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A q-Deformation of Wakimoto Modules, Primary Fields and Screening Operators

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Abstract: The q-vertex operators of Frenkel and Reshetikhin are studied by means of a q-deformation of the Wakimoto module for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ at an arbitrary level $k \neq 0, -2$. A Fock-module version of the q-deformed primary field of spin j is introduced, as well as the screening operators which (anti-)commute with the action of $U_q(\widehat{\mathfrak{sl}}_2)$ up to a total difference of a field. A proof of the intertwining property is given for the q-vertex operators corresponding to the primary fields of spin $j \notin \frac{1}{2}\mathbf{Z}_{\geqq 0}$. A sample calculation of the correlation function is also given.

1. Introduction

In a recent paper [FR], Frenkel and Reshetikhin constructed a certain q-deformation of the Wess–Zumino–Witten (WZW) model on the sphere in the operator formalism based on the representation theory of the quantum affine algebras. They defined q-deformed chiral vertex operators as certain intertwining operators, which give an analogue of the primary fields. In principle, the intertwining property characterizes them, however, it is not easy to find an explicit expression for them.

For the $\widehat{\mathfrak{sl}}_2$ WZW model, the following realization is known (cf. [FLMSS]). The standard $\mathfrak{sl}_2$ currents $J^{\pm}(z), J^0(z)$, screening operators $S(z), S^+(z)$ and the primary fields $\phi_{j,m}(z)$ of spin j are explicitly written as

$$J^{\pm}(z) = :\frac{1}{\sqrt{2}}[\sqrt{k+2}\,\partial\varphi_1(z) \pm i\sqrt{k}\,\partial\varphi_2(z)]\, e^{\pm\sqrt{\frac{2}{k}}[i\varphi_2(z)-\varphi_0(z)]}: ,$$

$$J^0(z) = -\sqrt{\frac{k}{2}}\,\partial\varphi_0(z) ,$$

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$$S(z) = :-\frac{1}{\sqrt{2}}[\sqrt{k+2}\partial\varphi_1(z) + i\sqrt{k}\partial\varphi_2(z)]\, e^{-\sqrt{\frac{2}{k+2}}\varphi_1(z)}:,$$

$$S^+(z) = :e^{\left[\sqrt{\frac{k+2}{2}}\varphi_1(z) + \sqrt{\frac{k}{2}}i\varphi_2(z)\right]}:$$

$$\phi_{j,m}(z) = :e^{\left[j\sqrt{\frac{2}{k+2}}\varphi_1(z) + m\sqrt{\frac{2}{k}}(i\varphi_2(z) - \varphi_0(z))\right]}:, \tag{1.1}$$

where $\varphi_i(z)$ are independent bosonic fields normalized as $\varphi_i(z)\varphi_i(w) \sim \log(z-w)$. Then the operator product of $S(z)$ or $S^+(z)$ with the current is a total derivative of a field, which is annihilated by integration on a closed cycle. Note that by fixing the picture of the Fock module [FMS] and restricting it to $\ker Q^+$, where $Q^+ = \oint S^+(z)dz$, one obtains the Wakimoto module [W]. On these Fock modules the primary fields act and a combination of them with the screening charge, which is an integration of the composite operator $S(t_1)\ldots S(t_r)$ on a certain cycle, gives rise to the chiral vertex operator of Tsuchiya and Kanie [TK], see also [BF].

The aim of the present article is to construct a q-deformation of this realization following the line of a previous work [M3]: the operators are deformed by changing the normalization of the modes of $\varphi_i(z)$ and by replacing the differential $\partial\varphi_i(z)$ with a certain difference. In this paper we show that $S(z)$, $S^+(z)$ and $\phi_{j,m}(z)$ are also deformed in the same spirit. Then the operator $Q^+ = \oint S^+(z)dz$ (anti)-commutes with the currents and the primary fields. We also fix the picture and restrict the Fock module to $\ker Q^+$. Thus we obtain a Fock-module version of the q-deformed chiral vertex operators of Frenkel and Reshetikhin. We show that the operator product of $S(z)$ with the current is a total p-difference of a field, where $p = q^{2(k+2)}$. This total difference is eliminated by the Jackson integral, a p-deformation of integration, thus we obtain a screening charge. Therefore, putting aside the problem of choosing of a cycle of the Jackson integral, we are able to obtain an explicit expression of the correlation function, which gives a Jackson-integral solution to the quantum (q-deformed) Knizhnik–Zamolodchikov equation.

The paper is organized as follows. In Sect. 2, we summarize some notions from representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}})_2$, and fix the notations in q-analysis. In Sect. 3 we review the result of [M3] in a modified form. Section 4 is devoted to a construction of a deformation of the screening operators $S(z)$ and $S^+(z)$. We also discuss the restriction of the Fock spaces. In Sect. 5, the primary fields of spin j are introduced and the intertwining property of the corresponding q-vertex operator is proved except for $2j \in \mathbf{Z}_{\geqq 0}$. In Sect. 6 we discuss a property of the correlation function, and give a sample calculation of it. A brief conclusion will be given in Sect. 7.

Recently Kato et al. [KQS] also discussed a free-field realization of q-vertex operators using Shiraishi's representation of $U_q(\widehat{\mathfrak{sl}}_2)$ [S]. A comment on [KQS] will also be given in Sect. 7.

2. Preliminaries

Let $\mathbf{C}$ denote the field of complex numbers and $\mathbf{C}^* = \mathbf{C} - \{0\}$ its multiplicative group. Let q be a complex number transcendental over $\mathbf{Q}$, the field of rational numbers. In the sequel, if necessary, we choose an appropriate branch of the complex power of the form q^μ, $\mu \in \mathbf{C}$.

