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The Quantum Group Structure of 2D Gravity and Minimal Models II: The Genus-Zero Chiral Bootstrap

Eugène Cremmer, Jean-Loup Gervais, Jean-François Roussel

Laboratoire de Physique Théorique de l'École Normale Supérieure[1], 24 rue Lhomond, F-75231 Paris Cedex 05, France

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Abstract: The chiral operator-algebra of the quantum-group-covariant operators (of vertex type) is completely worked out by making use of the operator-approach suggested by the Liouville theory, where the quantum-group symmetry is explicit. This completes earlier articles along the same line. The relationship between the quantum-group-invariant (of IRF type) and quantum-group-covariant (of vertex type) chiral operator-algebras is fully clarified, and connected with the transition to the shadow world for quantum-group symbols. The corresponding 3-j symbol dressing is shown to reduce to the simpler transformation of Babelon and one of the authors (J.-L. G.) in a suitable infinite limit defined by analytic continuation. The above two types of operators are found to coincide when applied to states with Liouville momenta going to ∞ in a suitable way. The introduction of quantum-group-covariant operators in the three dimensional picture gives a generalization of the quantum-group version of discrete three-dimensional gravity that includes tetrahedra associated with 3-j symbols and universal R-matrix elements. Altogether the present work and a previous parallel article gives the concrete realization of Moore and Seiberg's scheme that describes the chiral operator-algebra of two-dimensional gravity and minimal models.

1. Introduction

The holomorphic operator algebra that came out [1–4] by quantizing Liouville theory has been formulated in two equivalent bases. The original description of the references just given makes use of operators now denoted[2] $V_{m\hat{m}}^{(J\hat{J})}$. They are closely related with operators called IRF-chiral vertex operator in [9], which are associated with integrable models with solid-on-solid interactions around-the-face [10]. In the context

[1] Unité Propre du Centre National de la Recherche Scientifique, associée à l'École Normale Supérieure et à l'Université de Paris-Sud

[2] Our notations are the same as in previous articles [6–8]. We shall not spell them out again here. This work was supported in part by the European twinning Program, contract #540022

of Liouville theory, they were called [11] Bloch-wave operators since they diagonalise the monodromy matrix. The operators $V_{m\hat{m}}^{(J\hat{J})}(z)$ are of the type $(2\hat{J}+1, 2J+1)$ in the BPZ classification, they shift the zero-mode ϖ of the underlying Bäcklund free field by the fixed amount[3] $2m + 2\hat{m}\pi/h$. In a parallel article [5], we systematically studied the operator algebra of these Bloch-wave operators. The basic progress with respect to earlier discussions is that fusion was treated exactly to all orders using the general scheme [12] of Moore and Seiberg, and not to leading order as was done before. The fusion and braiding matrices of the V fields were shown to be given, up to coupling constants, by quantum 6-j symbols, where the above J's and m's appear as quantum group invariants. Thus the V fields should be regarded as quantum-group invariants. The aim of the present article is to extend this discussion to the other equivalent description of the operator algebra based on operators that are covariant under quantum group transformations. Previously, these operators were introduced in two seemingly different ways. First the invariant operators were "dressed with 3-j symbols" [9, 10]. In this approach, the quantum group covariance seems somewhat artificial and redundant, even though this construction allows [10] to relate integrable models of the IRF (interaction around-the-face) and vertex types. Another method [6, 7, 11, 13] was directly inspired by the operator approach [1–4] to Liouville theory which is explicitly quantum-group symmetric. A set of chiral primary fields noted $\xi_{M\hat{M}}^{(J,\hat{J})}(z)$ was constructed[4], such that the indices M and $\hat{M}$ transform covariantly under quantum group action. We shall concentrate mostly on the operators $\xi_M^{(J)} = \xi_{M0}^{(J,0)}$. They are deduced from the Bloch-wave operators by equations of the form [6, 11]

$$\xi_M^{(J)}(z) := \sum_{-J\le m\le J} |J,\varpi)_M^m \, E_m^{(J)}(\varpi) V_m^{(J)}(z)\,, \qquad -J \le M \le J\,, \tag{1.1}$$

where $|J,\varpi)_M^m$ are q-hypergeometric functions of $e^{ih(\varpi+m)}$, $E_m^{(J)}(\varpi)$ are normalization factors, and ϖ is the zero-mode. The braiding matrices of the ξ's coincide [6, 11] with the universal R-matrix of $U_q(sl(2))$. This construction is more economical than the dressing by 3-j symbols, since it does not involve any redundant quantum number. Thus we call it the *intrinsic transformation.* The leading-order fusion coefficients of the ξ fields were shown [6] to coincide with the quantum 3-j symbols, and it was stated [7] without proof that this is also true, up to a coupling constant, for every order. It is the purpose of the present work to complete that picture. At first, using the above relationship between ξ fields and V fields, we deduce, in Sect. 3, that the fusion of the former are given by

$$\begin{aligned}\xi_{M_1}^{(J_1)}(z_1)\xi_{M_2}^{(J_2)}(z_2) = &\sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} g_{J_1J_2}^{J_{12}} (J_1, M_1; J_2, M_2 \,|\, J_{12}) \\ &\times \sum_{\{\nu\}} \xi_{M_1+M_2}^{(J_{12},\{\nu\})}(z_2)\, \langle \varpi_{J_{12}}, \{\nu\}|\, V_{J_2-J_{12}}^{(J_1)}(z_1-z_2)\, |\varpi_{J_2}\rangle\,, \end{aligned} \tag{1.2}$$

where $(J_1, M_1; J_2, M_2 \,|\, J_{12})$ are the q-Clebsch-Gordan coefficients. Apart from the fact that the right-hand side involves one ξ field and one V field, this has the standard MS

[3] We consider, for simplicity the case where the quantum group parameter h/π is not rational

[4] There exists a related derivation of the universal R matrix in the Coulomb gas picture [14]. However it seems not to be so well suited for obtaining the complete fusion algebra

form [5] (the states $|\varpi_J, \{\nu\}\rangle$ span the Verma module with highest weight state $|\varpi_J\rangle$). It realizes our expectation that there must be fields such that the fusion corresponds to making q-tensor-products of representations. However there is the additional factor $g_{J_1 J_2}^{J_{12}}$. Since this form may be considered as governed by the q-deformed Wigner Eckart theorem, this justifies that g's be called coupling constants. It is thus clear that the ξ fields are quantum-group covariant. In the next section (3) we deepen our understanding of the transformation between V and ξ fields. We first use the fact that the right-hand side of Eq. (1.2) involves one ξ field and one V field to relate them operatorially. This is found equivalent to the dressing by 3-j symbols of [9, 10]. Since the ξ fields were defined by the intrinsic transformation [6, 11] [see Eq. (1.1)], we are able to establish the relationship between the two viewpoints: the dressing by 3-j symbols has an additional magnetic quantum number, and is shown to reduce to the intrinsic transformation, when this number tends to ∞ after a suitable analytic continuation. Using standard properties of the Clebsch-Gordan coefficients we thereby derive several very useful formulae for the coefficients of the intrinsic transformation Eq. (1.1). In particular this allows to give a general formula connecting the coupling constants g_{JK}^{L} with the coefficients $|J, \varpi)_M^m$ and $E_m^{(J)}$ of Eq. (1.1). These formulae are finally shown to prove consistency of the above discussion when it is applied to the $SL(2, C)$-invariant vacuum. This gives a quick way to rederive the coupling constants for the full operators with non-zero J and $\hat{J}$, already established in [8].

In Sect. 4, we discuss the connection between quantum group diagrams and the operator algebra of the V and ξ fields. They are shown to correspond to the shadow and real worlds of [16] respectively. We also display the three-dimensional aspect of the present scheme, extending the relationship [19] between 6-j symbols and discrete gravity in three dimensions. The new point is that the ξ operator-algebra introduces 3-j symbols and R-matrix elements which are represented by tetrahedra, at the same time as 6-j symbols. This gives a geometry of polyhedra with colored edges and faces.

Finally, in Sect. 5, we further develop the idea of understanding the connection between V and ξ fields from infinite limits. Inspired by a recent article of Witten [15] we show that the V and ξ fields coincide in the limit where the zero-mode ϖ goes to ∞, after suitable analytic continuation. This is explicitly proven from the intrinsic transformation summarized above [Eq. (1.1)]. The expressions of the fusing and braiding matrices of V and ξ fields in terms of q symbols then show that these symbols should be related by the same limit, and this is explicitly verified. This sheds light on the method followed in [15] to construct covariant vertex operators, although the conformal theory considered there is different. We also show that this limit has a particularly simple interpretation in the pictorial viewpoint of Sect. 4.

2. The Operator Algebra

In the parallel paper already mentioned [5] the final form of the operator-algebra for the V fields was given. For the operators $V_m^{(J)} \equiv V_{m0}^{(J0)}$, we gave two formulations. They are our starting point, and we summarize them, next. The notations are the same as before [8] and will not be spelled out again. In the Moore Seiberg form [12], the

[5] It already appears, without the g factor in [9]

fusing algebra reads

$$\begin{aligned}
&\langle \varpi_{J_{123}}, \{\nu_{123}\}|\, V^{(J_1)}_{J_{23}-J_{123}}(z_1)\, V^{(J_2)}_{J_3-J_{23}}(z_2)\,|\varpi_{J_3}, \{\nu_3\}\rangle \\
&\quad = \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} \frac{g^{J_{12}}_{J_1J_2}\, g^{J_{123}}_{J_{12}J_3}}{g^{J_{23}}_{J_2J_3}\, g^{J_{123}}_{J_1J_{23}}} \left\{\begin{matrix} J_1 & J_2 \\ J_3 & J_{123} \end{matrix}\middle|\begin{matrix} J_{12} \\ J_{23}\end{matrix}\right\} \\
&\quad \times \sum_{\{\nu_{12}\}} \langle \varpi_{J_{123}}, \{\nu_{123}\}|\, V^{(J_{12},\{\nu_{12}\})}_{J_3-J_{123}}(z_2)\,|\varpi_{J_3}, \{\nu_3\}\rangle \\
&\quad \times \langle \varpi_{J_{12}}, \{\nu_{12}\}|\, V^{(J_1)}_{J_2-J_{12}}(z_1-z_2)\,|\varpi_{J_2}\rangle\,. \qquad (2.1)
\end{aligned}$$

The coupling constants contain the contributions that are not trigonometric functions of h. They are given by

$$\begin{aligned}
g^{J_{12}}_{J_1J_2} &= (g_0)^{J_1+J_2-J_{12}} \prod_{k=1}^{J_1+J_2-J_{12}} \\
&\times \sqrt{\frac{F(1+(2J_1-k+1)h/\pi)F(1+(2J_2-k+1)h/\pi)F(-1-(2J_{12}+k+1)h/\pi)}{F(1+kh/\pi)}}\,, \qquad (2.2)
\end{aligned}$$

where we define $F(z) \equiv \Gamma(z)/\Gamma(1-z)$, and g_0 is a constant. In the operator formalism of the Liouville theory, this may be rewritten as

$$\begin{aligned}
&V^{(J_1)}_{m_1}(z_1)\, V^{(J_2)}_{m_2}(z_2) \\
&\quad = \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} \frac{g^{J_{12}}_{J_1J_2}\, g^{(\varpi-\varpi_0)/2}_{J_{12}(\varpi-\varpi_0+2m_1+2m_2)/2}}{g^{(\varpi-\varpi_0+2m_1)/2}_{J_2(\varpi-\varpi_0+2m_1+2m_2)/2}\, g^{(\varpi-\varpi_0)/2}_{J_1(\varpi-\varpi_0+2m_1)/2}} \\
&\quad \times \left\{\begin{matrix} J_1 & J_2 \\ (\varpi-\varpi_0+2m_1+2m_2)/2 & (\varpi-\varpi_0)/2 \end{matrix}\middle|\begin{matrix} J_{12} \\ (\varpi-\varpi_0+2m_1)/2 \end{matrix}\right\} \\
&\quad \times \sum_{\{\nu_{12}\}} V^{(J_{12},\{\nu_{12}\})}_{m_1+m_2}(z_2)\, \langle \varpi_{J_{12}}, \{\nu_{12}\}|\, V^{(J_1)}_{J_2-J_{12}}(z_1-z_2)\,|\varpi_{J_2}\rangle\,. \qquad (2.3)
\end{aligned}$$

Concerning this last formula, one should recall that ϖ is an operator such that

$$V^{(J)}_m \varpi = (\varpi + 2m)\, V^{(J)}_m \qquad (2.4)$$

so that the first two terms on the right-hand side do not commute with the third one. It is easy to check that this operator-expression is equivalent to Eq. (2.1), by computing the matrix element between the states $\langle \varpi_{J_{123}}, \{\nu_{123}\}|$, and $|\varpi_{J_3}, \{\nu_3\}\rangle$. Then, the additional spins of Eq. (2.1), as compared with Eq. (2.3), are given by

$$\begin{aligned}
J_{123} &= (\varpi - \varpi_0)/2\,, \\
J_{23} &= (\varpi - \varpi_0 + 2m_1)/2\,, \qquad (2.5)\\
J_3 &= (\varpi - \varpi_0 + 2m_1 + 2m_2)/2\,.
\end{aligned}$$

The explicit formula for the Racah-Wigner 6-j coefficients, which have the tetrahedral symmetry, are given in [16] by

$$\left\{\begin{matrix} a & b & e \\ d & c & f \end{matrix}\right\} = (\lfloor 2e+1\rfloor \lfloor 2f+1\rfloor)^{-1/2}(-1)^{a+b-c-d-2e}\left\{\begin{matrix} a & b \\ d & c \end{matrix}\middle| \begin{matrix} e \\ f \end{matrix}\right\}$$
$$= \Delta(a,b,e)\,\Delta(a,c,f)\,\Delta(c,e,d)\,\Delta(d,b,f)$$
$$\times \sum_{z\text{ integer}} (-1)^z \lfloor z+1\rfloor! [\lfloor z-a-b-e\rfloor!$$
$$\times \lfloor z-a-c-f\rfloor! \lfloor z-b-d-f\rfloor!$$
$$\times \lfloor z-d-c-e\rfloor! \lfloor a+b+c+d-z\rfloor!$$
$$\times \lfloor a+d+e+f-z\rfloor! \lfloor b+c+e+f-z\rfloor!]^{-1} \tag{2.6}$$

with

$$\Delta(l,j,k) = \sqrt{\frac{\lfloor -l+j+k\rfloor! \lfloor l-j+k\rfloor! \lfloor l+j-k\rfloor!}{\lfloor l+j+k+1\rfloor!}}$$

and

$$\lfloor n\rfloor! \equiv \prod_{r=1}^{n} \lfloor r\rfloor, \qquad \lfloor r\rfloor \equiv \frac{\sin(hr)}{\sin h}. \tag{2.7}$$

Next we summarize the formulae for the braiding [6]. In the MS form, it is given by

$$\langle \varpi_{J_{123}}, \{\nu_{123}\}|\, V^{(J_1)}_{J_{23}-J_{123}}(z_1)\, V^{(J_2)}_{J_3-J_{23}}(z_2)\, |\varpi_{J_3}, \{\nu_3\}\rangle$$
$$= \sum_{J_{13}} e^{\pm i\pi(\Delta_{J_{123}}+\Delta_{J_3}-\Delta_{J_{23}}-\Delta_{J_{13}})} \frac{g^{J_{13}}_{J_1 J_3} g^{J_{123}}_{J_{13} J_2}}{g^{J_{23}}_{J_2 J_3} g^{J_{123}}_{J_1 J_{23}}} \left\{\begin{matrix} J_1 & J_3 \\ J_2 & J_{123} \end{matrix}\middle| \begin{matrix} J_{13} \\ J_{23} \end{matrix}\right\}$$
$$\times \langle \varpi_{J_{123}}, \{\nu_{123}\}|\, V^{(J_2)}_{J_{13}-J_{123}}(z_2)\, V^{(J_1)}_{J_3-J_{13}}(z_1)\, |\varpi_{J_3}, \{\nu_3\}\rangle. \tag{2.8}$$

