c)(q^{-1}z))\right) : \\ &\quad = :{}_1\partial_z(\exp(-c(z)))\cdot\exp(-b(z;\mp 1)):, \end{aligned} \tag{B.2}$$

$$\begin{aligned} &\frac{1}{(q-q^{-1})z} : \left(\exp(\pm a_+(q^\alpha z)) - \exp(\pm a_-(q^{-\alpha} z))\right) : \\ &\quad = :{}_M\partial_z\left(\exp\left(\pm\left(\frac{1}{M}a\right)(z;\alpha)\right)\right)\cdot\exp\left(\mp\left(\frac{1}{M}a\right)(z;\alpha-M)\right):, \end{aligned} \tag{B.3}$$

$$\frac{1}{(q-q^{-1})z}:\left(\exp\left(b\left(q^{\alpha}z\right)\right)-\exp\left(b\left(q^{-\alpha}z\right)\right)\right):$$

$$=:_M\partial_z\left(\exp\left(\left(\frac{\alpha}{M}b\right)(z)\right)\right)\cdot\exp\left(\left(\frac{M-\alpha}{M}b\right)(z)\right):, \tag{B.4}$$

$$\left(\frac{\alpha}{M}b\right)(z)\pm\left(\frac{1}{M}b\right)(z;\pm\alpha+1)=\left(\frac{\alpha\pm1}{M}b\right)(z;1), \tag{B.5}$$

Eqs. (3.4), (3.6), (3.7) and (4.1) are rewritten as follows:

$$H^i(z)={}_1\partial_z\left(\sum_{j=1}^{i}\left(b^{j,i+1}\left(z;\frac{k}{2}+j-1\right)-b^{j,i}\left(z;\frac{k}{2}+j\right)\right)\right.$$
$$\left.+a^i\left(z;\frac{g}{2}\right)+\sum_{j=i+1}^{N}\left(b^{i,j}\left(z;\frac{k}{2}+j\right)-b^{i+1,j}\left(z;\frac{k}{2}+j-1\right)\right)\right), \tag{B.6}$$

$$E^{+,i}(z)=-\sum_{j=1}^{i}:\exp\left((b+c)^{j,i}(q^{j-1}z)\right)$$
$$\times{}_1\partial_z\left(\exp\left(-c^{j,i+1}\left(q^{j-1}z\right)\right)\right)\cdot\exp\left(-b^{j,i+1}\left(q^{j-1}z;-1\right)\right)$$
$$\times\exp\left(\sum_{\ell=1}^{j-1}\left(b_+^{\ell,i+1}\left(q^{\ell-1}z\right)-b_+^{\ell,i}\left(q^{\ell}z\right)\right)\right):, \tag{B.7}$$

$$E^{-,i}(z)=-\sum_{j=1}^{i-1}:\exp\left((b+c)^{j,i+1}(q^{-(k+j)}z)\right)$$
$$\times{}_1\partial_z\left(\exp\left(-c^{ji}\left(q^{-(k+j)}z\right)\right)\right)\cdot\exp\left(-b^{ji}\left(q^{-(k+j)}z;1\right)\right)$$
$$\times\exp\left(\sum_{\ell=j+1}^{i}\left(b_-^{\ell,i+1}\left(q^{-(k+\ell-1)}z\right)-b_-^{\ell,i}\left(q^{-(k+\ell)}z\right)\right)\right.$$
$$\left.+a_-^i\left(q^{-\frac{k+g}{2}}z\right)+\sum_{\ell=i+1}^{N}\left(b_-^{i,\ell}\left(q^{-(k+\ell)}z\right)-b_-^{i+1,\ell}\left(q^{-(k+\ell-1)}z\right)\right)\right):$$
$$+:{}_{k+g}\partial_z\left(\exp\left(\left(\frac{k+i}{k+g}(b+c)^{i,i+1}\right)(z)+\left(\frac{1}{k+g}a^i\right)\left(z;\frac{k+g}{2}\right)\right.\right.$$
$$\left.\left.+\sum_{\ell=i+1}^{N}\left(\left(\frac{1}{k+g}b^{i,\ell}\right)(z;k+\ell)-\left(\frac{1}{k+g}b^{i+1,\ell}\right)(z;k+\ell-1)\right)\right)\right)$$
$$\times\exp\left(\left(\frac{g-i}{k+g}(b+c)^{i,i+1}\right)(z)-\left(\frac{1}{k+g}a^i\right)\left(z;-\frac{k+g}{2}\right)\right.$$
$$\left.-\sum_{\ell=i+1}^{N}\left(\left(\frac{1}{k+g}b^{i,\ell}\right)(z;\ell-g)-\left(\frac{1}{k+g}b^{i+1,\ell}\right)(z;l-g-1)\right)\right):$$