The quantum affine algebra $U_q = U_q(\widehat{\mathfrak{sl}}_2)$ is the associative algebra generated over $\mathbf{C}$ by the letters $e_0, e_1, f_0, f_1, q^{\pm h_0}, q^{\pm h_1}, q^{\pm d}$ satisfying the following defining relations:

$$
\begin{aligned}
& q^{h_0}q^{h_1} = q^{h_1}q^{h_0}, \quad q^d q^{h_i} = q^{h_i} q^d, \quad q^{h_0}q^{-h_0} = q^{h_1}q^{-h_1} = q^d q^{-d} = 1\,,\\
& q^{h_i} e_i q^{-h_i} = q^2 e_i, \quad q^{h_i} e_j q^{-h_i} = q^{-2} e_j \quad (i \neq j)\,,\\
& q^{h_i} f_i q^{-h_i} = q^{-2} f_i, \quad q^{h_i} f_j q^{-h_i} = q^2 f_j \quad (i \neq j)\,,\\
& q^d e_i q^{-d} = q^{\delta_{i,0}} e_i, \quad q^d f_i q^{-d} = q^{-\delta_{i,0}} f_i\,,\\
& [e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\,,\\
& e_i^3 e_j - (q^2 + 1 + q^{-2}) e_i^2 e_j e_i + (q^2 + 1 + q^{-2}) e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (i \neq j)\,,\\
& f_i^3 f_j - (q^2 + 1 + q^{-2}) f_i^2 f_j f_i + (q^2 + 1 + q^{-2}) f_i f_j f_i^2 - f_j f_i^3 = 0 \quad (i \neq j)\,,
\end{aligned}
\tag{2.1}
$$

where we have used the standard notation:

$$
[m] = \frac{q^m - q^{-m}}{q - q^{-1}}\,. \tag{2.2}
$$

The subalgebra generated by $e_0, e_1, f_0, f_1, q^{\pm h_0}, q^{\pm h_1}$ is denoted by U_q'. This algebra admits another realization in terms of generators and relations, called the Drinfeld realization [D2], which will play a crucial role in our construction [M3]. However, since apparently it is not necessary in this paper, we omit it.

The algebra U_q becomes a bialgebra with the comultiplication defined by

$$
\begin{aligned}
\Delta(e_i) &= e_i \otimes 1 + q^{h_i} \otimes e_i\,,\\
\Delta(f_i) &= f_i \otimes q^{-h_i} + 1 \otimes f_i\,,\\
\Delta(q^{h_i}) &= q^{h_i} \otimes q^{h_i}\,,\\
\Delta(q^d) &= q^d \otimes q^d\,.
\end{aligned}
\tag{2.3}
$$

Thus we can consider the tensor product of two representations. Note that this choice of comultiplication coincides with that of [JMMN] or [DFJMN] and differs from that of [FR].

Let us introduce two kinds of representation of U_q for later use. The first kind is the highest-weight representations (cf. [L]) and the other is the evaluation representation (cf. [J]). The former are not necessarily integrable, and the latter are not necessarily the affinization of finite-dimensional modules.

A U_q-module M is said to be a highest-weight module of spin l at level k if there exists a vector $v_l \in M$ such that

$$
M = U_q v, \quad e_1 v = e_0 v = 0, \quad q^{h_1} v = q^{2l} v, \quad q^{h_0} v = q^{k-2l} v \quad \text{and} \quad q^d v = v\,. \tag{2.4}
$$

The vector v is called the highest-weight vector. Then the highest weight of M is $2l\Lambda_1 + (k-2l)\Lambda_0$, where Λ_i, $i = 0, 1$, are the usual fundamental weights. For instance the Verma module of U_q is a highest-weight module similarly defined as in the case of a Kac–Moody Lie algebra (cf. [L]). It is irreducible unless the highest weight satisfies a special condition.

The evaluation representation with which we are concerned in this paper is the following type: the vector space $V_j(z) = (\bigoplus_{l=0}^{\infty} \mathbf{C} v_{j,j-l}) \otimes \mathbf{C}[z, z^{-1}]$ equipped with the U_q-module structure defined by

$$\begin{aligned}
e_1 v_{j,m} \otimes z^n &= [j+m]\, v_{j,m-1} \otimes z^n\,,\\
e_0 v_{j,m} \otimes z^n &= [j-m]\, v_{j,m+1} \otimes z^{n+1}\,,\\
f_1 v_{j,m} \otimes z^n &= [j-m]\, v_{j,m+1} \otimes z^n\,,\\
f_0 v_{j,m} \otimes z^n &= [j+m]\, v_{j,m-1} \otimes z^{n-1}\,,\\
q^{h_1} v_{j,m} \otimes z^n &= q^{-2m} v_{j,m} \otimes z^n\,,\\
q^{h_0} v_{j,m} \otimes z^n &= q^{2m} v_{j,m} \otimes z^n\,,\\
q^{d} v_{j,m} \otimes z^n &= q^{n} v_{j,m} \otimes z^n\,.
\end{aligned} \tag{2.5}$$

The space $V_j(z)$ is also endowed with the natural $\mathbf{C}[z, z^{-1}]$-module structure. When $2j$ is a non-negative integer, then $(\bigoplus_{l=0}^{2j} \mathbf{C} v_{j,j-l}) \otimes \mathbf{C}[z, z^{-1}]$ is the affinization of a finite-dimensional U_q-module [J1].

Now let us turn to preliminaries from q-analysis. We define the difference operator $\frac{\partial_p}{\partial_p z}$ for a scalar $p \in \mathbf{C}^*$ by:

$$\frac{\partial_p}{\partial_p z} f(z) = \frac{f(p^{1/2} z) - f(p^{-1/2} z)}{(p^{1/2} - p^{-1/2}) z} \tag{2.6}$$

for a function $f(z)$ on $\mathbf{C}^*$. A function of the form $\frac{\partial_p}{\partial_p z} f(z)$ is called a *total* (p-)*difference* of a function $f(z)$.

To eliminate a total difference, we need some analogue of integration. For instance, the Jackson integral, defined by

$$\int_0^{s\infty} f(z) \frac{d_p t}{z} = (1-p) \sum_{p=-\infty}^{\infty} f(sp^n) \tag{2.7}$$

for a scalar $s \in \mathbf{C}^*$, satisfies

$$\int_0^{s\infty} \left\{ \frac{\partial_p}{\partial_p z} f(z) \right\} d_p z = 0 \tag{2.8}$$

if it is convergent. Another example is to take the residue at zero:

$$\frac{1}{2\pi\sqrt{-1}} \oint_{z=0} f(z) dz = [\text{residue of } f(z) \text{ at } z = 0]\,, \tag{2.9}$$

for a meromorphic function. Then we have

$$\frac{1}{2\pi\sqrt{-1}} \oint_{z=0} \frac{\partial_p}{\partial_p z} f(z) dz = 0\,. \tag{2.10}$$

We finally note that in this paper we shall freely use the technique of *operator product expansion*. Let A_n: $F_1 \to F_2$, $n \in \mathbf{Z}$, be linear maps of vector spaces F_1, F_2. Then the power series $A(z) = \sum_{n \in \mathbf{Z}} z^{-n-\Delta} A_n$, where Δ is a complex number, is simply called an operator or a field of dimension Δ. We will write by abuse of notation as $A(z)$: $F_1 \to F_2$. Now let $A(z)$: $F_2 \to F_3$ and $B(z)$: $F_1 \to F_2$ be operators. Then the composition $A(z)B(w)$: $F_1 \to F_3$ defined as a formal power series. When it can be analytically continued outside some locus, the notation $A(z)B(w) \sim Cz, w)$ means that $A(z)B(w) - C(z, w)$ is holomorphic on $\mathbf{C}^* \times \mathbf{C}^*$ in an appropriate sense. Note that the singular locus of $C(z, w)$ is not necessarily $z = w$ in our construction, unlike the $q = 1$ case. The reader who is not familiar with this technique should be referred to [TK] for a mathematical survey.