The equivalent operator-form is given by

$$V^{(J_1)}_{m_1}(z_1)\, V^{(J_2)}_{m_2}(z_2)$$
$$= \sum_{n_1+n_2=m_1+m_2} e^{\mp i h(2m_1 m_2 + m_2^2 - n_2^2 + \varpi(m_2-n_2))}$$
$$\times \left\{\begin{matrix} J_1 & (\varpi-\varpi_0+2m_1+2m_2)/2 \\ J_2 & (\varpi-\varpi_0)/2 \end{matrix}\middle| \begin{matrix} (\varpi-\varpi_0+2n_2)/2 \\ (\varpi-\varpi_0+2m_1)/2 \end{matrix}\right\}$$
$$\times \frac{g^{(\varpi-\varpi_0+2n_2)/2}_{J_1 (\varpi-\varpi_0+2n_1+2n_2)/2}\, g^{(\varpi-\varpi_0)/2}_{J_2 (\varpi-\varpi_0+2n_2)/2}}{g^{(\varpi-\varpi_0+2m_1)/2}_{J_2 (\varpi-\varpi_0+2m_1+2m_2)/2}\, g^{(\varpi-\varpi_0)/2}_{J_1 (\varpi-\varpi_0+2m_1)/2}}\, V^{(J_2)}_{n_2}(z_2)\, V^{(J_1)}_{n_1}(z_1). \tag{2.9}$$

The correspondence table is again given by Eq. 2.5, with, in addition,

$$J_{13} = (\varpi - \varpi_0 + 2n_2)/2. \tag{2.10}$$

[6] An equivalent form is given in [13], without connection to the 6-j

Let us recall that, as discussed in [5], the g factors may be absorbed by changing the normalization of the V fields. This eliminates them from the braiding, but not from the fusion since the latter depends upon the explicit values of the V matrix elements [last term of Eq. (2.1) or Eq. (2.3)]. In the operator-forms Eqs. (2.3), (2.9), one sees that the fusion and braiding matrices involve the operator ϖ, and thus do not commute with the V-operators. Such is the general situation of the operator-algebras in the MS formalism. This is in contrast with, for instance, the braiding relations for quantum group representations. In the following, and completing the results of [6, 11], we will change basis to the holomorphic operators ξ which are such that these ϖ dependences of the fusing and braiding matrices disappear. After the transformation, one is in the same situation as for quantum group, and its structure becomes more transparent. As already emphasized, the quantum numbers J and m of the $V_m^{(J)}$ operators should be regarded as quantum-group invariant. Indeed, the J's appear as total spins in $q-6j$ symbols, and the m's are given by differences of J's, as is clear from [5]. Thus the quantum group does not act on the V's. In [6, 11], other fields $\xi_M^{(J)}$ were defined which are quantum-group covariant. Following [6], this is done in two steps. A first change of field is performed by introducing ψ fields of the form

$$\psi_m^{(J)} \equiv E_m^{(J)}(\varpi) V_m^{(J)}, \tag{2.11}$$

such that the braiding matrix and the fusing coefficients for $\psi_\alpha^{(1/2)} \psi_m^{(J)} \to \psi_{m+\alpha}^{(J+1/2)}$ become trigonometric. Then the ξ fields are defined by expressions of the form

$$\xi_M^{(J)} = \sum_{-J\le m\le J} |J,\varpi)_M^m \psi_m^{(J)}, \qquad -J \le M \le J. \tag{2.12}$$

The explicit form of the coefficients $|J,\varpi)_M^m$ is given in Eq. (3.19), below. The formulae just written allow us to deduce the F and B matrices of the ξ fields from those of the V fields derived in the previous section. Indeed, it was already shown in [6] that the braiding matrix of the ξ field coincides with the universal R-matrix of $U_q(sl(2))$. Concerning the fusing matrices, we shall establish an explicit connection later on, by first relating the coefficients $|J,\varpi)_M^m$ to a limit of q-Clebsch-Gordan coefficients. At the present stage of the discussion, it is more enlightening to proceed in another way. We shall first transform the fusing matrix for the OPE of $V_{m_1}^{(1/2)}$ and $V_{m_2}^{(J_2)}$, which was the starting point of [8], and after, generalize the result using the associativity of the OPA.

Consider thus Eq. (2.3), with $J_1 = 1/2$, and make use of Eq. (2.2). Taking Eq. (2.4) into account, one sees that it is appropriate to multiply both sides by $E_{\pm 1/2}^{(1/2)}(\varpi) E_m^{(J)}(\varpi \pm 1)$. Using the recurrence relations satisfied by the functions C and D [see Eqs. (A.15), and (A.17) of [6]] one thereby derives the fusing relations

$$\begin{aligned}\psi_{\pm 1/2}^{(1/2)} \psi_m^{(J)} &= \sum_{\{\nu\}} \frac{\lfloor \varpi \mp J + m \rfloor}{\lfloor \varpi \rfloor} \psi_{m\pm 1/2}^{(J+1/2,\{\nu\})} \langle \varpi_J + 1, \{\nu\}| V_{-1/2}^{(1/2)} |\varpi_J\rangle \\ &\quad + \sum_{\{\nu\}} \frac{\lfloor \mp J + m \rfloor}{\lfloor \varpi \rfloor} \Gamma(1 + 2Jh/\pi) \Gamma(-1 - (2J+1)h/\pi) \\ &\quad \times \psi_{m\pm 1/2}^{(J-1/2,\{\nu\})} \langle \varpi_J - 1, \{\nu\}| V_{1/2}^{(1/2)} |\varpi_J\rangle .\end{aligned} \tag{2.13}$$

From now on we do not write the world-sheet variables explicitly any longer, since the dependence is always the standard one identical to Eqs. (2.1), (2.3), for fusions, and to Eqs. (2.8), (2.9) for braidings. The original motivation for introducing the ψ fields [6, 7] was that the braiding matrix, and the leading-order fusing coefficients for them are trigonometrical. The last equation written shows that this is not true for the other fusing coefficients $\psi_\alpha^{(1/2)}\psi_m^{(J)} \to \psi_{m+\alpha}^{(J-1/2)}$. As a result, the OPE is not associative if one forgets the contribution of the secondaries (more about this below). Next, comparing with the expression Eq. (2.2) of the g's, one sees that the last equation is naturally rewritten as

$$\psi_{\pm 1/2}^{(1/2)}\psi_m^{(J)} = \sum_{\varepsilon,\{\nu\}} g_{1/2,J}^{J+\varepsilon/2} N\left|{}^{1/2}_{\pm 1/2}\ {}^{J;}_{m;}\ {}^{J+\varepsilon/2}_{m\pm 1/2}; \varpi\right| \psi_{m\pm 1/2}^{(J+\varepsilon/2,\{\nu\})} \times \langle \varpi_J+\varepsilon,\{\nu\}|\, V_{-\varepsilon/2}^{(1/2)}\, |\varpi_J\rangle \tag{2.14}$$

where,

$$\begin{aligned} N\left|{}^{1/2}_{\pm 1/2}\ {}^{J;}_{m;}\ {}^{J+1/2}_{m\pm 1/2}; \varpi\right| &= \frac{\lfloor \varpi \mp J + m\rfloor}{\lfloor \varpi \rfloor}, \\ N\left|{}^{1/2,}_{\pm 1/2,}\ {}^{J;}_{m;}\ {}^{J-1/2}_{m\pm 1/2}; \varpi\right| &= \frac{i\pi g_0^{-1}}{\sqrt{\sin(2hJ)\sin((2J+1)h)}}\,\frac{\lfloor \mp J + m\rfloor}{\lfloor \varpi\rfloor}. \end{aligned} \tag{2.15}$$

One sees that the non-trigonometric part is entirely contained in the g's. It will be shown at the end of the section that Eq. (2.14) is a particular case of the general fusing algebra

$$\psi_{m_1}^{(J_1)}\psi_{m_2}^{(J_2)} = \sum_{J,\{\nu\}} g_{J_1,J_2}^{J} N\left|{}^{J_1}_{m_1}\ {}^{J_2;}_{m_2;}\ {}^{J}_{m_1+m_2}; \varpi\right| \psi_{m_1+m_2}^{(J_1,\{\nu\})} \times \langle \varpi_J,\{\nu\}|\, V_{J_2-J}^{(J_1)}\, |\varpi_{J_2}\rangle . \tag{2.16}$$

In addition, $N\left|{}^{J_1}_{m_1}\ {}^{J_2;}_{m_2;}\ {}^{J}_{m_1+m_2}; \varpi\right|$ will be related to a $q-6j$ symbol. For the time being we transform Eq. (2.14) further, in order to derive the fusion of the ξ fields. Their definition Eq. (2.12), together with the shift properties of the V fields [see Eq. (2.4)] are such that

$$\xi_\alpha^{(1/2)}\xi_M^{(J)} = \sum_{\eta=\pm 1/2, m} |1/2,\varpi)_\alpha^\eta\, |J,\varpi+2\eta)_M^m\, \psi_\eta^{(1/2)}\psi_m^{(J)} . \tag{2.17}$$

By a calculation which is similar to the one carried out in Appendix B of [6], one verifies that, if we choose $g_0 = 2\pi$, we have

$$\sum_{\eta=\pm 1/2, m, (\eta+m=n)} |1/2,\varpi)_\alpha^\eta\, |J,\varpi+2\eta)_M^m\, N\left|{}^{1/2,}_{\eta}\ {}^{J;}_{m;}\ {}^{J+\varepsilon/2}_{\eta\pm m}; \varpi\right| = (1/2,\alpha; J, M\,|\,J+\varepsilon/2)\,|J+\varepsilon/2,\varpi)_{M+\alpha}^n . \tag{2.18}$$

(Recall that $(1/2,\alpha; J, M\,|\,J+\varepsilon/2)$ denotes the 3-j symbols.) Equation (2.14) becomes

$$\xi_{\pm 1/2}^{(1/2)}\xi_m^{(J)} = \sum_{\varepsilon=\pm 1} g_{1/2,J}^{J+\varepsilon/2}(1/2,\alpha; J, M\,|\,J+\varepsilon/2) \times \sum_{\{\nu\}} \xi_{m\pm 1/2}^{(J+\varepsilon/2, m\{\nu\})} \langle \varpi_J+\varepsilon/2,\{\nu\}|\, V_{-\varepsilon/2}^{(1/2)}\, |\varpi_J\rangle . \tag{2.19}$$

Our next task is to generalize this last fusing identity. Note an important feature of this equation. As expected, it expresses the OPE of two ξ fields in terms of one ξ fields, (and its descendants), that is $\xi_{m\pm 1/2}^{(J+\varepsilon/2,\{\nu\})}$. However, the coefficients of this OPE are proportional to the matrix element of a V field. The basic reason is that the V matrix element, of the fusing equation for the V fields [Eq. (2.3)] contains no m_1 or m_2 dependence, and is thus unchanged when going from the ψ to the ξ fields. Thus we shall start from the general ansatz

$$\mathscr{P}_J \xi_{M_1}^{(J_1)} \xi_{M_2}^{(J_2)} = \mathscr{P}_J \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} \mathscr{F}(J_1, M_1, J_2, M_2, J_{12}, M_{12}, \varpi_J) \times \sum_{\{\nu\}} \xi_{M_1+M_2}^{(J_{12},\{\nu\})} \langle \varpi_{J_{12}}, \{\nu\} | V_{J_2-J_{12}}^{(J_1)} | \varpi_{J_2} \rangle , \tag{2.20}$$

where $\mathscr{P}_J$ is the projector on the Virasoro module with zero-mode $\varpi_J = \varpi_0 + 2J$ (see [5]). Our next task it to prove that the fusion coefficients are proportional to the CG coefficients and independent of ϖ_J:

$$\mathscr{F}(J_1, M_1, J_2, M_2, J_{12}, M_{12}, \varpi_J) = g_{J_1 J_2}^{J_{12}} (J_1, M_1; J_2, M_2 \,|\, J_{12}) . \tag{2.21}$$

To do this, we use the associativity equations for the ξ fields, that is, the pentagonal relation of the MS scheme. The proof is the same as the one for the V operators (see [5]), and we shall skip details (more about it soon, however). There is a difference yet, that we have to emphasize. In the demonstration given in [5], we pointed out that only four of the five fusing coefficients of the pentagonal relation really came from the fusion of the operators considered, as the other one – the third one on the left-hand side of the pentagonal relation – came from the fusion of the operators in the matrix elements, which had been restored as operators thanks to the closure relation. So, in the case of the ξ operators, the four fusion coefficients will become coefficients of the ξ, but **the other one will remain a fusing coefficient of V-fields**. Accordingly, we get a pentagonal relation of the form

$$\sum \mathscr{F}\mathscr{F} F = \mathscr{F}\mathscr{F} , \tag{2.22}$$

where F is the fusing matrix of the V fields as recalled on Eq. (2.1) or (2.3). Next, using the explicit expression Eq. (2.20), let us show that Eq. (2.21) is a solution of the associativity condition. Indeed, with this ansatz, the pentagonal relation becomes

$$\sum_{J_{23}} (J_2, M_2; J_3, M_3 \,|\, J_{23}) (J_1, M_1; J_{23}, M_{23} \,|\, J_{123}) \left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J_{123} \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\} = (J_1, M_1; J_2, M_2 \,|\, J_{12}) (J_{12}, M_{12}; J_3, M_3 \,|\, J_{123}) . \tag{2.23}$$

This is the basic identity that defines the $q-6j$ coefficients [16]. Once we know that Eq. (2.21) gives an associative algebra, it is easy to derive it by recursion from the particular case $J_1 = 1/2$, J_2 arbitrary [Eq. (2.19)] using a recurrence proof which is completely parallel to the one we gave for the V fields, in [5]. We do not go through it again, and consider Eq. (2.21) as established. □

An important feature of the result is that the fusing matrix does not depend upon ϖ_J. Thus the projector of Eq. (2.20) does not serve any purpose and may be removed.

The fusion algebra finally reads

$$\xi_{M_1}^{(J_1)}\xi_{M_2}^{(J_2)} = \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} g_{J_1J_2}^{J_{12}}(J_1,M_1;J_2,M_2\,|\,J_{12}) \times \sum_{\{\nu\}} \xi_{M_1+M_2}^{(J_{12},\{\nu\})} \langle \varpi_{J_{12}},\{\nu\}|\, V_{J_2-J_{12}}^{(J_1)}\,|\varpi_{J_2}\rangle\,. \tag{2.24}$$

For the coming discussion, we shall actually need the following generalization of this last relation to arbitrary descendants:

$$\xi_{M_1}^{(J_1,\{\gamma\})}\xi_{M_2}^{(J_2,\{\mu\})} = \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} g_{J_1J_2}^{J_{12}}(J_1,M_1;J_2,M_2\,|\,J_{12}) \times \sum_{\{\nu\}} \xi_{M_1+M_2}^{(J_{12},\{\nu\})} \langle \varpi_{J_{12}},\{\nu\}|\, V_{J_2-J_{12}}^{(J_1,\{\gamma\})}\,|\varpi_{J_2},\{\mu\}\rangle\,, \tag{2.25}$$

which holds according to the general principles of [12].

