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Fractal Wavelet Dimensions and Localization

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Abstract: In this paper we want to give a new definition of fractal dimensions as small scale behavior of the q-energy of wavelet transforms. This is a generalization of previous multi-fractal approaches. With this particular definition we will show that the 2-dimension (=correlation dimension) of the spectral measure determines the long time behavior of the time evolution generated by a bounded self-adjoint operator acting in some Hilbert space $\mathscr{H}$. It will be proved that for $\phi, \psi \in \mathscr{H}$ we have

$$\liminf_{T\to\infty} \frac{\log \int_0^T d\omega |\langle \psi | e^{-iA\omega}\phi\rangle|^2}{\log T} = -\kappa^+(2)$$

and that

$$\limsup_{T\to\infty} \frac{\log \int_0^T d\omega |\langle \psi | e^{-iA\omega}\phi\rangle|^2}{\log T} = -\kappa^-(2)\,,$$

where $\kappa^\pm(2)$ are the upper and lower correlation dimensions of the spectral measure associated with ψ and ϕ. A quantitative version of the RAGE theorem shall also be given.

1. Introduction

Let μ be a finite (signed) measure. A well known theorem of Wiener states that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T d\omega |\hat{\mu}(\omega)|^2 = \sum_{x\in\mathbb{R}} |\mu\{x\}|^2\,,$$

where the Fourier transform is given by

$$\hat{\mu}(\omega) = \int d\mu(t)\, e^{-i\omega t}\,.$$

Note that the sum is finite since μ is finite. Now let A be a self-adjoint operator acting in some Hilbert-space $\mathscr{H}$. For any state $\phi \in \mathscr{H}$ we shall be interested in the long time behavior of the unitary evolution $\phi \to e^{-itA}\phi$. More precisely look at ($\psi \in \mathscr{H}$)

$$\frac{1}{T}\int_0^T dt|\langle\psi|e^{-itA}\phi\rangle|^2 \quad \text{as } T \to \infty \,. \tag{1.1}$$

By the usual functional calculus the above integral is equal to

$$\frac{1}{T}\int_0^T d\omega\,|\hat{\mu}_{\psi,\phi}(\omega)|^2 \,,$$

where $\mu_{\psi,\phi}$ is the spectral measure associated with ψ and ϕ. Therefore in the case of an operator having only pure point-spectrum, the long time behavior of the averaged time-evolution (1.1) of a non-zero ϕ is given by Wieners theorem[1]:

$$\frac{1}{T}\int_0^T dt\,|\langle\psi|e^{-itA}\phi\rangle|^2 \sim 1 \quad (T \to \infty) \,.$$

On the other hand let $\phi \in \mathscr{H}_{\text{cont}}$ belong to the continuous spectrum of A (see [1] for the notation). Then the celebrated RAGE theorem (see e.g. [1]) states that for any compact operator B we have

$$\frac{1}{T}\int_0^T dt\,\|Be^{-itA}\phi\|^2 \to 0 \quad (\text{as } T \to \infty) \,.$$

In particular upon setting B: $\varphi \to \langle\psi|\varphi\rangle\psi$ we see that the mean evolution (1.1) for $\phi \in \mathscr{H}_{\text{cont}}$ is given by

$$\frac{1}{T}\int_0^T dt\,|\langle\psi|e^{-itA}\phi\rangle|^2 \to 0 \quad (T \to \infty) \,. \tag{1.2}$$

In this paper now we are concerned with a quantitative version of the last equation. It will turn out that the speed of decay towards 0 is determined by the fractal correlation dimension of the spectral measure.

The rest of the paper is organized as follows. We start by presenting our main analysis tool, which is the wavelet transform. After that we introduce the wavelet-dimensions for any tempered distribution. The last part considers the relation between the correlation dimension and the averaged time evolution.

2. Introduction to Wavelet Transforms

Consider the Schwarz space $S(\mathbb{R})$ that consists of those functions that together with all their derivatives decay at infinity faster than any polynomial. A topology is induced by the semi-norms

$$\|s\|_{S(\mathbb{R}),n,m} = \sup_{t\in\mathbb{R}} |t^n \partial_t^m s(t)| \,.$$

Let $S_+(\mathbb{R})$ be the subset of those functions in $S(\mathbb{R})$ whose Fourier transform is supported by the positive frequencies only. It is a closed subspace of $S(\mathbb{R})$ and we

[1] We write $s(t) \sim r(t)$ if there is a $c > 0$ such that $c < s(t)/r(t) < 1/c$ for all (considered) t

equip it with the topology induced by $S(\mathbb{R})$. Since the Fourier transform of any function $s \in S_+(\mathbb{R})$ is smooth at $\omega = 0$ and identically vanishing for $\omega \leqq 0$ we have that all moments for s vanish

$$\hat{s}(\omega) = O(\omega^n)(\omega \to 0) \Leftrightarrow \int_{-\infty}^{+\infty} dt\, t^n s(t) = 0 \quad \text{for all } n \in \mathbb{N}_0 . \tag{2.1}$$

The wavelet transform [2, 3, 4] of $s \in S_+(\mathbb{R})$ with respect to the wavelet $g \in S_+(\mathbb{R})$ is given by

$$\mathcal{W}_g s(b, a) = \int_{-\infty}^{+\infty} dt \frac{1}{a} \bar{g}\left(\frac{t-b}{a}\right) s(t) = \int_{-\infty}^{+\infty} dt\, \bar{g}_{b,a}(t)\, s(t), \quad a > 0, b \in \mathbb{R} .$$

It can easily be shown (e.g. [4, 5]), that for $s, g \in S_+(\mathbb{R})$ the wavelet transform of $s \in S_+(\mathbb{R})$ with respect to $g \in S_+(\mathbb{R})$ is a highly localized function over the half-plane $\mathbb{H} = \mathbb{R} \times \mathbb{R}^+$. More precisely let $S(\mathbb{H})$ be the space of all functions $\mathcal{T}$ over the half-plane that are localized such that

$$\| \mathcal{T}(b, a) \|_{S(\mathbb{H}), n} = \sup_{(b,a) \in \mathbb{H}} | \mathcal{T}(b, a) | (a + 1/a)^n (1 + |b|)^n < \infty \quad \text{for all } n > 0 .$$

These are actually semi-norm and $S(\mathbb{H})$ is a Fréchet space with the topology they induce. The wavelet transform is then a continuous map from $S_+(\mathbb{R})$ to $S(\mathbb{H})$.

