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Perturbative Renormalization of Massless ϕ_4^4 with Flow Equations

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Abstract: Perturbative renormalizability proofs in the Wilson-Polchinski renormalization group framework, based on flow equations, were so far restricted to massive theories. Here we extend them to Euclidean massless ϕ_4^4. As a by-product of the proof we obtain bounds on the singularity of the Green functions at exceptional momenta in terms of the exceptionality of the latter. These bounds seem to be new and are quite sharp.

1. Introduction

In recent years the authors have discussed the renormalization problem of perturbative field theory in a series of papers [1–5]. The method has been that of the renormalization group of Wilson as applied to perturbation theory in the form of differential flow equations by Polchinski [6]. We started by putting Polchinski's result on the renormalizability of massive ϕ_4^4 on a rigorous footing after simplifying the method of proof and included general (physical) renormalization conditions. The next step was to extend the method to QED [2] with a massive photon, where the main difficulty came from the fact that the regularization in the flow equation approach necessarily violates gauge invariance. (It is straightforward to convince oneself that the ϕ_4^4 proof works also for a general renormalizable (by power counting) massive Euclidean theory as long as there is no additional constraint, not respected by the regularization.) Furthermore we treated composite operator renormalization and the short distance expansion [3, 4], thus proving the method to be well-adapted also for more advanced and intricate issues in the field. Finally it turned out particularly suited for studying questions of convergence of the regularized theory to the renormalized one which go under the name of Symanzik's improvement programme [5].

Any method has, of course, its specific advantages and difficulties. In our framework we count among the latter that the regularization violates gauge invariance

⋆ Now at Department of Mathematics, University of Virginia, Charlottesville 22903 VA, USA. Supported by the Swiss National Science Foundation and also (more technically) that for a general theory the derivation of the flow equations and their specific form are somewhat lengthy.[1] In our opinion this is more than compensated by the conceptual advantages of the method and by the fact that all the difficult combinatorics of Feynman diagram types of proofs is completely sidestepped. This combinatorics is sometimes involved to a degree that the proofs in question are not written in detail, and the intellectual gaps to pass over remain quite impressive and sometimes prohibitive to the reader. We think the flow equation method has none of those difficulties in the sense that the proofs, if requiring careful reading, are accessible to a line by line analysis without leaving any gap or appealing to mathematical tools which are beyond standard graduate knowledge. Thus it seems worthwile trying to extend it further to cover the physically relevant models of particle physics and statistical mechanics. For a comprehensive renormalizability proof of the standard model of particle physics we also have to be able to pass over to Minkowski space, to cope with nonabelian gauge invariance and-as a prerequisite-to show how to treat massless fields. This paper is devoted to the last problem.

We want to prove the perturbative renormalizability of massless Euclidean ϕ_4^4, symmetric under $\phi \to -\phi$, which is the simplest example of a strictly renormalizable massless theory. Extension to more general theories of this kind will again be straightforward, some indications will be given in the end. As is well-known, massless theories with superrenormalizable couplings generally do not exist perturbatively. It is easy to see that e.g. a ϕ^3-term leads to infinities at two-loop order which cannot be cured by introducing a local counterterm (see e.g. [7]). The zero mass limit for QED needs additional care due to gauge invariance and will be treated separately.

We shall start by shortly recapitulating some results presented in [1] and [3] on the treatment of ϕ_4^4 in the UV region. This part will be in complete analogy with the massive case since, as long as we integrate out momenta above a certain positive scale only, no infrared (IR) singularities will appear anyway. So we will first impose general renormalization conditions (r.c.) at such a scale Λ_1 and treat the UV limit as in [1, 3]. Then we will single out the admissible renormalization conditions at scale $\Lambda = 0$ (where all momenta have been integrated out), convince ourselves that these are compatible with a subclass of the r.c. at Λ_1 and show that the connected amputated Green functions exist without IR cutoff for nonexceptional momenta, i.e. as long as no partial sum of momenta vanishes. The proof also allows to deduce bounds on the singularity of the Green functions at exceptional momenta as a function of the IR cutoff going to zero. We did not find such (or essentially equivalent) bounds in the literature. On inspection of examples the inverse power of the cutoff controlling this singularity in our proof is optimal in many cases.

2. A Short Reminder on ϕ_4^4 with IR Cutoff

The renormalization of massive ϕ_4^4 has been treated in [1, 3]. The treatment of the theory with vanishing bare mass is the same, as long as we keep a different IR cutoff, namely integrate out only momenta larger than some scale $\Lambda_1 > 0$. Thus the proofs may be taken essentially from [1, 3] for this case, and we will be rather short here.

We start from the UV-regularized theory, the cutoff being called Λ_0. Momenta between Λ_0 and Λ_1 will be integrated out with boundary conditions chosen such that

[1] Note however that for the renormalizability proof only the general structure, not the detailed form of the flow equation is important

we may take the limit $\Lambda_0 \to \infty$ in the end. We now introduce the objects of interest and fix the notation.

The regularized free Euclidean propagator is

$$C_\Lambda^{\Lambda_0}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2} (R(\Lambda_0,p) - R(\Lambda,p)), \quad 0 < \Lambda \le \Lambda_0 < \infty . \quad (1)$$

The Fourier transform will also be denoted as $C_\Lambda^{\Lambda_0}(p)$. In (1) we set

$$R(\Lambda,p) = K\left(\frac{p^2}{\Lambda^2}\right) \quad (\Lambda,p) \neq (0,0), \quad (2)$$

and K satisfies

$$\begin{aligned} &K \in C^\infty[0,\infty), \;\; 0 \le K \le 1, \\ &K(x) = 1 \text{ for } x \le 1, \quad K(x) = 0 \text{ for } x \ge 4. \end{aligned} \quad (3)$$

From (2, 3) we find $R(\Lambda,p) \in C^\infty(\mathbb{R}_+, \mathbb{R}^4)$. We also have for $\Lambda > 0$,

$$\partial^w \partial_\Lambda R(\Lambda,p) = 0 \quad \text{for } 0 \le |p| \le \Lambda \text{ or } 2\Lambda \le |p|, \quad (4)$$

where the multiindex w indicates momentum derivatives

$$\partial^w = \partial^{w_1} \cdots \partial^{w_4} = \frac{\partial^{w_1}}{\partial p_1^{w_1}} \cdots \frac{\partial^{w_4}}{\partial p_4^{w_4}} \quad \text{for } p = (p_1, \dots, p_4),\ w_i \in \mathbb{N}_0.$$

For smooth fields $\phi \in \mathscr{S}(\mathbb{R}^4)$ we then define the following quantities:

I. The functional Laplace operator

$$\Delta(\Lambda,\Lambda_0) = \tfrac{1}{2}\langle \delta_\phi, C_\Lambda^{\Lambda_0} \delta_\phi \rangle$$

with

$$\langle f_1, f_2 \rangle = \int f_1(x) f_2(x) d^4x, \quad \delta_\phi = \frac{\delta}{\delta\phi(x)}.$$

II. The lowest order interaction Lagrangian plus its counterterms at Λ_0 as a formal power series

$$l^{\Lambda_0} = \sum_{r \ge 1} g^r l_r^{\Lambda_0}$$

with

$$l_r^{\Lambda_0} = \int (a_r \phi^2 + b_r (\partial_\mu \phi)(\partial_\mu \phi) + c_r \phi^4),$$

where the formal power series coefficients a_r, b_r, c_r will be determined as functions of Λ_0, R and the r.c. The expansion parameter g is for standard r.c. the renormalized coupling.

