Lattice Topological Field Theory in Two Dimensions

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Abstract. The lattice definition of a two-dimensional topological field theory (TFT) is given generically, and the exact solution is obtained explicitly. In particular, the set of all lattice topological field theories is shown to be in one-to-one correspondence with the set of all associative algebras R, and the physical Hilbert space is identified with the center $Z(R)$ of the associative algebra R. Perturbations of TFT's are also considered in this approach, showing that the form of topological perturbations is automatically determined, and that all TFT's are obtained from one TFT by such perturbations. Several examples are presented, including twisted $N = 2$ minimal topological matter and the case where R is a group ring.

1. Introduction

Any consistent quantum field theory is expected to be realized as a continuum limit of a lattice model. Furthermore, the lattice definition is the only known method to investigate the non-perturbative structure of quantum field theories.

In this paper, we show that 2D topological field theories (TFT's), especially topological matter systems, can also be realized as lattice models, which will be called *lattice topological field theories* (LTFT's). The advantage of this approach to TFT over the conventional continuum field theoretic one [1] is in that this lattice definition makes much easier the understanding of geometric and algebraic structure of TFT. Moreover, since there should not be any dimensionful parameters in TFT or in LTFT, we do *not* need to take a continuum limit in our lattice model. This fact allows easy calculation of various quantities.

We first recall the basic axiom of TFT. Let $\hat{g}_{\mu\nu}$ be a background metric on a surface, on which matter field X_{matter} lives. The partition function $Z[\hat{g}_{\mu\nu}]$ is defined by

$$Z[\hat{g}] \equiv \int \mathscr{D}_{\hat{g}} X_{\text{matter}} \exp(-S[X_{\text{matter}}, \hat{g}]), \tag{1.1}$$

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with $S[X_{\text{matter}}, \hat{g}]$ the action. We also assume the existence of fermionic conserved quantity Q_{BRST} which generates all the symmetry of the theory and satisfies the nilpotency condition; $Q_{\text{BRST}}^2 = 0$. The theory is called a TFT if the energy-momentum tensor $T_{\mu\nu}$ is BRST-exact:

$$T_{\mu\nu} = \{Q_{\text{BRST}}, *\}. \tag{1.2}$$

Recall here that the energy-momentum tensor is defined by

$$T_{\mu\nu} \equiv \frac{-4\pi}{\sqrt{\hat{g}(x)}} \frac{\delta}{\delta \hat{g}^{\mu\nu}(x)} (-\ln Z[\hat{g}]). \tag{1.3}$$

Therefore, the condition (1.2) implies that our partition function $Z[\hat{g}]$ is invariant under *local* changes of background metric if we restrict ourselves to the "physical" Hilbert space $\mathscr{H}_{\text{phys}} \equiv \{|\text{phys}\rangle \mid Q_{\text{BRST}}|\text{phys}\rangle = 0\}$;

$$\frac{\delta Z[\hat{g}]}{\delta \hat{g}^{\mu\nu}(x)} \sim 0. \tag{1.4}$$

and thus any BRST-invariant quantity calculated over this physical space is topological.

How do we translate this property of TFT into lattice language? Intuitive consideration tells us that each background metric $g_{\mu\nu}$ in continuous theory should correspond to a triangulation T in our lattice framework. In fact, in 2D quantum gravity, the summation over all quantum fluctuations of metric can be replaced by the summation over all triangulations [2–6]:

$$\int \mathscr{D} g_{\mu\nu} \leftrightarrow \sum_{T\,:\,\text{triangulation}}. \tag{1.5}$$

Thus, we might wish to characterize our LTFF by the condition that the partition function of the lattice model is independent of triangulations. However, since the condition (1.4) is *local*, we should further require our LTFT to have the following property:

Ansatz 1. *The partition function of LTFF, which is first constructed with a given triangulation, should be invariant under any local changes of the triangulation.*

The present paper is organized as follows. In Sect. 2, we rewrite the above ansatz for LTFT into more concrete form by introducing 2D version of Mateev move. The general solutions to this ansatz are obtained explicitly in Sect. 3, where we find that there is a one-to-one correspondence between the set of all TFT's and the set of all associative algebras R. In Sect. 4, we then define physical operators and investigate the structure of their correlation functions. There we see that each physical operator in a given TFT has a one-to-one correspondence to an element of the center $Z(R)$ of the associative algebra R associated with the LTFT. We further show that these operators actually satisfy all the properties known in conventional TFT. The results of these two sections can be summarized schematically as follows:

$$\begin{array}{ccc} \text{LTFT} & \longrightarrow & \text{TFT} \\ \Updownarrow & & \Updownarrow \\ R & \longrightarrow & Z(R) \end{array}. \tag{1.6}$$

In Sect. 5, as an example, we consider the LTFT that corresponds to a group ring $R = \mathbf{C}[G]$ with G a group. The physical operators in this case have one-to-one correspondence to the conjugacy classes of G, and their correlation functions are calculated explicitly, showing the coincidnece with Witten's result obtained in

continuum approach [7]. In Sect. 6, we also study the perturbation of TFT's with introducing the concept of the moduli of TFT's. We there show that the form of perturbation is automatically determined in our lattice formulation upon requiring its locality and topological property to be preserved under the perturbation. We further show that every TFT can be obtained by perturbation from what will be called the standard topological field theory (STFT). As a simple example, the twisted $N = 2$ minimal topological matters [8, 9] are considered, and shown to live on the boundary of the moduli space of TFT's. Section 7 is devoted to the discussion about how to incorporate gravity into our lattice formulation.

   This paper was inspired by the work by Turaev and Viro [10] who constructed a series of three-dimensional topological invariants by using lattice approach (see also [11]).

2. Definition of LTFT

Let Σ_g be a closed oriented surface of genus g, T_g a triangulation of Σ_g. Then, the partition function of the lattice model associated with T_g is defined as follows: First, for an oriented triangle in T_g we make a coloring as preserving its orientation. That is, we give a set of color indices running from 0 through A to three edges of the triangle (see Fig. 1). We then assign a complex value C_{ijk} to a triangle with ordered color indices i, j, k. We here assume that C_{ijk} is symmetric under cyclic permutations of the indices:

$$C_{ijk} = C_{jki} = C_{kji} \, . \tag{2.1}$$

Note, however, that C_{ijk} is *not* necerssarily totally symmetric. Next, we glue these triangles by contracting their indices with $g^{ij} = g^{ji}$ (see Fig. 2). We further assume that g^{ij} has its inverse g_{ij}; $g_{ik}g^{kj} = \delta_i^j$, and raise or lower indices with these matrices. Thus, we have a complex-valued function of C_{ijk} and g^{ij} for each triangulation T_g, and we will interpret it as the partition function of our lattice model, denoting it by $Z(T_g)$. For example, the partition function for the triangulation of $\Sigma_0 = S^2$ depicted in Fig. 3 is expressed as