Now consider the wavelet synthesis $\mathcal{M}_h$: $S(\mathbb{H}) \to S_+(\mathbb{R})$ with respect to $h \in S_+(\mathbb{R})$. It is point-wise defined for $\mathcal{T} \in S(\mathbb{H})$ by

$$\mathcal{M}_h \mathcal{T}(t) = \int_0^\infty \frac{da}{a} \int_{-\infty}^{+\infty} db\, \mathcal{T}(b, a) \frac{1}{a} h\left(\frac{t-b}{a}\right) .$$

The wavelet synthesis is again a continuous map. It actually is essentially the inverse of the wavelet transform:

$$c_{g,h}^{-1} \mathcal{M}_h \mathcal{W}_g = \mathbb{1}, \quad s(t) = c_{g,h}^{-1} \int_0^\infty \frac{da}{a} \int_{-\infty}^{+\infty} db\, \mathcal{W}_g s(b, a) \frac{1}{a} h\left(\frac{t-b}{a}\right) \tag{2.2}$$

with any function $h \in S_+(\mathbb{R})$ satisfying

$$c_{g,h} = \int_0^\infty \frac{d\omega}{\omega} \hat{h}(\omega) \bar{\hat{g}}(\omega) \quad 0 < |c_{g,h}| < \infty . \tag{2.3}$$

Such a function h is called a reconstruction wavelet for g. In particular every non zero wavelet in $S_+(\mathbb{R})$ is its own reconstruction wavelet up to some constant trivial factor.

Upon reconstructing with g and analyzing with some other function h we obtain a simple relation between the wavelet transform with respect to h and the one with respect to g,

$$\mathcal{W}_h s(b, a) = \int_0^\infty \frac{da'}{a'} \int_{-\infty}^{+\infty} db' \frac{1}{a'} \Pi_{g,h}\left(\frac{b-b'}{a'}, \frac{a}{a'}\right) \mathcal{W}_g s(b', a') , \tag{2.4}$$

with $\Pi_{g,h}(b, a) = c_{g,h}^{-1} \mathcal{W}_h g(b, a)$. For $g = h$ this is the so-called reproducing kernel equation.

The Wavelet Transform of Distributions. Let η be a distribution in $S'(\mathbb{R})$. By the usual duality approach we can define the wavelet transform with respect to $g \in S_+$ via its action on $\mathscr{T} \in S(\mathbb{H})$,

$$(\mathscr{W}_g \eta)(\mathscr{T}) = \frac{1}{c_{g,h}} \eta(\mathscr{M}_h \mathscr{T}) .$$

It can be shown [5], that we may identify $\mathscr{W}_g \eta$ with a function over the half-plane given point-wise by

$$\mathscr{R}(b, a) = \eta(\bar{g}_{b,a}) .$$

The identification is made through the (absolutely convergent) "scalar product"

$$(\mathscr{W}_g \eta)(\mathscr{T}) = \langle \mathscr{R} | \mathscr{T} \rangle_{\mathbb{H}} = \int_0^\infty \frac{da}{a} \int_{-\infty}^{+\infty} db\, \bar{\mathscr{R}}(b, a)\, \mathscr{T}(b, a) .$$

The function $\mathscr{R}$ is a smooth function over the half-plane of at most polynomial growth in $b/(a + 1/a)$ as $|b|$ or $a + 1/a \to \infty$. For simplicity we shall denote this function $\mathscr{R}$ over the half-plane again by $\mathscr{W}_g \eta$. The action of $\eta \in S'(\mathbb{R})$ on $s \in S_+(\mathbb{R})$ may now be written as an absolutely convergent integral over the half-plane [5],

$$\eta(s) = c_{g,h}^{-1} \langle \mathscr{W}_g \eta | \mathscr{W}_h s \rangle_{\mathbb{H}}$$

and Eq. (2.4) is still point-wise valid in this case. For further reference we note that in Fourier space we have (in the sense of distributions)

$$\mathscr{W}_g \eta(b, a) = \frac{1}{2\pi} \int_0^\infty d\omega\, \bar{\hat{g}}(a\omega)\, e^{ib\omega} \hat{\eta}(\omega) .$$

This shows in particular that if $\hat{\eta}$ is a function and $\hat{\eta}(\omega) = O(\omega^n)$ as $\omega \to 0$ then

$$\mathscr{W}_g \eta(b, a) = O(1/a^{n+1}) \quad (a \to \infty) \tag{2.5}$$

uniformly in b.

Wavlet Analysis of Local Regularity. The wavelet transform may be seen as a sort of mathematical microscope [6, 7] whose position is given by b whereas a is the length-scale at which the object is examined. In particular it has been proved to be very useful in characterizing the local fractality (= local regularity) of functions [8] or even arbitrary distributions. For instance global Hölder regularity of degree α is characterized by a uniform decrease of the wavelet coefficients at small scale:

$$s(t) - s(u) = O(|t - u|^\alpha), \text{ with } 0 < \alpha < 1 \Leftrightarrow \mathscr{W}_g s(b, a) = O(a^\alpha) . \tag{2.6}$$

If $\alpha > 1$ and $n < \alpha < n + 1$ with $n \in \mathbb{N}$ then one has to replace s by ∂_t^n and the statement is still valid. But even point-wise information is available. Suppose now that s satisfies at some point t_0 at

$$s(t_0 + t) - s(t_0) = P_n(t) + O(t^\alpha), \quad (t \to 0)$$

with some polynomial P_n of degree n and some $\alpha \in (n, n + 1]$. Then

$$\mathscr{W}_g s(t_0 + b, a) = O(a^\alpha + b^\alpha) . \tag{2.7}$$

Vice versa if one knows that globally the function is of Hölder regularity with some arbitrary small regularity exponent $\varepsilon > 0$ and if the wavelet transform satisfies (2.7), with $\alpha \notin \mathbb{N}_0$, then there is a polynomial P_n of degree n such that [8, 9, 10]:

$$s(t_0 + t) - s(t_0) = P_n(t) + O(\log t\, t^\alpha)\ .$$

Thus the small-scale behavior of the wavelet transform reflects well the local fractality (= local regularity).