$$
\begin{aligned}
&+\sum_{j=i+2}^{N} :\exp\left((b+c)^{i,j}(q^{k+j-1}z)\right)\\
&\qquad\times {}_1\partial_z(\exp\left(-c^{i+1,j}\left(q^{k+j-1}z\right)\right))\cdot\exp\left(-b^{i+1,j}\left(q^{k+j-1}z;-1\right)\right)\\
&\qquad\times\exp\left(a_+^i\left(q^{\frac{k+g}{2}}z\right)+\sum_{\ell=j}^{N}\left(b_+^{i,\ell}\left(q^{k+\ell}z\right)-b_+^{i+1,\ell}\left(q^{k+\ell-1}z\right)\right)\right):,
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
S^i(z)=-:&\exp\left(-\left(\frac{1}{k+g}a^i\right)\left(z;\frac{k+g}{2}\right)\right)\\
&\times\sum_{j=i+1}^{N}\exp\left((b+c)^{i+1,j}(q^{N-j}z)\right)\\
&\times {}_1\partial_z\left(\exp\left(-c^{i,j}\left(q^{N-j}z\right)\right)\right)\cdot\exp\left(-b^{i,j}\left(q^{N-j}z;\,1\right)\right)\\
&\times\exp\left(\sum_{\ell=j+1}^{N}\left(b_-^{i+1,\ell}\left(q^{N-\ell+1}z\right)-b_-^{i,\ell}\left(q^{N-\ell}z\right)\right)\right):.
\end{aligned}
\tag{B.9}
$$

These expressions are adequate for taking the $q\to 1$ limit, because there is no denominator $q-q^{-1}$. In this limit ${}_\alpha\partial_z$, $\left(\frac{L_1}{M_1}\cdots\frac{L_r}{M_r}a\right)(z;\alpha)$, $\left(\frac{L_1}{M_1}\cdots\frac{L_r}{M_r}a_\pm\right)(z)$ become $\alpha\partial_z$, $\frac{L_1}{M_1}\frac{L_r}{M_r}a(z)$, 0 respectively. Equations (3.4), (3.6) and (3.7) become the bosonized version of the Wakimoto realization of $\widehat{sl_N}$ with level k [16, 17, 18]; $\beta^{ij}(z)$ and $\gamma^{ij}(z)$ are expressed in terms of $b^{ij}(z)$ and $c^{ij}(z)$ with $q=1$ as follows [33]:

$$
\beta^{ij}(z)=-:\partial_z\left(\exp\left(-c^{ij}(z)\right)\right)\cdot\exp\left(-b^{ij}(z)\right):,
\tag{B.10}
$$

$$
\gamma^{ij}(z)=:\exp\left((b+c)^{ij}(z)\right):.
\tag{B.11}
$$

Appendix C

In this appendix we give useful formulas.

First we give formulas for a boson a in Sect. 2 (see the footnote below Eq. (2.19)).