$$Z(T_0) = g^{ii'} g^{jj'} g^{kk'} g^{ll'} g^{mm'} g^{nn'} C_{ijk} C_{k'lm} C_{m'ni'} C_{j'n'l'} \, . \tag{2.2}$$
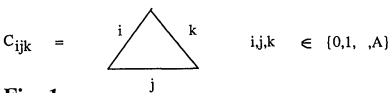

$$C_{ijk} \quad = \qquad \qquad i,j,k \ \in \ \{0,1, \ ,A\}$$

Fig. 1

$$= \ g^{mn}$$

$$= \ C_{ijm} \ g^{\,mn} C_{nkl}$$

Fig. 2 $= \ C_{ij}{}^{\,n} C_{nkl}$

Fig. 1. Colored triangle with a complex value C_{ijk}
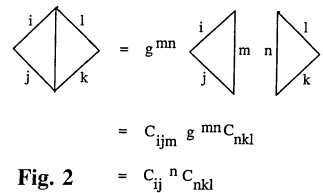
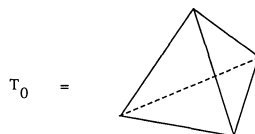**Fig. 2.** Gluing two triangles C_{ijm} and C_{nkl} with a propagator g^{mn}

$$T_0 \quad =$$

Fig. 3. A triangulation T_0 of sphere $\Sigma_0 = S^2$

Fig. 4   **Fig. 5**   **Fig. 6**
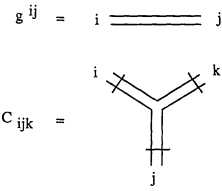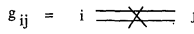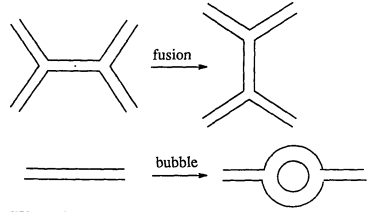
Fig. 4. Propagator g^{ij} and three-point vertex C_{ijk} in dual diagrams Cutting lines represent the truncation of external lines

Fig. 5. Two-point vertex g_{ij} Crossed lines represent that two external lines are truncated

Fig. 6. Fusion transformation and bubble transformation in dual diagrams

Note that in dual diagrams, g^{ij}, C_{ijk}, and g_{ij} are interpreted as propagator, 3-point vertex and 2-point vertex, respectively (see Fig. 4 and Fig. 5).

Now we consider LTFF and, by following the argument given in the previous section, we require that the partition function is invariant under local changes of triangulation, which will set some conditions on C_{ijk} and g^{ij}. There have been known several systematic methods to deal with these local moves, which can also generate all the triangulations with fixed topology. Among the best known are the so-called bond-flip method and the Alexander-move method, but it is difficult to find solutions when we require the invariance of our partition function $Z(T_g)$ under these moves. In this paper, we adopt instead the other one, 2D version of Matveev-move method, which can be expressed only in dual diagrams and consists of a sequence of two fundamental local moves; *fusion transformation* and *bubble transformation*, as shown in Fig. 6.[1] The reason why we adopt this is that we can easily find the general solutions when we require the invariance of the partition function under these 2D Matveev moves.

We conclude this section by summarizing our ansatz for the partition function of LTFT:

Ansatz 2. *Let $Z(T_g)$ be the partition function of LTFT associated with a triangulation T_g of genus-g closed surface. Then it should be invariant under any 2D Matveev moves acting on the triangulation T_g.*

3. General Solutions

In this section, we obtain general solutions of our ansatz, and show that our LTFT has a one-to-one correspondence to a (generally noncommutative) associative algebra.

First, the invariance of the partition function Z under fusion transformations is expressed in terms of g_{ij} and C_{ijk} as (see Fig. 7)

$$C_{ij}{}^{p}C_{pk}{}^{l} = C_{jk}{}^{p}C_{ip}{}^{l} . \tag{3.1}$$

This equation allows us to introduce an associative algebra R with a basis $\{\phi_i\}$ ($i = 0, 1, 2, \ldots, A$) and the structure constant $C_{ij}{}^{k}$; $\phi_i\phi_j = C_{ij}{}^{k}\phi_k$. It is easy to see that Eq. (3.1) ensures the associativity of our algebra R: $(\phi_i\phi_j)\phi_k = \phi_i(\phi_j\phi_k)$.

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[1] Although, as can be easily proven, 2D Matveev moves actually generate other kinds of local moves such as bond-flips or Alexander moves, 2D Matveev move cannot be obtained from these moves since 2D Matveev move can only be represented in dual diagrams
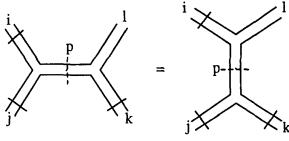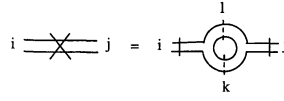
Fig. 7   **Fig. 8**

Fig. 7. Diagramatic representation of the invariance under fusion transformation [Eq. (3.1)]
Fig. 8. Diagramatric representation of the invariance under bubble transformation [Eq. (3 2)]

Then the invariance of the partition function Z under bubble transformations is now expressed as (see Fig. 8)

$$g_{ij} = C_{ik}{}^l C_{jl}{}^k. \tag{3.2}$$

We thus have obtained the map from LTFF to an associative algebra $R = \bigoplus_{i=0}^{A} \mathbf{C}\phi_i$ with the product $\phi_i \phi_j = C_{ij}{}^k \phi_k$ and the metric $g_{ij} = C_{ik}{}^l C_{jl}{}^k$. We can further show that this map is bijective. In fact, by introducing a metric by Eq. (3.2), we can define 3-point vertex $C_{ijk} \equiv C_{ij}{}^l g_{lk}$ and propagator $(g^{ij}) = (g_{ij})^{-1}$, which both satisfy the invariance conditions. Therefore, we have proven the following theorem:

Theorem 3. *The set of all* LTFT'*s defined above has a one-to-one correspondence to the set of semi-simple associative algebras* R.

A few remarks are now in order:
(1) The condition that the metric g_{ij} in (3.2) has its inverse is stated in the word *semi-simple* above. In fact, the necessary and sufficient condition for the metric to have its inverse turns out that the algebra is semi-simple. The sufficiency of the condition follows simply from Wedderburn's theorem [12], applied to the algebra R over $\mathbf{C}$, which says that the algebra is isomorphic to a direct sum of matrix rings over $\mathbf{C}$. On the other hand, to prove the necessity of the condition we note that Maschke's theorem [12] for finite group G extends naturally to our case of R. Maschke's theorem says that if an arbitrary finite dimensional G-module V over $\mathbf{C}$ contains W as its submodule then V decomposes into a direct sum as G-module; $V = W \oplus W'$, where W' is complement to W. The proof is essentially based on taking average of the projection map $P_W : V \to W$ over G:

$$\bar{P}_W = \frac{1}{|G|} \sum_{g \in G} \pi_W(g^{-1}) P_W \pi_V(g). \tag{3.3}$$

The average (3.3) naturally extends to our case of R as

$$\bar{P}_W = \sum_{i,j} \pi_W(\phi_i) g^{ij} P_W \pi_V(\phi_j), \tag{3.4}$$

if the metric has its inverse. Taking $V = R$ as left R-module we can conclude that the ring R is semi-simple. We, however, show in Sect. 6 that we can still define LTFT's which do not necessarily correspond to semi-simple associative algebras by using topological perturbations.