3. More on the $V - \xi$ Transformation

We have just shown that the complete fusion rule of two ξ fields is most naturally expressed in terms of one ξ field and one V field. Now, we use this fact to operatorially relate the fusing and braiding properties of V fields with those of the ξ fields. This will provide the general identities which relate their fusing and braiding matrices. The method is to apply the fusion algebra Eq. (2.25) repeatedly to the OPE of several ξ fields. In fact, the forthcoming calculation will explicitly verify some of the polynomial equations of the OPE of the ξ fields, and may also be regarded as a pedagogical explanation of the arguments given above to derive Eq. (2.25). For that purpose, one should of course deal with the descendant matrix-elements, and this is why Eq. (2.25) is needed. Consider the matrix element $\langle \varpi_J,\{\alpha\}|\,\xi_{M_1}^{(J_1)}\xi_{M_2}^{(J_2)}\xi_{M_3}^{(J_3,\{\varrho\})}\,|\varpi_K,\{\beta\}\rangle$. We apply Eq. (2.25) twice: $\xi_{M_3}^{(J_3,\{\varrho\})}$ and $\xi_{M_2}^{(J_2)}$ are fused first, and the result is then fused with $\xi_{M_1}^{(J_1)}$. One gets

$$\begin{aligned}
&\langle \varpi_J,\{\alpha\}|\,\xi_{M_1}^{(J_1)}\xi_{M_2}^{(J_2)}\xi_{M_3}^{(J_3,\{\varrho\})}\,|\varpi_K,\{\beta\}\rangle \\
&= \sum_{J_{23}\{\nu\}} g_{J_2J_3}^{J_{23}}(J_2,M_2;J_3,M_3\,|\,J_{23}) \\
&\quad\times \sum_{J_{123}\{\mu\}} g_{J_1J_{23}}^{J_{123}}(J_1,M_1;J_{23},M_2+M_3\,|\,J_{123}) \\
&\quad\times \langle \varpi_J,\{\alpha\}|\,\xi_{M_1+M_2+M_3}^{(J_{123},\{\mu\})}\,|\varpi_K,\{\beta\}\rangle \\
&\quad\times \langle \varpi_{J_{123}},\{\mu\}|\,V_{J_{23}-J_{123}}^{(J_1)}\,|\varpi_{J_{23}},\{\nu\}\rangle \\
&\quad\times \langle \varpi_{J_{23}},\{\nu\}|\,V_{J_3-J_{23}}^{(J_2)}\,|\varpi_{J_3},\{\varrho\}\rangle\,.
\end{aligned}$$

Performing the sum over $\{\nu\}$ gives

$$\begin{aligned}
&\langle \varpi_J, \{\alpha\}|\, \xi_{M_1}^{(J_1)} \xi_{M_2}^{(J_2)} \xi_{M_3}^{(J_3,\{\varrho\})} \,|\varpi_K, \{\beta\}\rangle \\
&= \sum_{J_{123} J_{23} \{\nu\}} g_{J_1 J_{23}}^{J_{123}} (J_1, M_1; J_{23}, M_2 + M_3 \mid J_{123}) g_{J_2 J_3}^{J_{23}} (J_2, M_2; J_3, M_3 \mid J_{23}) \\
&\quad \times \langle \varpi_J, \{\alpha\}|\, \xi_{M_1+M_2+M_3}^{(J_{123},\{\nu\})} \,|\varpi_K, \{\beta\}\rangle \\
&\quad \times \langle \varpi_{J_{123}}, \{\nu\}|\, V_{J_{23}-J_{123}}^{(J_1)} V_{J_3-J_{23}}^{(J_2)} \,|\varpi_{J_3}, \{\varrho\}\rangle \,. \qquad (3.1)
\end{aligned}$$

At this point it is convenient to consider the last equation as follows. The two fields $\xi_{M_1}^{(J_1)}$ and $\xi_{M_2}^{(J_2)}$ on the left-hand side have been converted into two V fields, that is $V_{J_{23}-J_{123}}^{(J_1)}$, and $V_{J_3-J_{23}}^{(J_2)}$, on the right-hand side. The third field $\xi_{M_3}^{(J_3,\{\varrho\})}$ plays the role of a background field which allows us to operatorially relate ξ fields to V fields by successive fusions. Its quantum numbers J_3 and $\{\varrho\}$ specify which matrix element of V operators will come out at the end. Its quantum number M_3 is arbitrary and does not appear in the final matrix element of the two V fields. We shall come back to it later on. It is easily seen that this procedure may be repeated for more than three ξ fields, and that the structure is similar. The right-most ξ field which is the only one not converted into a V field is to be considered as a background field. In fact, each V field is multiplied by a 3-j symbols, and this method naturally leads to the transformation through dressing by 3-j symbols of [10, 9], as we see next. Clearly, fusing or braiding the ξ fields and the corresponding V fields on each side of Eq. (3.1) will directly relate their F and B matrices. Since this relation is, to begin with, different from the one which comes out from the connection through the $|J,\varpi)_M^m$ coefficients, we next go through the derivations.

First consider the fusion. It follows from Eq. (2.1) that the right member of Eq. (3.1) may be rewritten as

$$\begin{aligned}
&\sum_{J_{123} J_{23} J_{12}} (J_1, M_1; J_{23}, M_2 + M_3 \mid J_{123}) (J_2, M_2; J_3, M_3 \mid J_{23}) \\
&\quad \times \left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J_{123} \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\} g_{J_1 J_2}^{J_{12}} g_{J_{12} J_3}^{J_{123}} \\
&\quad \times \sum_{\{\nu\}} \langle \varpi_J, \{\alpha\}|\, \xi_{M_1+M_2+M_3}^{(J_{123},\{\nu\})} \,|\varpi_K, \{\beta\}\rangle \\
&\quad \times \sum_{\{\gamma\}} \langle \varpi_{J_{123}}, \{\nu\}|\, V_{J_3-J_{123}}^{(J_{12},\{\gamma\})} \,|\varpi_{J_3}, \{\varrho\}\rangle \langle \varpi_{J_{12}}, \{\gamma\}|\, V_{J_2-J_{12}}^{(J_1)} \,|\varpi_{J_2}\rangle \,. \qquad (3.2)
\end{aligned}$$

On the left-hand side, we fuse $\xi_{M_1}^{(J_1)}$, and $\xi_{M_2}^{(J_2)}$, making use of Eq. (2.25). This gives

$$\begin{aligned}
&\sum_{J_{12}} g_{J_1 J_2}^{J_{12}} (J_1, M_1; J_2, M_2 \mid J_{12}) \\
&\quad \times \sum_{\{\gamma\}} \langle \varpi_J, \{\alpha\}|\, \xi_{M_1+M_2}^{(J_{12},\{\gamma\})} \xi_{M_3}^{(J_3,\{\varrho\})} \,|\varpi_K, \{\beta\}\rangle \\
&\quad \times \langle \varpi_{J_{12}}, \{\gamma\}|\, V_{J_2-J_{12}}^{(J_1)} \,|\varpi_{J_2}\rangle \,. \qquad (3.3)
\end{aligned}$$

Next the remaining two ξ fields are fused in their turn, and one gets

$$\sum_{J_{12},J_{123}} g_{J_1 J_2}^{J_{12}}(J_1, M_1; J_2, M_2 \mid J_{12}) g_{J_{12} J_3}^{J_{123}}(J_{12}, M_1 + M_2; J_3, M_3 \mid J_{123})$$
$$\times \sum_{\{\gamma\}\{\nu\}} \langle \varpi_J, \{\alpha\}| \xi_{M_1+M_2+M_3}^{(J_{123},\{\nu\})} |\varpi_K, \{\beta\}\rangle$$
$$\times \langle \varpi_{J_{123}}, \{\nu\}| V_{J_3-J_{123}}^{(J_{12},\{\gamma\})} |\varpi_{J_3}\{\varrho\}\rangle$$
$$\times \langle \varpi_{J_{12}}, \{\gamma\}| V_{J_2-J_{12}}^{(J_1)} |\varpi_{J_2}\rangle . \tag{3.4}$$

Comparing this last expression with Eq. (3.2), one sees that they coincide if the defining relation of the 6-j symbols [Eq. (2.23)] holds. Thus Eq. (3.1) does establish the correct correspondence between the fusion properties of the V and ξ fields.

Consider, next, the braiding. For the V fields, it is given by Eq. (2.8), or (2.9). Concerning the ξ fields, the braiding properties were derived in [6]. One has

$$\xi_{M_1}^{(J_1)} \xi_{M_2}^{(J_2)} = \sum_{-J_1 \le N_1 \le J_1;\, -J_2 \le N_2 \le J_2} (J_1, J_2)_{M_1 M_2}^{N_2 N_1} \xi_{N_2}^{(J_2)} \xi_{N_1}^{(J_1)} . \tag{3.5}$$

The symbol $(J_1, J_2)_{M_1 M_2}^{N_2 N_1}$ denotes the following matrix element of the universal R-matrix:

$$(J_1, J_2)_{M_1 M_2}^{M_2' M_1'} = (\langle\langle J_1, M_1| \otimes \langle\langle J_2, M_2|) R(|J_1, M_1'\rangle\rangle \otimes |J_2, M_2'\rangle\rangle) , \tag{3.6}$$

where $|J, M\rangle\rangle$ are group theoretic states which span the representation of spin J of $U_q(sl(2))$. The universal R-matrix R is given by

$$R = e^{-2ihJ_3 \otimes J_3} \sum_{n=0}^{\infty} \frac{(1 - e^{2ih})^n \, e^{ihn(n-1)/2}}{\lfloor n \rfloor !}$$
$$\times e^{-ihnJ_3}(J_+)^n \otimes e^{ihnJ_3}(J_-)^n , \tag{3.7}$$

$J_\pm$, and J_3 are the quantum-group generators. For later use we recall that the R-matrix-elements may be simply written in terms of CG coefficients, since the latter are "twisted" eigenvectors, namely,

$$\sum_{N_1 N_2} (J_2, N_2; J_1, N_1 \mid J_{12}) (J_1, J_2)_{M_1 M_2}^{N_2 N_1}$$
$$= e^{i\pi(\Delta_{J_{12}} - \Delta_{J_1} - \Delta_{J_2})}(J_1, M_1; J_2, M_2 \mid J_{12}) . \tag{3.8}$$

It follows from the orthogonality of the CG coefficients that

$$(J_1, J_2)_{M_1 M_2}^{N_2 N_1} = \sum_{J_{12}} (J_2, N_2; J_1, N_1 \mid J_{12}) e^{i\pi(\Delta_{J_{12}} - \Delta_{J_1} - \Delta_{J_2})}$$
$$\times (J_1, M_1; J_2, M_2 \mid J_{12}) . \tag{3.9}$$

Returning to our main line, we follow the same procedure as for fusion. We shall skip details since the present discussion goes in close parallel. One exchanges the first two ξ fields on the left-hand side of Eq. (3.1), and the two V fields on its right-hand side. Comparing the results, one derives the consistency condition

$$\sum_{N_1 N_2} (J_1, N_1; J_3, M_3 \mid J_{13})(J_2, N_2; J_{13}, N_1 + M_3 \mid J_{123}, M_{123})(J_1, J_2)_{M_1 M_2}^{N_2 N_1}$$
$$= \sum_{J_{23}} (J_2, M_2; J_3, M_3 \mid J_{23})(J_1, M_1; J_{23}, M_2 + M_3 \mid J_{123})$$
$$\times e^{i\pi(\Delta_{J_{123}} + \Delta_{J_3} - \Delta_{J_{23}} - \Delta_{J_{13}})} \left\{ \begin{matrix} J_1 & J_3 \\ J_2 & J_{123} \end{matrix} \middle| \begin{matrix} J_{13} \\ J_{23} \end{matrix} \right\}. \tag{3.10}$$

This last relation may be easily proven using an equation satisfied [16] by the 6-j symbols which shows that the braiding matrix Eq. (2.8) satisfies the Yang-Baxter equations. This defines the 6-j coefficient of the second type as was introduced in [18].

The outcome of the present discussion is that the defining relations for the two types of 6-j symbols [Eqs. (2.23) and (3.10)] may be considered as relating the braiding and fusing matrices of the V and ξ fields. Thus the connection is established in a way where the quantum-group meaning is transparent. Clearly, Eqs. (2.23), and (3.10) show that the connection is established via 3-j symbols. As a matter of fact, we have effectively re-derived the transformation through dressing by 3-j symbols of [9, 10]. This is in contrast with the intrinsic transformation of [6, 11], recalled in Eq. (2.12), which uses the $|J, \varpi)_M^m$ coefficients. Our next point is to establish the connection between these two transformations. the q-CG symbols involve 5 independent quantum numbers, and the $|J, \varpi)_M^m$ coefficients only 4. In this connection, we have remarked that, in Eq. (3.1), M_3 does not appear in the V-matrix element. It only appears in the two CG coefficients. The first one, that is $(J_1, M_1; J_{23}, M_2 + M_3 \mid J_{123})$, (resp. the second one, that is $(J_2, M_2; J_3, M_3 \mid J_{23})$) only involve J-quantum-numbers of the field $V^{(J_1)}_{J_{23}-J_{123}}$, (resp. $V^{(J_2)}_{J_3-J_{23}}$). These two sets of quantum numbers are treated on the same footing, but this is not the case for the M-quantum numbers, however, since the first CG coefficient contains M_1, $M_2 + M_3$, and the second M_2, M_3. To motivate the coming mathematical derivation, we may remark that the symmetry is restored if $M_2 + M_3 \sim M_3$, that is if M_3 is very large compared with M_2. One way to achieve this is to keep J_3 finite and to continue the quantum group for $M_3 > J_3$. This may be done rigorously, as we shall next show, since the $q - 3j$ symbols are given by q-deformed hypergeometric functions [7, 16]. In this limit one quantum number of the q-CG symbol drops out, and we will be left with the right number to identify the result with a $|J, \varpi)_M^m$ coefficient.