A Functional Calculus. Although we shall not make direct use of the results listed below we state them as instructive motivation for the rest of the paper.

Consider a self-adjoint operator A acting in some Hilbert space and let $R_z = (A - z)^{-1}$ be the resolvent of A. By the standard functional calculus

$$a^{n-1} R_z^n(A) = \frac{1}{a} \int \frac{dE_\lambda}{\left(\dfrac{\lambda - b}{a} + i\right)^n}, \quad (z = b + ia)\ ,$$

with dE_λ being the spectral family. Thus the powers of the resolvent can be seen as a wavelet transform of the spectral family with respect to the wavelet $h(t) = (t + i)^{-n}$ or by taking matrix elements as wavelet transform of the spectral measures $d\mu_{\phi,\psi}(\lambda) = \langle \phi | dE_\lambda \psi \rangle$.

$$\langle \psi | R_{b+ia}^n(A) \phi \rangle = a^{1-n} \mathcal{W}_h d\mu_{\phi,\psi}(b, a)\ .$$

The half-plane of the wavelet transform is now the complex-half plane where the imaginary part corresponds to the analyzed scale and the real part to the position of the mathematical microscope. Note however that h is not in $S_+(\mathbb{R})$ but this actually poses no extra difficulties.

Upon replacing t by A in the inversion formula (2.2) one might hope to get an expression for $s(A)$. This can actually be established in a mathematically correct way [11]. In particular it has been proved that in a weak sense we have for $n \geqq 2$, $g \in S_+(\mathbb{R})$ and A a self-adjoint operator (upon taking $s(t) = e^{-iTt}$)

$$e^{-iTA} = \frac{2\pi}{ic} \int_0^\infty da\, a^{n-1} \int_{-\infty}^{+\infty} db\, \bar{g}(aT) e^{-iTb} R_{b+ia}^n(A)$$

with $c = \int_0^\infty d\omega\, \omega^{n-2} \bar{\hat{g}}(\omega) e^{-\omega}$ and the integral over the half-plane is understood as $\lim_{\rho \to \infty} \int_{1/\rho}^{\rho} \frac{da}{a} \int_{-\rho}^{\rho} db$. This shows explicitly how the behavior for large T is determined by the behavior of R_z at $\Im z \sim 1/T$. Indeed suppose that the support of $\hat{g}$ is contained in an interval around $\omega = 1$, then the integral over the complex z half-plane actually runs only over a strip around $\Im z \sim 1/T$. Thus the long time behavior $t \to \infty$ of the time evolution is linked to the small scale behavior of some wavelet transform of the spectral measure, which in turn as we have seen is closely connected with local fractal properties (= local regularity properties) of the measure itself. This relation can actually be quantified as we shall see in the next sections.

3. The Definition of the Wavelet Dimensions

From now on we are only interested in local properties therefore we suppose that all analyzed distributions $\eta \in S'(\mathbb{R})$ are well behaved at infinity. More precisely we require that for all $s \in S(\mathbb{R})$ we have $s^*\eta \in S(\mathbb{R})$. This condition shall be assumed throughout. It is of a purely technical nature and may be relaxed considerably from case to case. In particular note that the Fourier transform of such a distribution actually is a smooth function (however it is not localized). In addition we assume that $\eta \neq 0$.

For every scale a we look at the mean q-energy at scale a,

$$G_g(a, q) = \| \mathscr{W}_g(\cdot, a) \|_q^q = \int_{-\infty}^{+\infty} db \, |\mathscr{W}_g \eta(b, a)|^q \quad \text{with } q \geqq 1.$$

Note that for $q = 2$ this actually is an energy since (e.g. [2])

$$\int_{-\infty}^{+\infty} dt \, |s(t)|^2 = \frac{1}{c_{g,g}} \int_0^{\infty} \frac{da}{a} \int_{-\infty}^{+\infty} db \, |\mathscr{W}_g s(b, a)|^2 \, . \tag{3.1}$$

At small scale a scaling behavior of the form $G(a, q) \sim a^{\kappa(q)}$ can be observed – at least for affine self-similar measures as e.g. the triadic Cantor set with Bernoulli measure [12] – giving rise to the definition of the fractal dimensions $\kappa(q)$. However we shall use a slightly modified definition. We set

$$\Gamma_g(a, q) = \int_a^1 \frac{d\alpha}{\alpha} G_g(\alpha, q) \, .$$

For every $q \geqq 1$ this is a monotone function of a. Therefore the limit $a \to 0$ exists, but may be infinite. This will always be the case if η is singular enough. In the opposite case when this limit is finite we subtract the constant $\int_0^1 \frac{d\alpha}{\alpha} G(\alpha, q)$ and we rather set

$$\Gamma_g(a, q) = \int_0^a \frac{d\alpha}{\alpha} G_g(\alpha, q) \, .$$

To summarize we have

$$\Gamma_g(a, q) = \min \left\{ \int_a^1 \frac{d\alpha}{\alpha} G_g(\alpha, q), \int_0^a \frac{d\alpha}{\alpha} G_g(\alpha, q) \right\} .$$

Note that $G_g(a, q) \sim a^{\kappa(q)}$ implies $\Gamma_g(a, q) \sim a^{\kappa(q)}$ unless $\kappa(q) = 0$. The generalized dimensions $\kappa^+(q)$ and $\kappa^-(q)$ are now defined as follows:

$$\kappa^+(q) = \limsup_{a \to 0} \frac{\log \Gamma_g(a, q)}{\log a}, \quad \kappa^-(q) = \liminf_{a \to 0} \frac{\log \Gamma_g(a, q)}{\log a} \, . \tag{3.2}$$

In the case that $\eta = 0$ we set $\kappa^\pm(q) = 0$. We shall refer to these numbers as the upper and lower q-wavelet dimensions. Note that in this form the dimensions are defined for any distribution, in particular for measures and functions. This kind of integrated wavelet transform approach to define fractal dimensions was introduced in [12], where it also was shown that for affine self-similar measures these dimensions coincide with the generalized fractal dimensions defined in [13]. More precisely we have $\kappa(q) = \tau(q) - (q-1)$, where $\tau(q)$ is defined by the box-counting procedure as described in [13] (see also Proposition (5.5)). Later variations of this approach have been considered [14, 15].

