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The Existence and Uniqueness of Solutions of Yang-Mills Equations with Bag Boundary Conditions

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Abstract: The Cauchy problem for the Yang-Mills equations in the Coulomb gauge is studied on a compact, connected and simply connected Riemannian manifold with boundary. An existence and uniqueness theorem for the evolution equations is proven for fields with Cauchy data in an appropriate Sobolev space. The proof is based the Hodge decomposition of the Yang-Mills fields and the theory of non-linear semigroups.

1. Introduction

Quantum theory is usually formulated in a way which depends on the global structure of space. On the other hand it is supposed to describe phenomena in the atomic and subatomic scale. Hence, it is of interest to study the quantum theory of systems of finite spatial extension, and the role played by the boundary conditions.

Yang-Mills theory is a non-linear generalization of electrodynamics. Yang-Mills fields are connections in a right principal fibre bundle $\mathscr{P}$ over the space time manifold $X = M \times \mathbb{R}$ with structure group G describing the internal symmetries of the theory. The canonical variables in the Yang-Mills theory can be described as a pair of $\mathfrak{g}$-valued, time dependent 1-forms $A = A_i dx^i$ and $E = E_i dx^i$ on a typical 3 dimensional Cauchy surface M, where $\mathfrak{g}$ is the Lie algebra of the structure group. We assume that $\mathfrak{g}$ is equipped with an ad-invariant metric.

The Yang-Mills equations split into the evolution equations and the constraint equations. The constraint equation is

$$\delta E + [A\cdot, E] = 0\,, \tag{1.1}$$

where δ denotes the co-differential with respect to a given Riemannian metric g on M, $[\cdot,\cdot]$ denotes the Lie bracket in $\mathfrak{g}$, and the dot denotes the scalar product of forms,

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that is $[A\cdot, E] = g^{ij}[A_i, E_j]$. The evolution equations can be written in the form

$$\dot{A} = E + d\phi - [\phi, A]\,, \tag{1.2}$$

$$\dot{E} = - * dB - *[A \wedge B] - [\phi, E]\,, \tag{1.3}$$

where d denotes the exterior derivative, $*$ denotes the Hodge star operator on M and $\wedge$ denotes the wedge product so that $*[A \wedge B] = \varepsilon_{ijk}[A_j, B_k]dx^i$. Furthermore ϕ is the scalar potential and

$$B = *dA + \frac{*[A\wedge, A]}{2} \tag{1.4}$$

is the dual of the curvature form of the connection defined by A.

There are no evolution equations for the scalar potential. Its arbitrariness reflects the gauge degrees of freedom of the theory. In order to obtain uniqueness of the solutions of the Cauchy problem for the evolution equations we have to assume a gauge condition determining the scalar potential.

The existence of solutions of the Cauchy problem for the evolution part of Yang-Mills equations in Minkowski space-time has been studied by several authors, [1–3], who used the temporal gauge condition $\phi = 0$. The aim of this paper is to extend their results to the situation when the Cauchy surface M is a connected and simply connected compact 3-manifold with smooth boundary ∂M, and the Cauchy data are supplemented by boundary conditions requiring the vanishing of the normal component nA of A, the normal component nE of E, and the tangential component tB of B. These conditions have physical singificance as they describe the boundary behavior of the gauge fields in the MIT bag model [4]. We find that the temporal gauge condition $\phi = 0$ is inadequate for this problem and use an analogoue of the Coulomb gauge.

The differentiability conditions on the fields involved can be expressed in terms of the Sobolev spaces H^k consisting of $\mathfrak{g}$-valued 1-forms on M which are square integrable together with their derivatives up to order k, where the scalar product is defined in terms of the metric g on M and the ad-invariant metric on $\mathfrak{g}$. In this setting we consider

$$\mathbf{D} = \{(A, E) \in H^2 \times H^1 \mid nA = 0,\ ndA = 0,\ nE = 0\} \tag{1.5}$$

as the phase space for the Yang-Mills fields with the bag boundary conditions. One should remark that the prescribed boundary conditions for (A, E) are in fact independent of the Riemannian metric on M. The results of this paper are summarized in the following:

Main Theorem. *Assume that M is a smooth, compact, connected and simply connected Riemannian 3-manifold with smooth boundary ∂M. Then, for every $(A_0, E_0) \in \mathbf{D}$, there exists $T > 0$ and a unique continuous differentiable curve $[0, T) \to \mathbf{D}.\, t \mapsto (A(t), E(t))$ satisfying the Yang-Mills evolution equations* (1.2), (1.3), *where ϕ is a solution of the Neumann problem*

$$\Delta\phi = -\delta E \quad \text{and} \quad nd\phi = 0\,, \tag{1.6}$$

and the initial conditions $A(0) = A_0$, $E(0) = E_0$. If the Cauchy data (A_0, E_0) satisfy the constraint equation (1.1), *then the solution $(A(t), E(t))$ satisfies the constraint equation for all $t \in [0, T)$.*

Yang-Mills equations are invariant under conformal rescalings of the Lorentzian metric in $M \times \mathbb{R}$,

$$dt^2 - g_{ij}dx^i dx^j \mapsto \varrho^2(dt^2 - g_{ij}dx^i dx^j)\,, \tag{1.7}$$

accompanied by the transformation of the fields $(A, E) \mapsto (A, \varrho^{-1}E)$, where the conformal factor ϱ is a positive function on $M \times \mathbb{R}$. Allowing conformal factors to vanish on the boundary corresponds to the conformal compactification of unbounded Cauchy surfaces, followed by attaching the sphere of directions at spatial infinity [5]. In this way one can transform the Yang-Mills equations in Minkowski space into the Yang-Mills equations in spatially bounded domains. In particular, our theorem gives rise to a corresponding existence and uniqueness result in Minkowski space. This approach yields results in weighted Sobolev spaces over $\mathbb{R}^3$ with the weight factor determined by the conformal factors.