$$
[A,B]\text{commute with } A,B\Rightarrow [A,e^B]=[A,B]e^B\,,
\tag{C.1}
$$

$$
e^Ae^B=e^{[A,B]}e^Be^A\,,
\tag{C.2}
$$

$$
\left[a_n,\left(\frac{L_1}{M_1}\cdots\frac{L_r}{M_r}a_\pm\right)(z)\right]=\pm\theta(\mp n>0)(q-q^{-1})\frac{[L_1n]\cdots[L_rn]}{[M_1n]\cdots[M_rn]}n\rho_a(n)z^n\,,
\tag{C.3}
$$

$$
\left[a_n,\left(\frac{L_1}{M_1}\cdots\frac{L_r}{M_r}a\right)(z;\alpha)\right]=\frac{[L_1n]\cdots[L_rn]}{[M_1n]\cdots[M_rn]}\frac{n}{[n]}\rho_a(n)q^{-\alpha|n|}z^n\,,
\tag{C.4}
$$

$$\left[\left(\frac{L_1}{M_1}\cdots\frac{L_r}{M_r}a_+\right)(z),\left(\frac{L'_1}{M'_1}\cdots\frac{L'_s}{M'_s}a_-\right)(w)\right]$$

$$= -(q-q^{-1})^2\sum_{n>0}\frac{[L_1n]\cdots[L_rn]}{[M_1n]\cdots[M_rn]}\frac{[L'_1n]\cdots[L'_sn]}{[M'_1n]\cdots[M'_sn]}n\rho_a(n)\left(\frac{w}{z}\right)^n, \tag{C.5}$$

$$\left[\left(\frac{L_1}{M_1}\cdots\frac{L_r}{M_r}a\right)(z;\alpha),\left(\frac{L'_1}{M'_1}\cdots\frac{L'_s}{M'_s}a_-\right)(w)\right]$$

$$= (q-q^{-1})\sum_{n>0}\frac{[L_1n]\cdots[L_rn]}{[M_1n]\cdots[M_rn]}\frac{[L'_1n]\cdots[L'_sn]}{[M'_1n]\cdots[M'_sn]}\frac{n}{[n]}\rho_a(n)q^{-\alpha n}\left(\frac{w}{z}\right)^n$$
$$+\frac{L_1\cdots L_r}{M_1\cdots M_r}\frac{L'_1\cdots L'_s}{M'_1\cdots M'_s}\rho_a\log q, \tag{C.6}$$

$$\left[\left(\frac{L'_1}{M'_1}\cdots\frac{L'_s}{M'_s}a_+\right)(z),\left(\frac{L_1}{M_1}\cdots\frac{L_r}{M_r}a\right)(w;\alpha)\right] = \text{Eq. (C.6)} \tag{C.7}$$

$$\left[\left(\frac{L_1}{M_1}\cdots\frac{L_r}{M_r}a\right)(z;\alpha),\left(\frac{L'_1}{M'_1}\cdots\frac{L'_s}{M'_s}a\right)(w;\beta)\right]$$

$$= -\sum_{n\neq 0}\frac{[L_1n]\cdots[L_rn]}{[M_1n]\cdots[M_rn]}\frac{[L'_1n]\cdots[L'_sn]}{[M'_1n]\cdots[M'_sn]}\frac{n}{[n]^2}\rho_a(n)q^{-(\alpha+\beta)|n|}\left(\frac{w}{z}\right)^n$$
$$-\frac{L_1\cdots L_r}{M_1\cdots M_r}\frac{L'_1\cdots L'_s}{M'_1\cdots M'_s}\rho_a\log\frac{w}{z}, \tag{C.8}$$

where $\theta(P)$ is a step function, $\theta(P)=1(0)$ when the proposition P is true (false). These are formal power series equations.

Next we give specific formulas often used in proofs. For calculation of $[H^i_n,*]$,

$$[a^i_n,a^j_\pm(z)] = \pm\theta(\mp n>0)(q-q^{-1})\frac{1}{n}[(k+g)n][a_{ij}n]z^n, \tag{C.9}$$

$$\left[a^i_n,\left(\frac{1}{k+g}a^j\right)(z;\alpha)\right] = \frac{1}{n}[a_{ij}n]q^{-\alpha|n|}z^n, \tag{C.10}$$

$$[b_n,b_\pm(z)] = \mp\theta(\mp n>0)(q-q^{-1})\frac{1}{n}[n]^2z^n, \tag{C.11}$$

$$[b_n,b(z)] = -\frac{1}{n}[n]z^n, \tag{C.12}$$

where we suppress the superscript of b^{ij}. For OPE calculation,

$$\exp(\alpha b_+(z))\exp(\beta b_-(w))$$
$$\simeq\left(\frac{(z-w)^2}{(z-q^2w)(z-q^{-2}w)}\right)^{\alpha\beta}\exp(\beta b_-(w))\exp(\alpha b_+(z)), \tag{C.13}$$