The fractal dimensions $\kappa^{\pm}(q)$ are actually well defined as is shown by the following theorem.

(3.3) Theorem. *The dimensions $\kappa^{\pm}(q)$ do not depend on the analyzing wavelet $g \in S_+(\mathbb{R})$ for $q \geqq 1$, provided $g \neq 0$.*

Proof. We start by rephrasing expressions like (3.2). Only the first identity of this lemma is needed in this proof now, the rest however is for later reference.

(3.4) Lemma. *Let $s(t) > 0$ be a positive function defined for $t > 0$. Then we have*

$$\liminf_{t\to 0} \frac{\log s(t)}{\log t} = \sup\{\gamma \in \mathbb{R}: s(t) = O(t^\gamma), (t \to 0)\} ,$$

$$\limsup_{t\to 0} \frac{\log s(t)}{\log t} = \inf\{\gamma \in \mathbb{R}: t^\gamma = O(s(t)), (t \to 0)\} ,$$

$$\liminf_{t\to \infty} \frac{\log s(t)}{\log t} = \sup\{\gamma \in \mathbb{R}: t^\gamma = O(s(t)), (t \to \infty)\} ,$$

$$\limsup_{t\to \infty} \frac{\log s(t)}{\log t} = \inf\{\gamma \in \mathbb{R}: s(t) = O(t^\gamma), (t \to \infty)\} .$$

Proof. (Although this is well known we give the proof anyway.) Consider the first equation. Call the left-hand side α and the right-hand side β. For any $\gamma < \beta$ there is a constant $c > 0$ such that for t small enough we have

$$s(t) \leqq ct^\gamma .$$

Therefore for $0 < t < 1$ we have

$$\frac{\log s(t)}{\log t} \geqq \frac{c}{\log t} + \gamma ,$$

and thus upon passing to the limit $t \to 0$ we get $\alpha \geqq \gamma$ and since $\gamma < \beta$ was arbitrary we have

$$\alpha \geqq \beta .$$

On the other hand for every $\gamma > \beta$ there is a sequence $t_n \to 0$ such that

$$t_n^{-\gamma} s(t_n) \to \infty , \quad (n \to \infty) .$$

Therefore $(\log s(t_n) - \gamma \log t_n) \to \infty$ and thus in particular if n is large enough

$$\log s(t_n) > \gamma \log t_n .$$

Therefore we have

$$\alpha = \liminf_{t\to 0} \frac{\log s(t)}{\log t} \leqq \liminf_{t_n\to 0} \frac{\log s(t_n)}{\log t_n} \leqq \gamma .$$

Since $\gamma > \beta$ was arbitrary we have $\alpha \leqq \beta$, and we are done. The other formulas are proved in the same way, and we leave them to the reader. □

Therefore to prove the independency of $\kappa^-(q)$ on g it is enough to show that $\Gamma_g(a, q) = O(a^\gamma)$ implies $\Gamma_h(a, q) = O(a^\gamma)$ for any pair of wavelets $g, h \in S_+(\mathbb{R})$. This shall be done now.

We start by modifying a little the definition of the partition function Γ_g. Here we have to distinguish two cases according to whether $\lim_{a\to 0} \int_a^1 \frac{d\alpha}{\alpha} G_g(\alpha, q)$ is infinite or not. Consider first the case that it is infinite. Note that we may suppose for all $m > 0$,

$$\hat{\eta}(\omega) \leqq O(\omega^m) \quad (\omega \to 0) . \tag{3.5}$$

Indeed the fractal dimensions of η and $\eta + s$ with $s \in S(\mathbb{R})$ are the same because of the fast decay of $\mathscr{W}_g s$ at small scale $a \to 0$ (see Eq. (2.6) for α arbitrarily large). Therefore if we chose some $\psi \in S(\mathbb{R})$ satisfying for all m at $\psi(\omega) = 1 + O(\omega^m)$ as $\omega \to 0$, we may replace η by $\eta - \psi * \eta$ without modifying the dimensions. Now this later distribution satisfies at (3.5). Therefore by (2.5) the wavelet transform decays fast as $a \to \infty$ and thus we may replace $\int_a^1$ by $\int_a^\infty$ in the definition of Γ_g; that is we may consider

$$\Gamma_g(a, q) = \int_a^\infty \frac{d\alpha}{\alpha} \| \mathscr{W}_g \eta(\cdot, \alpha) \|_q^q .$$

We now attack the estimates. Note that from Eq. (2.4) it follows that with

$$K_{a', a}(b) = \frac{1}{\alpha'} \Pi_{g,h}\left(\frac{b}{a'}, \frac{a}{a'}\right)$$

the passage from $\mathscr{W}_g\eta$ to $\mathscr{W}_h\eta$ reads

$$\mathscr{W}_h \eta(\cdot, a) = \int_0^\infty \frac{da'}{a'} K_{a', a} * \mathscr{W}_g \eta(\cdot, a') .$$

However we have to make sure that $K_{a',a}$ is well defined. The only possible obstruction to this is the constant $c_{g,h}$ as defined in (2.3) that may vanish. (Note that it is never ∞ for $g, h \in S_\pm(\mathbb{R})$.) However it cannot vanish for all the dilated and translated versions $g_{\beta,\alpha} = \alpha^{-1} g([\cdot - \beta]/\alpha)$ of g since this would merely mean that the wavelet transform of h with respect to g vanishes, which is impossible for $h, g \neq 0$. Now replacing g by one of its dilated and translated versions $g_{\beta,\alpha}$ amounts to replace $\mathscr{W}_g(b, a)$ by

$$\mathscr{W}_{g_{\beta,\alpha}} \eta(b, a) = \frac{1}{\alpha} \mathscr{W}_g \eta\left(\frac{b - \beta}{\alpha}, \frac{a}{\alpha}\right),$$

and therefore the dimensions computed with $g_{\beta,\alpha}$ instead of g are the same. We therefore may suppose that $c_{g,h} \neq 0$.