(2) It is easy to show that C_{ijk} is totally symmetric if and only if R is commutative. In such a case, however, the partition function Z has a value which is independent of topology, so that the model only has trivial structure.[2]
(3) If we introduce the regular representation (π, V) of algebra R by $(\pi(\phi_i))^k{}_j = C_{ij}{}^k$, then g_{ij} and C_{ijk} can be simply expressed as follows:

$$g_{ij} = \mathrm{tr}_V \, \pi(\phi_i)\pi(\phi_j) \,, \tag{3.5}$$
$$C_{ijk} = \mathrm{tr}_V \, \pi(\phi_i)\pi(\phi_j)\pi(\phi_k) \,. \tag{3.6}$$

This representation is useful in constructing LTFT directly from a given algebra (see Sect. 5).
(4) If we set $\phi_0 = 1$ (unit element of R), then we have

$$C_{0i}{}^j = C_{i0}{}^j = \delta_i^j \,, \tag{3.7}$$

since $\phi_0\phi_i = \phi_i\phi_0 = \phi_i$.

4. Physical Observables and Their Correlation Functions

In the previous section, we found that our LTFT has a one-to-one correspondence with an associative algebra R. In this section, we investigate the structure of physical observables, and show that all information we need can be reduced to the center $Z(R)$ of the algebra R, and further show that our method actually reproduces the well-known results of continuous TFT.

We first define operators $\mathcal{O}_i$ $(i = 0, 1, 2, \ldots, A)$ by interpreting the insertion of $\mathcal{O}_i$ into correlation functions as creating a loop boundary with color index i, and we denote the correlation function of $\mathcal{O}_{i_1}, \ldots, \mathcal{O}_{i_n}$ on genus-g closed surface by $\langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle_g$.

Let us consider 2-point function of $\mathcal{O}_i$ and $\mathcal{O}_j$ on sphere; $\eta_{ij} = \langle \mathcal{O}_i \mathcal{O}_j \rangle_0$, and investigate the property of physical operators. The simplest triangulation for η_{ij} is depicted in Fig. 9, and written as

$$\eta_{ij} = C_{ik}{}^l C_{lj}{}^k \,. \tag{4.1}$$

Furthermore, due to its independence of triangulation, η_{ij} can also be calculated from another graph shown in Fig. 10, which yields an important identity; $\eta_{ij} = \eta_i{}^k \eta_{kj}$, or
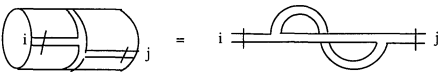
$$\eta_i{}^j = \eta_i{}^k \eta_k{}^j \,. \tag{4.2}$$
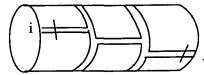

Fig. 9. A triangulation of η_{ij}

Fig. 10. Another triangulation of η_{ij}

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[2] We, however, can construct nontrivial theories by perturbation from such a trivial LTFT that corresponds to a commutative algebra (see Sect 6)
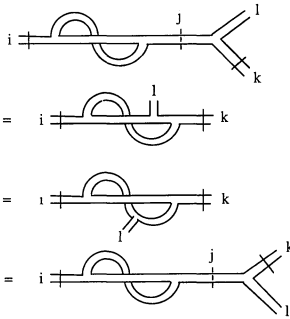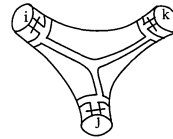
Fig. 11   **Fig. 12**

Fig. 11. Graphical proof of $\eta_i{}^j C_{jk}{}^l = \eta_i{}^j C_{kj}{}^l$

Fig. 12. A triangulation of the three-point function N_{ijk}

Thus, we know that the operator $\eta = (\eta_i{}^j)$ acting on R is idempotent; $\eta^2 = \eta$, and so expect that η is a kind of projection map. In fact, we can prove the following theorem:

Theorem 4. $\eta = (\eta_i{}^j)$ *is the surjective projector from R to its center* $Z(R) = \{\tilde\phi \in R \,|\, \phi\tilde\phi = \tilde\phi\phi \text{ for } \forall\phi \in R\}$.

To prove this, we first show that η is a map from R into its center $Z(R)$. We only have to show that $\tilde\phi_i\phi_k = \phi_k\tilde\phi_i$ $(\forall i, k)$ with $\tilde\phi_i \equiv \eta_i{}^j\phi_j$, and this is easily seen from the relation $\eta_i{}^j C_{jk}{}^l = \eta_i{}^j C_{kj}{}^l$ as depicted in Fig. 11. Moreover, we can also show that

$$\eta\tilde\phi = \tilde\phi \quad \text{for } \forall\tilde\phi \in Z(R), \tag{4.3}$$

which asserts that this map $\eta : R \to Z(R)$ is surjective. We thus proved that $\eta = (\eta_i{}^j)$ is the surjective projector from R to its center $Z(R)$.

Proof of Eq. (4.3). For $\phi = c^i\phi_i \in Z(R)$, we have a relation $c^i C_{ik}{}^l = c^i C_{ki}{}^l$ since $\tilde\phi\phi_k = \phi_k\tilde\phi$. Thus, we have

$$\begin{aligned}
\eta\tilde\phi &= \eta_i{}^j c^i \phi_j \\
&= C_{ik}{}^l C^{jk}{}_l c^i \phi_j \\
&= C_{ki}{}^l C^{jk}{}_l c^i \phi_j \\
&= c^j \phi_j \\
&= \tilde\phi. \quad \text{[Q.E.D.]} \tag{4.4}
\end{aligned}$$

Next, we study 3-point function on sphere, $N_{ijk} \equiv \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle_0$. The simplest triangulation is shown in Fig. 12, and evaluated as

$$N_{ijk} \equiv \eta_i{}^{i'} \eta_j{}^{j'} \eta_k{}^{k'} C_{i'j'k'}. \tag{4.5}$$

Note that the indices i, j, and k in C_{ijk} are subject to the projection of η, and so we know that N_{ijk} is now totally symmetric even though C_{ijk} is not so.