$$\exp(\alpha b_+(z)):\exp(\beta b(w)):\simeq\left(\frac{z-qw}{qz-w}\right)^{\alpha\beta}:\exp(\beta b(w)):\exp(\alpha b_+(z)), \tag{C.14}$$

$$
\begin{aligned}
:\exp(\alpha b(z)):\exp(\beta b_-(w)) &\simeq \left(\frac{z-qw}{qz-w}\right)^{\alpha\beta}\exp(\beta b_-(w)):\exp(\alpha b(z)): \\
&= \left(\frac{z-qw}{qz-w}\right)^{\alpha\beta} q^{\alpha\beta}:\exp(\alpha b(z)+\beta b_-(w)):\ , \qquad \text{(C.15)}
\end{aligned}
$$

$$
\begin{aligned}
&\exp\left(a^i_+\left(q^{\frac{k+g}{2}}z\right)\right)\exp\left(a^j_-\left(q^{-\frac{k+g}{2}}w\right)\right) \\
&\simeq \frac{z-q^{a_{ij}}w}{z-q^{-a_{ij}}w}\,\frac{z-q^{-a_{ij}-2(k+g)}w}{z-q^{a_{ij}-2(k+g)}w}\exp\left(a^j_-\left(q^{-\frac{k+g}{2}}w\right)\right) \\
&\quad\times\exp\left(a^i_+\left(q^{\frac{k+g}{2}}z\right)\right)\ , \qquad \text{(C.16)}
\end{aligned}
$$

$$
\begin{aligned}
&\exp\left(a^i_+\left(q^{\frac{k+g}{2}}z\right)\right):\exp\left(-\left(\frac{1}{k+g}a^j\right)\left(w;\frac{k+g}{2}\right)\right): \\
&\simeq \frac{z-q^{a_{ij}-(k+g)}w}{q^{a_{ij}}z-q^{-(k+g)}w}:\exp\left(-\left(\frac{1}{k+g}a^j\right)\left(w;\frac{k+g}{2}\right)\right): \\
&\quad\times\exp\left(a^i_+\left(q^{\frac{k+g}{2}}z\right)\right)\ , \qquad \text{(C.17)}
\end{aligned}
$$

$$
\begin{aligned}
&:\exp\left(-\left(\frac{1}{k+g}a^i\right)\left(z;\frac{k+g}{2}\right)\right):\exp\left(a^j_-\left(q^{-\frac{k+g}{2}}w\right)\right) \\
&\simeq \frac{z-q^{a_{ij}-(k+g)}w}{q^{a_{ij}}z-q^{-(k+g)}w}\exp\left(a^j_-\left(q^{-\frac{k+g}{2}}w\right)\right) \\
&\quad\times:\exp\left(-\left(\frac{1}{k+g}a^i\right)\left(z;\frac{k+g}{2}\right)\right): \\
&= \frac{z-q^{a_{ij}-(k+g)}w}{q^{a_{ij}}z-q^{-(k+g)}w}q^{a_{ij}}:\exp\Bigg(-\left(\frac{1}{k+g}a^i\right)\left(z;\frac{k+g}{2}\right) \\
&\quad+a^j_-\left(q^{-\frac{k+g}{2}}w\right)\Bigg):\ , \qquad \text{(C.18)}
\end{aligned}
$$

where α and β are parameters and $\simeq$ means equality in the OPE sense (analytic continuation sense).

$\exp(b+c)'s$ commute each other because $\rho_b(n)+\rho_c(n)=0$.