We start from the explicit expression of the CG coefficients [7, 16], that is,

$$(J_1, M_1; J_2, M_2 \mid J) = B(J_1, J_2, J, M_1)$$
$$\times \sqrt{\lfloor J_2 - M_2 \rfloor ! \lfloor J_2 + M_2 \rfloor ! \lfloor J - M_1 - M_2 \rfloor ! \lfloor J + M_1 + M_2 \rfloor !}$$
$$\times e^{ihM_2 J_1} \sum_{\mu=0}^{J_1+J_2-J} \left\{ \frac{e^{-ih\mu(J+J_1+J_2+1)}(-1)^\mu}{\lfloor \mu \rfloor ! \lfloor J_1 + J_2 - J - \mu \rfloor !} \right.$$
$$\left. \times \frac{1}{\lfloor J_1 - M_1 - \mu \rfloor ! \lfloor J - J_2 + M_1 + \mu \rfloor ! \lfloor J_2 + M_2 - \mu \rfloor ! \lfloor J - J_1 - M_2 + \mu \rfloor !} \right\}, \tag{3.11}$$

where

$$B(J_1, J_2, J, M_1) \\ = e^{ih(J_1+J_2-J)(J_1+J_2+J+1)/2} e^{-ihM_1 J_2} \\ \times \sqrt{\lfloor J_1 - M_1 \rfloor! \lfloor J_1 + M_1 \rfloor!} \sqrt{\lfloor 2J+1 \rfloor} \\ \times \sqrt{\frac{\lfloor J_1 + J_2 - J \rfloor! \lfloor -J_1 + J_2 + J \rfloor! \lfloor J_1 - J_2 + J \rfloor!}{\lfloor J_1 + J_2 + J + 1 \rfloor!}} . \tag{3.12}$$

The terms in M_2 are conveniently rewritten as

$$\sqrt{\frac{\lfloor J_2 - M_2 \rfloor!}{\lfloor J - J_1 + \mu - M_2 \rfloor!}} \sqrt{\frac{\lfloor J - M_1 - M_2 \rfloor!}{\lfloor J - J_1 + \mu - M_2 \rfloor!}} \\ \times \sqrt{\frac{\lfloor J_2 + M_2 \rfloor!}{\lfloor J_2 - \mu + M_2 \rfloor!}} \sqrt{\frac{\lfloor J + M_1 + M_2 \rfloor!}{\lfloor J_2 - \mu + M_2 \rfloor!}} . \tag{3.13}$$

We shall take the limit by giving an imaginary part to M_2, thus we have to continue the above formulae in this variable. The last formula contains all the M_2 dependence. It has been written as a product of square roots of ratios of q-deformed factorials. Consider each term one-by-one. The differences between the arguments of numerators and denominators are

$$\begin{cases} J_2 - M_2 - (J - J_1 + \mu - M_2) = J_1 + J_2 - J - \mu \\ J - M_1 - M_2 - (J - J_1 + \mu - M_2) = J - M_1 - \mu \\ J_2 + M_2 - (J_2 - \mu + M_2) = \mu \\ J + M_1 + M_2 - (J_2 - \mu + M_2) = J + M_1 - J_2 + \mu . \end{cases}$$

The right members are independent from M_2. In the expression Eq. (3.11), the actual range of summation is dictated by the fact that a factorial with negative argument is infinite, so that each factorial may only have a non-negative argument. This immediately shows that the right-hand sides of the last set of equations are non-negative integers. As in [7], let us introduce (ν is a positive integer, and a arbitrary)

$$\lfloor a \rfloor_\nu \equiv \lfloor a \rfloor \lfloor a+1 \rfloor \dots \lfloor a + \nu - 1 \rfloor = \frac{\lfloor a + \nu - 1 \rfloor!}{\lfloor a - 1 \rfloor!} . \tag{3.14}$$

Equation (3.13) may be rewritten as

$$\sqrt{\lfloor J - J_1 + \mu - M_2 + 1 \rfloor_{J_1 + J_2 - J - \mu} \lfloor J - J_1 + \mu - M_2 + 1 \rfloor_{J - M_1 - \mu}} \\ \times \sqrt{\lfloor J_2 - \mu + M_2 + 1 \rfloor_\mu \lfloor J_2 - \mu + M_2 + 1 \rfloor_{J + M_1 - J_2 + \mu}} . \tag{3.15}$$

According to the definition Eq. (3.14), each term involves a number of factors which is independent from M_2, so that the last expression makes sense for arbitrary complex M_2. The limit is taken with an imaginary part, since, otherwise, the functions $\sin[h(M_2 + \alpha)]$, α constant, which appear in Eq. (3.15), would not have a well defined limit. Of course, with the imaginary part, one exponential of trigonometric functions blows up while the other vanishes. The choice of sign is such that

$$\lim_{M_2\to\infty} \lfloor \alpha \pm M_2 \rfloor = \mp \frac{1}{2i\sin h}\, e^{-ih(M_2\pm\alpha)}, \tag{3.16}$$

$$\frac{\lfloor \alpha \pm M_2 \rfloor!}{\lfloor \beta \pm M_2 \rfloor!} \sim (-1)^{(\alpha-\beta)(1\pm1)/2}(2i\sin h)^{\beta-\alpha}\, e^{ih(\beta-\alpha)M_2}\, e^{\mp ih(\alpha-\beta)(\alpha+\beta+1)/2}\,. \tag{3.17}$$

Substitute into Eq. (3.11), one gets

$$\begin{aligned}(J_1, M_1; J_2, M_2 \mid J) &\\ \sim e^{ih[-M_1(J+1/2)+(J_2^2-J_1^2-J^2+2J_1J_2)/2+(J_1+J_2-J)/2]} &\\ \times B(J_1, J_2, J, M_1)\, \frac{e^{i\pi(J-J_2+M_1)/2}}{(2i\sin h)^{J_1}} &\\ \times \sum_{\mu=0}^{J_1+J_2-J} \left\{ \frac{e^{-2ih\mu(J+J_2+1)}}{\lfloor \mu \rfloor!\, \lfloor J_1+J_2-J-\mu \rfloor!} \right. &\\ \left. \times \frac{1}{\lfloor J_1-M_1-\mu \rfloor!\, \lfloor J-J_2+M_1+\mu \rfloor!} \right\}. &\end{aligned} \tag{3.18}$$

Note that the limit is perfectly finite, since all exponentials in M_2 cancel out. On the other hand, the explicit expression of $|j,\varpi)^m_M$ is [6] [7]

$$\begin{aligned}|j,\varpi)^m_M = \sqrt{\binom{2j}{j+M}}\, e^{ihm/2} \sum_{(J-M+m-t)/2 \text{ integer}} &\\ \times \frac{e^{iht(\varpi+m)} \lfloor j-M \rfloor!\, \lfloor j+M \rfloor!}{\left\lfloor \frac{j-M+m-t}{2} \right\rfloor!\, \left\lfloor \frac{j-M-m+t}{2} \right\rfloor!\, \left\lfloor \frac{j+M+m+t}{2} \right\rfloor!\, \left\lfloor \frac{j+M-m-t}{2} \right\rfloor!}, &\end{aligned} \tag{3.19}$$

and, letting $\mu = (j+M-m-t)/2$,

$$\begin{aligned}|j,\varpi)^m_M = \sqrt{\binom{2J}{J+M}}\, e^{ihm/2}\, e^{ih((\varpi+m)(j+M-m)} &\\ \times \sum_{\mu} \frac{e^{-2ih\mu(\varpi+m)} \lfloor j-M \rfloor!\, \lfloor j+M \rfloor!}{\lfloor \mu \rfloor!\, \lfloor j-m-\mu \rfloor!\, \lfloor m-M+\mu \rfloor!\, \lfloor j+M-\mu \rfloor!}\,. &\end{aligned} \tag{3.20}$$

Comparing Eqs. (3.18), and (3.20), one sees that the variables should be related by

$$\varpi = \varpi_0 + 2J_2\,, \qquad J_1 = j\,, \qquad M = -M_1\,, \qquad m = J - J_2\,. \tag{3.21}$$

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[7] $\binom{n}{p}$ denotes the q-deformed binomial coefficients $\binom{n}{p} \equiv \lfloor n \rfloor! / \lfloor p \rfloor!\, \lfloor n-p \rfloor!$

One gets altogether,

$$(J_1, M_1; J_2, M_2 \,|\, J) \sim \sqrt{\frac{\lfloor 2J+1 \rfloor}{\begin{pmatrix} 2J_1 \\ J_1+J_2-J \end{pmatrix} \lfloor J_2+J-J_1+1 \rfloor_{2J_1+1}}} \times e^{i\pi[(J-J_2+M_1)/2+J+M_1-J_1-J_2]} \times \frac{e^{ih(M_1+J_2-J)/2}}{(2i\sin h)^{J_1}} \,|J_1, \varpi_0+2J_2)^{J-J_2}_{-M_1} \,. \tag{3.22}$$

Recall that one has [7]

$$\sum_{M=-j}^{j} (-1)^{j-M} e^{ih(j-M)} \,|j,\varpi)^m_M \,|j,\varpi+2p)^{-P}_{-M} = \delta_{m,p} C^{(j)}_m(\varpi) \,, \tag{3.23}$$

where

$$C^{(j)}_m(\varpi) = (-1)^{j-m} (2i\sin h)^{2j} e^{ihj} \begin{pmatrix} 2j \\ j-m \end{pmatrix} \frac{\lfloor \varpi - j + m \rfloor_{2j+1}}{\lfloor \varpi + 2m \rfloor} \,. \tag{3.24}$$

Equation (3.22) may be re-written as

$$(J_1, M_1; J_2, M_2 \,|\, J) \sim \sqrt{\frac{e^{ih(J_1+J_2-J+M_1)} e^{i\pi(J_1+M_1)}}{C^{(J_1)}_{J-J_2}(\varpi_0+2J_2)}} \,|J_1, \varpi_0+2J_2)^{J-J_2}_{-M_1} \,. \tag{3.25}$$

This is consistent with the orthogonality of the CG coefficients:

$$\sum_{M_1} \left\{ \left(J_1, M_1; \frac{\varpi-\varpi_0}{2}, M_2 \,\middle|\, \frac{\varpi+2m-\varpi_0}{2} \right) \times \left(J_1, M_1; \frac{\varpi-\varpi_0}{2}, M_2 \,\middle|\, \frac{\varpi+2p-\varpi_0}{2} \right) \right\} = \delta_{m,p} \,. \tag{3.26}$$

In the limit $M_2 \to \infty$ this gives

$$\sum_{M_1} (-1)^{J_1+M_1} e^{ih(J_1+M_1)} \,|J_1,\varpi)^m_{-M_1} \,|J_1,\varpi)^p_{-M_1} = \delta_{m,p} C^{(J_1)}_m(\varpi) e^{ihm} \,. \tag{3.27}$$

This is equivalent to Eq. (3.23) since $|J_1, \varpi+2p)^{-p}_{-M_1} = e^{-ihp} \,|J_1,\varpi)^p_{M_1}$. Finally we use the defining relation for the 6-j symbols [Eq. (2.23)], that is in the present notations,

$$\sum_{j_{23}} (j_2, \mu_2; j_3, \mu_3 \,|\, j_{23}) (j_1, \mu_1; j_{23}, \mu_{23} \,|\, j_{123}) \left\{ \begin{matrix} j_1 & j_2 \\ j_3 & j_{123} \end{matrix} \middle| \begin{matrix} j_{12} \\ j_{23} \end{matrix} \right\} = (j_1, \mu_1; j_2, \mu_2 \,|\, j_{12}) (j_{12}, \mu_{12}; j_3, \mu_3 \,|\, j_{123}) \,. \tag{3.28}$$

For the present purpose, it is convenient to let

$$j_2 = J_1 \,, \quad \mu_2 = -M_1 \,, \quad j_3 = \frac{\varpi-\varpi_0}{2} \,, \quad j_{23} = \frac{\varpi-\varpi_0+2m_1}{2} \,,$$
$$j_1 = J_2 \,, \quad \mu_1 = -M_2 \,, \quad j_{123} = \frac{\varpi-\varpi_0+2m_1+2m_2}{2} \,. \tag{3.29}$$

In the limit one gets

$$\sum_{m_1+m_2=m_{12}} |J_1,\varpi)^{m_1}_{M_1}\,|J_2,\varpi+2m_1)^{m_2}_{M_2}\sqrt{\frac{e^{i(h+\pi)(J_1+J_2)}}{C^{(J_1)}_{m_1}(\varpi)C^{(J_2)}_{m_2}(\varpi+2m_1)}}$$
$$\times\left\{\begin{matrix} J_2 & J_1 \\ \frac{\varpi-\varpi_0}{2} & \frac{\varpi+2m_1+2m_2-\varpi_0}{2}\end{matrix}\left|\begin{matrix} J_{12} \\ \frac{\varpi+2m_1-\varpi_0}{2}\end{matrix}\right.\right\}$$
$$=(J_1,M_1;J_2,M_2\,|\,J_{12})\,|J_{12},\varpi)^{m_{12}}_{M_1+M_2}\sqrt{\frac{e^{i(h+\pi)J_{12}}}{C^{(J_{12})}_{m_{12}}(\varpi)}}\,, \tag{3.30}$$

where we used that fact that

$$(J_2,-M_2;J_1,-M_1\,|\,J_{12})=(J_1,M_1;J_2,M_2\,|\,J_{12})\,. \tag{3.31}$$

This relation may be verified on the explicit expression Eq. (3.11). Equation (3.30) is the generalization of Eq. (2.18). Indeed, starting from the fusing algebra of the ξ fields [Eq. (2.25)], and taking account of the relationship between ξ and ψ fields [Eq. (2.12)], one concludes that the fusing of the ψ fields is of the form [8]

$$\psi^{(J_1)}_{m_1}\psi^{(J_2)}_{m_2}=\sum_{J,\{\nu\}} g^{J}_{J_1,J_2}N\left|\begin{matrix}J_1 & J_2; & J\\ m_1 & m_2; & m_1+m_2\end{matrix};\varpi\right|\psi^{(J,\{\nu\})}_{m_1+m_2}$$
$$\times\langle\varpi_J,\{\nu\}|\,V^{(J_1)}_{J_2-J}\,|\varpi_{J_2}\rangle\,, \tag{3.32}$$

where

$$N\left|\begin{matrix}J_1 & J_2; & J_{12}\\ m_1 & m_2; & m_1+m_2\end{matrix};\varpi\right|$$
$$=\sqrt{\frac{e^{i\pi(J_{12}-J_1-J_2)}C^{(J_{12})}_{m_{12}}(\varpi)}{C^{(J_1)}_{m_1}(\varpi)C^{(J_2)}_{m_2}(\varpi+2m_1)}}\;e^{ih(J_1+J_2-J_{12})/2}$$
$$\times\left\{\begin{matrix} J_2 & J_1 \\ \frac{\varpi-\varpi_0}{2} & \frac{\varpi+2m_1+2m_2-\varpi_0}{2}\end{matrix}\left|\begin{matrix} J_{12} \\ \frac{\varpi+2m_1-\varpi_0}{2}\end{matrix}\right.\right\}. \tag{3.33}$$

Our next point is a cross check of the whole section. We re-derive the fusing matrix of the ψ fields by starting from that of the V fields [Eq. (2.1)], and applying the transformation Eq. (2.11). The result reads

$$\langle\varpi_{J_{123}},\{\nu_{123}\}|\,\psi^{(J_1)}_{J_{23}-J_{123}}\psi^{(J_2)}_{J_3-J_{23}}\,|\varpi_{J_3},\{\nu_3\}\rangle$$
$$=\sum_{J_{12}=|J_1-J_2|}^{J_1+J_2}\frac{g^{J_{12}}_{J_1J_2}g^{J_{123}}_{J_{12}J_3}}{g^{J_{23}}_{J_2J_3}g^{J_{123}}_{J_1J_{23}}}\,\frac{E^{(J_1)}_{J_{23}-J_{123}}(\varpi_{J_{123}})E^{(J_2)}_{J_3-J_{23}}(\varpi_{J_{23}})}{E^{(J_{12})}_{J_3-J_{123}}(\varpi_{J_{123}})}$$
$$\times\left\{\begin{matrix}J_1 & J_2\\ J_3 & J_{123}\end{matrix}\left|\begin{matrix}J_{12}\\ J_{23}\end{matrix}\right.\right\}$$
$$\times\sum_{\{\nu_{12}\}}\langle\varpi_{J_{123}},\{\nu_{123}\}|\,V^{(J_{12},\{\nu_{12}\})}_{J_3-J_{123}}\,|\varpi_{J_3}\{\nu_3\}\rangle\langle\varpi_{J_{12}},\{\nu_{12}\}|\,V^{(J_1)}_{J_2-J_{12}}\,|\varpi_{J_2}\rangle\,. \tag{3.34}$$

[8] Already announced on Eq. (3.1)

Comparing with Eq. (3.33), we conclude that

$$\frac{g^{J_{123}}_{J_{12}J_3}}{g^{J_{23}}_{J_2J_3}\, g^{J_{123}}_{J_1J_{23}}} = e^{-ih(J_1+J_2-J_{12})/2}\sqrt{\frac{C^{(J_{12})}_{J_3-J_{123}}(\varpi_{J_{123}})e^{i\pi(J_{12}-J_1-J_2)}}{C^{(J_1)}_{J_{23}-J_{123}}(\varpi_{J_{123}})C^{(J_2)}_{J_3-J_{23}}(\varpi_{J_{23}})}} \times \frac{E^{(J_{12})}_{J_3-J_{123}}(\varpi_{J_{123}})}{E^{(J_1)}_{J_{23}-J_{123}}(\varpi_{J_{123}})E^{(J_2)}_{J_3-J_{23}}(\varpi_{J_{23}})}\,. \tag{3.35}$$