III. The effective Lagrangian

$$L^{\Lambda,\Lambda_0} = \sum_{r \ge 1} g^r L_r^{\Lambda,\Lambda_0}$$

is introduced as follows: We define S^{Λ,Λ_0} via

$$\exp(-S^{\Lambda,\Lambda_0}) = \exp(\Delta(\Lambda,\Lambda_0)) \exp(-l^{\Lambda_0}), \quad \text{where} \quad S^{\Lambda,\Lambda_0} = \sum_{r \ge 1} g^r S_r^{\Lambda,\Lambda_0}.$$

Then we set

$$L^{\Lambda,\Lambda_0} = S^{\Lambda,\Lambda_0} - I^{\Lambda,\Lambda_0}\,.$$

Here I^{Λ,Λ_0} collects the terms in S^{Λ,Λ_0} which are independent of the field ϕ. (Note that strictly speaking the volume has to be kept finite until these terms have been subtracted.) The previous definitions imply

$$L^{\Lambda,\Lambda_0} = l^{\Lambda_0}\,.$$

From III. we find that L^{Λ,Λ_0} satisfies the differential flow equation

$$\partial_\Lambda L^{\Lambda,\Lambda_0} + \text{field independent terms} = (\partial_\Lambda \Delta(\Lambda,\Lambda_0))L^{\Lambda,\Lambda_0} - 1/2\langle \delta_\phi L^{\Lambda,\Lambda_0}, \partial_\Lambda C_\Lambda^{\Lambda_0}\delta_\phi L^{\Lambda,\Lambda_0}\rangle\,. \tag{5}$$

An important proposition states that:
L^{Λ,Λ_0} is the *generating functional of* the order $r \geq 1$ perturbative amputated connected *Green functions* of the Euclidean field theory defined by the propagator $C_\Lambda^{\Lambda_0}$ and the vertices from $l_r^{\Lambda_0}$. To obtain these Green functions themselves in momentum space we may write

$$L_r^{\Lambda,\Lambda_0}(\phi) = \sum_{n\geq 2}\int \mathscr{L}_{r,n}^{\Lambda,\Lambda_0}(p_1,\ldots,p_{n-1})\hat{\phi}(p_1)\ldots\hat{\phi}(p_n)\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_{n-1}}{(2\pi)^4} \tag{6}$$
$$(p_n = -p_1 - \cdots - p_{n-1})\,.$$

Then $\mathscr{L}_{r,n}^{\Lambda,\Lambda_0}$ is the r^{th} order contribution to the connected amputated n-point function. We have:

i. $\mathscr{L}_{r,n}^{\Lambda,\Lambda_0}$ may be assumed symmetric under permutations of $p_1,\ldots,p_n$.

ii. $\mathscr{L}_{r,n}^{\Lambda,\Lambda_0} \equiv 0$ if $n > 2r+2$ (connectedness !),
$\mathscr{L}_{r,2k+1}^{\Lambda,\Lambda_0} \equiv 0$ (due to the symmetry $\phi \to -\phi$).

iii. $\mathscr{L}_{r,n}^{\Lambda,\Lambda_0}$ are invariant under $O(4)$-transformations of the p_i.

iv. $\mathscr{L}_{r,n}^{\Lambda,\Lambda_0}$ is in $C^\infty((0,\Lambda_0]\times\mathbb{R}^{4(n-1)})$ as a function of Λ and $p_1,\ldots,p_{n-1}$, since for any $\Lambda > 0$ the propagator $C_\Lambda^{\Lambda_0}$ vanishes for $|p| < \Lambda$.

We can rewrite the flow equation (5) for the expansion coefficients $\mathscr{L}_{r,n}^{\Lambda,\Lambda_0}$ $(n \geq 2)$, since $\hat{\phi} \in \mathscr{S}$ is arbitrary,

$$\begin{aligned}
&\partial^w\partial_\Lambda \mathscr{L}_{r,n}^{\Lambda,\Lambda_0}(p_1,\ldots,p_{n-1})\\
&= -\binom{n+2}{1}\int\frac{d^4p}{(2\pi)^4}\frac{\partial_\Lambda R(\Lambda,p)}{p^2}\partial^w \mathscr{L}_{r,n+2}^{\Lambda,\Lambda_0}(p,-p,p_1,\ldots,p_{n-1})\\
&+ \sum_{\substack{w'+w''+w'''=w\\ r'+r''=r\,,\; n'+n''=n+2}} \frac{n'n''}{2}\Bigg[\left(\partial^{w'}\frac{\partial_\Lambda R(\Lambda,q')}{q'^2}\right)(\partial^{w''}\mathscr{L}_{r',n'}^{\Lambda,\Lambda_0}(p_1,\ldots,p_{n'-1}))\\
&\times(\partial^{w'''}\mathscr{L}_{r'',n''}^{\Lambda,\Lambda_0}(-q',p_{n'},\ldots,p_{n-1}))\Bigg]_{\text{sym}}\,.
\end{aligned} \tag{7}$$

Here we have directly written the equation where $|w|$ momentum derivatives have been taken on both sides. As regards notation, we set

$$w \in \mathbb{N}_0^{4(n-1)} \quad w = (w_1, \ldots, w_{n-1}) \quad w_i = (w_{i,1}, \ldots, w_{i,4}),$$
$$|w| = \sum |w_{i_\nu}| \quad \partial^w = \partial^{w_{1,1}} \ldots \partial^{w_{n-1,4}} = \frac{\partial^{w_{1,1}}}{\partial p_{1,1}^{w_{1,1}}} \cdots \frac{\partial^{w_{n-1,4}}}{\partial p_{n-1,4}^{w_{n-1,4}}}.$$

$[\ ]_{\text{sym}}$ means symmetrization w.r.t. $p_1, \ldots, p_n, q' := -p_1 - \ldots - p_{n'-1}$.

The r.h.s. of (7) is seemingly ill-defined for momentum configurations $p_1, \ldots, p_n$ such that one of the partial sums q' vanishes. More rigorously, the derivation of (7) shows that $[\ldots]_{\text{sym}} = 0$ for such momenta, a fact which is a consequence of our choice of regularization, specifically (4).

The renormalizability proof is performed by inductively estimating the r.h.s. of (7), where the induction proceeds upwards in r and for given r downwards in n (see also below, proof of Theorem 1). We shall need in particular an estimate of the derivatives of R:

$$|\partial^w R(\Lambda, p)| < c\Lambda^{-|w|}, \quad \left|\partial^w \partial_\Lambda R(\Lambda, p)\frac{1}{p^2}\right| < c\Lambda^{-3-|w|}, \tag{8}$$

where c denotes some constant and $\Lambda > 0$. Both inequalities immediately follow from (2)–(4), the second also holds for $p^2 \to 0$ by continuity.

In this paper we do not intend to give bounds on the large momentum behaviour of the Green functions. Therefore we will always assume that

$$|p_i| \leq M, \quad 1 \leq i \leq n-1, \tag{9}$$

where M is some arbitrary large but fixed constant, whenever we deal with Green functions $\mathscr{L}_{r,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_{n-1})$, $0 \leq \Lambda \leq 1$.

We want to estimate the solutions of the flow equation. To do so we need boundary conditions (b.c.). For $\Lambda = \Lambda_0$ these are fixed for nearly all w, n by the structure of $l_r^{\Lambda_0}$ (see above). The values of the remaining terms are fixed by the b.c. (11) imposed at $\Lambda = \Lambda_1 = 1$:

$$\underline{\Lambda = \Lambda_0}: \quad \partial_p^w \mathscr{L}_{r,n}^{\Lambda_0,\Lambda_0} \equiv 0, \quad \text{if } n + |w| \geq 5, \tag{10}$$

$$\begin{aligned} \underline{\Lambda = \Lambda_1}: \quad & \mathscr{L}_{r,2}^{1,\Lambda_0}(0) = a_r^1, \\ & \partial_{p_\mu}\partial_{p_\nu}\mathscr{L}_{r,2}^{1,\Lambda_0}(p = k)\big|_{\delta_{\mu\nu}} = 2b_r^1, \\ & \mathscr{L}_{r,4}^{1,\Lambda_0}(p_1 = k_1, p_2 = k_2, p_3 = k_3) = c_r^1. \end{aligned} \tag{11}$$

The a_r^1, b_r^1, c_r^1 are (real) coefficients independent of Λ_0. Due to $O(4)$-invariance we have

$$\partial_{p_\mu}\partial_{p_\nu}\mathscr{L}_2(k) = A(k^2)\delta_{\mu\nu} + B(k^2)k_\mu k_\nu,$$

and we defined $\partial_{p_\mu}\partial_{p_\nu}\mathscr{L}_{r,2}(k)\big|_{\delta_{\mu\nu}} \equiv A(k^2)$. The momenta k, k_1, k_2, k_3 may be freely chosen at this stage. For later convenience we demand, however,

$$k, k_1, k_2, k_3, k_1 + k_2, k_1 + k_3, k_2 + k_3, k_1 + k_2 + k_3 \neq 0, \tag{12}$$

$$\begin{aligned} &(\text{e.g.} \quad k = (\mu, 0, 0, 0) \neq 0, \ k_i \cdot k_j = \frac{\mu^2}{4}(4\delta_{ij} - 1), \\ &1 \leq i, j \leq 4, k_4 = -k_1 - k_2 - k_3). \end{aligned}$$