Using Minkowki's and Hölder's inequalities we now may write

$$\begin{aligned} \| \mathscr{W}_h \eta(\cdot, a) \|_q^q &\leqq \left\{ \int_0^\infty \frac{da'}{a'} \| K_{a',a} * \mathscr{W}_g \eta(\cdot, a') \|_q \right\}^q \\ &\leqq \left\{ \int_0^\infty \frac{da'}{a'} \| K_{a',a} \|_1 \, \| \mathscr{W}_g \eta(\cdot, a') \|_q \right\}^q . \end{aligned}$$

Now we have

$$\|K_{a',a}\|_1 = \int_{-\infty}^{+\infty} db \frac{1}{a'} \left| \Pi_{g,h}\left(\frac{b}{a'}, \frac{a}{a'}\right) \right| = H(a/a') ,$$

with

$$H(a) = \int_{-\infty}^{+\infty} db \, |\Pi_{g,h}(b, a)| .$$

This is a nonnegative function that is rapidly decaying as $a + 1/a$ gets large. It now comes using Jensen's inequality

$$\| \mathscr{W}_h \eta(\cdot, a) \|_q^q \leqq \left\{ \int_0^\infty \frac{da'}{a'} H(a/a') \right\}^{q-1} \int_0^\infty \frac{da'}{a'} H(a/a') \, \| \mathscr{W}_g \eta(\cdot, a') \|_q^q .$$

By the high localization of H the first integral is a finite constant and thus

$$\begin{aligned} \Gamma_h(\varepsilon, q) &\leqq O(1) \int_\varepsilon^\infty \frac{da}{a} \int_0^\infty \frac{da'}{a'} H(a/a') \;\; \| \mathscr{W}_g \eta(\cdot, a') \|_q^q \\ &= O(1) \int_0^\infty \frac{da'}{a'} H(1/a') \int_\varepsilon^\infty \frac{da}{a} \, \| \mathscr{W}_g \eta(\cdot, aa') \|_q^q \\ &= O(1) \int_0^\infty \frac{da'}{a'} H(1/a') \int_{\varepsilon a'}^\infty \frac{da}{a} \, \| \mathscr{W}_g \eta(\cdot, a) \|_q^q . \end{aligned}$$

Thus we have

$$\Gamma_h(\varepsilon, q) \leqq O(1) \int_0^\infty \frac{da}{a} H(\varepsilon/a) \, \Gamma_g(a, q) .$$

The same type of relation holds for g and h exchanged. Therefore the theorem will be now an immediate consequence of the next lemma that deals with such mean values.

(3.6) Lemma. *Let $s(t)$, $t > 0$, be a nonnegative, monotone (= either nondecreasing or nonincreasing) function of at most polynomial growth near* 0,

$$s(t) = O(t^{-m}), \quad (t \to 0), \quad \textit{for some } m > 0 .$$

Near ∞ we assume that it is bounded

$$s(t) = O(1), \quad t \geqq 1 .$$

Let further $H(t)$ be a nonnegative function that is arbitrarily well polynomially localized, namely

$$H(t) = O((t + 1/t)^{-n}), \quad \textit{for all } n > 0 .$$

Then for the mean values

$$r(t) = \int_0^\infty \frac{da}{a} H(t/a) \, s(a) ,$$

we have the following estimate:

$$\liminf_{t\to 0} \frac{\log r(t)}{\log t} = \liminf_{t\to 0} \frac{\log s(t)}{\log t},$$

$$\limsup_{t\to 0} \frac{\log r(t)}{\log t} = \limsup_{t\to 0} \frac{\log s(t)}{\log t}.$$

Proof. Suppose $s(t) \leqq O(t^\gamma)$ as $t \to 0$. Then

$$r(t) \leqq O(t^\gamma) \int_0^\infty \frac{da}{a} H(1/a) a^\gamma .$$

The integral on the right converges due to the high localization of H and thus $r(t) = O(t^\gamma)$ too as was to be shown for the first estimation.

We now come to the second estimation. Suppose that the right-hand side of the second inequality is ∞. In this case clearly there is nothing to prove. Therefore we may suppose now that there is a $\lambda > 0$ and a constant $c > 0$ such that

$$s(t) \geqq ct^\lambda, \quad (0 < t < 1). \tag{3.7}$$

We now pick ε, $0 < \varepsilon < 1$, and keep it fixed. For $0 < t < 1$ we split the integral defining $r(t)$ into three parts,

$$r(t) = \left\{ \int_0^{t^{1+\varepsilon}} + \int_{t^{1+\varepsilon}}^{t^{1-\varepsilon}} + \int_{t^{1-\varepsilon}}^{\infty} \right\} \frac{da}{a} H(t/a) s(a) = X_1 + X_2 + X_3 .$$

In the last term we may estimate $s(t) = O(1)$ and thus

$$X_3 \leqq O(1) \int_{1/t^\varepsilon}^{\infty} \frac{da}{a} H(1/a) .$$

Since $H(t)$ is arbitrarily well polynomially localized it follows that $X_3 = O(t^n)$ for all $n > 0$.

In X_1 we may estimate $s(t) \leqq t^{-m}$ and thus it comes

$$X_1 \leqq O(1) t^{-m} \int_0^{t^\varepsilon} \frac{da}{a} H(1/a) a^{-m} .$$

Since H is arbitrarily well polynomially localized the integral is rapidly decaying and thus again $X_1 = O(t^n)$ for all $n > 0$.

The main contribution remains which is the middle term X_2. Since $s(t)$ is monotone as $t \to 0$ we may estimate $s(a) \gtreqqless s(t^{1+\varepsilon})$ or $s(a) \gtreqqless s(t^{1-\varepsilon})$ for $t^{1+\varepsilon} \leqq a \leqq t^{1-\varepsilon}$ depending on whether s is nongrowing or nondecreasing. Therefore we end up with

$$X_2 \gtreqqless C' s(t^{1\pm\varepsilon}) \int_{t^{1+\varepsilon}}^{t^{1-\varepsilon}} \frac{da}{a} H(1/a) \quad \text{respectively} .$$

The last integral is again convergent and therefore $X_2 \geqq C'' s(t^{1\pm\varepsilon})$ for t small enough. Because of (3.7) it follows that this estimate also holds for the sum of the three contributions and thus for all $\varepsilon > 0$ we have

$$r(t) \gtreqqless C''' s(t^{1\pm\varepsilon}) \quad \text{respectively} .$$

Since ε was arbitrary the lemma is proved. □

Since we may exchange the roles of g and h the theorem is proved. The proof remains the same if $\lim_{a\to 0} \Gamma_g(s, q) < \infty$. □

Remark. The proof needed the high localization of the kernel $\Pi_{g,h}$, which in turn reflects that the wavelets must be very regular with all moments vanishing. However for a fixed q, and a given $\varepsilon > 0$ only some regularity is needed and some moments have to vanish in order to compute the dimension $\kappa^-(q)$ without error and the dimensions $\kappa^+(q)$ with error at most ε.