Appendix D

In this appendix we give how poles cancel each other in OPE of $E^{+,i}(z)$ and $E^{-,j}(w), E^{-,i}(z)$ and $E^{-,j}(w), E^{\pm,i}(z)$ and $S^j(w)$. Let us denote each term of Eqs. (3.6), (3.7), (4.1) as follows:[5]

$$
E^{+,i}(z)=\sum_{j=1}^{i}\left(E^{+,i(j,1)}(z)+E^{+,i(j,2)}(z)\right)\ , \qquad \text{(D.1)}
$$

---

[5] For example, $E^{+,i(j,2)}(z)=\frac{-1}{(q-q^{-1})z}:\exp\left((b+c)^{j,i}(q^{j-1}z)\right)\times(-1)\exp\Bigg(b^{j,i+1}_-(q^{j-1}z)$ $-(b+c)^{j,i+1}(q^{j-2}z)\Bigg)\times\exp\left(\sum_{l=1}^{j-1}\left(b^{\ell,i+1}_+(q^{\ell-1}z)-b^{\ell,i}_+(q^{\ell}z)\right)\right):.$

$$
\begin{aligned}
E^{-,i}(z) = \sum_{j=1}^{i-1} & \left(E^{-,i(j,1)}(z) + E^{-,i(j,2)}(z)\right) \\
& + E^{-,i(i,1)}(z) + E^{-,i(i,2)}(z) + \sum_{j=i+2}^{N} (E^{-,i(j,1)}(z) \\
& + E^{-,i(j,2)}(z)) ,
\end{aligned} \tag{D.2}
$$

$$
S^i(z) = \sum_{j=i+1}^{N} \left(S^{i(j,1)}(z) + S^{i(j,2)}(z)\right) , \tag{D.3}
$$

I. $E^{+,i}(z)E^{-,j}(w)$.

For $i = j$, OPE $E^{+,i}(z)E^{-,j}(w)$ has poles at $z = q^k w$ and $z = q^{-k} w$. They come from $E^{+,i(i,1)}(z)E^{-,j(j,2)}(w)$ and $E^{+,i(1,2)}(z)E^{-,j(1,1)}(w)$ respectively.

Some terms of $E^{+,i}(z)E^{-,j}(w)$ have other poles but all these poles cancel in pairs. We give these poles $(z = q^\alpha w)$ and pairs $\left(E^{+,i(A)}(z)E^{-,j(B)}(w)\right.$ and $\left.E^{+,i(C)}(z)E^{-,j(D)}(w)\right)$.

		α	(A)	(B)	(C)	(D)
(i)	$j = i$	$-k-2\ell$	$(\ell,1)$	$(\ell,2)$	$(\ell+1,2)$	$(\ell+1,1)$
						$1 \leqq \ell \leqq i-1$
(ii)	$j = i+1$	$-k-2\ell+1$	$(\ell,1)$	$(\ell,2)$	$(\ell,2)$	$(\ell,1)$
						$1 \leqq \ell \leqq i$
(iii)	$j = i-1$	$k+1$	$(i-1,1)$	$(j+2,1)$	$(i,1)$	$(j,2)$
		$k+1$	$(i-1,1)$	$(j+2,2)$	$(i,2)$	$(j,2)$
		$k+1$	$(i,1)$	$(j+2,2)$	$(i,2)$	$(j+2,1)$
(iv)	$j \leqq i-2$	$k+i-j$	$(j,1)$	$(i+1,1)$	$(j+1,1)$	$(i,1)$
		$k+i-j$	$(j,1)$	$(i+1,2)$	$(j+1,2)$	$(i,1)$
		$k+i-j$	$(j+1,1)$	$(i+1,2)$	$(j+1,2)$	$(i+1,1)$.

II. $E^{-,i}(z)E^{-,j}(w)$.

$E^{-,i}(z)E^{-,j}(w)$ has poles at $z = q^{-a_{ij}} w$. Some terms of this OPE have extra poles. But these extra poles $(z = q^\alpha w)$ cancel in the following pairs $\left(E^{-,i(A)}(z)E^{-,j(B)}(w)\right.$ and $\left.E^{-,i(C)}(z)E^{-,j(D)}(w)\right)$.