For g to be the renormalized coupling in standard renormalization we have to demand

$$a_1^1\,, b_1^1 = 0\,, \quad c_1^1 = \tfrac{1}{4!}\,. \tag{13}$$

The higher order contributions (i.e. a_r^1, b_r^1, c_r^1, $r > 1$) will be in one-to-one correspondence to the r.c. which we impose at $\Lambda = 0$, in the next section. Therefore it is important that the following inequalities hold for *arbitrary* given a_r^1, b_r^1, c_r^1 in (11) and uniformly for momenta (9):

$$\begin{aligned}
&(\textit{boundedness}) && |\partial^w \mathscr{L}_{r,n}^{\Lambda,\Lambda_0}| \le \Lambda^{4-n-|w|}\, P\log\Lambda\,, \\
&(\textit{renormalizability}) && |\partial_{\Lambda_0}\partial^w \mathscr{L}_{r,n}^{\Lambda,\Lambda_0}| \le \Lambda_0^{-2}\Lambda^{5-n-|w|}\, P\log\Lambda_0\,, \\
& && (1 = \Lambda_1 \le \Lambda \le \Lambda_0 < \infty)\,.
\end{aligned} \tag{14}$$

$P\log\Lambda$ is each time it appears a possibly new polynomial in $\log\Lambda$ with nonnegative coefficients. So for $\Lambda \to 1$ it goes to a finite constant > 0. The bounds (14) imply that the perturbative connected amputated Green functions of massless ϕ_4^4 with fixed IR cutoff $\Lambda \ge 1$ and without UV cutoff

$$\mathscr{L}_{r,n}^{\Lambda}(p_1,\ldots,p_{n-1}) \equiv \lim_{\Lambda_0\to\infty} \mathscr{L}_{r,n}^{\Lambda,\Lambda_0}(p_1,\ldots,p_{n-1}) \tag{15}$$

exist and are C^∞-functions of the momenta for arbitrary (finite) values of the latter.

3. The Infrared Limit

In this section we want to show that with certain standard restrictions on the r.c. the IR limit $\Lambda \to 0$ can be taken. It is well-known and can be proven within the framework of BPHZ renormalization [8] that the connected Green functions of strictly renormalizable Euclidean massless field theories (i.e. those which contain only dimension 0 couplings) exist in momentum space if
1. The external momenta are nonexceptional, i.e. no partial sum of the external momenta vanishes.
2. The renormalization conditions are imposed such that
   - the dimension 4 terms are arbitrarily fixed at some nonexceptional momenta
   - the renormalized amputated Green functions of dimension < 4, i.e. in symmetric ϕ_4^4 the two point-function, vanish at 0 momentum.

When the external momenta become exceptional the Green functions will generally be singular. The problem of characterizing the degree of this singularity in terms of the exceptionality of the external momenta, as the mass goes to 0, can be rephrased as the question on the asymptotic behaviour of the massive Green functions in the limit, when certain momenta become large. This question has been extensively studied in the literature and turned out to be of considerable technical difficulty, if treated in the framework of renormalized Feynman diagrams on a rigorous level, see e.g. [9, 10]. For a recent overview with a lot of references, see [11]. Finally we note that the treatment of perturbative massless ϕ_4^4 using phase space expansions leads not only to a BPHZ type proof of renormalizability, but also to (presumably essentially optimal) factorial bounds on the large order behaviour of perturbation theory [12]. The object of analysis in all these papers is the individual Feynman diagram. In the present approach we want to prove directly the existence of the r^{th} order Green functions as a whole under the conditions 1,2 above. The method – though still restricted to perturbation theory – is thus closer in spirit to constructive field theory. We would like to point out that in the framework of the latter the IR problem of ϕ_4^4 could also be solved nonperturbatively: There are nonperturbative existence proofs for massless ϕ_4^4 with UV-cutoff based on the IR asymptotic freedom of the model [13–15]. [14] includes the statement that the perturbation series is Borel summable.

Our treatment will also yield an estimate on the degree of singularity of the perturbative Green functions at exceptional momenta in terms of inverse powers and logarithms of Λ for $\Lambda \to 0$. It is not hard to see how the question of exceptional momenta arises in our framework:
a) the r.h.s. of (7) contains the Green function $\mathscr{L}_{r,n+2}(p,-p,p_1,\dots,p_{n-1})$. This momentum configuration is exceptional, even if $p_1,\dots,p_n$ are not.
b) The second term on the r.h.s. of (7) may become singular if $\{p_1,\dots p_n\}$ is exceptional since then the momentum q' may vanish. This tells us that

a) we have to deal with exceptional momenta, even if we are only interested in Green functions at nonexceptional points,
b) Green functions at exceptional momenta will generally be singular.

The essential step in constructing the limit $\Lambda \to 0$ will thus consist in establishing a relation between the singularity of the Green functions and the exceptionality of momentum sets, which is compatible with inductively estimating solutions of the flow equation.

This leads us to the following definitions:

Definition 1. *A set of momenta*[2] p_i, $i = 1,\dots,n$ *is called admissible (*w.r.t. *symmetric* ϕ_4^4*) if*

$$\text{(i)}\quad n \in 2\mathbb{N}\,,\qquad n \geq 4\,,$$
$$\text{(ii)}\quad \sum_{i=1}^{n} p_i = 0\,.$$

The restriction to $n \geq 4$ is due to the special role of the two-point function. For $P_1 \subset P$, $P_2 = P - P_1$, with P an admissible set of momenta, we define

$$\tilde{P}(p) = \{p,-p\} \cup P\,,\quad P' = P_1 \cup \{p'\}\,,\quad P'' = P_2 \cup \{p''\} \tag{16}$$
$$\left(\text{where } p' = -\sum_{P_1} p_i\,,\quad p'' = -\sum_{P_2} p_i = -p'\,,\quad \sum_{P_1} := \sum_{p_i \in P_1} \text{ etc.}\right).$$

Definition 2. *An admissible momentum set* (a.m.s.) P *is called exceptional, if there exists* $Q \subset P$, $\emptyset \neq Q \neq P$ *such that* $\sum_{Q} p_i = 0$.

Definition 3. *A partition* $Z(P)$ *of an* a.m.s. P *is a system of nonempty subsets* $E_\nu \subset_{\neq} P$, $\nu = 1,\dots,N$ *such that*

$$\text{(i)}\ P = \bigcup_{\nu=1}^{N} E_\nu\,,\qquad \text{(ii)}\ E_\nu \cap E_\mu = \emptyset\,,\ \text{if } \nu \neq \mu\,,$$
$$\text{(iii)}\ \sum_{E_\nu} p_i = 0\,,\qquad \text{(iv)}\ |E_\nu| = 1 \text{ for at most one } \nu \in \{1,\dots,N\}\,.$$

[2] We regard p_i and p_j ($i \neq j$) as different entities, even if $p_i = p_j$ as elements of $\mathbb{R}_4$, since they belong to different fields or external lines. p_i may be thought of as a mapping $i \mapsto p_i$, we do not develop this point explicitly, however

For any partition $Z(P)$ we define the subsets

$$A(Z) = \{E_\nu \in Z \mid |E_\nu| > 2\}, \quad B(Z) = \{E_\nu \in Z \mid |E_\nu| = 2\}, \\ C(Z) = \{E_\nu \in Z \mid |E_\nu| = 1\} \tag{17}$$

and the indices

$$d(Z) = |A(Z)|, \quad e(Z) = |B(Z)|, \quad f(d) = [3/2d], \tag{18}$$

where $[x]$ is the largest integer $\leq x$. In the following we will often omit the argument Z.

Definition 4. *The* IR *index of a partition $Z(P)$ is defined as:*

$$g_Z(P) = \begin{cases} e + f(d-1), & \text{if } |C| = 1 \\ \sup(0, e + f(d) - 2), & \text{if } |C| = 0 \end{cases}.$$

Definition 5. *The* IR-*index of an* a.m.s. *P is defined to be*

$$g(P) = \begin{cases} \sup\limits_{Z \in \mathscr{Z}} g_Z(P), & \text{if } \mathscr{Z} \neq \emptyset \\ 0, & \text{if } \mathscr{Z} = \emptyset, \end{cases}$$

where $\mathscr{Z}$ is the set of all partitions Z of P.