Such a graphical consideration can be easily generalized to the case of other multi-point functions and of higher genuses, and we see that every insertion of operator $\mathcal{O}_i$ is necessarily subject to the projection of η. Thus, we obtain the following theorem:
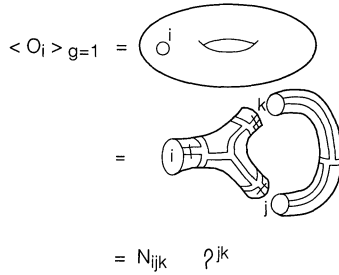
Fig. 13. One-point function $\langle \mathcal{O}_i \rangle_{g=1}$ on torus

Theorem 5. *The set of physical operators is in one-to-one correspondence with the center $Z(R)$ of the associative algebra R associated with the* LTFT *we consider. In particular, the number of independent physical operators is equal to the dimension of $Z(R)$.*

To get correlation functions, we only have to combine N_{ijk}'s by contracting their indices with η^{ij}, as exemplified in Fig. 13. In the following, we relabel the indices of basis $\{\phi_i\}$ ($i = 0, 1, \ldots, A$) of R in such a way that the first $(K+1)$ indices represent a basis of $Z(R)$:

$$R = \bigoplus_{i=0}^{A} \mathbf{C}\phi_i$$

$$= Z(R) \oplus Z^C(R)$$

$$= \left(\bigoplus_{\alpha=0}^{K} \mathbf{C}\phi_\alpha \right) \oplus \left(\bigoplus_{p=K+1}^{A} \mathbf{C}\phi_p \right). \tag{4.6}$$

Since $\eta = (\eta_i{}^j)$ ($i, j = 0, 1, \ldots, A$) is the projector onto $Z(R)$ and the relation (4.3) holds, η has the following form under the above decomposition (4.6):

$$\eta_{ij} = \begin{bmatrix} \eta_{\alpha\beta} = g_{\alpha\beta} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\eta_i{}^j = \begin{bmatrix} \eta_\alpha{}^\beta = \delta_\alpha{}^\beta & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.7}$$

$$\eta^{ij} = \begin{bmatrix} \eta^{\alpha\beta} = g^{\alpha\beta} & 0 \\ 0 & 0 \end{bmatrix},$$

and the relation $\eta_{ik}\eta^{kj} = \eta_i{}^j$ implies that $(\eta^{\alpha\beta})$ is the inverse to $(\eta_{\alpha\beta})$ if we restrict their defining region to $Z(R)$:

$$\eta_{\alpha\gamma}\eta^{\gamma\beta} = \delta_\alpha{}^\beta . \tag{4.8}$$

Note also that

$$N_{\alpha\beta\gamma} = C_{\alpha\beta\gamma} . \tag{4.9}$$

Fig. 14. (a) Cylinder $\eta^{\alpha\beta}$ and (b) diaper $N_{\alpha\beta\gamma}$

Equations (4.7)–(4.9) simplify the calculation of correlation functions since we only have to sum over indices $\alpha = 0, 1, \ldots, K$ of $Z(R)$ in glueing.[3] In summary,

Theorem 6. *All correlation functions are obtained by connecting cylinder $\eta^{\alpha\beta}$ and diaper $N_{\alpha\beta\gamma}$ (see Fig. 14).*

In the rest of this section, we show that our LTFT actually satisfies all the known properties in continuous TFT. Recall that due to our redefinition of indices (4.6), physical operators $\mathscr{O}_\alpha$ ($\alpha = 0, 1, \ldots, K$) correspond to a basis ϕ_α of the center $Z(R)$.

Let $A(\mathscr{O})$ be a function of physical operators (e.g. $A(\mathscr{O}) = \mathscr{O}_{\alpha_1}\mathscr{O}_{\alpha_2}\ldots\mathscr{O}_{\alpha_n}$). Then we have the following theorem:

Theorem 7. *Calculation of correlation functions with genus g can always be reduced to that with genus 0 by using the handle operator H:*

$$\langle A(\mathscr{O})\rangle_g = \langle A(\mathscr{O})H^g\rangle_0 \tag{4.10}$$

with

$$H \equiv w^\alpha \mathscr{O}_\alpha, \qquad w^\alpha \equiv N^{\alpha\beta}{}_\beta. \tag{4.11}$$

Proof. Correlation function with genus g is calculated

$$\langle A(\mathscr{O})\rangle_g = \langle A(\mathscr{O})\mathscr{O}_{\alpha_1}\mathscr{O}_{\alpha_2}\ldots\mathscr{O}_{\alpha_g}\rangle_0 w^{\alpha_1}w^{\alpha_2}\ldots w^{\alpha_g} \tag{4.12}$$

as shown in Fig. 15. Thus, if we introduce H as in Eq. (4.11), then we have $\langle A(\mathscr{O})\rangle_g = \langle A(\mathscr{O})H^g\rangle_0$. [Q.E.D.]
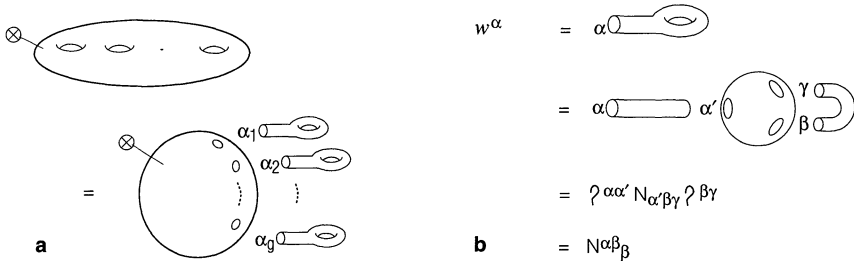

Fig. 15. Calculation of correlation functions with genus g is reduced to that with genus 0

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[3] Since physical states of a topological field theory are in one-to-one correpondence with elements of a commutative algebra (the center of some associative algebra), it might seem that we can construct *lattice* topological field theories only by using totally symmetric couplings $N_{\alpha\beta\gamma}$, that represent interactions between physical states. However, we also have to introduce nonphysical states in order to insure local topological invariance (see remark (3) at the end of Sect 3) This fact reminds us of the spurious states in continuum topological field theories that must be introduced to insure local topological invariance, and are usually eliminated from physical Hilbert space under the projection of BRST charge
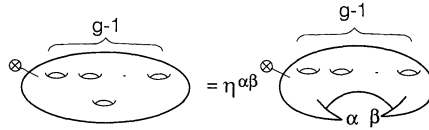
Fig. 16. Strong factorization property

Furthermore, we can show

Theorem 8. *Operators $\mathcal{O}_\alpha$ satisfy the following OPE:*

$$\mathcal{O}_\alpha \mathcal{O}_\beta = N_{\alpha\beta}{}^\gamma \mathcal{O}_\gamma \,, \qquad (N_{\alpha\beta}{}^\gamma \equiv N_{\alpha\beta\gamma'}\eta^{\gamma'\gamma}) \tag{4.13}$$

as an identity in any correlation functions.