		α	(A)	(B)	(C)	(D)
(i)	$j = i-1$	$2k+i+j$	$(i-1,2)$	$(j,2)$	$(i,1)$	$(j+2,2)$
(ii)	$j \leqq i-2$	$2k+i+j$	$(j,2)$	$(i,1)$	$(j+1,1)$	$(i+1,2)$
		$2k+i+j$	$(j,2)$	$(i,2)$	$(j+1,2)$	$(i+1,2)$
		$2k+i+j$	$(j+1,2)$	$(i,1)$	$(j+1,1)$	$(i,2)$.

III. $E^{+,i}(z)S^j(w)$.

Poles $(z = q^\alpha w)$ cancel in the following pairs $\left(E^{+,i(A)}(z)S^{j(B)}(w)\right.$ and $\left.E^{+,i(C)}(z)S^{j(D)}(w)\right)$.

		α	(A)	(B)	(C)	(D)
(i)	$j = i$	$N-i-j$	$(i,1)$	$(j+1,2)$	$(i,2)$	$(j+1,1)$
(ii)	$j \leqq i-1$	$N-i-j$	$(j,1)$	$(i,2)$	$(j+1,2)$	$(i+1,1)$
		$N-i-j$	$(j,1)$	$(i+1,2)$	$(j,2)$	$(i+1,1)$
		$N-i-j$	$(j,2)$	$(i,2)$	$(j+1,2)$	$(i+1,2)$.

IV. $E^{-,i}(z)S^j(w)$.

For $i=j$, OPE $E^{-,i}(z)S^j(w)$ has poles at $z=q^{k+g}w$ and $z=q^{-(k+g)}w$. They come from $E^{-,i(N,1)}(z)S^{j(N,2)}(w)$ and $E^{-,i(i,1)}(z)S^{j(j+1,1)}(w)$ respectively.

Some terms of $E^{-,i}(z)S^j(w)$ have other poles but all these poles cancel in pairs. Poles ($z=q^\alpha w$) and pairs ($E^{-,i(A)}(z)S^{j(B)}(w)$ and $E^{-,i(C)}(z)S^{j(D)}(w)$) are

	α	(A)	(B)	(C)	(D)
(i) $j=i$	$k-N+2i+2$	$(i,2)$	$(j+1,2)$	$(i+2,2)$	$(j+2,1)$
	$k-N+2\ell$	$(\ell,1)$	$(\ell,2)$	$(\ell+1,2)$	$(\ell+1,1)$
					$i+2\leqq\ell\leqq N-1$
(ii) $j=i+1$	$k-N+2\ell-1$	$(\ell,1)$	$(\ell,2)$	$(\ell,2)$	$(\ell,1)$
					$i+1\leqq\ell\leqq N$
(iii) $j\leqq i-1$	$-k-g+i-j$	$(j,1)$	$(i+1,1)$	$(j+1,1)$	$(i,1)$
	$-k-g+i-j$	$(j,2)$	$(i,1)$	$(j,1)$	$(i,2)$
	$-k-g+i-j$	$(j,2)$	$(i+1,1)$	$(j+1,1)$	$(i,2)$.

Note Added. To ensure the intertwining property of the vertex operators for more general finite dimensional representations than vector representation, we must slightly modify $\phi^{\bar\lambda}(z;\alpha)$ in (4.20) as follows:

$$\phi^{\bar\lambda}(z)\ =:\exp\left(\sum_{i,j=1}^{N-1}\left(\frac{\ell^i}{k+g}\frac{\min(i,j)}{N}\frac{N-\max(i,j)}{1}a^j\right)\left(zq^{\xi_i};-\frac{k+g}{2}\right)\right):\,,$$

where

$$\xi_i=k+g+\sum_{j=1}^{i-1}\ell^j-\sum_{j=i+1}^{N-1}(\ell^j+1)-\sum_{j=1}^{N-2}(N-j-1)\ell^j+(N-2)\mu\,,$$

with an arbitrary constant $\mu\in\mathbf{C}$.

This vertex operator defines the type I vertex operator $\Phi_{V(\mu)}^{V(\nu)V_\lambda}(z)$: $V(\mu)\to V(\nu)\otimes V_{\lambda z}$ with the finite dimensional representation $V_{\lambda z}$ of $U_q(\widehat{sl_N})$ of [34].

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