So in particular $g(P) = 0$, if P is nonexceptional. If P is exceptional, $g(P)$ will bound the degree of singularity of the Green functions $\mathscr{L}^{\Lambda}_{r,n}(P)$ in terms of inverse powers of Λ^2 (see Theorem 1) as $\Lambda \to 0$. The following lemma is the key to our subsequent estimates of the $\mathscr{L}^{\Lambda}_{r,n}(P)$ for $\Lambda \to 0$, since it allows to bound the IR indices of the momentum sets occurring on the r.h.s. of the flow equation (7) in terms of that on the l.h.s. with sufficient precision. We may state it as

Lemma 1. *Let P be an* a.m.s. *Define $\tilde{P}$, P', P'' as in* (16). *Suppose that P', P'' are also* a.m.s.. *Then we have*

(a) $g(\tilde{P}) \leq g(P) + 1$, if all p_i vanish or, for $\sup |p_i| > 0$, if $|p| \leq \eta$,

where $\eta > 0$ is defined as $\eta(P) = \frac{1}{2} \inf_J \eta_J$, and the inf *is over all sets J with $J \subset_{\neq} \{1, \ldots, n\}$ such that $\left| \sum\limits_{i \in J} p_i \right| =: \eta_J > 0$.*

(b) $$g(\tilde{P}) \leq g(P) + 2.$$

(c) $$g(P') + g(P'') + 1 \leq g(P), \text{ if } \sup |p_i| = 0 \text{ or if } |p'| \leq \eta(P).$$

(d) $$g(P') + g(P'') \leq g(P). \tag{19}$$

Remarks. The partitions of P, P', P'', $\tilde{P}$ will be denoted as Z, Z', Z'', $\tilde{Z}$. Similarly we will write $A, A', \ldots, d, d', \ldots$ for the respective sets and indices. The elements of Z', Z'', $\tilde{Z}$ containing p', p'' and $p, -p$ will be called E', E'', $\tilde{E}_1$, $\tilde{E}_2$. If $\tilde{E}_1 = \tilde{E}_2$ we write $\tilde{E}$. The proof is by completely elementary but somewhat tedious case-by-case analysis.

Proof. (a), (b) Let $\tilde{Z}$ be such that $\tilde{g} := g(\tilde{P}) = g_{\tilde{Z}}(\tilde{P})$ and without restriction we assume $\tilde{g} \geq 1$. We have to distinguish various cases as regards the form of $\tilde{E}$ or $\tilde{E}_i$. For all cases we will then show that there exists a partition Z of P with $g_Z(P)+1 \geq \tilde{g}$ for (a) and with $g_Z(P)+2 \geq \tilde{g}$ for (b). For the indices of this partition we will write for shortness g, d, e, f, for those of $\tilde{Z}$ we write $\tilde{g}, \ldots$ (by slight abuse of notation). For all the different cases appearing we shall write the form of $\tilde{E}$, $\tilde{E}_i$, the definition of Z and the relations of the indices e, d, g. Sometimes we also have to distinguish whether a partition has $|C| = 1$ or $|C| = 0$ (17).

(1) $\tilde{E} = \{p, -p\},\ Z = \tilde{Z} \setminus \{\tilde{E}\},\ \tilde{e} = e+1,\ \tilde{d} = d \Rightarrow \tilde{g} = g+1$.

(2) $|\tilde{E}| = 3$.

(2i) $\tilde{E} = \{p, -p, p_{i_1}\},\ |\tilde{C}| = 0,\ Z = (\tilde{Z} \setminus \{\tilde{E}\}) \cup \{p_{i_1}\}$,
$\tilde{e} = e,\ \tilde{d} = d+1 \Rightarrow \tilde{g} = e + f(d+1) - 2 = e + f(d-1) + 1 = g+1$.

(2ii) $\tilde{E} = \{p, -p, p_{i_1}\},\ |\tilde{C}| = 1,\ Z = (\tilde{Z} \setminus \{\tilde{E}\}) \cup \{p_{i_1}, p_{i_2}\}$, where p_{i_2} is such that $\{p_{i_2}\} \in \tilde{Z},\ \tilde{e} = e-1,\ \tilde{d} = d+1 \Rightarrow \tilde{g} = e - 1 + f(d) = e + f(d) - 2 + 1 = g+1$.

(3) $|\tilde{E}| > 4$. In this case the partition $\tilde{Z}'$ generated from $\tilde{Z}$ by subdividing $\tilde{E}$ into $\{p, -p\}$ and its complement has $\tilde{g}' \geq \tilde{g}$. We therefore consider this partition instead, which leads back to (1).

Now we go through the cases, where p and $-p$ belong to different sets $\tilde{E}_1$ and $\tilde{E}_2$. Cases which differ only by interchange of $p, -p$ are of course equivalent. $E_1 := \tilde{E}_1 \setminus \{p\},\ E_2 := \tilde{E}_2 \setminus \{-p\}$.

(4) $\tilde{E}_2 = \{-p\}$ which implies $p = 0$.

(4i) $\tilde{E}_1 = \{p, p_{i_1}\}$. This case is equivalent to (1) on interchange of $-p$ and p_{i_1}.

(4ii) $|\tilde{E}_1| = 3,\ Z = (\tilde{Z} \setminus \{\tilde{E}_1, \tilde{E}_2\}) \cup \{E_1\}$,
$\tilde{e} = e-1,\ \tilde{d} = d+1,\ \tilde{g} = e - 1 + f(d) = e + f(d) - 2 + 1 = g+1$.

(4iii) $|\tilde{E}_1| > 4$. Choose $\tilde{Z}'$ instead of $\tilde{Z}$, where $\tilde{E}_2' = \{p, -p\},\ \tilde{E}_1' = \tilde{E}_1 \setminus \{p\}$. Then $\tilde{g}' \geq \tilde{g}$ and $\tilde{Z}'$ is treated in (1).

(5) $\tilde{E}_2 = \{p_{i_1}, -p\}$. On interchange of p and p_{i_1} this leads back to (1).

(6) $|\tilde{E}_2| = 3$.

(6i) $\tilde{E}_1 = \{p, p_{i_1}, p_{i_2}\},\ Z = (\tilde{Z} \setminus \{\tilde{E}_1, \tilde{E}_2\}) \cup \{E_1 \cup E_2\}$ for case b),
$\tilde{e} = e,\ \tilde{d} = d+1,\ \tilde{g} \leq g+2$ for $|C| = 0$ and $|C| = 1$.

Under the additional restriction of (a) we must have $p_{i_1} + p_{i_2} = 0$ (which implies $p = 0$) and equivalently for the momenta in $\tilde{E}_2$. Therefore we set $Z = (\tilde{Z} \setminus \{\tilde{E}_1, \tilde{E}_2\}) \cup \{E_1, E_2\}$ for (a) and find $\tilde{e} = e-2,\ \tilde{d} = d+2,\ \tilde{g} = g+1$ for $|C| = 0$ and $|C| = 1$.

(6ii) $|\tilde{E}_1| \geq 4$. Choose Z for (b) as in (6i), $\tilde{e} = e,\ \tilde{d} = d+1,\ \tilde{g} \leq g+2$. Choose Z for (a) as in (6i), $\tilde{e} = e-1,\ \tilde{d} = d+1,\ \tilde{g} \leq g+1$.

(7) $|\tilde{E}_1| > 4,\ |\tilde{E}_2| > 4$. Choose Z for (b) as in (6i), $\tilde{e} = e,\ \tilde{d} = d+1,\ \tilde{g} \leq g+2$. Choose Z for (a) as in (6i), $\tilde{e} = e,\ \tilde{d} = d,\ \tilde{g} = g$.

(c): $|p'| \leq \eta(P)$ and $p' = -\sum_{P_1} p_i$ necessitates $p' = 0$ by definition of η. So we assume $p' = 0$ in (c). Z, Z' are supposed to be partitions of P, P' such that

$g' := g(P') = g_{Z'}(P')$ and similarly for g''. We write $E_1 = E' \backslash \{p'\}$, $E_2 = E'' \backslash \{p''\}$. Without restriction $g' \geq g''$. We go again through the cases and define always some Z for P such that $g := g_Z(P)$ fulfills the inequality of the lemma.

(1) $g' = g'' = 0$, set $Z = \{P_1, P_2\} \Rightarrow e = 0,\ d = 2,\ |C| = 0,\ g = 1$.

(2) $g' \geq 1,\ g'' = 0$.

(2i) $E' = \{p'\}$, $Z = (Z' \setminus \{E'\}) \cup \{P_2\}$, we have $|C'| = 1$, $|C| = 0$ and $g' + 1 = e' + f(d') - 1 < e' + f(d') = g$, since $e' = e,\ d' = d - 1$.