4. Localization and the Dimension $\kappa(2)$

In this section we want to establish a relation between the behavior to the Fourier transform of η and the fractal wavelet dimension $\kappa(2)$. Some evidence for such a relation has been given in [16]. This question is of relevance whenever one considers the long time behavior $T \to \infty$ of e^{-iTA}, where A is a self adjoint operator. Indeed in this case we have

$$\langle \psi | e^{-iTA} \phi \rangle = \hat{\mu}_{\psi,\phi}(T) ,$$

where $\mu_{\psi,\phi}$ is the spectral measure associated to ψ and ϕ. We are mainly interested in the time averages of the form

$$\int_0^T d\omega |\langle \psi | e^{-i\omega A} \phi \rangle|^2 = \int_0^T d\omega |\hat{\mu}_{\psi,\phi}(\omega)|^2 .$$

The next theorem shows how this time average is related to the dimension $\kappa^\pm(2)$. We will prove the following theorem not only for measures but also for essentially any tempered distribution.

(4.1) Theorem. *Let $\eta \in S'(\mathbb{R})$, $\eta \neq 0$, satisfy at $s * \eta \in S(\mathbb{R})$ for all $s \in S(\mathbb{R})$. Suppose that $\eta \notin L^2(\mathbb{R})$. Then $\kappa^-(2) \leqq 0$ and it follows that*

$$-\kappa^+(2) = \liminf_{T\to\infty} \frac{\log \int_0^T d\omega |\hat{\eta}(\omega)|^2}{\log T} \leqq \limsup_{T\to\infty} \frac{\log \int_0^T d\omega |\hat{\eta}(\omega)|^2}{\log T} = -\kappa^-(2) . \quad (4.2)$$

If $\eta \in L^2(\mathbb{R})$ then $\kappa^-(2) \geqq 0$ and the long-time behavior is trivial in the sense that the integral in (4.2) *tends to a finite constant. The speed of convergence is given by*

$$-\kappa^+(2) = \liminf_{T\to\infty} \frac{\log \int_T^\infty d\omega |\hat{\eta}(\omega)|^2}{\log T} \leqq \limsup_{T\to\infty} \frac{\log \int_T^\infty d\omega |\hat{\eta}(\omega)|^2}{\log T} = -\kappa^-(2) .$$

This shows that if the spectral measure of a bounded self-adjoint operator A has a certain 2 wavelet dimension then the long time behavior of e^{-iTA} is governed by this number. In particular for all $\gamma < -\kappa^+(2)$ there is a constant $c > 0$ such that for T large enough

$$\int_0^T d\omega |\langle \psi | e^{-i\omega A} \phi \rangle|^2 \geqq c T^\gamma .$$

This shows that the decay (1.2) may be very slow if the spectral measure has an upper 2-wavelet-dimension that is close to -1, which would be the 2-wavelet dimension of the pure point spectrum. For a similar relation see [17] and [18]. For a different approach see [19].

Proof. Let $g \in S_+(\mathbb{R})$ and consider $\mathcal{W}_g\eta$. As we may, we suppose that $\hat{g}$ is compactly supported. Again we may suppose in addition that $\hat{\eta}(\omega) = O(\omega^m)$ for all m and that therefore the wavelet transform decays fast at large scale.

Suppose $\eta \notin L^2(\mathbb{R})$. This implies (see (3.1)) that $\int_0^\infty \frac{da}{a} \int_{-\infty}^{+\infty} db\, |\mathcal{W}_g\eta(b, a)|^2 = \infty$ and thus

$$\Gamma_g(a, 2) = \int_a^\infty \frac{d\alpha}{\alpha} \int_{-\infty}^{+\infty} db\, |\mathcal{W}_g\eta(b, \alpha)|^2 .$$

We therefore consider now $\kappa^-(2) \leqq 0$ and thus $\int_\alpha^1 \frac{da}{a} \int_{-\infty}^{+\infty} db\, |\mathcal{W}_g\eta(b, a)|^2 \to \infty$ as $\alpha \to 0$. A direct application of Parsevals equation shows that we have

$$\int_{-\infty}^{+\infty} db\, |\mathcal{W}_g\eta(b, a)|^2 = \int_0^\infty d\omega\, |\hat{g}(a\omega)|^2\, |\hat{\eta}(\omega)|^2 .$$

Then let

$$H(\omega) = \int_\omega^\infty \frac{da}{a} |\hat{g}(a)|^2 = \int_1^\infty \frac{da}{a} |\hat{g}(a\omega)|^2 ,\ \omega > 0$$

By a simple exchange of integration we have

$$\begin{aligned}\int_{1/T}^\infty \frac{da}{a} \int_{-\infty}^{+\infty} db\, |\mathcal{W}_g\eta(b, a)|^2 &= \int_{1/T}^\infty \frac{da}{a} \int_0^\infty d\omega\, |\hat{g}(a\omega)|^2\, |\hat{\eta}(\omega)|^2 \\ &= \int_0^\infty d\omega\, H(\omega/T)\, |\hat{\eta}(\omega)|^2 .\end{aligned}$$

And finally

$$\int_0^\infty d\omega\, H(\omega/T)\, |\hat{\eta}(\omega)|^2 = \int_{1/T}^\infty \int_{-\infty}^{+\infty} db\, |\mathcal{W}_g\eta(b, a)|^2 = \Gamma_g(T^{-1}, 2) .$$

Since H is non-negative, $H(0) > 0$, and of compact support (since $\hat{g}$ is), we can find numbers $\lambda > 0$ and $\Lambda > 0$ such that

$$\lambda \chi_{[0, \lambda]}(\omega) \leqq H(\omega) \leqq \Lambda \chi_{[0, \Lambda]}(\omega) ,$$

where χ_I is the characteristic function of I. Therefore

$$\lambda \int_0^{\lambda T} d\omega\, |\hat{\eta}(\omega)|^2 \leqq \Gamma_g(T^{-1}, 2) \leqq \Lambda \int_0^{\Lambda T} d\omega\, |\hat{\eta}(\omega)|^2 ,$$

and the theorem follows.