Proof. If we introduce the regular representation (ϱ, W) of commutative algebra $Z(R)$ as $\varrho(\phi_\alpha)^\gamma{}_\beta \equiv N_{\alpha\beta}{}^\gamma = C_{\alpha\beta}{}^\gamma$, then $\varrho(\phi_\alpha)$ satisfies the product law: $\varrho(\phi_\alpha)\varrho(\phi_\beta) = N_{\alpha\beta}{}^\gamma \varrho(\phi_\gamma)$. On the other hand, as can be seen graphically, the expectation value of $A(\mathcal{O})$ with $g = 1$ (torus) is represented as a trace over this representation space $W : \langle A(\mathcal{O})\rangle_{g=1} = \mathrm{tr}_W A(\varrho(\phi))$. We thus have the following relation:

$$\begin{aligned}
\langle \mathcal{O}_\alpha \mathcal{O}_\beta a(\mathcal{O})\rangle_g &= \langle \mathcal{O}_\alpha \mathcal{O}_\beta A(\mathcal{O}) H^{g-1}\rangle_1 \\
&= \mathrm{tr}_W \, \varrho(\phi_\alpha)\varrho(\phi_\beta)A(\varrho(\phi))\varrho(H)^{g-1} \\
&= N_{\alpha\beta}{}^\gamma \, \mathrm{tr}_W \, \varrho(\phi_\gamma)A(\varrho(\phi))\varrho(H)^{g-1} \\
&= N_{\alpha\beta}{}^\gamma \langle \mathcal{O}_\gamma A(\mathcal{O})\rangle_g \,. \quad \text{[Q.E.D.]}
\end{aligned} \tag{4.14}$$

By using this OPE, we can further show that our model has a *strong factorization property* [4] (see Fig. 16):

$$\langle A(\mathcal{O})\rangle_g = \eta^{\alpha\beta}\langle A(\mathcal{O})\mathcal{O}_\alpha \mathcal{O}_\beta\rangle_{g-1}\,, \tag{4.15}$$

since $H = w^\alpha \mathcal{O}_\alpha = \eta^{\alpha\beta}\mathcal{O}_\alpha \mathcal{O}_\beta$.

5. Example: $R = \mathbf{C}[g]$

In this section, we deal with the special case where R is a group ring:

$$R = \mathbf{C}[G] = \bigoplus_{x \in G} \mathbf{C}\phi_x \,, \tag{5.1}$$

with the product induced from the group multiplication; $\phi_x \phi_y = \phi_{xy}$. Here we assume that $G = \{x, y, z, \ldots, g, h, \ldots\}$ is a finite group, for simplicity. Extension to continuous group is straightforward, but yields more fruitful structure in the obtained theory, as will be reported elsewhere.

In order to calculate 2- and 3-point vertices, it is useful to use the regular representation (π, V) of $R = \mathbf{C}[g]$:

$$\pi(\phi_x)^z{}_y = C_{xy}{}^z = \delta(xy, z)\,, \tag{5.2}$$

---

[4] For the reason why we call Eq. (4.14) strong factorization, see Sect. 7

where

$$\delta(x,y) \equiv \begin{cases} 1 & (x = y) \\ 0 & (\text{otherwise}). \end{cases} \tag{5.3}$$

Thus, if we use Eqs. (3.5) and (3.6) together with the following formula:

$$\text{tr}\,\pi(\phi_x) = |G|\delta(x,1), \tag{5.4}$$

we have

$$g_{xy} = |G|\delta(xy,1), \tag{5.5}$$
$$C_{xyz} = |G|\delta(xyz,1).$$

By using these equations, we easily obtain

$$\eta_{xy} = \langle \mathcal{O}_x \mathcal{O}_y \rangle_0 = \frac{|G|}{h_{[x]}}\,\delta_{[x],[y^{-1}]}. \tag{5.6}$$

Here $[x]$ denotes the conjugacy class of x; $[x] \equiv \{y \in G \,|\, y = gxg^{-1},\, \exists g \in G\}$, and $h_{[x]}$ is the number of the elements of $[x]$; $h_{[x]} = \#([x])$. In the following, we denote $[x^{-1}]$ by $\widehat{[x]}$, and label conjugacy classes by Greek letters. Note that $h_{\hat\alpha} = h_\alpha$.

Let us investigate the property of the projection operator $\eta = (\eta_x{}^y)$;

$$\eta_x{}^y = \eta_{xz}g^{zy} = \frac{1}{h_{[x]}}\,\delta_{[x]}^{[y]}. \tag{5.7}$$

By operating η on R, we obtain

$$\tilde\phi_x = \sum_{y \in G} \eta_x{}^y \phi_y = \frac{1}{h_{[x]}} \sum_{y \in [x]} \phi_y, \tag{5.8}$$

and thus know that $Z(R = \mathbf{C}[G])$ is spanned by the obits of conjugacy classes:[5]

$$Z(R) = \bigoplus_\alpha \mathbf{C}C_\alpha, \qquad C_\alpha \equiv \frac{1}{\sqrt{h_\alpha}} \sum_{x \in \alpha} \phi_x. \tag{5.9}$$

We here normalize the basis $\{C_\alpha\}$ by the factor $\sqrt{h_\alpha}$ for later convenience. In this basis, $\eta_{\alpha\beta}$ is represented by

$$\eta_{\alpha\beta} = \frac{1}{\sqrt{h_\alpha h_\beta}} \sum_{x \in \alpha} \sum_{y \in \beta} \frac{|G|}{h_{[x]}} \delta_{[x],[y^{-1}]}$$
$$= |G|\delta_\alpha^\beta = |G|\delta_\beta^{\hat\alpha}. \tag{5.10}$$

$N_{\alpha\beta}{}^\gamma$ is now easily read out from the following relation:

$$C_\alpha C_\beta = N_{\alpha\beta}{}^\gamma C_\gamma, \tag{5.11}$$

and found to be

$$N_{\alpha\beta}{}^\gamma = \frac{1}{\sqrt{h_\alpha h_\beta h_\gamma}} \sum_{x \in \alpha} \sum_{x \in \beta} \delta_{[xy]}^{-\gamma}. \tag{5.12}$$

---

[5] In the previous section, C_α was written as ϕ_α We, however, use different symbol here in order to avoid confusion

This expression in turn gives us 3-point function on sphere:

$$
\begin{aligned}
N_{\alpha\beta\gamma} &= \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \rangle_0 \\
&= N_{\alpha\beta}{}^{\gamma'} \eta_{\gamma'\gamma} \\
&= \frac{|G|}{\sqrt{h_\alpha h_\beta h_\gamma}} \sum_{x \in \alpha} \sum_{x \in \beta} \delta^{\hat{\gamma}}_{[xy]} \,.
\end{aligned}
\tag{5.13}
$$

These forms of $\eta_{\alpha\beta}$ and $N_{\alpha\beta\gamma}$, however, are not so useful for direct calculation, and so in the following we rewrite them into more convenient form. Since $\eta_{\alpha\beta}$ and $N_{\alpha\beta\gamma}$ both are functions of conjugacy classes, these must be expanded with respect to irreducible characters. In fact, short algebraic calculation shows that

$$
\begin{aligned}
\eta_{\alpha\beta} &= \sum_j \chi_j(C_\alpha)\chi_j(C_\beta) \,, \\
N_{\alpha\beta\gamma} &= \sum_j \frac{\chi_j(C_\alpha)\chi_j(C_\beta)\chi_j(C_\gamma)}{d_j} \,.
\end{aligned}
\tag{5.14}
$$

Here χ_j is the character of an irreducible representation j, and its defining region is extended to $\mathbf{C}[G]$ by linearity. Furthermore, d_j stands for the dimension of the representation j. Recall that $d_j = \chi_j(1)$.