(2ii) $|E'| \geq 2$; in this case we always assume $|C'| = 0$. The case $|C'| = 1$ can be reduced to (2i) by interchanging two momenta of modulus 0 (namely p' and the one in $C'(Z')$). Then we set $Z = (Z' \setminus \{E'\}) \cup \{P_2\} \cup \{E_1\}$ and find for the cases $|E'| = 2, 3$ $g = g' + 1$ and for $|E'| \geq 4$ $g \geq g' + 1$.

(3) $g', g'' \geq 1$ and without restriction $|E'| \geq |E''|$.

(3i) $|E'|,\ |E''| = 1$, set $Z = (Z' \setminus \{E'\}) \cup (Z'' \setminus \{E''\})$, $|C| = 0$, $e = e' + e''$, $d = d' + d''$, $g' + g'' + 1 = e' + e'' + f(d' - 1) + f(d'' - 1) + 1 \leq e + f(d) - 2 = g$.

(3ii) $|E'| \geq 2,\ |E''| = 1$, set $Z = (Z' \setminus \{E'\}) \cup (Z'' \setminus \{E''\}) \cup \{E_1\}$, $|C'| = 0$ (see (2ii)), $|C''| = 1$; $|C| = 1$ for $|E'| = 2$, $|C| = 0$ for $|E'| > 2$. We again verify for these cases $g' + g'' + 1 \leq g$. From (3iii) to (3v) we assume $|C'| = |C''| = 0$ as in (2ii) since otherwise interchanging momenta leads back to (2i), (3i) or (3ii).

(3iii) $|E'| = 2,\ |E''| = 2$, $Z = (Z' \setminus \{E'\}) \cup (Z'' \setminus \{E''\}) \cup \{E_1 \cup E_2\}$, $g' + g'' + 1 = e' + e'' + f(d') + f(d'') - 3 \leq e + f(d) - 2 = g$.
(3iv) $|E'| \geq 3,\ |E''| = 2$, Z as in (3iii), $g' + g'' + 1 \leq g$.
(3v) $|E'|,\ |E''| \geq 3$, $Z = (Z' \backslash \{E'\}) \cup (Z'' \backslash \{E''\}) \cup \{E_1\} \cup \{E_2\}$, $g' + g'' + 1 \leq g$. (For $|E'|,\ |E''| \geq 4$ we even find $g' + g'' + 1 \leq g - 1$.)

(d) The treatment is similar to (c). The notation is the same. Due to (c) we may restrict to $p' \neq 0$. All cases where $|E'| = 1$ or $|E''| = 1$ are then forbidden. The case $g', g'' = 0$ is now trivial. We assume w.r. $g' \geq g''$.

(1) $g' \geq 1,\ g'' = 0$, $|E'|,\ |E''| \geq 2$, $Z = (Z' \setminus \{E'\}) \cup \{P_2 \cup E_1\}$, $e = e' - 1$, $d = d' + 1$ for $|E'| = 2$, $e = e'$, $d = d'$ for $|E'| > 2$, in both cases $g' \leq g$.

(2) $g',\ g'' \geq 1$, $|E'|,\ |E''| \geq 2$, $Z = (Z' \setminus \{E'\}) \cup (Z'' \setminus \{E''\}) \cup \{E_1 \cup E_2\}$, if $|C'| + |C''| \leq 1$, $Z = (Z' \setminus (\{E'\} \cup \{E'_0\})) \cup (Z'' \setminus (\{E''\} \cup \{E''_0\})) \cup \{E_1 \cup E_2\} \cup \{E'_0 \cup E''_0\}$ if $|C'| + |C''| = 2$, where $E'_0 \in Z'$, $E''_0 \in Z''$ and $|E'_0| = |E''_0| = 1$.

We consider in detail the case $|E'|,\ |E''| > 2$:

i) $|C'| = |C''| = 0$: $e = e' + e''$, $d = d' + d'' - 1$, $g' + g'' = e' + e'' + f(d') - 2 + f(d'') - 2 \leq e + f(d) - 2 = g$, where equality holds if d' and d'' are even.

ii) $|C'| = 1,\ |C''| = 0$ ($|C'| = 0,\ |C''| = 1$ is analogous): $e = e' + e''$, $d = d' + d'' - 1$, $g' + g'' = e' + e'' + f(d' - 1) + f(d'') - 2 \leq e + f(d - 1) = g$, where equality holds if d' is odd and d'' is even.

iii) $|C'| = |C''| = 1$: $e = e' + e'' + 1$, $d = d' + d'' - 1$, $g' + g'' = e' + e'' + f(d' - 1) + f(d'' - 1) \leq e + f(d) - 2 = g$, where equality holds if d' and d'' are odd.

The cases where $|E'| = 2$ or $|E''| = 2$ (or both) are less dangerous. Analogous considerations always give $g' + g'' + 1 \leq g$. This finishes the proof of d) and of Lemma 1. □

We shall also need:

Lemma 2. *For any* a.m.s. $P = \{p_1, \ldots, p_n\}$ *there exists* $\varepsilon(P) > 0$ *and a neighbourhood*

$$U_\varepsilon(P) = \left\{(q_1, \ldots, q_n) \mid (q_i - p_i)^2 \le \varepsilon^2 ,\ 1 \le i \le n ,\ \sum_{i=1}^{n} q_i = 0\right\},$$

such that for any $Q = (q_1, \ldots, q_n)$ *with* $(q_1, \ldots, q_n) \in U_\varepsilon(P)$*:* $g(Q) \le g(P)$.

Proof. Take all the partitions of P. All subsets $S \subset_{\neq} P$ which are not an element of any $Z(P)$ have $\sum_{p_i \in S} p_i \neq 0$. Take ε so small that all these inequalities still hold in $U_\varepsilon(P)$, e.g. $\varepsilon = \frac{1}{n}\eta(P)$ (19). □

As a last prerequisite we define the sets of nonexceptional momenta M_n, $n \in 2\mathbb{N}$ as subsets of $\mathbb{R}^{4(n-1)}$:

$$M_n := \left\{(p_1, \ldots, p_{n-1}) \in \mathbb{R}^{4(n-1)} \mid \sum_{i \in J} p_j \neq 0 \text{ for all } J \subset_{\neq} \{1, \ldots, n\}\right\} \quad (20)$$

$$(\text{as usual } p_n = -p_1 - \ldots - p_{n-1})$$

The sets M_n are obviously open in $R^{4(n-1)}$.

Now we are able to prove

Theorem 1.

(a) *The (connected amputated) renormalized Green functions of perturbative massless* ϕ_4^4

$$\mathscr{L}_{r,n}(p_1, \ldots, p_{n-1}) := \lim_{\Lambda \to 0} \mathscr{L}^{\Lambda}_{r,n}(p_1, \ldots, p_{n-1}) \quad (21)$$

exist in $C^\infty(M_n)$ *(see (20)). For* $n = 2$ *we also have*

$$\mathscr{L}_{r,2} \in C^\infty(\mathbb{R}^4 \setminus \{0\}) \cap C^1(\mathbb{R}^4) . \quad (22)$$

(b) *They obey the boundary conditions* (10) *(where in* (10) *the limit* $\Lambda_0 \to \infty$ *has been taken). The admissible renormalization conditions i.e. those for which (a) can be shown to be true are as follows:*

$$\text{(i)}\ \ \mathscr{L}_{r,2}(0) = 0 , \quad \text{(ii)}\ \ \partial_\mu \partial_\nu \mathscr{L}_{r,2}(k)|_{\delta_{\mu,\nu}} = 2b_r^R ,$$

$$\text{(iii)}\ \ \mathscr{L}_{r,4}(k_1, k_2, k_3) = c_r^R .$$

Here b_r^R, c_r^R *are arbitrary (real) numbers. Their choice together with* (i) *uniquely fixes* b_r^1, c_r^1, a_r^1 (11). *The momenta* k, (k_1, k_2, k_3, k_4) *are chosen nonexceptional.*

(c) *Let* $P = \{p_1, \ldots, p_n\}$ *be an a.m.s. (Def. 1).*

(c1) *If* P *is nonexceptional or, for* $n = 2$, *if* $p_1 \neq 0$ *we have*

$$\partial^w \mathscr{L}^{\Lambda}_{r,n}(p_1, \cdots, p_{n-1}) = \lim_{\Lambda \to 0} \partial^w \mathscr{L}^{\Lambda}_{r,n}(p_1, \ldots, p_{n-1}) . \quad (23)$$

(c2) *If* P *is such that no partial sum over an odd number of momenta vanishes then*

$$|\partial^w \mathscr{L}^{\Lambda}_{r,n}(q_1, \ldots, q_{n-1})| \le \Lambda^{-2g(P)} P \log \Lambda^{-1} , \quad 0 < \Lambda \le 1 , \quad (24)$$

where $Q = \{q_1, \ldots, g_n\} \in U_\varepsilon(P)$. *The coefficients in* $P\log$ *depend on* P, r, n, w, *but the statement is uniform in* $U_\varepsilon(P)$, ε *sufficiently small.*

(c3) *Generally we have*

$$|\partial^w \mathscr{L}^\Lambda_{r,n}(q_1,\ldots,q_{n-1})| \leq \Lambda^{-2g(P)-|w|}\, P \log \Lambda^{-1} \tag{25}$$

(in the same sense as in (c2)).
(c4) *For* $n = 2$ *and* $0 < \Lambda \leq 1$ *we have*

$$|\partial \mathscr{L}^\Lambda_{r,2}(p)| = \Lambda^{2-|w|}\, P \log \Lambda^{-1} \tag{26}$$

for $|w| \geq 2$, *and for* $|w| \leq 1$ *if* $|p| \leq 2\Lambda$ *with coefficients depending on* w, r.