3. Free Field Representation of $U_q(\widehat{\mathfrak{sl}}_2)$

Let $\{\alpha_n, \bar{\alpha}_n, \beta_n \mid n \in \mathbf{Z}\}$ be a set of operators satisfying the following commutation relations for $m \neq 0$:

$$[\alpha_m, \alpha_{-m}] = \frac{[2m]\,[km]}{m} ,$$

$$[\bar{\alpha}_m, \bar{\alpha}_{-m}] = -\frac{[2m]\,[km]}{m} ,$$

$$[\beta_m, \beta_{-m}] = \frac{[2m]\,[(k+2)m]}{m} . \tag{3.1}$$

Suppose that the other commutators are zero. By renormalizing these operators, we have the usual modes of three bosonic fields. In other words they form the direct sum of three Heisenberg algebras with infinite generators.

We define

$$N_+ = \mathbf{C}[\alpha_m, \bar{\alpha}_m, \beta_m]_{m>0} ,$$

$$N_- = \mathbf{C}[\alpha_m, \bar{\alpha}_m, \beta_m]_{m<0} . \tag{3.2}$$

The left Fock module F_{l,m_1,m_2} is uniquely characterized by the following properties: there exists a vector $|l, m_1, m_2\rangle$ in F_{l,m_1,m_2} such that

$$\beta_n |l, m_1, m_2\rangle = 2l\delta_{n,0} |l, m_1, m_2\rangle ,$$

$$\alpha_n |l, m_1, m_2\rangle = 2m_1\delta_{n,0} |l, m_1, m_2\rangle ,$$

$$\bar{\alpha}_n |l, m_1, m_2\rangle = -2m_2\delta_{n,0} |l, m_1, m_2\rangle , \quad \text{and}$$

$$N_- |l, m_1, m_2\rangle \text{ is a free } N_-\text{-module of rank 1} . \tag{3.3.L}$$

The right Fock module $F^\dagger_{l,m_1,m_2}$ is similarly characterized by

$$\langle l, m_1, m_2 | \beta_n = 2l\delta_{n,0} \langle l, m_1, m_2 | ,$$

$$\langle l, m_1, m_2 | \alpha_n = 2m_1\delta_{n,0} \langle l, m_1, m_2 | ,$$

$$\langle l, m_1, m_2 | \bar{\alpha}_n = -2m_2\delta_{n,0} \langle l, m_1, m_2 | , \quad \text{and}$$

$$\langle l, m_1, m_2 | N_+ \text{ is a free } N_+\text{-module of rank 1} . \tag{3.3.R}$$

In the sequel we will be mainly concerned with the left Fock modules. Each statement will have a right-module counterpart.

For each triple of complex numbers r, s_1 and s_2, we define the operator $e^{2r\beta+2s_1\alpha+2s_2\bar{\alpha}}$ by the mapping $|l, m_1, m_2\rangle \mapsto |l+r, m_1+s_1, m_2+s_2\rangle$ such that it commutes with the action of $N_\pm$. The normal ordering $::$ is defined according to $\alpha < \alpha_0, \bar{\alpha} < \bar{\alpha}_0, \beta < \beta_0$ and $N_- < N_+$.

Consider the operators $X^\pm(z): F_{l,m_1,m_2} \to F_{l,m_1\pm 1,m_2\pm 1}$ defined by

$$X^+(z) = \frac{1}{(q-q^{-1})z} :Y^+(z)\{Z_+(q^{-\frac{k+2}{2}}z)W_+(q^{-\frac{k}{2}}z) - W_-(q^{\frac{k}{2}}z)Z_-(q^{\frac{k+2}{2}}z)\}:,$$

$$X^-(z) = \frac{-1}{(q-q^{-1})z} \times :Y^-(z)\{Z_+(q^{\frac{k+2}{2}}z)W_+(q^{\frac{k}{2}}z)^{-1} - W_-(q^{-\frac{k}{2}}z)^{-1}Z_-(q^{-\frac{k+2}{2}}z)\}:, \tag{3.4}$$

where

$$Y^+(z) = \exp\left\{\sum_{m=1}^\infty q^{-\frac{km}{2}} \frac{z^m}{[km]}(\alpha_{-m} + \bar{\alpha}_{-m})\right\} \times e^{2(\alpha+\bar{\alpha})} z^{\frac{1}{k}(\alpha_0+\bar{\alpha}_0)} \exp\left\{-\sum_{m=1}^\infty q^{-\frac{km}{2}} \frac{z^{-m}}{[km]}(\alpha_m + \bar{\alpha}_m)\right\},$$

$$Y^-(z) = \exp\left\{-\sum_{m=1}^\infty q^{\frac{km}{2}} \frac{z^m}{[km]}(\alpha_{-m} + \bar{\alpha}_{-m})\right\} \times e^{-2(\alpha+\bar{\alpha})} z^{-\frac{1}{k}(\alpha_0+\bar{\alpha}_0)} \exp\left\{\sum_{m=1}^\infty q^{\frac{km}{2}} \frac{z^{-m}}{[km]}(\alpha_m + \bar{\alpha}_m)\right\}, \tag{3.5}$$

$$Z_+(z) = \exp\left\{-(q-q^{-1})\sum_{m=1}^\infty z^{-m}\frac{[m]}{[2m]}\bar{\alpha}_m\right\} q^{-\frac{1}{2}\bar{\alpha}_0}, \tag{3.6}$$

$$Z_-(z) = \exp\left\{(q-q^{-1})\sum_{m=1}^\infty z^{m}\frac{[m]}{[2m]}\bar{\alpha}_{-m}\right\} q^{\frac{1}{2}\bar{\alpha}_0}, \tag{3.7}$$

$$W_+(z) = \exp\left\{-(q-q^{-1})\sum_{m=1}^\infty z^{-m}\frac{[m]}{[2m]}\beta_m\right\} q^{-\frac{1}{2}\beta_0}, \tag{3.8}$$

$$W_-(z) = \exp\left\{(q-q^{-1})\sum_{m=1}^\infty z^{m}\frac{[m]}{[2m]}\beta_{-m}\right\} q^{\frac{1}{2}\beta_0}. \tag{3.9}$$