It is further convenient to introduce the following symbol:

$$
\begin{aligned}
\chi_j &\leftrightarrow \langle \chi_j | \,, \\
C_\alpha &\leftrightarrow |\alpha\rangle .
\end{aligned}
\tag{5.15}
$$

Then, the first, and the second orthogonality relation of irreducible characters are expressed in the following form:

$$
\begin{aligned}
\langle \alpha | \beta \rangle = |G| \delta^\alpha_\beta \,, \qquad & \langle \chi_j | \chi_k \rangle = \delta^j_k \,, \\
\frac{1}{|G|} \sum_\alpha |\alpha\rangle \langle \alpha| = \mathbf{1} \,, \qquad & \sum_j |\chi_j\rangle\langle\chi_j| = \mathbf{1} \,.
\end{aligned}
\tag{5.16}
$$

Note that

$$
\langle \chi_j | \hat{\alpha} \rangle = \langle \alpha | \chi_j \rangle \,,
\tag{5.17}
$$

since $\langle \chi_j | \hat{\alpha} \rangle = \chi_j(C_{\hat{\alpha}}) = \chi_j(C_\alpha)^* = \langle \alpha | \chi_j \rangle$. Thus, we have the following expression for $\eta_{\alpha\beta}$, $N_{\alpha\beta\gamma}$, and $N_{\alpha\beta}{}^\gamma$:

$$
\begin{aligned}
\eta_{\alpha\beta} &= \sum_j \langle \chi_j | \alpha \rangle \langle \chi_j | \beta \rangle \\
&= \langle \hat{\beta} | \alpha \rangle = \langle \hat{\alpha} | \beta \rangle \,, \\
N_{\alpha\beta\gamma} &= \sum_j \frac{\langle \chi_j | \alpha \rangle \langle \chi_j | \beta \rangle \langle \chi_j | \gamma \rangle}{\langle \chi_j | 0 \rangle} \,, \\
N_{\alpha\beta}{}^\gamma &= \frac{1}{|G|} \sum_j \frac{\langle \chi_j | \alpha \rangle \langle \gamma | \chi_j \rangle \langle \chi_i | \beta \rangle}{\langle \chi_j | 0 \rangle} \,.
\end{aligned}
\tag{5.18}
$$

Here we denote the conjugacy class of identity by 0; $C_0 = 1$, and so we have $d_j = \langle \chi_j | 0 \rangle$.

We now can calculate all correlation functions explicitly. Following the prescription given in the previous section, we first introduce the regular representation (ϱ, W) of
$$Z(R) = \bigoplus_{\alpha=0}^{K} \mathbf{C}C_{\alpha};^{6}$$

$$\varrho(C_\alpha)^\gamma{}_\beta \equiv N_{\alpha\beta}{}^\gamma . \tag{5.19}$$

We then get the following formula:
$$\langle \mathscr{O}_{\alpha_1} \ldots \mathscr{O}_{\alpha_n} \rangle_g = \langle \mathscr{O}_{\alpha_1} \ldots \mathscr{O}_{\alpha_n} H^{g-1} \rangle_1$$
$$= \mathrm{tr}_W \, \varrho(C_{\alpha_1}) \ldots \varrho(C_{\alpha_n}) \varrho(H)^{g-1} . \tag{5.20}$$

Here, $\varrho(H) = w^\alpha \varrho(C_\alpha)$ is calculated as
$$\varrho(H)^\gamma{}_\beta = N^{\alpha\delta}{}_\delta N_{\alpha\beta}{}^\gamma$$
$$= \frac{1}{|G|} \sum_j \frac{\langle \gamma \,|\, \chi_j \rangle \langle \chi_j \,|\, \beta \rangle}{\langle \chi_j \,|\, 0 \rangle^2} , \tag{5.21}$$

and, by substituting this equation into Eq. (5.20) and using Eq. (5.16), we finally obtain
$$\langle \mathscr{O}_{\alpha_1} \ldots \mathscr{O}_{\alpha_n} \rangle_g = \sum_j \frac{\langle \chi_j \,|\, \alpha_1 \rangle \ldots \langle \chi_j \,|\, \alpha_n \rangle}{\langle \chi_j \,|\, 0 \rangle^{2g-2+n}}$$
$$= \sum_j \frac{\chi_j(C_{a_1}) \ldots \chi_j(C_{\alpha_n})}{d_j{}^{2g-2+n}} . \tag{5.22}$$

This has the same form as Witten's result calculated by using continuous TFT [7].

6. Moduli of TFT's and Their Perturbation

In this section, we investigate the moduli space of TFT's. In particular, we show that every TFT can be obtained by perturbation from the *standard topological field theory* (STFT) to be defined below. The following discussions are inspired by [13].

6.1. Standard Basis and Standard Topological Field Theory. As has been shown in preceding sections, a TFT with $(K + 1)$ independent physical operators has been realized by a commutative algebra $\tilde{R}$ of dimension $(K + 1)$, which can be regarded as the center of an associative algebra R in our lattice language; $\tilde{R} = Z(R)$. In particular, the physical operators $\mathscr{O}_\alpha$ correspond to a basis ϕ_α of $\tilde{R}$. In the following, we further investigate these correspondences in order to introduce the concept of the moduli of TFT's.

We again consider the regular representation (ϱ, W) of $\tilde{R}$; $\varrho(\phi_\alpha)^\gamma{}_\beta = N_{\alpha\beta}{}^\gamma$ with $\varrho(\phi_\alpha)\varrho(\phi_\beta) = N_{\alpha\beta}{}^\gamma \varrho(\phi_\gamma)$. Since $\tilde{R}$ is commutative, the following relation holds:
$$\varrho(\phi_\alpha)\varrho(\phi_\beta) = \varrho(\phi_\beta)\varrho(\phi_\alpha) , \tag{6.1}$$

---

[6] $(K + 1)$ is the dimension of center, and equal to the number of conjugacy classes, which is also equal to the number of irreducible representations, as is clear from the orthogonality relations of irreducible characters (5.16)

from which we know that the $\varrho(\phi_\alpha)$'s ($\alpha = 0, 1, \ldots, K$) are simultaneously diagonalizable:

$$\varrho(\phi_\alpha) = \begin{bmatrix} \lambda_0^{(\alpha)} & & 0 \\ & \ddots & \\ 0 & & \lambda_K^{(\alpha)} \end{bmatrix}, \tag{6.2}$$

that is, $N_{\alpha\beta}{}^\gamma = \lambda_\beta^{(\alpha)}\delta_\beta^\gamma$. Moreover, since $N_{\alpha\beta}{}^\gamma = N_{\beta\alpha}{}^\gamma$, we can further make a transformation of the basis in such a way that $N_{\alpha\beta}{}^\gamma$ has the following form:

$$N_{\alpha\beta}{}^\gamma = \lambda_\alpha \delta_\alpha^\gamma \delta_\beta^\gamma. \tag{6.3}$$