Thus, we should have

$$g^{J_{23}}_{J_2J_3} = e^{ih(J_2+J_3-J_{23})/2}\sqrt{e^{i\pi(J_{23}-J_2-J_3)}\,C^{(J_2)}_{J_3-J_{23}}(\varpi_{J_{23}})} \times E^{(J_2)}_{J_3-J_{23}}(\varpi_{J_{23}})\beta^{J_{23}}_{J_2J_3}\,, \tag{3.36}$$

where β is a solution of the equation

$$\beta^{J_{123}}_{J_{12}J_3} = \beta^{J_{23}}_{J_2J_3}\,\beta^{J_{123}}_{J_1J_{23}} \tag{3.37}$$

that ensures that it disappears from Eq. (3.35). In this last relation, no summation over J_{23} is understood. Since the left-hand side is independent of J_1, this must be true for the right-hand side, and it follows that, in general, β^K_{LJ} is independent from the first lower index L. This shows that the general solution of Eq. (3.37) is of the form

$$\beta^K_J = f(K)/f(J)\,. \tag{3.38}$$

The remaining unknown function $f(I)$ is determined from the condition $g^{J_2+J_3}_{J_2J_3} = 1$, and one finally gets

$$g^{J_{23}}_{J_2J_3} = e^{ih(J_{23}-J_2-J_3)/2}\sqrt{\frac{e^{i\pi(J_{23}-J_2-J_3)}C^{(J_2)}_{J_3-J_{23}}(\varpi_{J_{23}})C^{(J_3)}_{-J_3}(\varpi_{J_3})}{C^{(J_{23})}_{-J_{23}}(\varpi_{J_{23}})}} \times \frac{E^{(J_2)}_{J_3-J_{23}}(\varpi_{J_{23}})E^{(J_3)}_{-J_3}(\varpi_{J_3})}{E^{(J_{23})}_{-J_{23}}(\varpi_{J_{23}})}\,. \tag{3.39}$$

It is straightforward, but a bit lengthy, to verify that this is equivalent to our previous expression Eq. (2.2). Next we illustrate the meaning of this new expression for the coupling constants. It simply ensures that the fusion properties of the ξ fields, which do not depend upon ϖ, may be indeed rederived by applying the product of ξ fields simply to the right $SL(2,C)$-invariant vacuum $|\varpi_0\rangle$, with $\varpi_0 = 1+\pi/h$. Let us compute $\langle\varpi_{J_3}|\,\xi^{(J_1)}_{M_1}(z)\xi^{(J_2)}_{M_2}(0)\,|\varpi_0\rangle$ in two different ways

(1) directly using the relation between $\xi^{(J)}_M$ and $V^{(J)}_M$ as well as Eq. (2.10),

$$\langle\varpi_{J_3}|\,\xi^{(J_1)}_{M_1}(z)\xi^{(J_2)}_{M_2}(0)\,|\varpi_0\rangle = E^{(J_1)}_{J_2-J_3}(\varpi_{J_3})E^{(J_2)}_{-J_2}(\varpi_{J_2})\,|J_1,\varpi_{J_3})^{J_2-J_3}_{M_1}\,|J_2,\varpi_{J_2})^{-J_2}_{M_2}\,z^{(\Delta_3-\Delta_1-\Delta_2)}\,; \tag{3.40}$$

(2) using first the short-distance expansion

$$\xi^{(J_1)}_{M_1}(z)\xi^{(J_2)}_{M_2}(0) \cong \sum_{J_{12}} z^{\Delta_{12}-\Delta_1-\Delta_2}[G^{J_1J_2J_{12}}_{M_1M_2M_1+M_2}\xi^{J_{12}}_{M_1+M_2}(0)+\ldots]\,, \tag{3.41}$$

where G is to be determined. Then

$$\begin{aligned}&\langle \varpi_{J_3} |\, \xi^{(J_1)}_{M_1}(z)\xi^{(J_2)}_{M_2}(0)\, |\varpi_0\rangle \\ &\quad = G^{J_1 J_2 J_3}_{M_1 M_2 M_1+M_2} E^{(J_3)}_{-J_3}(\varpi_{J_3})\, |J_3, \varpi_{J_3})^{-J_3}_{M_1+M_2}\, z^{(\Delta_3-\Delta_1-\Delta_2)} + \dots .\end{aligned} \tag{3.42}$$

One gets immediately

$$G^{J_1 J_2 J_3}_{M_1 M_2 M_1+M_2} = \frac{E^{(J_1)}_{J_2-J_3}(\varpi_{J_3}) E^{(J_2)}_{-J_2}(\varpi_{J_2})}{E^{(J_3)}_{-J_3}(\varpi_{J_3})} \frac{|J_1, \varpi_{J_3})^{J_2-J_3}_{M_1}\, |J_2, \varpi_{J_2})^{-J_2}_{M_2}}{|J_3, \varpi_{J_3})^{-J_3}_{M_1+M_2}} . \tag{3.43}$$

Applying Eq. (3.30) with $m_{12} = -J_3$, $\varpi = \varpi_{J_3}$ and noting that due to triangular inequalities the non-symmetric 6-j coefficient $\left\{ \begin{matrix} J_2 & J_1 \\ J_3 & 0 \end{matrix} \middle| \begin{matrix} J_3 \\ J_3 + m_1 \end{matrix} \right\}$ is different from 0 only for $m_1 = J_2 - J_3$ we get

$$\begin{aligned}&\frac{|J_1, \varpi_{J_3})^{J_2-J_3}_{M_1}\, |J_2, \varpi_{J_2})^{-J_2}_{M_2}}{|J_3, \varpi_{J_3})^{-J_3}_{M_1+M_2}} \\ &\quad = e^{[ih(J_3-J_1-J_2)+i\pi(J_3-J_1-J_2)]/2} \\ &\quad\times \sqrt{\frac{C^{(J_1)}_{J_2-J_3}(\varpi_{J_3}) C^{(J_2)}_{-J_2}(\varpi_{J_2})}{C^{(J_3)}_{-J_3}(\varpi_{J_3})}}\, (J_1 M_1 J_2 M_2 \,|\, J_3) .\end{aligned} \tag{3.44}$$

As expected, this gives

$$G^{J_1 J_2 J_{12}}_{M_1 M_2 M_1+M_2} = g^{J_{12}}_{J_1 J_2} (J_1 M_1 J_2 M_3 \,|\, J_{12}) , \tag{3.45}$$

where $g^{J_{12}}_{J_1 J_2}$ is given by Eq. (3.39). It is worth to note that, instead of using the short distance expansion Eq. (3.41), it is consistent to use the exact Eq. (2.24) and we get the same result due to the fact that $\langle \varpi_{J_3} |\, V^{(J_{12},\nu)}_{m_1+m_2}(0)\, |\varpi_0\rangle$ is different from 0 only for the highest weight operator $V^{(J_{12})}_{m_1+m_2}(\nu = 0)$.

So far, we only dealt with the operators connected with the quantum group parameter h. Until the end of this section, we consider the most general operators $\xi^{(J\hat{J})}_{M\hat{M}}$ that involve the two quantum group parameters h and $\hat{h}$. It is straightforward to extend the full discussion just given to this case, since the fusing and braiding matrices of the $V^{(J)}_m$ (or equivalently of the $\xi^{(J)}_M$) with the $V^{(\hat{J})}_{\hat{m}}$ (or equivalently with the $\xi^{(\hat{J})}_{\hat{M}}$) operators are simple phases. We shall not do it explicitly to keep the length of this paper within reasonable limits. We shall simply show that the calculation just mentioned also applies to the fusion of the $\xi^{(J\hat{J})}_{M\hat{M}}$ fields using now [6]

$$\xi^{(J\hat{J})}_{M\hat{M}} = e^{4i\pi J\hat{J}}\, e^{-i\pi(M\hat{J}+\hat{M}J)} \sum_{m,\hat{m}} |J, \varpi)^m_M\, \widehat{|\hat{J}, \hat{\varpi})}^{\hat{m}}_{\hat{M}}\, E^{(J\hat{J})}_{m\hat{m}}(\varpi) V^{(J\hat{J})}_{m\hat{m}} \tag{3.46}$$

and

$$\left| J, \varpi + 2\hat{l}\,\frac{\pi}{h} \right)^m_M = e^{2i\pi\hat{l}m}\, e^{-2i\pi\hat{l}(J-M)}\, |J, \varpi)^m_M . \tag{3.47}$$

The short distance expansion of the product of two $\xi_{M\hat{M}}^{J\hat{J}}$ fields is written as

$$\xi_{M_1\hat{M}_1}^{(J_1\hat{J}_1)}(z)\xi_{M_2\hat{M}_2}^{(J_2\hat{J}_2)}(0) \cong \sum_{J_3,\hat{J}_3} z^{(\Delta_{J_3\hat{J}_3}-\Delta_{J_1\hat{J}_1}-\Delta_{J_2\hat{J}_2})} \times [G_{M_1\hat{M}_1M_2\hat{M}_2M_1+M_2\hat{M}_1+\hat{M}_2}^{J_1\hat{J}_1J_2\hat{J}_2J_3\hat{J}_3}\xi_{M_1+M_2\hat{M}_1+\hat{M}_2}^{(J_3\hat{J}_3)}(0)+\ldots]\,. \tag{3.48}$$

The final result reads

$$G_{M_1\hat{M}_1M_2\hat{M}_2M_1+M_2\hat{M}_1+\hat{M}_2}^{J_1\hat{J}_1J_2\hat{J}_2J_3\hat{J}_3} = g_{J_1\hat{J}_1J_2\hat{J}_2}^{J_3\hat{J}_3}(J_1M_1J_2M_2\,|\,J_3)\hat{(}\hat{J}_1\hat{M}_1\hat{J}_2\hat{M}_2\,\hat{|}\,\hat{J}_3\hat{)} \times e^{i\pi[M_1\hat{J}_2-M_2\hat{J}_1+\hat{M}_1J_2-\hat{M}_2J_1+(M_1-M_2)(\hat{J}_3-\hat{J}_1-\hat{J}_2)+(\hat{M}_1-\hat{M}_2)(J_3-J_1-J_2)]} \tag{3.49}$$

with

$$g_{J_1\hat{J}_1J_2\hat{J}_2}^{J_3\hat{J}_3} = g_{J_1J_2}^{J_3}\,\hat{g}_{\hat{J}_1\hat{J}_2}^{\hat{J}_3}\,h_{J_1\hat{J}_1J_2\hat{J}_2}^{J_3\hat{J}_3}\,, \tag{3.50}$$

where a careful calculation gives

$$h_{J_1\hat{J}_1J_2\hat{J}_2}^{J_3\hat{J}_3} = (-1)^{p\hat{p}+2p\hat{J}_3+2\hat{p}J_3}\prod_k\prod_{r,\hat{r}\in X_k}\left(r\sqrt{h/\pi}+\hat{r}\sqrt{\pi/h}\right)^{\varepsilon_k}, \tag{3.51}$$

here the X_k's are the direct product of intervals $[a_k,b_k]\otimes[\hat{a}_k,\hat{b}_k]$ and given by

$$X_k = \begin{cases} [1,2J_1]\otimes[1,2\hat{J}_1] \\ [1,2J_1-p]\otimes[1,2\hat{J}_1-\hat{p}] \\ [1,2J_2]\otimes[1,2\hat{J}_2] \\ [1,2J_2-p]\otimes[1,2\hat{J}_2-\hat{p}] \\ [2,2J_3+1]\otimes[2,2\hat{J}_3+1] \\ [2,2J_3+p+1]\otimes[2,2\hat{J}_3+\hat{p}+1] \\ [1,p]\otimes[1,\hat{p}] \end{cases}, \quad \text{with} \quad \varepsilon_k = \begin{cases} +1 \\ -1 \\ +1 \\ -1 \\ +1 \\ -1 \\ -1 \end{cases}$$

and

$$p = J_1+J_2-J_3\,, \qquad \hat{p} = \hat{J}_1+\hat{J}_2-\hat{J}_3\,. \tag{3.52}$$

We note that G is symmetric under the exchange of 1 and 2. This is in agreement with the consistency of the fusion rules and the braiding.

Using the properties of the Γ function, the factor h can be absorbed partially or completely. A particular form which exhibits the symmetry between $(J_1,\hat{J}_1)$, $(J_2,\hat{J}_2)$ and $(-J_3-1,-\hat{J}_3-1)$ is

$$g_{J_1\hat{J}_1J_2\hat{J}_2}^{J_3\hat{J}_3} = (i/2)^{p+\hat{p}}\,\frac{H_{p\hat{p}}(J_1,\hat{J}_1)H_{p\hat{p}}(J_2,\hat{J}_2)H_{p\hat{p}}(-J_3-1,-\hat{J}_3-1)}{H_{p\hat{p}}(p/2,\hat{p}/2)} \tag{3.53}$$

with

$$H_{p\hat{p}}(J, \hat{J}) = \frac{\sqrt{\prod_{r=1}^{p} F(2\hat{J} + 1 + (2J - r + 1)h/\pi) \prod_{\hat{r}=1}^{\hat{p}} F(2J + 1 + (2\hat{J} - \hat{r} + 1)\pi/h)}}{\prod_{r=1}^{p} \prod_{\hat{r}=1}^{\hat{p}} [(2J - r + 1)\sqrt{h/\pi} + (2\hat{J} - \hat{r} + 1)\sqrt{\pi/h}]} \tag{3.54}$$

and $p = 1 + J_1 + J_2 + (-J_3 - 1)$ is symmetric. These expressions are equivalent to the one derived in [8].

4. The General (3D) Structure

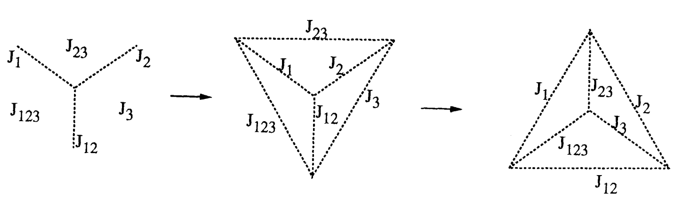
In this section we discuss the general structure of the bootstrap equations. We shall not give all details, but rather establish the connection with earlier works [9, 10, 16, 21] where the $U_q(sl(2))$ quantum group structure was discussed in other contexts, and show the generalization brought about by the introduction of ξ fields. In [16], quantum-group diagrams were introduced which involve two different "worlds": the "normal" one and the "shadow" one. Adopting this terminology from now on, we are going to verify that the OPA of the V and ξ fields is in exact correspondence with these diagrams, if the ξ and V OPE's are associated with the normal and shadow worlds respectively. At the same time we shall discuss the associated three dimensional aspect. For the V fields it is already known, since it corresponds to the quantum-group version of the Regge-calculus approach to the discrete three-dimensional gravity [19] or to the discussion of [15], for instance. This case will serve as an introduction to the novel structure that comes out when V and ξ fields are considered together.

In the pictorial representations, we omit the g coefficients. Thus, we actually make use of the operator-algebra expressed in terms of the $\tilde{V}$ fields defined by

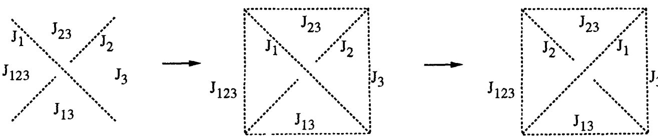
$$\mathscr{P}_{J_{12}} \tilde{V}^{J_1}_{J_2 - J_{12}} \equiv g^{J_{12}}_{J_1 J_2} \mathscr{P}_{J_{12}} V^{J_1}_{J_2 - J_{12}}, \tag{4.1}$$

instead of the V's.