Now $\alpha_0 + \bar{\alpha}_0$ acts on $F_{l,m,m}$ by zero. Therefore the expansion of the form

$$X^\pm(z) = \sum_{m\in\mathbf{Z}} z^{-m-1} x_m^\pm \tag{3.10}$$

makes sense on $F_{l,m,m}$. Moreover let us introduce the operator p^δ for a scalar $p \in \mathbf{C}^*$ characterized by

$$p^\delta \alpha_m p^{-\delta} = p^m \alpha_m, \quad p^\delta \bar{\alpha}_m p^{-\delta} = p^m \bar{\alpha}_m, \quad p^\delta \beta_m p^{-\delta} = p^m \beta_m, \tag{3.11}$$

and

$$p^\delta |l, m_1, m_2\rangle = 1 \quad \text{for any } l, m_1, m_2. \tag{3.12}$$

Then we have

Proposition 3.1 ([M3]). *There exists a representation* $\pi_l: U_q \to \mathrm{End}(\bigoplus_{m=l+\mathbf{Z}} F_{l,m,m})$ *such that*

$$\pi_l(e_1) = x_0^+, \qquad \pi_l(f_1) = x_0^-, \qquad \pi_l(q^{h_1}) = q^{\alpha_0},$$

$$\pi_l(e_0) = x_1^- q^{-\alpha_0}, \quad \pi_l(f_0) = q^{\alpha_0} x_{-1}^+, \quad \pi_l(q^{h_0}) = q^k q^{-\alpha_0}, \quad \pi_l(q^d) = q^\delta .$$

Proof. Straightforward. Q.E.D.

Note. A similar Fock representation of U_q is also constructed in [ABE], [S] and [Ki], see also [Bo].

Remark 1. To make (3.4) be single-valued, $\frac{1}{k}(\alpha_0 + \bar{\alpha}_0)$ is necessarily an integer. The constraint $\alpha_0 + \bar{\alpha}_0 = 0$ means that we have fixed a *picture* of the Fock representation, see [FMS] and [FLMSS].

Remark 2. Let L_0 be the operator defined by

$$L_0 = \sum_{m>0} \frac{m^2}{[2m][km]}(\alpha_{-m}\alpha_m - \bar{\alpha}_{-m}\bar{\alpha}_m) + \frac{1}{4k}(\alpha_0^2 - \bar{\alpha}_0^2)$$

$$+ \sum_{m>0} \frac{m^2}{[2m][(k+2)m]} \beta_{-m}\beta_m + \frac{1}{4(k+2)}(\beta_0^2 + 2\beta_0) . \quad (3.13)$$

Then we have $\delta + L_0 = \frac{l(l+1)}{k+2}$ on F_{l,m_1,m_2}. When $q = 1$, this is the usual L_0 of the energy-momentum tensor $T(z) = \sum_{m \in \mathbf{Z}} z^{-m-2} L_m$ (cf. [FLMSS]).

In the sequel we drop π_l when we write the action of U_q.

Proposition 3.2. *The vector* $|l, l, l\rangle$ *for any* $l \in C$ *satisfies* $e_1 |l, l, l\rangle = e_0 |l, l, l\rangle = 0$.

Proof. Straightforward. Q.E.D.

4. Screening Operators and Structure of Fock Spaces

In this section we define an analogue of screening operators. We set

$$H(z) = \sum_{m \in \mathbf{Z}} z^{-m-1} \alpha_m \quad (4.1)$$

for convenience.

Consider the operator $S^+(z)$: $F_{l,m_1,m_2} \to F_{l+\frac{k+2}{2}, m_1, m_2+\frac{k}{2}}$ defined by:

$$S^+(z) = \exp\left\{ \sum_{m=1}^{\infty} z^m \frac{1}{[2m]} (q^{\frac{k}{2}m} \beta_{-m} + q^{\frac{k+2}{2}m} \bar{\alpha}_{-m}) \right\} e^{(k+2)\beta + k\bar{\alpha}} z^{\frac{1}{2}(\beta_0 + \bar{\alpha}_0)}$$

$$\times \exp\left\{ -\sum_{m=1}^{\infty} z^{-m} \frac{1}{[2m]} (q^{\frac{k}{2}m} \beta_m + q^{\frac{k+2}{2}m} \bar{\alpha}_m) \right\} . \quad (4.2)$$

Lemma 4.1. *We have the following relations:*

$$X^+(z)S^+(w) = -S^+(w)X^+(z) \sim \frac{\partial_{q^2}}{\partial_{q^2}w}\left\{\frac{1}{z-w} Y^+(z)S_1^+(z)\right\},$$
$$X^-(z)S^+(w) = -S^+(w)X^-(z) \sim 0,$$
$$H(z)S^+(w) = S^+(w)H^-(z) \sim 0,$$
$$q^d S^+(w)q^{-d} = q^{m_2-l}S^+(q^{-1}w),$$

where

$$S_1^+(z) = \exp\left\{\sum_{m=1}^{\infty} z^m \frac{1}{[2m]}(q^{\frac{k+2}{2}m}\beta_{-m} + q^{\frac{k+4}{2}m}\bar{\alpha}_{-m})\right\} e^{(k+2)\beta+k\bar{\alpha}} z^{\frac{1}{2}(\beta_0+\bar{\alpha}_0)}$$
$$\times \exp\left\{-\sum_{m=1}^{\infty} z^{-m}\frac{1}{[2m]}(q^{\frac{k+2}{2}m}\beta_m + q^{\frac{k+4}{2}m}\bar{\alpha}_m)\right\}.$$

Proof. Straightforward. Q.E.D.

Note that $S^+(z)$ is single-valued on F_{l,m_1,m_2} provided $l - m_2$ is an integer. Let Q^+ be defined by:

$$Q^+ = \frac{1}{2\pi\sqrt{-1}}\oint S^+(z)dz\,. \tag{4.3}$$

Then we immediately see

Proposition 4.2. *The operator Q^+ commutes with the action of U_q' up to sign and with q^d up to a scalar multiple.*

Now we are in a position to consider a restriction of the Fock spaces. We define

$$F_l = \bigoplus_{m=l+\mathbf{Z}} \ker(Q^+ : F_{l,m,m} \to F_{l+\frac{k+2}{2},m,m+\frac{k}{2}})\,. \tag{4.4}$$

Because of Proposition 4.2, U_q acts on F_l. Moreover the following holds:

Proposition 4.3. *The character of F_l is same as the Verma module of spin l at level k.*