Finally the proof for the case $\eta \in L^2(\mathbb{R})$ is the same we only have to use

$$\Gamma_g(a, 2) = \int_0^a \frac{da}{a} \int_{-\infty}^{+\infty} db\, |\mathcal{W}_g\eta(b, a)|^2 ,$$

and to adapt the limits of integration accordingly. □

As an easy corollary of this theorem we have the following quantitative version of the RAGE theorem. Let A be a bounded self-adjoint operator acting in some Hilbert space $\mathscr{H}$ and let dE_λ be its spectral family. Let further B be a Hilbert–Schmidt operator with singular decomposition

$$B\colon s \mapsto \sum_{n\in\mathbb{N}} \gamma_n \langle \phi_n | s \rangle \psi_n$$

with respect to the ortho-normal sets $\{\phi_n\}$ and $\{\psi_n\}$. Since

$$\| Be^{-i\omega A}\varphi \|^2 = \sum_{n\in\mathbb{N}} |\gamma_n|^2 |\langle \phi_n | e^{-i\omega A}\varphi \rangle|^2$$

we have

$$|\gamma_n|^2 \, |\langle \phi_n | e^{-i\omega A}\varphi \rangle|^2 \leqq \| Be^{-i\omega A}\varphi \|^2$$

and thus one can use Theorem (4.1) to obtain quantitative estimates for the speed of decay of

$$\frac{1}{T}\int_0^T d\omega \, \| Be^{-i\omega A}\varphi \|^2$$

in terms of the 2-wavelet dimensions of the spectral measures $d\mu_{\phi_n,\varphi}(\lambda) = \langle \phi_n | dE_{\lambda\varphi} \rangle$.

5. Appendix: Some Estimates and Explicit Formulas

We note some easy estimates of the dimensions $\kappa^-(q)$.

We start by considering the effect of deriving a distribution.

(5.1) Proposition. *Let $\eta \in S'(\mathbb{R})$ be a distribution such that for all $s \in S(\mathbb{R})$ we have $s * \eta \in S(\mathbb{R})$. Then if we replace η by $\partial\eta$ we have to replace $\kappa^\pm(q)$ by $\kappa(q)^\pm - q$.*

Proof. This follows from the identity

$$\mathscr{W}_{\partial g}\eta(b,a) = -\, a\mathscr{W}_g \partial\eta(b,a) \, . \tag{5.2}$$

If some global regularity of s is known we have a lower bound for the dimensions of s.

(5.3) Proposition. *If s is a function of Hölder regularity with exponent α*

$$|s(t) - s(u)| = O(|t-u|^\alpha) \, ,$$

then

$$\kappa^+(q) \geqq \kappa^-(q) \geqq q\alpha \, .$$

Proof. This follows from (2.7) and the rephrasing of the definition of the dimensions given in Lemma (3.4). □

Another estimate of this type is well adapted to the situation where the support of the distribution does not contain any interval but rather has some Cantor set-like structure. Here now both the fractality of the support together with the local regularity exponent may be used to obtain an estimate for $\kappa^-(q)$. As an example for what we mean by that look at δ and $\partial\delta$. Both are supported by the origin; however the former is more regular. This is clearly mirrored in the dimensions where the former gives $\kappa^{\pm}(q) = -q$, whereas the other is $\kappa^{\pm}(q) = -2q$.

First we need a suited definition of dimension for a set $\Lambda \subset \mathbb{R}$. For all $\varepsilon > 0$ consider the set of points at distance from Λ smaller than or equal to ε,

$$M(\Lambda, \varepsilon) = \{t \in \mathbb{R} : |t - u| \leqq \varepsilon \text{ for some } u \in \Lambda\} .$$

Suppose now that Λ is compact. Then M is of finite Lebesgue measures, denoted by $|M|$. The cover dimension $D(\Lambda)$ is now defined through the small scale behavior of $M(\Lambda, \varepsilon)$,

$$D(\Lambda) = \liminf_{\varepsilon \to 0} \frac{\log |M(\Lambda, \varepsilon)|}{\log \varepsilon} .$$

(5.4) Proposition. *Let $\eta \subset S'(\mathbb{R})$ have compact support $\Lambda \subset \mathbb{R}$. Suppose in addition that uniformly in b we have*

$$\mathscr{W}_g \eta(b, a) = O(a^{\alpha}) \qquad (a \to 0) .$$

Suppose further that η is supported by some compact set Λ with lower cover dimension $D(\Lambda)$. Then we have

$$\kappa^{+}(q) \geqq \kappa^{-}(q) \geqq D(\Lambda) + \alpha q .$$

Proof. Following the remark after the proof of Theorem (3.3) for every q we may compute the dimensions with a wavelet that is regular enough and has sufficient moments vanishing but that need not be in $S_{+}(\mathbb{R})$. In particular we may suppose that the wavelet we use has compact support, that is, say, contained in $[-1/2, +1/2]$. For a given scale a the support of $\mathscr{W}_g \eta(\cdot, a)$ is then contained in $M(\Lambda, a)$. For all $\varepsilon > 0$ we therefore may estimate

$$\int_{-\infty}^{+\infty} db\, |\mathscr{W}_g \eta(b, a)|^q = O(1) \int_{M(\Lambda, a)} db\, a^{-\alpha q} = O(a^{D(\Lambda) - \alpha q - \varepsilon}) .$$

Thus we are done because of Lemma (3.4). □

There is at least one class of distributions with support on a Cantor set, where all the dimensions can be computed explicitly. These are the affine self-similar measures, which we shall present now.

Consider $N > 1$ finite closed, pairwise disjoint intervals I_n. The disjointness ensures that every interval is separated by a gap from its neighbors. Denote by I the smallest interval that contains all the I_n. Let T_n: $\mathbb{R} \to \mathbb{R}$ be affine maps whose restriction to I is onto I_n. If we write T_n: $t \to \lambda_n t + \beta_n$ we assume $\lambda_n > 0$. Now let N positive numbers p_n with $\sum p_n = 1$ be given. They are usually called the "a priori" probabilities, since the distribution we shall construct is a probability measure. It can be shown that there is exactly one probability measure that satisfies the following self-similarity equation for any Borel set K,

$$\mu(K) = \sum_{n=1}^{N} p_n \mu(T_n^{-1} K) .$$

It is called an affine self-similar measure with length-scales λ_n and probabilities p_n.