Though of great importance for operator-product expansion, the coupling constants g define a pure gauge for the polynomial equations or knot-theory viewpoints. We could draw other figures including g coefficients, to show how they cancel out of those equations, but this would be cumbersome. The basic fusing and braiding operations on the $\tilde{V}$ operators have three equivalent representations



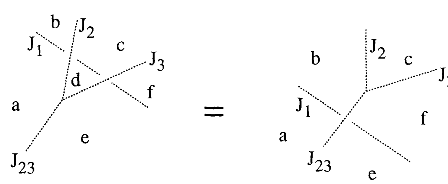
$$= \left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J_{123} \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\} \quad \text{(fusion of } \tilde{V} \text{ operators)}, \tag{4.2}$$



$$= e^{i\pi(\Delta_{J_{123}}+\Delta_{J_3}-\Delta_{J_{23}}-\Delta_{J_{13}})} \left\{ \begin{matrix} J_1 & J_3 \\ J_2 & J_{123} \end{matrix} \middle| \begin{matrix} J_{13} \\ J_{23} \end{matrix} \right\}$$

(braiding of $\tilde{V}$ operators). (4.3)

First consider the left diagrams, and the associated Eqs. (2.1), and (2.8). Apart from the $\tilde{V}$-matrix element on the right-hand side of the fusing relation, which has no specific representative, each operator $\tilde{V}_m^{(J)}$ is represented by a dashed line carrying the label J. The spins on the faces display the zero-modes of the Verma modules on which the $\tilde{V}_m^{(J)}$ operators act. Thus the m's are differences between the spin-labels of the two neighbouring faces. For the braiding diagram, the spins on the edges are unchanged at crossings, and, for given J_1, J_2, the braiding diagram has the form of a vertex of an interaction-around-the-face (IRF) model. These diagrams are two-dimensional (2D). The appearance of spins on the faces reflect the fact that the fusion and braiding properties depend upon the Verma module on which the operators act. It is easily seen that, when they are used as building blocks, the above drawings generate diagrams which have the same structure as the quantum-group ones of [16] in the shadow world[9]. The polynomial equations can be viewed as link-invariance conditions. For instance,



(4.4)

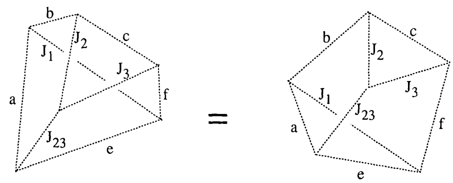
gives the pentagonal relation of the V fields discussed in [8], after cancellation of the phases (with a change of indices).

The middle diagrams of Figs. 4.2 and 4.3 are obtained from the left ones (first arrow) by enclosing the 2D figures with extra dashed lines carrying the spin labels which were previously on the faces. In this way, one gets three-dimensional (3D) tetrahedra, with spin labels only on the edges. The right figures are obtained from the middle ones by dualisation: the face, surrounded by the edges J_a, J_b, J_c, becomes the vertex where the edges J_a, J_b, J_c join, and conversely, a vertex becomes a face. An edge joining two vertices becomes the edge between the two dual faces. There is one triangular face for each $\tilde{V}$ field, including the $\tilde{V}$ matrix-element of the fusing relation Eq. (2.1). On the dualised polyhedra, the triangular inequalities give the addition rules for spins. The main point of the middle and right diagrams is that, as a consequence of the basic MS properties of the OPA, they are really 2D projections of three-dimensional diagrams which may be rotated at essentially no cost[10]. For instance, the MS relation between fusing and braiding matrices simply corresponds to the fact that they are represented by tetrahedra which may be identified after a rigid 3D rotation.

We used dashed lines to agree with the conventions of [16]

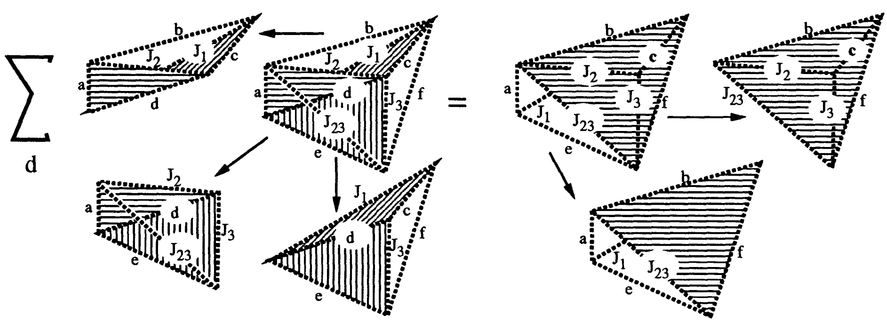
10 We use 6-j symbols which do not have the full tetrahedral symmetry, so that two edges should be distinguished. This will be discussed at the end of this section

We shall illustrate the general properties of the 3D diagrams on the example of the pentagonal relation. In the same way as we closed the basic figures in Eqs. (4.2), (4.3), the rule to go to 3D is to close the composite Fig. 4.4. It gives a polyhedron which has vertices with three edges only, which we call type V3E. The two-dimensional Eq. (4.4) now simply corresponds to viewing the V3E polyhedron from two different angles:



(4.5)

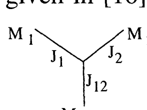
Dualisation gives a polyhedron, with only triangular faces, which we call F3E. The polynomial equations are recovered by decomposing a F3E polyhedron into tetrahedra (this correspondence only works with F3E polyhedra, this is the reason for dualisation). In parallel with the two different fusing-braiding decompositions of each side of Eq. (4.4), there are two 3D decompositions of the F3E polyhedron. This is represented in split view on the next figure, where the internal faces are hatched for clarity:



(4.6)

In general, the rule is to take a polyhedron with triangular faces and to decompose it in tetrahedra in different ways. Substituting the associated 6-j symbols yields the polynomial identities [11].

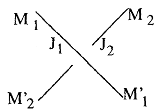
In quantum-group diagrams [16], a second world was introduced – the normal one – which is represented by solid lines. In this world, the quantum numbers are spins J and magnetic numbers M both on the lines. The label M changes at the crossings. We now show how our ξ operator algebra is a realisation of this normal world. Indeed, the fusing and braiding matrices of the fields $\xi_M^{(J)}$ do not depend upon the Verma module on which they act, so that it is consistent that the quantum numbers J and M be attached to the corresponding line. Comparing Eqs. (2.24), and (3.5) with the formulae given in [16], one sees that one has the following 2D representation [12]



$$= (J_1, M_1; J_2, M_2 \mid J_{12}) = \text{fusion of } \xi \text{ operators,} \qquad (4.7)$$

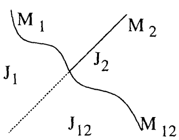
[11] We restrict ourselves to polyhedra which are orientable surfaces

[12] For Eq. (2.24), the factor $g_{J_1 J_2}^{J_{12}}$ is absorbed by going from V to $\tilde{V}$ matrix element on the right-hand side



$$= (J_1, J_2)_{M_1 M_2}^{M_2' M_1'} = \text{braiding of } \xi \text{ operators}, \tag{4.8}$$

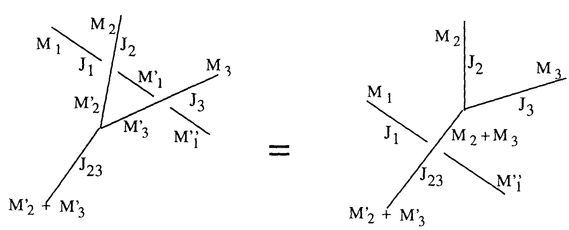
which coincides with the corresponding quantum-group vertices of [16]. Clearly, the braiding diagram Eq. (4.8) should be regarded as an interaction of the vertex type. In the same way as for the $\tilde{V}$ fields, the fusing vertex has only three legs so that the $\tilde{V}$ matrix element is not represented per se. On the other hand, we have shown, using repeated fusions [Eqs. (3.1)–(3.10)] that the fusing equation also provides the transitions between $\tilde{V}$ and ξ operator-product algebras. From this viewpoint, the result of the discussion just recalled may be pictorially represented by introducing transitions between the two worlds based on the graph



$$= (J_1, M_1; J_2, M_2 \mid J_{12}) \text{ (from normal to shadow world)}, \tag{4.9}$$

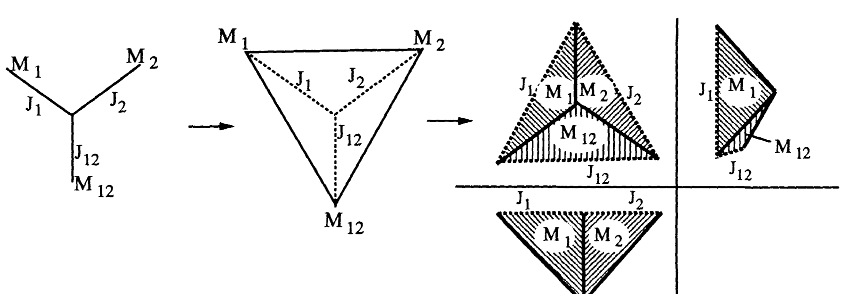
which coincides with the one introduced in [16].

The polynomial equations involving $\tilde{V}$, and/or ξ fields are summarized by the link-invariance of the diagrams constructed out of the building bocks just given. For instance, the example previously given for the $\tilde{V}$ operators becomes, in the normal world,

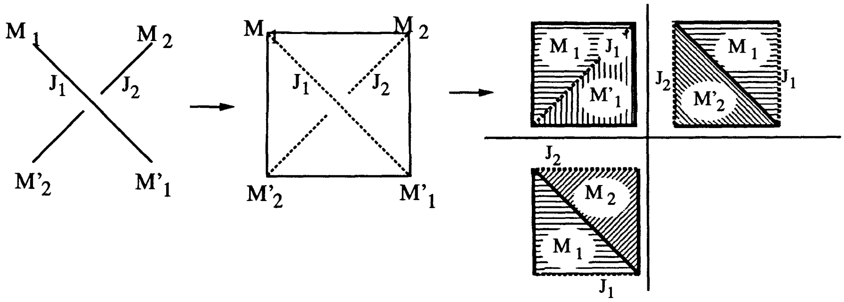


(4.10)

Let us establish a three-dimensional representation involving ξ operators. We propose the following

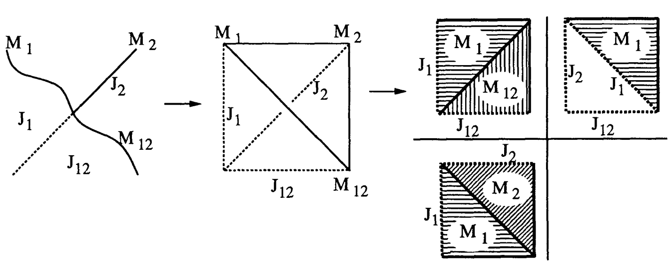


(4.11)



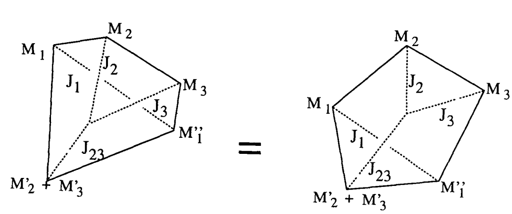
(4.12)

When the left diagrams are enclosed, we put the M's at the vertices. The surrounding lines are drawn as solid, while the lines which already existed become dashed. This ensures consistency with the tetrahedral representation of the $\tilde{V}$ operator-product algebra given above, since dashed lines have a J label in agreement with the previous convention – contrary to the solid ones. In the dualisation, the dashed lines are transformed as before, while the M's naturally go on the faces. These come out of two types. In the fusion, there is one face which has no M and is surrounded by three dashed lines. It represents the $\tilde{V}$ field which appears in Eq. (2.24). All other faces in the two above diagrams are similar: surrounded by two solid lines with no label, and a dashed line with a J label; they carry the corresponding magnetic number M. Each of them represents a ξ field. In the dualised diagrams, the face associated with $\tilde{V}$ fields is drawn as transparent, while the ξ faces are hatched. For them we also give a top-view drawn on the lower-left quarter plane, and a left-view drawn on the upper-right quarter plane. The tetrahedral representation of the diagram Eq. (4.9) is chosen so that its 3D representations agrees with the one of the fusion up to a rotation since they are given by the same 3-j symbols. Thus we let



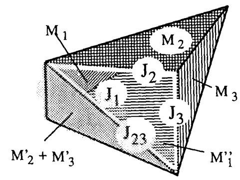
(4.13)

The 3D aspect now summarizes the general polynomial equations. The relation displayed by Fig. 4.10 is transformed into



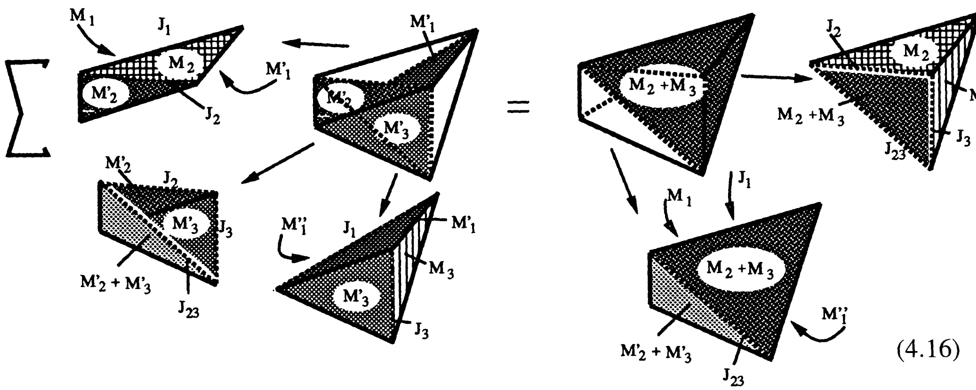
(4.14)

The dualised polyhedron is



(4.15)

where all the faces are hatched except the face J_2, J_3, J_{23}, of the $\tilde{V}$ type, which is transparent, according with the general convention. We next draw its decomposition, altering the general convention for visibility: the external quantum number are not written, and some external faces are made transparent



(4.16)

Consider in general a higher 2D diagram with one separation between a shadow and a real part. The enclosure proceeds as follows. In each world the rule is as indicated above. Concerning the separation line, one follows the prescription suggested by Fig. 4.13, namely, the separation line becomes solid, and thus carries no label. As a result, the higher 3D diagram before dualisation is again of the V3E type, with any number of dashed lines, and one closed loop of solid lines. Thus there are only two types of vertices: with three dashed lines, or with one dashed and two solid lines. After dualisation, one may obtain any polyhedron of the F3E type with the two kinds of faces introduced above: faces of the $\tilde{V}$ type (three dashed lines, each with a J, around a – transparent – face), and faces of the ξ type (two solid lines with no number, a dashed line with a J, and an M on the face – which is hatched). A J has to be interpreted as the length of the corresponding edge, and an M as the difference of length of the two surrounding solid edges. This gives all the spin addition rules and relations between M's as triangular inequalities. Of course, the J value of a dashed line is common to the two adjacent faces. The polynomial equations are derived by splitting a general F3E polyhedron in tetrahedra. Since there are two types of faces, one can only obtain three types of tetrahedra. There is a first type of tetrahedron, with three $\tilde{V}$ and one ξ faces, (see Fig. 4.11 or 4.13), its value is the corresponding Clebsch-Gordan coefficient. The tetrahedra of the second type have four ξ faces, (see Fig. 4.12), the values are the corresponding R-matrix elements. The third type tetrahedron has four $\tilde{V}$ faces (see Fig. 4.2 or 4.3), its value is the corresponding 6-j coefficient. In the case of the third type tetrahedron, changing the orientation yields an extra phase as we already mentioned.