Proof. Let us first make the following observation: putting

$$S^+(z) = :\exp\chi(z):,$$

where

$$\chi(z) = \sum_{m=1}^{\infty} z^m\left\{\frac{q^{\frac{k}{2}m}}{[2m]}\beta_m + \frac{q^{\frac{k+2}{2}m}}{[2m]}\bar{\alpha}_m\right\} + (k+2)\beta + k\bar{\alpha}$$
$$+\frac{1}{2}\log(z)(\beta_0+\bar{\alpha}_0) - \sum_{m=1}^{\infty} z^{-m}\left\{\frac{q^{\frac{k}{2}m}}{[2m]}\beta_m + \frac{q^{\frac{k+2}{2}m}}{[2m]}\bar{\alpha}_m\right\},$$

we have $\chi(z)\chi(w) \sim \log(z-w)$. Therefore we may understand Q^+ as the zero mode η_0 of the fermionic ghost system (η, ξ) of dimension $(1, 0)$:

$$\eta(z) = \sum_{m\in\mathbf{Z}} z^{-m-1}\eta_m = :e^{\chi(z)}:, \quad \xi(z) = \sum_{m\in\mathbf{Z}} z^{-m}\xi_m = :e^{-\chi(z)}:.$$

Since we have $\eta_0^2 = 0$ and $\eta_0\xi_0 + \xi_0\eta_0 = 1$, we obtain the following exact sequence:

$$0 \to F_l \to \bigoplus_m F_{l,m,m} \xrightarrow{Q^+} \bigoplus_m F_{l+\frac{k+2}{2},m,m+\frac{k}{2}} \xrightarrow{Q^+} \dots .$$

By using this sequence, we can compute the character of F_l, which is shown to be same as the Verma module with spin l at level k. Q.E.D.

In particular we have

Corollary 4.4. *Let l be a general complex number such that the Verma module with spin l at level k is irreducible. Then it is isomorphic to F_l with the highest weight vector $|l, l, l\rangle$.*

Note. When $q=1$, the result is also found in the physics literature, see [FMS], [FLMSS]. In this limit, F_l is isomorphic to the Wakimoto module of $\widehat{\mathfrak{sl}}_2$. This situation allows us to claim that our representation is a q-deformation of the Wakimoto module. We refer the reader to [FF] for some background on the character of F_l.

Remark. There exists another screening operator $S^-(z)$ defined by

$$S^-(z) = \exp\left\{-\sum_{m=1}^{\infty} z^m \frac{1}{[2m]}(q^{-\frac{k}{2}m}\beta_{-m} - q^{-\frac{k+2}{2}m}\bar{\alpha}_{-m})\right\}$$
$$\times e^{-(k+2)\beta+k\bar{\alpha}} z^{\frac{1}{2}(-\beta_0+\bar{\alpha}_0)} \exp\left\{\sum_{m=1}^{\infty} z^{-m} \frac{1}{[2m]}(q^{-\frac{k}{2}m}\beta_m - q^{-\frac{k+2}{2}m}\bar{\alpha}_m)\right\}.$$

Now until the end of this section, we consider the other screening operator. Consider the operator $S(z)$: $F_{l,m_1,m_2} \to F_{l-1,m_1,m_2}$ defined as follows :

$$S(z) = \frac{-1}{(q-q^{-1})z} : U(z)\{Z_+(q^{-\frac{k+2}{2}}z)^{-1} W_+(q^{-\frac{k}{2}}z)^{-1}$$
$$- W_-(q^{\frac{k}{2}}z)^{-1} Z_-(q^{\frac{k+2}{2}}z)^{-1}\} : , \tag{4.5}$$

where $Z_\pm(z)$ and $W_\pm(z)$ are defined by (3.6) and (3.7) respectively, and

$$U(z) = \exp\left\{-\sum_{m=1}^{\infty} z^m \frac{q^{-\frac{k+2}{2}m}}{[(k+2)m]}\beta_{-m}\right\}$$
$$\times e^{-2\beta} z^{-\frac{1}{k+2}\beta_0} \exp\left\{\sum_{m=1}^{\infty} z^{-m} \frac{q^{-\frac{k+2}{2}m}}{[(k+2)m]}\beta_m\right\}. \tag{4.6}$$

Lemma 4.5. *We have the following relations*

$$S(z)S^+(w) = S^+(w)S(z) \sim \frac{\partial_{q^2}}{\partial_{q^2}w}\left\{\frac{1}{z-w} U(z)S_2^+(z)\right\},$$

where

$$S_2^+(z) = \exp\left\{\sum_{m=1}^{\infty} z^m \frac{1}{[2m]}(q^{\frac{k-2}{2}m}\beta_{-m} + q^{\frac{k}{2}m}\bar{\alpha}_{-m})\right\}$$
$$\times e^{(k+2)\beta+k\bar{\alpha}} z^{\frac{1}{2}(\beta_0+\bar{\alpha}_0)} \exp\left\{-\sum_{m=1}^{\infty} z^{-m} \frac{1}{[2m]}(q^{\frac{k-2}{2}m}\beta_m + q^{\frac{k}{2}m}\bar{\alpha}_m)\right\},$$

$$X^+(z)S(w) = S(w)X^+(z) \sim 0$$
$$X^-(z)S(w) = S(w)X^-(z) \sim [k+2]\frac{\partial_p}{\partial_p w}\left\{\frac{1}{z-w}Y^-(z)U_1(z)\right\},$$
$$H(z)S(w) = S(w)H(z) \sim 0\,,$$
$$p^d S(w) p^{-d} = p^{-\frac{2l}{k+2}+1} S(p^{-1}w)\,,$$

where $p = q^{2(k+2)}$, $Y^-(z)$ *is defined by* (3.5), *and*

$$U_1(z) = \exp\left\{-\sum_{m=1}^{\infty} z^m \frac{q^{\frac{k+2}{2}m}}{[(k+2)m]}\beta_{-m}\right\}$$
$$\times e^{-2\beta} z^{-\frac{1}{k+2}\beta_0} \exp\left\{\sum_{m=1}^{\infty} z^{-m}\frac{q^{\frac{k+2}{2}m}}{[(k+2)m]}\beta_m\right\}.$$

Proof. Straightforward. Q.E.D.

Thus $S(z)$ defines an operator

$$S(z): F_l \to F_{l-1}\,, \tag{4.7}$$

and we have

Proposition 4.6. *The operator* $S(z)$ *commutes with the action of* U_q' *up to total p-difference of a field.*

Remark. Let L_0 be the operator defined by (3.13). Then we have

$$p^{L_0} S(w) p^{-L_0} = pS(pw) = S(w) + (p-1)\frac{\partial_p}{\partial_p w}\left\{wS(p^{\frac{1}{2}}w)\right\}.$$

Therefore the operator $S(w)$ commutes with p^{L_0} modulo a total p-difference. When $q = 1$, it corresponds to $[L_0, S(w)] = \dfrac{\partial}{\partial w}\{wS(w)\}$, which comes from the operator product of $S(w)$ with the energy-momentum tensor $T(z) = \sum_{m\in\mathbf{Z}} z^{-m-2} L_m$.