Thus, the physical operators $\{\mathscr{O}_\alpha\}$ now have the following OPE [13]:

$$\mathscr{O}_\alpha \mathscr{O}_\beta = \lambda_\alpha \delta_{\alpha\beta} \mathscr{O}_\alpha. \tag{6.4}$$

Let $\mathscr{M}_{\mathrm{TFT}}$ be the moduli space of TFT's, which is nothing but the set of all commutative algebras. For the physical operators of *almost all* TFT's in $\mathscr{M}_{\mathrm{TFT}}$, all the λ_α's in Eq. (6.4) have nonvanishing values. Thus, by properly normalizing $\mathscr{O}_\alpha$, we have the following OPE:

$$\mathscr{O}_\alpha \mathscr{O}_\beta = \delta_{\alpha\beta} \mathscr{O}_\alpha. \tag{6.5}$$

We will call $\{\mathscr{O}_\alpha\}$ with this OPE the standard basis of the TFT we consider. Since this form of OPE completely determines the basis $\{\phi_\alpha\}$ up to their permutation, and any correlation functions are uniquely calculated from their one-point functions on sphere;

$$v_\alpha \equiv \langle \mathscr{O}_\alpha \rangle_0, \tag{6.6}$$

we now know [13] that $\mathscr{M}_{\mathrm{TFT}}$ is parametrized by the number $(K + 1)$ of physical operators (the dimension of the algebra $\tilde{R}$) and their one-point functions $\{v_\alpha\}$ ($\alpha = 0, 1, \ldots, K$). Note that for this standard basis, the handle operator H [Eq. (4.11)] is expressed as

$$H = \sum_\alpha \frac{\mathscr{O}_\alpha}{v_\alpha}. \tag{6.7}$$

We, in particular, call the TFT where $v_\alpha \equiv 1$ ($\alpha = 0, 1, \ldots, K$) the Kth *standard topological field theory* (STFT).

Example. $R = \mathbf{C}[G]$. We follow the notation in Sect. 5: G is a finite group, and α (resp. j) labels conjugacy classes (resp. irreducible representations) of G. We can always construct the standard basis in the LTFT correponding to $R = \mathbf{C}[G]$, group ring of G. In fact, if we make a transformation of basis as

$$\bar{O}_j \equiv \frac{d_j}{|G|} \sum_\alpha \langle \alpha | \chi_j \rangle \mathscr{O}_\alpha$$

$$= \frac{d_j}{|G|} \sum_\alpha \chi_j(C_\alpha)^* \mathscr{O}_\alpha, \tag{6.8}$$

then $\{\bar{O}_j\}$ satisfies the following OPE:

$$\bar{O}_j \bar{O}_k = \delta_{jk} \bar{O}_j. \tag{6.9}$$

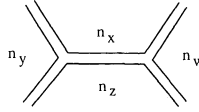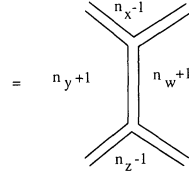**Fig. 17**   **Fig. 18**

Fig. 17. There are five traingles around a vertex x, $n_x = 5$
Fig. 18. Invariance under fusion transformation

The one-point functions are easily calculated to be found

$$v_j \equiv \langle \bar{O}_j \rangle_0 = d_j^2 \,. \tag{6.10}$$

Recall here that $d_j \equiv 1$ $(\forall j)$ for commutative groups. Thus, K^{th} STFT can be realized by the LTFT that corresponds to a group ring $R = \mathbf{C}[G]$ of commutative group G with order $|G| = K + 1$.

6 2. Perturbation of TFT. In this and the next subsections, we show that every TFT can be obtained from STFT by perturbation. In particular, we see that the TFT's which have vanishing λ_α for some α can also be expressed by this perturbation.

Suppose that we have chosen a TFT, and let us perturb it by adding δS to the original action. Perturbed correlation functions to be denoted with prime are thus calculated by inserting the operator $\exp(-\delta S)$ into the original correlation function:

$$\langle \dots \rangle' \equiv \langle \dots e^{-\delta S} \rangle \,. \tag{6.11}$$

In the following, we show that the possible form of δS can be determined automatically if we require its locality and topological property. We first fix a triangulation T_g. Then locality condition leads to the following form of δS:

$$\delta S = \sum_\alpha \sum_x f_\alpha(n_x) \mathscr{O}_\alpha \,, \tag{6.12}$$

where x parametrizes vertices in the triangulation, and n_x stands for the number of triangles around the vertex x (see Fig. 17). [7] Then, by the invariance of $\exp(-\delta S)$ under the fusion and bubble transformations, $f_\alpha(n_x)$ is determined to have the form

$$f_\alpha(n_x) = A_\alpha(n_x - 6) \quad (A_\alpha: \text{constant}) \,, \tag{6.13}$$

which implies that $f_\alpha(n_x)$ is proportional to the deficit angle around the vertex x.

Proof. First, the invariance under the fusion transformation (Fig. 18) yields the following identity:

$$\begin{aligned}
f_\alpha(n_x) &+ f_\alpha(n_y) + f_\alpha(n_z) + f_\alpha(n_w) \\
&= f_\alpha(n_x - 1) + f_\alpha(n_y + 1) + f_\alpha(n_z - 1) + f_\alpha(n_w + 1) \,,
\end{aligned} \tag{6.14}$$

---

[7] In the language of continuum theory, this ansatz corresponds to requiring that δS has the following form:

$$\sum_\alpha \int d^2 x \sqrt{g} f_\alpha(R) \mathscr{O}_\alpha(x)$$
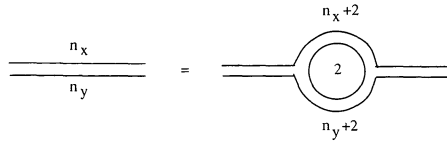
with R scalar curvature

Fig. 19. Invariance under bubble transformation

from which $f_\alpha(n_x)$ is known to be a linear function of n_x; $f_\alpha(n_x) = A_\alpha n_x + B_\alpha$. Next, from the invariance under the bubble transformation, we have the relation (see Fig. 19)

$$f_\alpha(n_x) + f_\alpha(n_y) = f_\alpha(n_x + 2) + f_\alpha(n_y + 2) + f_\alpha(2), \qquad (6.15)$$

which gives $B_\alpha = -6A_\alpha$. [Q.E.D.]

Thus, by setting $A_\alpha = (1/12)\mu_a$, we have[8]

$$e^{-\delta S} = \exp\left(-\frac{1}{12} \sum_\alpha \mu_\alpha \sum_x (n_x - 6)\mathcal{O}_\alpha \right). \qquad (6.16)$$

If we insert this operator into genus-g correlation function $\langle \ldots \rangle_g$, we then obtain

$$e^{-\delta S} = \exp\left\{ (1 - g) \sum_\alpha \mu_\alpha \mathcal{O}_\alpha \right\}. \qquad (6.17)$$

since $\mathcal{O}_\alpha$, which corresponds to an element of commutative algebra, is independent of its location. We here also used the Gauss-Bonnet theorem: $\sum_x (n_x - 6) = -12(1 - g)$. In particular, if $\{\mathcal{O}_\alpha\}$ is the standard basis of the TFT we consider, then we have

$$e^{-\delta S} = 1 + \sum_\alpha (e^{(1-g)\mu_\alpha} - 1)\mathcal{O}_\alpha \qquad (6.18)$$

in genus-g correlation functions.