For completeness, we have to add that the 6-j, R-matrix or Clebsch-Gordan have less symmetries than the tetrahedra by which they are represented, and therefore that these symmetries must be broken by adding extra characteristics to the tetrahedra, so that the correspondence be one-to-one. We give them briefly. First, the faces of the tetrahedra must be either "incoming" or "outgoing." Each tetrahedron has two incoming faces (the faces J_1, J_{23}, J_{123} and J_2, J_3, J_{23} in Figs. 4.2 and 4.3, the faces M_1 and M_2 in Figs. 4.11, 4.12 and 4.13) and two outgoing ones (the two other ones). When splitting a composite polyhedron in tetrahedra, the internal faces, common to two tetrahedra, are outgoing for one tetrahedron and incoming for the other one. This rule allows to single out the two particular J of the non-symmetric non-RW 6-j (the J between the two incoming faces, and the J between the two outgoing ones), to distinguish M_1 and M_2 from M_{12} for the CG, and to distinguish between the M_i's and M_i''s of the R-matrix. It is clear that the incoming faces represent the operators on which the fusing or braiding are performed, and the outgoing faces the resulting operators. Secondly, the M_i's on the faces are oriented quantities. Like the m_i's, they should be considered as differences of lengths of the two solid lines surrounding the face M_i, which thus must be supplemented by an ordering. This ordering must always be from left to right (for instance) on the 2D projection. If we rotate the tetrahedron of Fig. 4.12 representing $(J_1, J_2)_{M_1 M_2}^{M_2' M_1'}$, by π around a vertical axis in the plane of the page, this exchanges 1 and 2, but the orderings as well. The signs of the M_i's must therefore be changed to restore the left-right ordering of the 2D projection. The resulting value is then $(J_2, J_1)_{-M_2 -M_1}^{-M_1' -M_2'}$, which is indeed equal to $(J_1, J_2)_{M_1 M_2}^{M_2' M_1'}$. So, these tetrahedra with two outgoing and two incoming faces, and oriented M_i's, are in one-to-one correspondence with the 6-j, R-matrix or Clebsch-Gordan coefficients.

5. ξ's as Limits of $\tilde{V}$'s

In Sect. 4, we showed that the V braiding diagrams could be interpreted as IRF model vertices and the ξ braiding diagrams as vertex model interactions. Witten showed that one could get vertex models as limit of IRF models [15], when letting the spins on the faces go to infinity with fixed differences. In this part we shall apply this method directly to our operators. The spins on the faces are the ones corresponding to zero-modes. We shall denote by I_ϖ the left-most one (it is defined by $\varpi \equiv \varpi_0 + 2I_\varpi$), and let ϖ go to infinity. Hence, getting a vertex model as limit of an IRF model is equivalent to obtaining ξ operators as limits of $\tilde{V}$ operators. More precisely, let us prove that

$$\lim_{\varpi \to \infty} \mathscr{P}_{I_\varpi} \tilde{V}_m^{(J)} / \beta_{I_\varpi + m}^{I_\varpi} = (2i)^m e^{-ihm} \xi_m^{(J)} . \tag{5.1}$$

The role of the β coefficients is to remove the Γ functions which would have no well defined limit. They will cancel out of the braiding or fusing identities thanks to their property Eq. (3.37). $\mathscr{P}_{I_\varpi}$ is the projector on the Verma module of spin I_ϖ whose definition was recalled in [5]. Moreover, we have to give an imaginary part to I_ϖ (or ϖ) so that the limit $e^{\pm ih\varpi}$ be well defined: we choose a negative imaginary part, which makes $e^{-ih\varpi}$ go to zero and $e^{ih\varpi}$ to infinity.

Using the expression Eq. (3.36) of g and

$$\tilde{V}_m^{(J)} = (g_{J I_\varpi + m}^{I_\varpi} / E_m^{(J)}(\varpi)) \sum_M (J, \varpi|_m^M \xi_M^{(J)} ,$$

where $(J,\varpi|_m^M$ is the inverse matrix of $|J,\varpi)_M^m$, Eq. (5.1) is equivalent to

$$\lim_{\varpi\to\infty} \mathscr{P}_{I_\varpi}\, e^{ih(J+m)/2}\sqrt{e^{i\pi(J+m)}C_m^{(J)}(\varpi)}\,\sum_M (J,\varpi|_m^M\,\xi_M^{(J)}$$
$$= (2i)^m\, e^{-ihm}\xi_m^{(J)}\,. \tag{5.2}$$

Let us prove now that the matrix $|J,\varpi)_M^m$ has a leading term proportional to the identity matrix when $\varpi\to\infty$, and consequently, such is the leading term of its inverse matrix. Recall Eq. (3.19). The coefficient $|J,\varpi)_M^m$ is a polynomial in $e^{ih\varpi}$. As $e^{ih\varpi}$ goes to infinity, the maximum value of t dominates in the sum. The boundaries on t are given by the condition that the q-factorials arguments be positive. Thus, we get

$$\lim_{\varpi\to\infty} e^{-ihJ\varpi}\,|J,\varpi)_M^m = \sqrt{\begin{pmatrix}2J\\ J+M\end{pmatrix}}\, e^{ihM(J+1/2)}\delta_{m,M}\,. \tag{5.3}$$

This gives the limit of the inverse matrix directly. In view of Eq. (5.2) we also need the limit of $C_m^{(J)}(\varpi)$. Since the complex exponential with positive argument is dominant in $\sin(h\varpi)$, we get

$$\lim_{\varpi\to\infty} e^{-ih2J\varpi}C_m^{(J)}(\varpi) = (-1)^{-J-m}\, e^{ihJ}\begin{pmatrix}2J\\ J-m\end{pmatrix} e^{ih(2J-1)m}\,, \tag{5.4}$$

which leads to Eq. (5.1).

Now, we take this limit $\varpi\to\infty$ in the braiding or fusing equations. Begin with the fusing Eq. (2.1). Let us write it down in terms of three $\tilde V$ and one V

$$\mathscr{P}_{I_\varpi}\tilde V_{m_1}^{(J_1)}\tilde V_{m_2}^{(J_2)} = \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} g_{J_1J_2}^{J_{12}}\left\{\begin{matrix}J_1 & J_2\\ I_\varpi+m_1+m_2 & I_\varpi\end{matrix}\,\middle|\,\begin{matrix}J_{12}\\ I_\varpi+m_1\end{matrix}\right\}$$
$$\times\sum_{\{\nu_{12}\}}\mathscr{P}_{I_\varpi}\tilde V_{m_1+m_2}^{(J_{12},\{\nu_{12}\})}\langle\varpi_{J_{12}},\{\nu_{12}\}|\,V_{J_2-J_{12}}^{(J_1)}\,|\varpi_{J_2}\rangle\,. \tag{5.5}$$

The β coefficients introduced by Eq. (5.1) cancel out thanks to the property Eq. (3.37), and, in the limit, we do get the fusing Eq. (2.24) of the ξ operators, provided that

$$\lim_{\varpi\to\infty}\left\{\begin{matrix}J_1 & J_2\\ I_\varpi+m_1+m_2 & I_\varpi\end{matrix}\,\middle|\,\begin{matrix}J_{12}\\ I_\varpi+m_1\end{matrix}\right\} = (J_1,m_1;J_2,m_2\,|\,J_{12})\,. \tag{5.6}$$

Here again, we have to give an imaginary part to ϖ so that the limits of the complex exponentials be well defined. But the 6-j coefficients are only defined for positive half-integer spins. So, we have to extend this definition to non-integer I_ϖ before going to the limit. Such was not the case for the limit of $\mathscr{P}_{I_\varpi}\tilde V_m^{(J)}$ considered above, since everything was defined for any ϖ. This gives us the condition that our extended definition of the 6-j must be coherent with the limit of the $\tilde V$, i.e. it must obey Eq. (5.6).

The expression of the 6-j coefficients is given in Eq. (2.6). The ambiguity of the extension lies in the range of summation. For half-integer spins the boundaries are given by the condition that the arguments of the q-factorial in the denominator must not be negative integers. For half integer I_ϖ the initial definition is strictly equivalent to the following one, obtained by the change of index $\mu = z - 2I_\varpi - J_{12} - m_1 - m_2$:

$$\left\{\begin{array}{cc|c} J_1 & J_2 & J_{12} \\ I_\varpi + m_1 + m_2 & I_\varpi & I_\varpi + m_1 \end{array}\right\}$$
$$= (-1)^{J_{12}-J_1-J_2} \sqrt{\lfloor 2J_{12}+1\rfloor \lfloor 2I_\varpi + 2m_1 + 1\rfloor}$$
$$\times \Delta(J_1, J_2, J_{12}) \Delta(J_1, I_\varpi, I_\varpi + m_1)$$
$$\times \Delta(I_\varpi + m_1 + m_2, I_\varpi, J_{12}) \Delta(I_\varpi + m_1 + m_2, J_2, I_\varpi + m_1)$$
$$\times \sum_{\mu \text{ integer}} (-1)^\mu \lfloor 2I_\varpi + J_{12} + m_1 + m_2 + \mu + 1\rfloor!$$
$$\times [\lfloor 2I_\varpi - J_1 - J_2 + m_1 + m_2 + \mu\rfloor! \lfloor J_{12} - J_1 + m_2 + \mu\rfloor!$$
$$\times \lfloor J_{12} - J_2 - m_1 + \mu\rfloor! \lfloor \mu\rfloor! \lfloor J_1 + J_2 - J_{12} - \mu\rfloor!$$
$$\times \lfloor J_1 + m_1 - \mu\rfloor! \lfloor J_2 - m_2 - \mu\rfloor!]^{-1}. \tag{5.7}$$

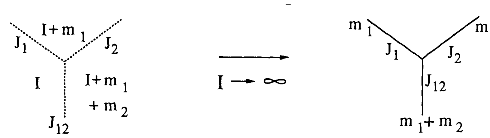
But, when we give an imaginary part to I_ϖ, the q-factorials with a I_ϖ have complex arguments and yield no restriction. Such is the case of the last six factorials of the sum with the definition of Eq. (2.6), and, of the first two with the definition of Eq. (5.7). The first definition rapidly appears inadequate as it yields no upper boundary for z. The second definition [Eq. (5.7)] leads to well defined boundaries for the index μ and to a finite limit thanks to

$$\frac{\lfloor \alpha + 2I_\varpi\rfloor!}{\lfloor \beta + 2I_\varpi\rfloor!} \sim (2i \sin h)^{\beta-\alpha}\, e^{ih(\alpha-\beta)2I_\varpi}\, e^{ih(\alpha-\beta)(\alpha+\beta+1)/2} \tag{5.8}$$

for $\varpi \to \infty$ with negative imaginary part, similar to Eq. (3.17) which was for positive imaginary part.

We use this limit in the sum and in the Δ prefactors of Eq. (5.7), and recognize the expression of the Clebsch-Gordan coefficient given in Eq. (3.11). This justifies our extended definition of the 6-j.

Pictorially [13], this reads



(5.9)

We come now to the case of braiding. It works like fusing. From Eq. (5.1) and the properties of the β coefficients Eq. (3.37), we see that we only have to prove that when J goes to infinity with negative imaginary part

$$\lim_{\varpi\to\infty} \left\{\begin{array}{cc|c} J_1 & I_\varpi + m & I_\varpi + m_2' \\ J_2 & I_\varpi & I_\varpi + m_1 \end{array}\right\}$$
$$\times e^{i\pi(\Delta_{I_\varpi+m} + \Delta_{I_\varpi} - \Delta_{I_\varpi+m_2'} - \Delta_{I_\varpi+m_1})} = (J_1, J_2)_{m_1 m_2}^{m_1' m_2'} \tag{5.10}$$

[13] For a better readability, we omit the index ϖ of I_ϖ in the drawings

and

$$\lim_{\varpi\to\infty}\left\{\begin{matrix} J_1 & I_\varpi+m \\ J_2 & I_\varpi \end{matrix}\middle|\begin{matrix} I_\varpi+m_2' \\ I_\varpi+m_1 \end{matrix}\right\}$$
$$\times e^{-i\pi(\Delta_{I_\varpi+m}+\Delta_{I_\varpi}-\Delta_{I_\varpi+m_2'}-\Delta_{I_\varpi+m_1})}=\overline{(J_1,J_2)}_{m_1m_2}^{m_1'm_2'}, \tag{5.11}$$

where $m\equiv m_1+m_2=m_1'+m_2'$ and (J_1,J_2) (resp. $\overline{(J_1,J_2)}$) is the universal R-matrix R (resp. $\bar{R}$), following the conventions of [6] [see Eq. (3.6)]. We only deal in details with the case of the R-matrix R. We compute its matrix element from its universal form given in Eq. (3.7),

$$(J_1,J_2)_{m_1m_2}^{m_2'm_1'}=e^{-2ihm_1m_2}\frac{(-1)^n(2i\sin(h))^n e^{ihn(n+1)}}{\lfloor n\rfloor!}e^{-ihnm_1}e^{ihnm_2}$$
$$\times\sqrt{\frac{\lfloor J_1+m_1\rfloor!\,\lfloor J_1-m_1'\rfloor!\,\lfloor J_2-m_2\rfloor!\,\lfloor J_2+m_2'\rfloor!}{\lfloor J_1-m_1\rfloor!\,\lfloor J_1+m_1'\rfloor!\,\lfloor J_2+m_2\rfloor!\,\lfloor J_2-m_2'\rfloor!}}, \tag{5.12}$$

for $n\equiv m_2'-m_2\geq 0$, and 0 otherwise. It is an upper triangular matrix.

To begin with, we examine the behavior of the 6-j coefficient when ϖ goes to infinity with a negative imaginary part. This time, the suitable change of summation index to define the 6-j for non-integer spins is $x=2I_\varpi+J_1+J_2+m_1+m_2-z$ (basically, this is the same definition of the extension as before, which amounts to taking $z-2I_\varpi$ integer, the extra integer shifts by J_i+m_i being trivial). Here, only the first and eighth factorials of the sum have infinite arguments. After using Eq. (5.8), we get

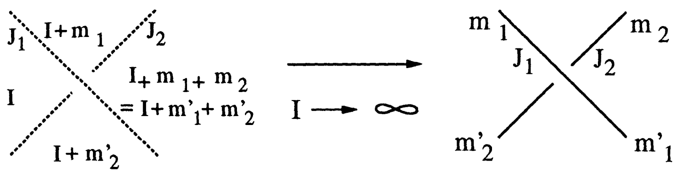
$$\left\{\begin{matrix} J_1 & I_\varpi+m \\ J_2 & I_\varpi \end{matrix}\middle|\begin{matrix} I_\varpi+m_2' \\ I_\varpi+m_1 \end{matrix}\right\}\sim e^{-ih2I_\varpi n}e^{ih(m_1+m_2'+1)}$$
$$\times\sqrt{\lfloor J_1+m_1\rfloor!\,\lfloor J_1-m_1\rfloor!\,\lfloor J_1+m_1'\rfloor!\,\lfloor J_1-m_1'\rfloor!}$$
$$\times\sqrt{\lfloor J_2+m_2\rfloor!\,\lfloor J_2-m_2\rfloor!\,\lfloor J_2+m_2'\rfloor!\,\lfloor J_2-m_2'\rfloor!}$$
$$\times(2i\sin(h))^n e^{-ih/2(1+n)(m+m_1+m_2'+2)}\sum_{x\text{ integer}}$$
$$\times\frac{e^{-4ihI_\varpi x}(2i\sin(h))^{2x}e^{-ihx(m+m_1+m_2'+2)}(-1)^x}{\lfloor J_1-m_1-x\rfloor!\,\lfloor J_1+m_1'-x\rfloor!\,\lfloor J_2+m_2-x\rfloor!\,\lfloor J_2-m_2'-x\rfloor!\,\lfloor x\rfloor!\,\lfloor n+x\rfloor!}. \tag{5.13}$$

As $e^{-ihI_\varpi}\to 0$, due to the factor $e^{-4ihI_\varpi x}$ the term of the sum with x minimal will be dominant. The lower boundary for x is given by $\lfloor x\rfloor!$ and $\lfloor n+x\rfloor!$: for $n\geq 0$, the minimal allowed x is 0, and for $n\leq 0$, it is n. Hence, summarizing both cases, the dominant behavior of the 6-j is $e^{-ih2I_\varpi|n|}$.