5. Primary Fields and q-Vertex Operators

In this section we construct a Fock-module version of q-vertex operators. As for the formulation of intertwining operators, we follow [J2] (cf. [JMMN, IIJMNT]). For any complex number j, let $\phi_j(z): F_{r,m_1,m_2} \to F_{r+j,m_1+j,m_2+j}$ be the operator defined by

$$\phi_j(z) = \exp\left\{\sum_{m=1}^{\infty}(q^{k+2}z)^m \frac{q^{\frac{km}{2}}[2jm]}{[km][2m]}(\alpha_{-m}+\bar{\alpha}_{-m})\right\}$$
$$\times e^{2j(\alpha+\bar{\alpha})} z^{\frac{j}{k+2}(\alpha_0+\bar{\alpha}_0)} \exp\left\{-\sum_{m=1}^{\infty}(q^{k+2}z)^{-m}\frac{q^{\frac{km}{2}}[2jm]}{[km][2m]}(\alpha_m+\bar{\alpha}_m)\right\}$$
$$\times \exp\left\{\sum_{m=1}^{\infty}(q^{k+2}z)^m \frac{q^{\frac{k+2}{2}m}[2jm]}{[2m][(k+2)m]}\beta_{-m}\right\}$$
$$\times e^{2j\beta} z^{\frac{j}{k+2}\beta_0} \exp\left\{-\sum_{m=1}^{\infty}(q^{k+2}z)^{-m}\frac{q^{\frac{k+2}{2}m}[2jm]}{[km][(k+2)m]}\beta_m\right\}. \tag{5.1}$$

The primary fields of spin j are defined inductively by

$$\phi_{j,m-1}(z) = \frac{1}{[j-m+1]}\{\phi_{j,m}(z)x_0^- - q^{2m}x_0^-\phi_{j,m}(z)\},$$
$$\phi_{j,j}(z) = \phi_j(z). \tag{5.2}$$

We understand $\phi_{j,m}(z) = 0$ if $j - m \notin \mathbf{Z}_{\geq 0}$.

Proposition 5.1. *The primary field $\phi_{j,m}(z)$ commutes with the action of Q^+ up to sign.*

Proof. The statement follows from Corollary 4.3 and $S^+(z)\phi_j(w) = \phi_j(w)S^+(z) \sim 0$, which is proved by a straightforward calculation. Q.E.D.

Therefore the image of F_l by each component of $\phi_{j,m}(z)$ is contained in F_{l+j} for any $l \in \mathbf{C}$. Thus we have obtained an operator

$$\phi_{j,m}(z)\colon F_l \to F_{l+j}. \tag{5.3}$$

Now let us consider the intertwining property of the primary fields.

Lemma 5.2. *By analytic continuation, we have the following:*

(1) $$X^+(w)\phi_j(z) = \phi_j(z)X^+(w) \sim 0,$$

(2) $$(q^{k+2}z - q^{2j}w)\phi_j(z)X^-(w) = (q^{2j+k+2}z - w)X^-(w)\phi_j(z) \sim 0.$$

Proof. Straightforward. Q.E.D.

Proposition 5.3. *We have the following relations:*

(1) $$\phi_{j,m}(z)e_1 = e_1\phi_{j,m}(z) + [j+m+1]q^{h_1}\phi_{j,m+1}(z),$$

(2) $$\phi_{j,m}(z)e_0 = e_0\phi_{j,m}(z) + z[j-m+1]q^{h_0}\phi_{j,m-1}(z),$$

(3) $$\phi_{j,m}(z)f_1 = q^{2m}f_1\phi_{j,m}(z) + [j-m+1]\phi_{j,m-1}(z),$$

(4) $$\phi_{j,m}(z)f_0 = q^{-2m}f_0\phi_{j,m}(z) + z^{-1}[j+m+1]\phi_{j,m+1}(z),$$

(5) $$\phi_{j,m}(z)q^{h_1} = q^{-2m}q^{h_1}\phi_{j,m}(z),$$

(6) $$\phi_{j,m}(z)q^{h_0} = q^{2m}q^{h_0}\phi_{j,m}(z),$$

(7) $$\phi_{j,m}(z)q^d = q^{d-\frac{2jl}{k+2}}\phi_{j,m}(qz).$$

Proof. The relation (3) is obvious by the definition, and (5), (6) and (7) are directly proved. The relations (1), (2) and (4) for $m = j$ are checked by Lemma 5.2. Then the relation (2) for $m < j$ is proved by induction on m. Finally (1) and (4) are derived from (2) and (3). Q.E.D.

Let us put

$$\Delta_l = \frac{l(l+1)}{k+2}. \tag{5.4}$$

As a consequence of Proposition 5.3, we have the following theorem.

Theorem 5.4. *Suppose that $2j \notin \mathbf{Z}_{\geqq 0}$. Then the operator*

$$\tilde{\Phi}_{l,j}^{l+j}(z) = z^{\Delta_l + \Delta_j - \Delta_{l+j}} \sum_{m=0}^{\infty} \phi_{j,j-m}(z) \otimes v_{j,j-m}$$

gives rise to an intertwining operator $F_l \to F_{l+j} \otimes V_j(z)$ of U_q-modules.

Remark 1. This theorem asserts that $\tilde{\Phi}^{l+j}_{j,l}(z)$ is a Fock-module version of the q-deformed chiral vertex operators of Frenkel and Reshetikhin. Note that to prove the intertwining property when $2j\in\mathbf{Z}_{\geqq 0}$ one must verify $\phi_{j,-j}(z)f_1 = q^{-2j}f_1\phi_{j,-j}(z)$ and $\phi_{j,-j}(z)e_0 = e_0\phi_{j,-j}(z)$.

Remark 2. We formally set

$$\Phi^{l+j-r}_{l,j}(z) = z^{\Delta_l+\Delta_j-\Delta_{l+j-r}}\sum_{m=0}^{\infty}\int_0^{s_1\infty} S(t_1)d_pt_1\ldots\int_0^{s_r\infty} S(t_r)d_pt_r\phi^{(r)}_{j,j-m}(z)\otimes v_{j,j-m}$$

for an appropriate choice of scalars $s_1,\ldots,s_r$. Then it would give an intertwining operator

$$\tilde{\Phi}^{l+j-r}_{l,j}(z)\colon (F_l\to F_{l+j-r}\otimes V_j(z)$$

for an arbitrary $l\in\mathbf{C}^*$. To understand it mathematically, we need to study a choice of the q-cycle of the Jackson integral rigorously in the operator formalism.