Now we have the general form of the perturbation operator $\exp(-\delta S)$, it is easy to see that every TFT can be obtained from STFT by perturbation. In fact, we have the following formula for the standard basis of STFT [13]:

$$\begin{aligned}
v'_\alpha &\equiv \langle \mathcal{O}_\alpha \rangle'_0 \\
&= \langle \mathcal{O}_\alpha e^{-\delta S} \rangle_0 \\
&= e^{\mu_\alpha} \langle \mathcal{O}_\alpha \rangle_0 \\
&= e^{\mu_\alpha} v_\alpha .
\end{aligned} \qquad (6.19)$$

Thus, if we, in particular, start from STFT where $v_\alpha \equiv 1$, we then have $v'_\alpha = e^{\mu_\alpha}$, and so can obtain all values of v'_α by adjusting the parameters μ_α. On the other hand, the form of OPE is preserved under perturbation. Therefore, we know that every TFT which can have standard basis is obtained from STFT by perturbation. Moreover, as will be shown in the following examples, TFT's which do *not* have the standard basis

---

[8] This corresponds to

$$\delta S = -\frac{1}{2} \sum_\alpha \mu_\alpha \int d^2x \sqrt{g} R\mathcal{O}_\alpha(x),$$

the form of which is the same with the one given in [13]

are also obtained from STFT by perturbations in a suitable limit of the perturbation parameters μ_α and with infinite renormalization of physical operators. In this sense, such TFT's live on the boundary of $\mathscr{M}_{\text{TFT}}$.

6.3. Examples. In the following, we consider some examples, and explain how to obtain those TFT's by perturbations which do not necessarily have the standard basis.

Example 1. TFT *associated with* $A_K^{(1)}$ WZW *of level* 1. Let ω be the primitive $(K+1)^{\text{th}}$ root of unity, and $\{\mathscr{O}_\alpha\}$ the standard basis of Kth STFT. If we make a transformation of the basis into the following form:

$$A_j \equiv \sum_{\alpha=0}^{K} \omega^{j\alpha} \mathscr{O}_\alpha \quad (j = 0, 1, \ldots, K), \tag{6.20}$$

then it is easy to see that the following OPE holds:

$$A_j A_k = A_{[j+k]} \tag{6.21}$$

with the one-point function

$$\langle A_j \rangle_0 = \sum_{\alpha=0}^{K} \omega^{j\alpha} = (K+1)\delta_{j,0}. \tag{6.22}$$

Here $[l]$ stands for l modulo $(K+1)$. Thus, we now know that the Kth STFT is nothing but the TFT associated with $A_K^{(1)}$ WZW of level 1.

Example 2. Twisted $N = 2$ minimal topological matter of level K. This theory is characterized by the following OPE and the vacuum expectation value of physical operators σ_j $(j = 0, 1, \ldots, K)$ [8,9]:

$$\sigma_j \sigma_k = \theta(j + k \leq K)\sigma_{j+k}, \tag{6.23}$$

$$\langle \sigma_j \rangle_0 = \delta_{j,K}. \tag{6.24}$$

What is special in this case is that we cannot introduce the standard basis in the commutative algebra corresponding to this theory, since Eq. (6.23) means that the matrix $\varrho(\sigma_j)$ in the regular representation has some vanishing eigenvalues. However, we can realize the theory as a limit of perturbed theory. In fact, if we define the operators $\sigma_j^{(\varepsilon)}$ from the operators A_j in Example 1 as

$$\sigma_j^{(\varepsilon)} \equiv \varepsilon^j A_j = \varepsilon^j \sum_{\alpha=0}^{K} \omega_{j\alpha} \mathscr{O}_\alpha, \tag{6.25}$$

and set the perturbation parameters μ_α as

$$e^{\mu_\alpha} = \varepsilon^{-K} \frac{1}{K+1} \omega^\alpha, \tag{6.26}$$

then we have the desired form of OPE and vacuum expectation values in the limit of $\varepsilon \to 0$:

$$\sigma_j^{(\varepsilon)} \sigma_k^{(\varepsilon)} = \theta(j + k \leq K)\sigma_{j+k}^{(\varepsilon)} + O(\varepsilon), \tag{6.27}$$

$$\langle \sigma_j^{(\varepsilon)} \rangle_0' = \delta_{j,K}. \tag{6.28}$$

We thus know that the twisted $N = 2$ minimal topological matter is obtained at the boundary of the moduli space $\mathscr{M}_{\text{TFT}}$.

7. Discussion

In this paper, we give the lattice definition of topological matter system, find its explicit solution and investigate physical consequences, with emphasis on the algebraic structure of lattice topological field theory.

What still remains to be invetigated is how to incorporate gravity, especially topological gravity, in our formalism. There seem to be the following two possibilities: (1) "Topological gravity can also be treated within our framework without any essential modification." In fact, gravity can also be regarded as a matter field if we expand metric $g_{\mu\nu}$ around a background metric $\hat{g}_{\mu\nu}$:

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \delta g_{\mu\nu}\,, \tag{7.1}$$

under some proper gauge condition on $\delta g_{\mu\nu}$. For example, in the conformal gauge gravitational quantum fluctuations are represented by the Liouville field, which is in turn regarded as a conformal matter on a Riemann surface with fixed background metric $\hat{g}_{\mu\nu}$ [4]. However, to go ahead in this direction, we need more machinery than we now have. In fact, we should set the dimension of $Z(R)$ to infinity ($K \to \infty$), since there are infinitely many physical observables in topological gravity [5]. Moreover, the Schwinger-Dyson equation of gravity [6] shows that its quantum theory only has the weak factorization property. That is, factorization of a surface along trivial cycles is necessarily accompanied by factorization along nontrivial cycles, while the topological matter system has the strong factorization property in the sense that the geometry can be factorized along any cycles independently. However, the limiting procedure of $K \to \infty$ requires some regularization, which might reduce the strong factorization property to the weak one.
(2) "Quantum fluctuations of gravity can only be described by summing over different geometries." If this is the case, the results obtained in this paper will not work directly for any quantum gravity. However, it will then be interesting to incorporate our lattice model in the Kontsevich model [14], and to investigate whether the resulting model is equivalent to the so-called generalized Kontsevich model given in [15] (see also [16]).

Investigtions along these two lines would be interesting.

Note added in proof. After submitting this paper, we were informed by Costas Bachas and Marios Petropoulos that they had also proposed the classification of lattice topological field theories in terms of associative algebras, and identified the physical operators of a given lattice model with the central elements of the corresponding associative algebra [17]

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