Remarks. The symbol ε will always denote a positive number, chosen sufficiently small case per case (we do not introduce ε', ε'', ...) such that the respective estimate holds uniformly in $U_\varepsilon(\ldots)$. c always denotes some positive Λ-independent constant. As noted before, all independent momenta are restricted to be smaller than some arbitrary, large, but finite constant $M(9)$. For a given momentum set $P = \{p_1,\ldots,p_n\}$ we denote by $Q = \{q_1,\ldots,q_n\}$ a momentum set in $U_\varepsilon(P)$ and by $\tilde{P}(p)$ or shortly P the set $\{p, -p, p_1,\ldots,p_n\}$.

Proof. The technique of proof is the usual one in the flow equation framework: We proceed inductively, upwards in r and for given r downwards in n using (6, ii). This is the induction scheme appropriate to estimate the l.h.s. of (7) in terms of the r.h.s.. Note that the limit $\Lambda_0 \to \infty$ has already been taken (15). Our induction hypothesis are the statements (a)–(c) of the theorem for a given pair (r, n).

(A), $r = 1$ The only nonvanishing $\mathscr{L}^\Lambda_{1,n}$ are those with $n = 4, 2$ (6, ii). Using (7) we find that $\mathscr{L}^\Lambda_{1,4}$ is independent of Λ, using (10) it is then also momentum independent and given by (11) so that $c^1_1 = c^R_1$ which may be chosen freely $\left(\text{e.g. } \frac{1}{4!}\right)$. For $\mathscr{L}^\Lambda_{1,2}(p_1)$ we now find from (7)

$$\partial_\Lambda \mathscr{L}^\Lambda_{1,2} \equiv -6c^R_1 \int \frac{d^4p}{(2\pi)^4} \frac{\partial_\Lambda R(\Lambda, p)}{p^2} \tag{27}$$

independently of p_1. This implies (see (8))

$$|\partial_\Lambda \mathscr{L}^\Lambda_{1,2}| \leq c\Lambda\,.$$

Therefore for $\Lambda \geq 0$

$$\mathscr{L}^\Lambda_{1,2}(p_1) = \mathscr{L}^1_{1,2}(p_1) - \int_\Lambda^1 \partial_t \mathscr{L}^t_{1,2}\, dt\,. \tag{28}$$

Choosing $\mathscr{L}^1_{1,2}(0)$, i.e. a^1_1 as

$$a^1_1 = \int_0^1 \partial_t \mathscr{L}^t_{1,2}\, dt\,, \tag{29}$$

where the r.h.s. is given by (27), we find

$$\mathscr{L}_{1,2}(0) = 0\,, \quad \mathscr{L}_{1,2}(p_1) = \mathscr{L}^1_{1,2}(p_1) - a^1_1\,. \tag{30}$$

Since $\mathscr{L}^1_{1,2}(p_1) = a^1_1 + b^1_1 p_1^2$ due to (10), (11) we obtain

$$\mathscr{L}_{1,2}(p_1) = b^1_1 p_1^2\,, \quad \mathscr{L}^\Lambda_{1,2}(p_1) = a(\Lambda) + b^1_1 p_1^2 \;\text{ with }\; a(\Lambda) = \int_0^\Lambda \partial_t \mathscr{L}^t_{1,2}\, dt\,.$$

To resume we have found

$$\mathscr{L}^{\Lambda}_{1,4} = c_1^R = c_1^1 \text{ independent of } \Lambda, p_1, p_2, p_3\,, \tag{31}$$
$$\mathscr{L}^{\Lambda}_{1,2}(p_1) = a(\Lambda) + b_1^R p_1^2\,, \quad a(0) = 0\,,\ b_1^R = b_1^1\,,$$
$$\mathscr{L}^{\Lambda}_{1,n} \equiv 0 \text{ for } n \geq 6\,.$$

This set of functions satisfies all assertions of Theorem 1 to order 1, in particular (30) uniquely fixes a_1^1 (29) and using (4), (8) we find $|a(\Lambda)| \leq c \int_0^\Lambda dt \int_t^{2t} t^{3-3} dt \leq c\Lambda^2$ which is sufficient to verify (c4).

(B), $r > 1$ We assume the theorem to hold for (r', n') with $r' < r$, $n' \in \mathbb{N}$, and for (r, n') with $n' > n$ for some $n \in \mathbb{N}$. We want to prove it for (r, n). All statements are trivial for $n > 2r + 2$. So we assume $n \leq 2r + 2$ and to start with

(B1), $n > 2$ We apply the induction hypothesis to the r.h.s. of (7). Using Lemma 1, Lemma 2 we find

$$|\partial^w \mathscr{L}^{\Lambda}_{r,n+2}(q, -q, q_1, \ldots, q_{n-1})| \leq \Lambda^{-2g(P)-2-|w|}\, P\log \Lambda^{-1} \tag{32}$$

for $\tilde{Q} = \{q, -q, q_1, \ldots, q_n\} \in U_\varepsilon(\tilde{P}(p))$, if $|p| \leq \eta(P)$ (see (19)), and

$$|\partial^w \mathscr{L}^{\Lambda}_{r,n+2}(q, -q, q_1, \ldots, q_{n-1})| \leq \Lambda^{-2g(P)-4-|w|}\, P\log \Lambda^{-1} \tag{33}$$

as in (32), but without restriction on $|p|$. Furthermore

$$\left| \left[\partial^w \frac{\partial_\Lambda R(\Lambda, q')}{q'^2} \mathscr{L}^{\Lambda}_{r',n'}(q_1, \ldots, q_{n'-1}) \mathscr{L}^{\Lambda}_{r'',n''}(-q', q_{n'}, \ldots, q_{n-1}) \right]_{\text{sym}} \right|$$
$$\leq \Lambda^{-2g(P)-1-|w|}\, P\log \Lambda^{-1} \text{ for } Q = \{q_1, \ldots, q_n\} \text{ in } U_\varepsilon(P(p))\,. \tag{34}$$

Here we used (8) and Lemma 1(c). Those contributions from the r.h.s. of (7) for which the respective $p' \neq 0$ are bounded by $\eta^{-2g-1-|w|}\, P\log \eta^{-1}$ and may be absorbed in $P\log \Lambda^{-1}$. Note that (34) also holds for $n' = 2$ or $n'' = 2$ due to (c4).

The bounds (32),(33) hold for given p, $P \in U_\varepsilon(\tilde{P}(p))$. By a standard compactness argument they thus hold uniformly in $\{(q, -q) \mid |q| \leq \eta\} \times U_\varepsilon(P)$ (32) and $\{(q, -q) \mid |q| \leq 2\} \times U_\varepsilon(P)$ (33) respectively.

We may now estimate the r.h.s. of (7). Using (4, 8, 32, 33) the momentum integral is bounded by

$$\int_\Lambda^{2\Lambda} dt\, t^3 t^{-3} (\Theta(\Lambda - \eta)\eta^{-2g-4-|w|}\, P\log \eta^{-1} + t^{-2g-2-|w|}\, P\log t^{-1})$$
$$\leq \Lambda^{-2g-1-|w|}\, P\log \Lambda^{-1}\,, \quad 0 < \Lambda \leq 1 \tag{35}$$

(where the first term has been absorbed in the second by a (η-dependent) redefinition of the constants in $P\log$).