6. Correlation Functions

Let $j_0,\ldots,j_N,j_\infty$ be a set of complex numbers satisfying

$$j_0+\cdots+j_N-j_\infty = M \tag{6.1}$$

for some non-negative integer M. For each $a=1,\ldots,N$ let $\pi_a\colon U_q\to V_{j_a}(z)$ be the evaluation representation and $\tilde{\Phi}^m_{l,j_a}(z)\colon F_l\to F_m\otimes V_{j_a}(z)$ be an intertwining operator. Put

$$\Phi^m_{l,j_a}(z) = z^{\Delta_m-\Delta_l}\tilde{\Phi}^m_{l,j_a}(z)\,.$$

In this section we shall be concerned with a formal calculation of the n-point correlation functions:

$$\langle\Phi^{j_\infty}_{l_{N-1}j_N}(z_N)\Phi^{l_{N-1}}_{l_{N-2},j_{N-1}}(z_{N-1})\ldots\Phi^{l_2}_{l_1,j_2}(z_2)\Phi^{l_1}_{j_0,j_1}(z_1)\rangle \tag{6.2}$$

for a choice of $l_l,\ldots,l_N$. Here $\langle A\rangle = (\langle j_\infty|, A|j_0\rangle)$ is given by the canonical pairing $F^\dagger_{j_\infty,j_\infty,j_\infty}\times F_{j_\infty,j_\infty,j_\infty}\to\mathbf{C}$.

However, since the operator $\tilde{\Phi}^m_{l,j}(z)$ for $m\neq l+j$ is not defined rigorously at present, we shall use the following function instead:

$$\begin{aligned}&\mathscr{F}(t_1,\ldots,t_M,z_1,\ldots,z_N)\\&\quad=\langle S(t_1)\ldots S(t_M)\Phi^{j_\infty+M}_{l_{N-1},j_N}(z_N)\Phi^{l_{N-1}}_{l_{N-2},j_{N-1}}(z_{N-1})\ldots\Phi^{l_2}_{l_1,j_2}(z_2)\Phi^{l_1}_{j_0,j_1}(z_1)\rangle\,,\end{aligned} \tag{6.3}$$

where $l_a = j_0+\cdots+j_a$.

Let $\mathscr{R}\in U_q\hat{\otimes}U_q$ be the universal R-matrix and put

$$R_{V_aV_b}(z_a/z_b) = (\pi_a\otimes\pi_b)(\sigma(\mathscr{R}^{-1}))\,,$$

where $\sigma(u\otimes v)=v\otimes u$ (cf. [FR]). It acts on the a^{th} and b^{th} components of $V_{j_1}\otimes\cdots\otimes V_{j_N}$ as a formal power series of z_a/z_b. Let $T_af(z_1,\ldots,z_N) = f(z_1,\ldots,pt_a,\ldots,z_N)$. Then we have the following:

Proposition 6.1. The function $\mathscr{F}(t,z)=\mathscr{F}(t_1,\ldots,t_M,z_1,\ldots,z_N)$ valued in $V_{j_1}\otimes\cdots\otimes V_{j_N}$ satisfies the quantum (q-deformed) Knizhnik–Zamolodchikov equation:

$$T_a \mathscr{F}(t,z) = R_{V_a V_{a-1}}\left(\frac{pz_a}{z_{a-1}}\right) \cdots R_{V_a V_1}\left(\frac{pz_a}{z_1}\right)$$

$$\pi_a(q^{h_1})^{j_0+j_\infty+1} R_{V_N V_a}\left(\frac{z_N}{z_a}\right)^{-1} \cdots R_{V_{a+1}V_a}\left(\frac{z_a+1}{z_a}\right)^{-1} \mathscr{F}(t,z)\,,$$

$$a = 1, \ldots, N\,,$$

modulo total p-difference of a function with respect to $t_1, \ldots, t_M$, where $p = q^{2(k+2)}$ as before.

Proof. It is proved in the same way as [FR, J2 or IIJMNT] if we note the following:

(1) The Drinfeld Casimir operator acts on F_l by a scalar p^{Δ_l}.

(2) The operator $S(t)$ commutes with the action of U_q' modulo a total difference.

Here (1) for general l follows from Corollary 4.4 and the result is continued to arbitrary l, and (2) is nothing else but Proposition 4.6. Q.E.D.

Now let us explicitly calculate the correlation function when the number of the screening operator is one, using the explicit expression of $\Phi_{l,j}^{l+j}(z)$ given in Sect. 5. Let us prepare the following notations:

$$(x;p)_\infty = \prod_{m=0}^{\infty}(1-p^m x), \quad (x;p,q^4)_\infty = \prod_{m_1=0}^{\infty}\prod_{m_2=0}^{\infty}(1-p^{m_1}q^{4m_2}x)\,.$$

At first, by operator product expansion, we have

$$\langle S(t)\phi_{j_N,j_N}(z_N)\ldots\phi_{j_{a+1},j_{a+1}}(z_{a+1})X^-(x)\phi_{j_a,j_a}(z_a)\ldots\phi_{j_1,j_1}(z_1)\rangle$$

$$= D(t,z_1,\ldots,z_N)\psi_a(t,x,z_1,\ldots,z_N)\,,$$

where

$$D(t,z_1,\ldots,z_N) = \prod_{a=0}^{N} t^{-\frac{2j_a}{k+2}} \prod_{a=1}^{N}\frac{(pq^{2j_a}t/z_a;p)_\infty}{(pq^{-2j_a}t/z_a;p)_\infty} \prod_{0\leqq a<b\leqq N} z_b^{\frac{2j_aj_b}{k+2}}$$

$$\times \prod_{1\leqq a<b\leqq N}\frac{(pq^{2j_a+2j_b+2}z_b/z_a;p,q^4)_\infty(pq^{-2j_a-2j_b+2}z_b/z_a;p,q^4)_\infty}{(pq^{2j_a-2j_b+2}z_b/z_a;p,q^4)_\infty(pq^{-2j_a+2j_b+2}z_b/z_a;p,q^4)_\infty}\,,$$

and

$$\psi_a(t,x,z_1,\ldots,z_N) = \frac{1}{(q-q^{-1})^2 tx}$$

$$\times\left\{q^{-1}\prod_{m=1}^{a}\frac{q^{j_m}x-q^{k+2}z_m}{x-q^{k+2+j_m}z_m} - \frac{q^{-1}t-q^{-k-1}x}{t-q^{-k-2}x}\prod_{m=1}^{a}q^{-j_m}\prod_{m=a+1}^{N}\frac{z_m-q^{-k-2-j_m}x}{z_m-q^{-k-2+j_m}x}\right.$$

$$\left.-\frac{qt-q^{k+1}x}{t-q^{k+2}x}\prod_{m=1}^{a}\frac{q^{j_m}x-q^{k+2}z_m}{x-q^{k+2+j_m}z_m} + q\prod_{m=1}^{a}q^{-j_m}\prod_{m=a+1}^{N}\frac{z_m-q^{-k-2-j_m}x}{z_m-q^{-k-2+j_m}x}\right\}.$$

Here each term of $\psi_a(t, x, z_1, \ldots, z_N)$ is understood to be a power series, by expanding as

$$\frac{u - q^a v}{u - q^b v} = (1 - q^a v/u) \sum_{m=0}^{\infty} (q^b v/u)^m, (|q^b v/u| < 1),$$

which is analytically continued.