(5.5) Proposition. *The q-wavelet dimensions of an affine self-similar measure with length-scales λ_n and probabilities p_n are the solution of the following implicit equation $\kappa(q) = \kappa^{\pm}(q)$:*

$$\sum_{n=1}^{N} p_n^q \lambda_n^{1-q-\kappa(q)} \equiv 1 .$$

Proof. Let T_n: $t \to \lambda_n t + \beta_n$. We then have by direct computation,

$$\mathscr{W}_g\mu(b, a) = \sum_{n=1}^{N} p_n \mathscr{W}_g\mu(T_n^{-1} b, a/\lambda_n)/\lambda_n .$$

Now following the remark after the proof of Theorem (3.3), for every q we may assume that the wavelet we use for the computation of the dimensions is compactly supported. Since the support of μ is contained in the union of the non-intersecting I_n, for a small enough, the support of the wavelet will with at most one of the I_n have at non-empty intersection. Therefore for a small enough, for each b the sum consists of at most one term and we may write

$$\begin{aligned} G(a, q) &= \int_{-\infty}^{+\infty} db\, |\mathscr{W}_g\mu(b, a)|^q \\ &= \sum_{n=1}^{N} \int_{-\infty}^{+\infty} db\, |p_n \mathscr{W}_g\mu(T_n^{-1} b, a/\lambda_n)/\lambda_n|^q \\ &= \sum_{n=1}^{N} p_n \lambda_n^{1-q}\, G(a/\lambda_n, q) . \end{aligned}$$

Now instead of considering μ we shall look at some m^{th} primitive of μ and use Proposition (5.1) afterwards. Because of (5.2) we obtain an equation of the same type for the new q-energy that we shall again denote by $G(a, q)$,

$$G(a, q) = \sum_{n=1}^{N} p_n^q \lambda_n^{mq+1-q}\, G(a/\lambda_n, q) .$$

For m sufficiently large we have

$$\Gamma(a, q) = \int_0^a \frac{d\alpha}{\alpha} G(\alpha, q) ,$$

and thus we have again a scaling equation for Γ

$$\Gamma(a, q) = \sum_{n=1}^{N} p_n^q \lambda_n^{mq+1-q}\, \Gamma(a/\lambda_n, q) .$$

The proposition follows now from the next lemma.

(5.6) Lemma. *Let $s(t)$ be a positive function defined for $t \in (0, 1]$ that is bounded away from 0 and infinity,*

$$\forall t \in (0, 1] \quad 0 < \inf_{u \in [t, 1]} s(u) \leq \sup_{u \in [t, 1]} s(u) < \infty ,$$

and that satisfies pointwise at

$$s(t) = \sum_{n=1}^{N} \alpha_n \, s(t/\beta_n) \quad t \in (0, \min\{\beta_n\}]$$

with some constants $\alpha_n > 0$, $\beta_n \in (0, 1)$. *Then*

$$\limsup_{t\to 0} \frac{\log s(t)}{\log t} = \liminf_{t\to 0} \frac{\log s(t)}{\log t} = \delta$$

is the unique solution of

$$\sum_{n=1}^{N} \alpha_n \beta_n^{-\delta} = 1 \; .$$

Proof. We first want to show that

$$\sum_{n=1}^{N} \alpha_n < 1 \Rightarrow \lim_{t\to 0} s(t) = 0 \; ,$$

$$\sum_{n=1}^{N} \alpha_n > 1 \Rightarrow \lim_{t\to 0} s(t) = \infty \; .$$

The lemma follows then by considering $t^{-\gamma} s(t)$ that satisfies the same type of equation as s but where α_n is replaced by $\alpha_n \beta_n^{-\gamma}$.

Now consider

$$s^+(t) = \min\left\{ \sup_{u\in[t,1]} s(u), \quad \sup_{u\in[0,t]} s(u) \right\} ,$$

$$s^-(t) = \max\left\{ \inf_{u\in[t,1]} s(u), \quad \inf_{u\in[0,t]} s(u) \right\} .$$

They are the smallest (largest) monotone functions that majorize (minorize) s for $t \in (0, 1]$. For $t \in (0, 1]$ we have

$$0 < s^-(t) \leqq s^+(t) < \infty ,$$

since suppose $s^+(t) = \infty$, say, then $\sup_{u\in[t,1]} s(u) = \infty$ which contradicts our hypothesis on s. We may estimate for $0 < t \leqq \min\{\beta_n\}$,

$$s(t) = \sum_{n=1}^{N} \alpha_n \, s(t/\beta_n) \leqq \sum_{n=1}^{N} \alpha_n \, s^+(t/\beta_n) \; ,$$

$$s(t) = \sum_{n=1}^{N} \alpha_n \, s(t/\beta_n) \geqq \sum_{n=1}^{N} \alpha_n \, s^-(t/\beta_n) \; .$$

The right-hand sides are again monotone functions that majorize (minorize) s and thus since $s^\pm$ is optimal we have for $0 < t < \min\{\beta_n\}$,

$$s^+(t) \leqq \sum_{n=1}^{N} \alpha_n \, s^+(t/\beta_n) \; ,$$

$$s^-(t) \geqq \sum_{n=1}^{N} \alpha_n \, s^-(t/\beta_n) \; .$$

By monotony we may write

$$s^{+}(t) \leqq \left(\sum_{n=1}^{N} \alpha_n \right) s^{+}(t/\beta^{+}) \,,$$

$$s^{-}(t) \geqq \left(\sum_{n=1}^{N} \alpha_n \right) s^{+}(t/\beta^{-}) \,,$$

where $\beta^{\pm}$ are given by either $\min\{\beta_n\}$ or $\max\{\beta_n\}$ respectively depending on whether $s^{\pm}$ is non-decreasing or non-growing. Since $\beta^{\pm} \in (0, 1)$ we are done. $\square$

This shows the proposition. $\square$

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7. References

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