Then, the extra factor due to $e^{\pm i\pi(\Delta_{I_\varpi+m}+\Delta_{I_\varpi}-\Delta_{I_\varpi+m_2'}-\Delta_{I_\varpi+m_1})}$, which is $e^{\pm ih2I_\varpi n}\times$ (finite term), must be taken into account. Altogether, it gives $e^{ih2I_\varpi(\pm n-|n|)}$, i.e. $e^{-ih4I_\varpi|n|}$ when $n\leq 0$, and 1 when $n\geq 0$, in the upper case (+). So, when $e^{-ih2I_\varpi}\to 0$, the limit is an upper triangular matrix, as it should. (In the lower case (−) it gives 1 when $n\leq 0$, and $e^{-ih4I_\varpi|n|}$ when $n\geq 0$ and so, a lower triangular matrix in the limit.)

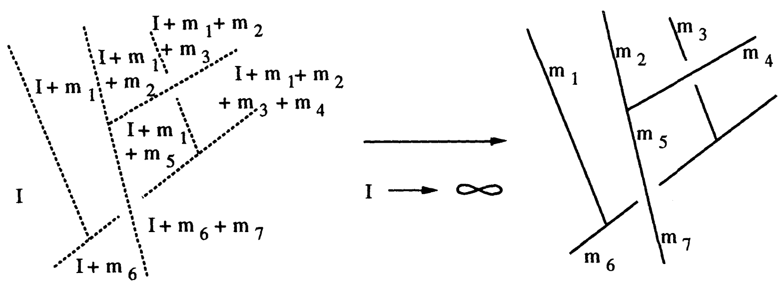
It is then straightforward to check that the limit agrees with Eq. (5.12), and this terminates the proof of Eq. (5.10).

The pictorial representation of this is



(5.14)

We can use these limits on more complex braiding-fusing identities of $\tilde{V}$ fields, thereby proving the same identities for ξ fields. For instance, we have the following limit:

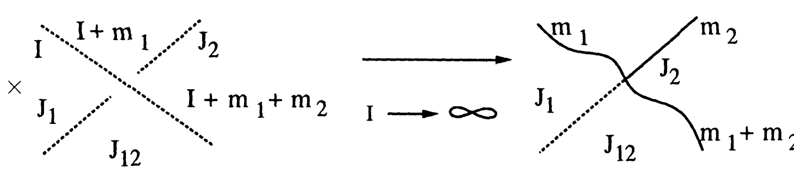


(5.15)

where we did not write the spins J_i on the lines as they are not affected by the limit, and where the missing m_i on the r.h.s. can be deduced from conservation of $\sum_i m_i$ at each vertex. Making this limit on both sides of identities in the shadow world yields identities in the normal world.

We can as well take this limit only on a part of the figure. Equation (5.6) gives as well

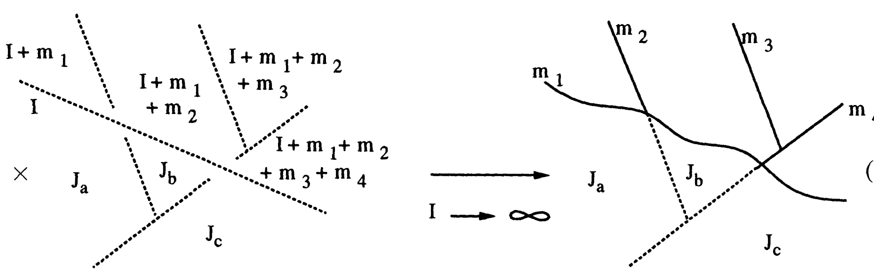
$$\frac{e^{-i\pi(\Delta_{I+m_1+m_2}-\Delta_{J_{12}})}}{e^{-i\pi(\Delta_{I+m_1}-\Delta_{J_1})}}$$



(5.16)

which gives on a composite figure

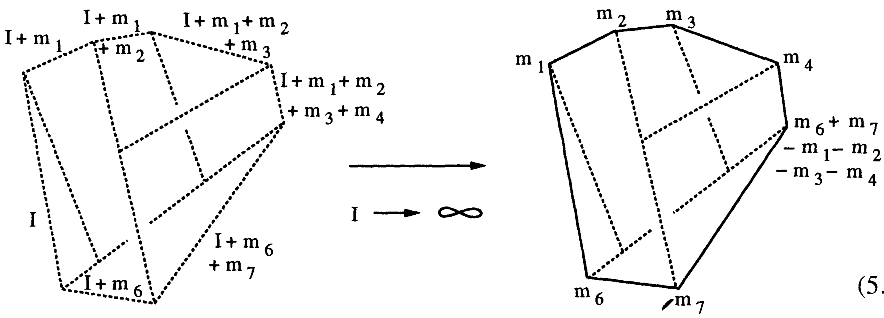
$$\frac{e^{-i\pi(\Delta_{I+m_1+m_2+m_3+m_4}-\Delta_{J_c})}}{e^{-i\pi(\Delta_{I+m_1}-\Delta_{J_a})}}$$



(5.17)

The extra phases in Eq. (5.16) cancel out between two neighbouring basic figures all along the wavy line, and only the exterior ones remain as in Eq. (5.17). Hence, they do not spoil equalities between transformed figures since the exterior spins of both sides are the same.

To conclude with this part, we show how this limit can be understood in the case of the 3D representation by V3E polyhedra. As an example, we close the 2D Figs. 5.15 and 5.17 and get the 3D representation

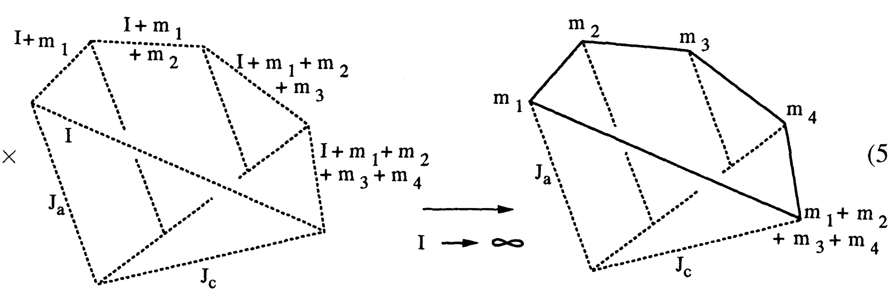


(5.18)

and

$$\frac{e^{-i\pi(\Delta_{I+m_1+m_2+m_3+m_4}-\Delta_{J_c})}}{e^{-i\pi(\Delta_{I+m_1}-\Delta_{J_a})}}$$

$\times$



(5.19)

where again the spins on the internal lines, which are unchanged by the limit have not been written down.

We see on these examples that the partial or global limits are not fundamentally different. On the V3E polyhedra, the normal world is obtained from the shadow world by adding I to all the edges of an arbitrarily chosen closed loop, and then letting I go to infinity. In this limit, only the differences between the values on the edges remain finite and relevant; they go on the vertices. When projecting in two dimensions, the part of the drawing inside the projection of the closed loop naturally becomes the normal world, as the part outside becomes the shadow world. Hence, depending on whether the closed loop is at the exterior of the drawing or not, there is only a normal world (3D in Fig. 5.18, 2D in Fig. 5.15) or both worlds (3D in Fig. 5.19, 2D in Fig. 5.17), but fundamentally, this is not different.

Following the remark of the end of Part 4.1, we note that we need to orient the M_i's at the vertices of the V3E polyhedra (and then on the faces of the F3E polyhedra) in order to know how to order the difference of the two neighbouring spins in the infinite limit.

6. Conclusion

One outcome of the present paper is the general fusion formula [14] the ξ fields,

$$\begin{aligned}
&\xi^{(J_1,\hat{J}_1)}_{M_1\hat{M}_1}(z_1)\,\xi^{(J_2,\hat{J}_2)}_{M_2\hat{M}_2}(z_2)\\
&= \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2}\ \sum_{\hat{J}_{12}=|\hat{J}_1-\hat{J}_2|}^{\hat{J}_1+\hat{J}_2}\\
&\times g^{J_{12}\hat{J}_{12}}_{J_1\hat{J}_1J_2\hat{J}_2}\, e^{i\pi f_\xi(J_1,\hat{J}_1,M_1,\hat{M}_1,J_2,\hat{J}_2,M_2,\hat{M}_2,J_{12},\hat{J}_{12})}\\
&\times (J_1,M_1;J_2,M_2\,|\,J_{12})\,\widehat{(}\hat{J}_1,\hat{M}_1;\hat{J}_2,\hat{M}_2\,\widehat{|}\,\hat{J}_{12}\widehat{)}\\
&\times \sum_{\{\nu\}} \xi^{(J_{12},\hat{J}_{12},\{\nu\})}_{M_1+M_2,\hat{M}_1+\hat{M}_2}(z_2)\\
&\times \langle \varpi_{J_{12},\hat{J}_{12}},\{\nu\}|\, V^{(J_1,\hat{J}_1)}_{J_2-J_{12},\hat{J}_2-\hat{J}_{12}}(z_1-z_2)\,|\varpi_{J_2,\hat{J}_2}\rangle\,, \qquad (6.1)
\end{aligned}$$

with [see Eq. (3.49)]

$$\begin{aligned}
&f_\xi(J_1,\hat{J}_1,M_1,\hat{M}_1,J_2,\hat{J}_2,M_2,\hat{M}_2,J_{12},\hat{J}_{12})\\
&\quad= M_1\hat{J}_2 - M_2\hat{J}_1 + \hat{M}_1J_2 - \hat{M}_2J_1\\
&\qquad - (M_1-M_2)(\hat{J}_{12}-\hat{J}_1-\hat{J}_2)\\
&\qquad - (\hat{M}_1-\hat{M}_2)(J_{12}-J_1-J_2)\,. \qquad (6.2)
\end{aligned}$$

The general braiding relation was already known [7, 17], but we recall it for completeness,

$$\begin{aligned}
&\xi^{(J_1,\hat{J}_1)}_{M_1\hat{M}_1}(z_1)\,\xi^{(J_2,\hat{J}_2)}_{M_2\hat{M}_2}(z_2)\\
&= \sum_{M_1',\hat{M}_1',M_2',\hat{M}_2'} e^{\pm i\pi b_\xi(J_1,\hat{J}_1,M_1,\hat{M}_1,M_1',\hat{M}_1',J_2,\hat{J}_2,M_2,\hat{M}_2,M_2',\hat{M}_2')}\\
&\times (J_1,J_2)^{M_2'M_1'}_{M_1M_2}\,\widehat{(}\hat{J}_1,\hat{J}_2\widehat{)}^{\hat{M}_2'\hat{M}_1'}_{\hat{M}_1\hat{M}_2}\,\xi^{(J_2,\hat{J}_2)}_{M_2',\hat{M}_2'}(z_2)\,\xi^{(J_1,\hat{J}_1)}_{M_1',\hat{M}_1'}(z_1)\,, \qquad (6.3)
\end{aligned}$$

with

$$\begin{aligned}
&b_\xi(J_1,\hat{J}_1,M_1,\hat{M}_1,M_1',\hat{M}_1',J_2,\hat{J}_2,M_2,\hat{M}_2,M_2',\hat{M}_2')\\
&\quad= \hat{J}_2(M_1+M_1') - \hat{J}_1(M_2+M_2')\\
&\qquad + J_2(\hat{M}_1+\hat{M}_1') - J_1(\hat{M}_2+\hat{M}_2') - 2(J_1\hat{J}_2+J_2\hat{J}_1)\,. \qquad (6.4)
\end{aligned}$$

[14] Formulae for fusion were already given at the level of primaries in [7, 17]. The present expression does not quite agree with them. Reference [7] was based on a co-product which was not co-associative. This point was corrected in [17]. In the formula given there [Eq. (5.12)], the complete g was wrongly supposed to factorize [ignoring the h factor of Eq. (3.50)], and a different overall sign convention was used (in addition, the first term in the second line has a misprint)

One may verify that all the polynomial equations are satisfied. In particular the pentagonal relations follow from Eq. (2.23) – together with its hatted counterpart – and from the equation

$$\begin{aligned}
& f_\xi(J_1, \hat{J}_1, M_1, \hat{M}_1, J_2, \hat{J}_2, M_2, \hat{M}_2, J_{12}, \hat{J}_{12}) \\
& \quad + f_\xi(J_{12}, \hat{J}_{12}, M_{12}, \hat{M}_{12}, J_3, \hat{J}_3, M_3, \hat{M}_3, J_{123}, \hat{J}_{123}) \\
& = f_\xi(J_2, \hat{J}_2, M_2, \hat{M}_2, J_3, \hat{J}_3, M_3, \hat{M}_3, J_{23}, \hat{J}_{23}) \\
& \quad + f_\xi(J_1, \hat{J}_1, M_1, \hat{M}_1, J_{23}, \hat{J}_{23}, M_{23}, \hat{M}_{23}, J_{123}, \hat{J}_{123}) \\
& \quad + f_V(J_1, \hat{J}_1, J_2, \hat{J}_2, J_{123}, \hat{J}_{123}, J_{23}, \hat{J}_{23}, J_3, \hat{J}_3, J_{12}, \hat{J}_{12}) \bmod 2 , \qquad (6.5)
\end{aligned}$$

where the last term is the additional phase of the V fusion (see [5]). Checking this equation is straightforward but a bit cumbersome.

As was pointed out several times [6, 7, 20], the $U_q(sl(2))$ quantum-group structure remarkably comes out of the holomorphic OPA of Liouville theory. With the present work we have fully established its interplay with the MS bootstrap formalism, where the basic principle is that the operator algebra is associative, contrary to (q) tensor products of representations (where the 6-j's precisely encode the non-associativity). It is interesting to think about a sort of reciprocal. Given the quantum group quantities – CG coefficients, 6-j symbols, universal R matrix –, the tensor product of representations defines a "multiplication" of representations which is not associative. Then, one may consider introducing additional quantum numbers such that the "product" becomes associative. A solution for this is given by the indices $\{\nu\}$ that characterize the descendants, and then the "product" becomes the OPE of holomorphic fields we have displayed. Indeed, the only way to have Eq. (2.22) satisfied is that F be equal to a 6-j symbol. This forces the existence of the V's and fixes their OPE. Then some of the J's of the 6-j symbols have a natural interpretation as zero-modes which are shifted by the V fields as shown on Eq. (2.4). It is likely that this is the unique possibility, although a proof of this fact is beyond the scope of this article. In any case, our present study indicates that the relationship between $U_q(sl(2))$ and the holomorphic OPA of 2D gravity goes even more deeply than previously thought.

One may forsee future developments of the present work in several directions. The most interesting one at this point concerns the strong-coupling regime $1 < C < 25$, with complex h, where the present method is the only available so far. In [7] a unitary-truncation theorem was derived for $C = 7$, 13, and 19, by only considering leading-order OPE's where it is sufficient to deal with primaries. This discussion may now be completed using the result of the present paper. Moreover, the solutions of the MS relations just displayed should allow us to check that the associated non-critical string theories are consistent as such, namely, that they satisfy duality relations between crossed channels similar to the ones of the Veneziano model. Another point is that, now that the full bootstrap is at hand, one may re-consider, with a much greater insight, the extension to negative J, handled in [7] by means of a symmetry principle between spins J and $-J-1$. Dealing with negative spins is unavoidable since they are the holomorphic components of exponentials of Liouville field with positive weights. These must be introduced for proper dressing by 2D gravity [20].

At a more ambitious level, understanding the connection between quantum group and 2D gravity more deeply is certainly a step toward unravelling the non-commutative geometry of the latter theory.

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