Let $\oint \frac{dx}{2\pi\sqrt{-1}}$ denote the integration around $x = 0$ according to the expansion above, and $\mathrm{Res}_{x=x_0}$ denote the residue at $x = x_0$ of a rational function after continued. Since $\mathrm{Res}_{x=0}\, \psi_a(t, x, z_1, \ldots, z_N) = 0$, we have

$$\begin{aligned}
&\oint \frac{dx}{2\pi\sqrt{-1}} \psi_a(t, x, z_1, \ldots, z_N) \\
&= \frac{1}{(q - q^{-1})^2} \oint \frac{dx}{2\pi\sqrt{-1}} \frac{1}{tx} \left\{ q^{-1} - \frac{qt - q^{k+1}x}{t - q^{k+2}x} \right\} \prod_{m=1}^{i} \frac{q^{j_m}x - q^{k+2}z_m}{x - q^{k+2+j_m}z_m} \\
&= -\frac{1}{(q - q^{-1})} \oint \frac{dx}{2\pi\sqrt{-1}} \frac{1}{(t - q^{k+2}x)x} \prod_{m=1}^{a} \frac{q^{j_m}x - q^{k+2}z_m}{x - q^{k+2+j_m}z_m}.
\end{aligned}$$

Now $\mathrm{Res}_{x=\infty}\, \psi_a(t, x, z_1, \ldots, z_N) = 0$, and

$$\mathrm{Res}_{x=q^{-k-2}t}\, \psi_a(t, x, z_1, \ldots, z_N) = -\frac{1}{(q - q^{-1})t} \prod_{m=1}^{a} \frac{q^{-k-2+j_m}t - q^{k+2}z_m}{q^{-k-2}t - q^{k+2+j_m}z_m}.$$

Therefore, by the residue theorem, we have

$$\oint \frac{dx}{2\pi\sqrt{-1}} \psi_a(t, x, z_1, \ldots, z_N) = \frac{1}{(q - q^{-1})t} \prod_{m=1}^{a} \frac{q^{j_m}t - q^{2(k+2)}z_m}{t - q^{2(k+2)+j_m}z_m}.$$

Hence we conclude

$$\begin{aligned}
&\langle S(t)\phi_{j_N, j_N}(z_N) \ldots \phi_{j_a-1, j_a}(z_a) \ldots \phi_{j_1, j_1}(z_1) \rangle \\
&= D(t, z_1, \ldots, z_N) \oint \frac{dx}{2\pi\sqrt{-1}} \{ \psi_{a-1}(t, x, z_1, \ldots, z_N) - q^{j_a}\psi_a(t, x, z_1, \ldots, z_N) \} \\
&= D(t, z_1, \ldots, z_N) \frac{1}{(q - q^{-1})t} \left\{ 1 - q^{j_a} \frac{q^{j_a}t - q^{2(k+2)}z_a}{t - q^{2(k+2)+j_a}z_a} \right\} \prod_{m=1}^{a-1} \frac{q^{j_m}t - q^{2(k+2)}z_m}{t - q^{2(k+2)+j_m}z_m} \\
&= D(t, z_1, \ldots, z_N) \frac{(1 - q^{2j_a})}{q - q^{-1}} \frac{1}{t - q^{2(k+2)+j_a}z_a} \prod_{m=1}^{a-1} \frac{q^{j_m}t - q^{2(k+2)}z_m}{t - q^{2(k+2)+j_m}z_m}.
\end{aligned}$$

The correlation function is given by a Jackson integral of this function, which gives rise to a solution to the quantum Knizhnik–Zamolodchikov equation. This expression of the integrand is similar to the one considered previously by the author [M1] up to normalization and transformation.

7. Discussion

In this paper, we have constructed a q-deformation of the screening operators and primary fields acting on Fock modules F_{l,m_1,m_2}. The screening operator $S^+(z)$ is used to define the small Fock space $F_l = \ker Q^+$, which is the same as the Fock space of the Wakimoto module. The commutator of the screening operator $S(z)$ with U'_q is a total difference, which is eliminated by a Jackson integral in the sense of Sect. 2. Thus the correlation functions of our primary fields and screening operators satisfy the quantum (q-deformed) Knizhnik–Zamolodchikov equation. When the number of the screening operator is one, the expression for the correlation function essentially coincides with the one introduced in [M1]. In this simple case, the calculation is easy because only the primary fields of type $\phi_{j,j}(z)$ or $\phi_{j,j-1}(z)$ are necessary. In principle this calculation can be generalized to the case where the number of screening operator is larger than one. It will be discussed elsewhere. It is also desirable to study the case of $2j \in \mathbf{Z}_{\geqq 0}$, especially for an integral level. It is worthwhile to mention that, to construct the q-vertex operators of irreducible U_q-modules, the BRST analysis of Bernard–Felder [BF] seems to be necessary.

Finally we shall make a comment on the work by Kato et al. [KQS], which also gives an expression of the q-vertex operators using the Fock modules constructed by J. Shiraishi [S]. Their preprint seems to contain a serious confusion in the following sense. The primary fields are a priori operators among the large Fock modules. In their formulation the small Fock spaces are by definition $F_l = U_q|l\rangle$. This is unsatisfactory since it is not obvious that the primary field is an operator among them; it could fail when the spin l is special. Moreover, since the Fock module is reducible, the existence and uniqueness of the q-vertex operators are not guaranteed. It is not certain whether the properties (3.16)–(3.18) in [KQS] uniquely determine the expression for the primary field. To see it one must know the structure of the Fock module embedded in the large Fock space in some detail. Consequently their construction of the q-vertex operators seems to be incomplete. In the present paper we have overcome these difficulties by introducing the screening operator $S^+(z)$. The situation is similar to the $q = 1$ case (cf. [FLMSS]).

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