Using (34, 35) we may now estimate $\partial^w \mathscr{L}^{\Lambda}_{r,n}$ by integrating the r.h.s. of (7) from 1 to Λ:

$$|\partial^w \mathscr{L}^{\Lambda}_{r,n}(p_1, \ldots, p_{n-1})| \leq |\partial^w \mathscr{L}^{1}_{r,n}(p_1, \ldots, p_{n-1})| + \Lambda^{-2g-|w|}\, P\log \Lambda^{-1}\,, \tag{36}$$

which proves (25) and (c3).

If P is restricted as in (c2), we find for any odd partial sum q'_π of at least 3 and at most $n-1$ momenta in $U_\varepsilon(P)$,

$$q'^2_\pi > \eta^2(P) > 0 . \tag{37}$$

Then the bound (34) may be replaced by

$$\eta^{-2g-1-|w|}\, P \log \eta^{-1} \leq \text{const}(P) \tag{38}$$

(remember (4))
and the induction hypothesis allows to replace (32) by

$$|\partial^w \mathscr{L}^{\Lambda}_{r,n+2}(q,-q,q_1,\dots,q_{n-1})| \leq \Lambda^{-2g(P)-2}\, P \log \Lambda^{-1} \tag{39}$$

with the same restrictions as in (32). This is true since the assumption (c2) follows also for $\tilde{Q} \in U_\varepsilon(\tilde{P})$ if $|p|$ is such that $\Lambda \leq |p| \leq \eta$. For the momentum integral on the r.h.s. of (7) we now get the bound

$$\int_{\Lambda}^{2\Lambda} dt\, t^3 t^{-3} (\Theta(\Lambda-\eta)\eta^{-2g-4-|w|}\, P \log \eta^{-1} + t^{-2g-2}\, P \log t^{-1})$$
$$\leq \Lambda^{-2g-1}\, P \log \Lambda^{-1} \tag{40}$$

and integrating this bound and the bound (38) from 1 to Λ we arrive at (c2), (24) as before at (c3), (25).

Now we assume P to be nonexceptional. (38) holds a fortiori as an estimate of the second term on the r.h.s. of (7), with $g=0$. We also have $g(\tilde{Q})=0$ for $|p| \leq \eta$ and may therefore estimate

$$|\partial^w \mathscr{L}^{\Lambda}_{r,n+2}(q,-q,q_1,\dots,q_{n-1})| \leq P \log \Lambda^{-1} \tag{41}$$

in $\{(q,-q) \mid |q| \leq \eta\} \times U_\varepsilon(P)$, using the same compactness argument as above. Thus the r.h.s. of (7) is bounded by $\Lambda P \log \Lambda^{-1} + c$, uniformly in $U_\varepsilon(P)$, and integrating over Λ shows for $0 < \Lambda,\ \Lambda' \leq 1$

$$|\partial^w \mathscr{L}^{\Lambda}_{r,n}(q_1,\dots,q_{n-1}) - \partial^w \mathscr{L}^{\Lambda'}_{r,n}(q_1,\dots,q_{n-1})| \leq c|\Lambda-\Lambda'| \tag{42}$$

uniformly in $U_\varepsilon(P)$ for given $w \in \mathbb{N}_0^{4(n-1)}$.

This assures the smoothness of $\partial^w \mathscr{L}_{r,n}$ and yields

$$\partial^w \mathscr{L}_{r,n}(p_1,\dots,p_{n-1}) = \lim_{\Lambda \to 0} \partial^w \mathscr{L}^{\Lambda}_{r,n}(p_1,\dots,p_{n-1}) \tag{43}$$

which finishes the proof of (c1) and of (a) for $n>2$. The last statement to verify for $n>2$ is (b) (iii). We have

$$c^R_r = \mathscr{L}_{r,4}(k_1,k_2,k_3) = c^1_r - \int_0^1 \partial_t \mathscr{L}^t_{r,4}(k_1,k_2,k_3)\, dt ,$$

where the last term is known by induction, independently of the choice of c^1_r. Thus c^R_r may be chosen freely and it uniquely fixes the choice of c^1_r.

(B2), $n=2$ For nonexceptional p_1, i.e. $p_1 \neq 0$ we verify (23) as before for nonexceptional P (see (41)–(43)) and find

$$\mathscr{L}_{r,2} \in C^\infty(\mathbb{R}^4 \setminus \{0\}) . \tag{44}$$

Now we regard $p_1 = 0$, $q_1 \in U_\varepsilon(p_1)$. The inductive estimate of the r.h.s. of (7) gives for $0 < \Lambda \leq 1$,

$$|\partial_\Lambda \partial^w \mathcal{L}_{r,2}^\Lambda(q_1)| \leq \int_\Lambda^{2\Lambda} dt\, t^3 t^{-3-|w|}\, P \log t^{-1} + \Lambda^{4-3-|w|}\, P \log \Lambda^{-1}\,, \qquad (45)$$

since by induction and the definition of $g(P)$ we get for the first term

$$|\partial^w \mathcal{L}_{r,4}^\Lambda(p, -p, q_1)| \leq \Lambda^{-|w|}\, P \log \Lambda^{-1} \qquad (46)$$

uniformly in $\{(q, -q) \mid |q| \leq 2\} \times U_\varepsilon(0)$ by the usual compactness argument. To estimate the second term on the r.h.s. of (7) by $\Lambda^{4-3-|w|}\, P \log \Lambda^{-1}$ we used (c4) (note r', $r'' < r$ in (7)).

Note again that this term vanishes for $|q'| < \Lambda$, $|q'| > 2\Lambda$ ($q' = q_1$ for $n = 2$) due to (4). Integrating (45) over Λ yields

$$|\partial^w \mathcal{L}_{r,2}^\Lambda(q_1)| \leq \Lambda^{2-|w|}\, P \log \Lambda^{-1}\,, \quad |w| \geq 2\,,\ 0 < \Lambda \leq 1\,,$$

$$|\partial^w \mathcal{L}_{r,2}^\Lambda(q_1) - \partial^w \mathcal{L}_{r,2}^{\Lambda'}(q_1)| \leq \int_\Lambda^{\Lambda'} dt\, t^{1-|w|}\, P \log t^{-1}\,. \qquad (47)$$

All estimates are uniform in $U_\varepsilon(0)$ and therefore also uniform for $|q_1| \leq 2$, cf. (44). The last one implies thus

$$\partial_\mu \mathcal{L}_{r,2}(0) = \lim_{\Lambda \to 0} \partial_\mu \mathcal{L}_{r,2}^\Lambda(0) = 0 \qquad (48)$$

(where we also used $O(4)$-invariance for $\Lambda > 0$)
and, using (44, 47) this implies

$$\mathcal{L}_{r,2} \in C^1(\mathbb{R}^4)\,. \qquad (49)$$

We have

$$\mathcal{L}_{r,2}(0) = \mathcal{L}_{r,2}^1(0) - \int_0^1 \partial_t \mathcal{L}_{r,2}^t(0)\, dt\,. \qquad (50)$$

Choosing $\mathcal{L}_{r,2}(0) = 0$ uniquely fixes $a_r^1 = \mathcal{L}_{r,2}^1(0)$ to be equal to $\int_0^1 \partial_t \mathcal{L}_{r,2}^t(0) dt$, and the integrand is already known by induction from (7).

We also have

$$2b_r^R = \partial_\mu \partial_\nu \mathcal{L}_{r,2}(k)|_{\delta_{\mu,\nu}} = \partial_\mu \partial_\nu \mathcal{L}_{r,2}^1(k)|_{\delta_{\mu,\nu}} - \int_0^1 \partial_\mu \partial_\nu \partial_t \mathcal{L}_{r,2}^t(k)|_{\delta_{\mu,\nu}} dt\,. \qquad (51)$$

Choosing freely b_r^R thus fixes b_r^1-uniquely, since the second term on the r.h.s. is known by induction and independent of b_r^1.

It remains to establish (c4) for $|w| \leq 1$. We know (47, 49)

$$|\partial^w \mathcal{L}_{r,2}^\Lambda(q_1)| \leq P \log \Lambda^{-1}\,,\ |w| = 2\,,\ 0 < \Lambda \leq 1\,,\ |q_1| \leq 2\,,\ \mathcal{L}_{r,2}^\Lambda \in C^1(\mathbb{R}^4)\,,\ \Lambda \geq 0\,.$$

From this we find with the aid of

$$\partial_\mu \mathscr{L}^\Lambda_{r,2}(p) = \partial_\mu \mathscr{L}^\Lambda_{r,2}(0) + p_\nu \int_0^1 \partial_\nu \partial_\mu \mathscr{L}^\Lambda_{r,2}(\lambda p)\, d\lambda$$

and with the aid of (48) that

$$|\partial_\mu \mathscr{L}^\Lambda_{r,2}(p)| \le |p|\, P \log \Lambda^{-1} \tag{52}$$

uniformly for $|p| \le 2$. Now

$$\mathscr{L}_{r,2}(p) = \mathscr{L}_{r,2}(0) + p_\nu \int_0^1 \partial_\nu \mathscr{L}_{r,2}(\lambda p)\, d\lambda$$

implies

$$|\mathscr{L}_{r,2}(p)| \le |p|^2\, P \log \Lambda^{-1}\,, \tag{53}$$

since $\mathscr{L}_{r,2}(0) = 0$. Equations (52, 53) imply in particular (c4) for $|w| \le 1$. □

The same technique of proof may be used to obtain sharper statements on the behaviour of $\mathscr{L}^\Lambda_{r,n}$ in momentum space near exceptional points. As an example we join

Proposition 2. *For an* a.m.s. $P = \{p_1, \dots, p_n\}$ *and nonempty* $P_1 \subset \{p_1, \dots, p_{n-1}\}$ *let* $\zeta(P_1) = \inf_{P_2}\{|\sum_{p_i \in P_2} p_i|\}$, *where* $P_2 \subset \{p_1, \dots, p_{n-1}\}$, $P_2 \cap P_1 \neq \emptyset$, *and* $\rho(P_1, \Lambda) :=$ sup$(\zeta(P_1), \Lambda)$. *For given* P_1 *we write* ∂^{w_1} *for a product of derivatives w.r.t. momenta* p_{i_μ} *with* $p_i \in P_1$. *For the* $\mathscr{L}^\Lambda_{r,n}$ *of Theorem 1 we assert:*
(a) $|\partial^{w_1} \mathscr{L}^\Lambda_{r,n}(p_1, \dots, p_{n-1})| \le \Lambda^{-2g(P)}\, P \log \Lambda^{-1} \rho^{-|w_1|}(P_1, \Lambda)\,, \quad n > 2,$
$|\partial^{w} \mathscr{L}^\Lambda_{r,2}(p)| \le \Lambda^2\, P \log \Lambda^{-1} (\sup(|p|, \Lambda))^{-|w|}$, $n = 2$.
The estimates are uniform in $U_\varepsilon(P)$.
(b) $|\mathscr{L}_{r,2}(p)| \le p^2\, P \log |p|^{-1}, \quad |\partial_\mu \mathscr{L}_{r,2}(p)| \le |p|\, P \log |p|^{-1}$.
The notations and assumptions are as in Theorem 1.

Proof. We apply the same method of proof as previously for the theorem. So we will be very short. We also use the same conventions and the same induction scheme. For $r = 1$ (a), (b) are obvious from (31).

$r > 1$, (a): We use (7) to estimate $\partial_\Lambda \partial^{w_1} \mathscr{L}^\Lambda_{r,n}$. The r.h.s. of (7) is bounded by induction through (see (32)–(35) for an analogous estimate)

$$\int_\Lambda^{2\Lambda} dt\, [(\Theta(\Lambda - \eta))\eta^{-2g-4}\, P \log \eta^{-1} + t^{-2g-2}\, P \log t^{-1})(\rho(P_1, t))^{-|w_1|}]$$
$$+ \Lambda^{-2g-1} \rho^{-|w_1|}(P_1, \Lambda)\, P \log \Lambda^{-1}\,. \tag{54}$$

In the second term we used (4, 8) which assure that any derivative from ∂^{w_1} applied to $\frac{\partial_\Lambda R(\Lambda, q')}{(q')^2}$ gives up to a Λ- and momenta-independent constant at most an additional ρ^{-1}. The same holds obviously by induction for derivatives applied to $\mathscr{L}^\Lambda_{r',n'}, \mathscr{L}^\Lambda_{r'',n''}$.

From (54) we find

$$|\partial_\Lambda \partial^{w_1} \mathscr{L}^\Lambda_{r,n}(p_1,\dots,p_{n-1})| \le \Lambda^{-2g-1}\rho^{-|w_1|}(P_1,\Lambda)\,P\log\Lambda^{-1}\,,$$
$$|\partial^{w_1} \mathscr{L}^\Lambda_{r,n}(p_1,\dots,p_{n-1})| \le \Lambda^{-2g}\rho^{-|w_1|}(P_1,\Lambda)\,P\log\Lambda^{-1} \tag{55}$$

as in Theorem 1 (35, 36).

$r > 1$, (b): We proceed as in Theorem 1. We have from the flow equation (7) and (a) for $|w| = 2$,

$$|\partial_\Lambda \partial^w \mathscr{L}^\Lambda_{r,2}(p)| \le (\Theta(|p| - \Lambda)\frac{\Lambda}{p^2} + \Theta(\Lambda - |p|)\Lambda^{-1})\,P\log\Lambda^{-1} \tag{56}$$

(where constants of order 1 have been absorbed in $P\log$).

This implies on integrating from 1 to Λ

$$|\partial_\mu\partial_\nu \mathscr{L}^\Lambda_{r,2}(p)| \le \Theta(|p| - \Lambda)\,P\log|p|^{-1} + \Theta(\Lambda - |p|)\,P\log\Lambda^{-1} \tag{57}$$

Taylor-expanding $\partial_\mu \mathscr{L}^\Lambda_{r,2}$ now gives, instead of (52),

$$|\partial_\mu \mathscr{L}^\Lambda_{r,2}(p)| \le |p|(\Theta(|p| - \Lambda)\,P\log|p|^{-1} + \Theta(\Lambda - |p|)\,P\log\Lambda^{-1})\,, \tag{58}$$

and therefore, for $\Lambda \to 0$

$$|\partial_\mu \mathscr{L}_{r,2}(p)| \le |p|\,P\log|p|^{-1}\,. \tag{59}$$

Taylor-expanding again (see (53)) now yields

$$|\mathscr{L}_{r,2}(p)| \le p^2\,P\log|p|^{-1}\,, \tag{60}$$

which proves (b). □

We think these statements show how the simple induction scheme applied to the flow equations leads to straightforward proofs of IR-properties of massless ϕ_4^4. The necessary input is of course some intuition or information on the perturbative IR behaviour to find a useful induction hypothesis. It should be clear now how to treat other massless theories whose symmetries are respected by the momentum space regularization. In a Yukawa-theory with Lagrangian

$$\mathscr{L} = \bar\psi i\partial\!\!\!/\psi + 1/2(\partial_\mu\phi)(\partial_\mu\phi) + \lambda_1\bar\psi\psi\phi + \lambda_2\phi^4\,,$$

one has to derive the flow equations as before and restrict the r.c. such that all terms of dimension < 4, i.e.

$$\bar\psi\psi\,,\ \phi\,,\ \phi^2\,,\ \phi^3$$

are fixed to vanish at 0 momentum. Then it is necessary to prove a statement corresponding to Lemma 1, whereon estimates as in Theorem 1 are straightforward by analogy. Even though one has to cope with an unfortunate bulk of notation in this case, we hope that the reader is convinced of the generality of our method. As regards gauge invariance, in particular QED, we refer to a future publication.

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Note added in proof. After submitting this paper we came across the paper: Perturbative renormalization and infrared finiteness in the Wilson renormalization group: the massless scalar case, by M. Bonini et al. which now appeared in Nucl. Phys. B **409**, 441–464 (1993). Unfortunately the proof of IR-finiteness given in this paper is wrong: It is based on an inductive estimate of one-particle irreducible functions and makes the induction hypothesis, that those are finite, if there are no *pairs* $q, -q$ of exceptional momenta (see Eq. (32a) of the paper). This assumption is obviously wrong already at one loop (regard e.g. an eight point function where two partial sums of four external momenta vanish). Such functions may be arbitrarily divergent. We call attention to the fact that the mistake cannot be remedied without major changes: There seems to be no way of proving IR finiteness with flow equations without having an analogue of our Lemma 1, which of course would have to be adapted to the one particle irreducible case and which is completely missing in that paper. This is because complete knowledge of the behaviour at exceptional momenta is required also for the proof at nonexceptional momenta, since the flow equations (Eq. (11), (12b) in their paper) imply integration over momenta as in our case.