The proof of this theorem is based on the theory of non-linear semigroups, [6]. In Sect. 2 we review elements of the Hodge decomposition and apply it to the Yang-Mills equations. The existence and uniqueness of solutions of the linearized evolution equations is studied in Sect. 3. The full non-linear evolution equations are discussed in Sect. 4. In Sect. 5 the conservation of the constraint equation is studied. Proofs are given in the Appendix.

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2. Hodge Decomposition

The Hodge decomposition of the space L^2 of square integrable 1-forms on M is

$$L^2 = \mathscr{E} \oplus \mathscr{D} \oplus \mathscr{H}\,, \quad \text{where} \quad \begin{aligned} \mathscr{E} &= \{*du \,|\, u \in H^1,\, tu = 0\}\,, \\ \mathscr{D} &= \{df \,|\, f \in H^1,\, f|_{\partial M} = 0\}\,, \\ \mathscr{H} &= \{u \in L^2 \,|\, du = 0,\, \delta u = 0\}\,, \end{aligned} \tag{2.1}$$

are the spaces of exact 1-forms, of co-exact 1-forms, and of harmonic fields, respectively. The decomposition (2.1) is orthogonal in the L^2 scalar product. We remark that $nv = 0$ for all $v \in \mathscr{E}$. Similarly, we have a direct sum decomposition,

$$H^1 = (\mathscr{E} \cap H^1) \oplus (\mathscr{D} \cap H^1) \oplus (\mathscr{H} \cap H^1)\,, \tag{2.2}$$

cf. [7]. It is convenient to combine exact and harmonic fields together and obtain what is called the Helmholtz decomposition [8] for any 1-form $v \in L^2$ into its longitudinal and transverse component,

$$v = v^L + v^T\,, \quad \text{where} \quad v^L \in \mathscr{D} \oplus \mathscr{H} \quad \text{and} \quad v^T \in \mathscr{E}\,. \tag{2.3}$$

Observing that $*dB \in \mathscr{E}$, since $tB = 0$ by the boundary conditions for A and by (1.4), and that $d\phi$ is L^2-orthogonal to $\mathscr{E}$, we can decompose the evolution part of the Yang-Mills equations this way to obtain

$$\begin{aligned} \dot{A}^L &= E^L + d\phi - [\phi, A]^L\,, \\ \dot{A}^T &= E^T - [\phi, A]^T\,, \\ \dot{E}^L &= -(*[A\wedge, B])^L - [\phi, E]^L\,, \\ \dot{E}^T &= -*dB - (*[A\wedge, B])^T - [\phi, E]^T\,. \end{aligned} \tag{2.4}$$

The scalar potential ϕ we choose as the unique solution of the Neumann problem

$$\Delta\phi = -\delta E \quad \text{and} \quad nd\phi = 0\,, \tag{2 5}$$

which is L^2-orthogonal to harmonic functions on M. Since M is connected and simply connected these functions are just the constants, so the last condition becomes

$$\int_M \phi dV = 0\,, \tag{2.6}$$

and this choice of ϕ together with the boundary condition $nE = 0$ yields

$$E^L = -d\phi\,. \tag{2.7}$$

Since the operator δd coincides on $\mathscr{E}$ with the Laplacian Δ and $*d* = \delta$, we can rewrite the evolution equations as

$$\begin{aligned} \dot{A}^L &= -[\phi, A]^L\,, \\ \dot{A}^T &= E^T - [\phi, A]^T\,, \\ \dot{E}^L &= -(*[A\wedge, B])^L - [\phi, E]^L\,, \\ \dot{E}^T &= -\Delta A^T - \tfrac{1}{2}\,\delta[A\wedge, A] - (*[A\wedge, B])^T - [\phi, E]^T\,. \end{aligned} \tag{2.8}$$

3. Linearized Equations

Linearizing the evolution equations given by (2.8) we obtain

$$\begin{aligned} \dot{A}^L = 0 \quad &\text{and} \quad \dot{E}^L = 0\,, \\ \dot{A}^T = E^T \quad &\text{and} \quad \dot{E}^T = -\Delta A^T\,. \end{aligned} \tag{3.1}$$

To study the linearized dynamics we introduce the Hilbert space

$$\mathbf{H} = \{(A, E) \in H^1 \times L^2 \mid A^L \in H^2,\ E^L \in H^1;\quad nA^L = nE^L = 0\}\,, \tag{3.2}$$

endowed with the scalar product

$$\begin{aligned} \langle (A, E), (\tilde{A}, \tilde{E}) \rangle_{\mathbf{H}} = \langle A^L, \tilde{A}^L \rangle_{H^2} + \langle *dA^T, *d\tilde{A}^T \rangle_{L^2} \\ + \langle E^L, \tilde{E}^L \rangle_{H^1} + \langle E^T, \tilde{E}^T \rangle_{L^2}\,, \end{aligned} \tag{3.3}$$

and show:

Proposition 1. *Equations* (3.1) *define an operator* $\mathscr{S}$, *given by*

$$\mathscr{S}(A, E) = (E^T, -\Delta A^T)\,, \tag{3.4}$$

which is skew adjoint in $\mathbf{H}$ *and has as its domain*

$$\mathbf{D} = \{(A, E) \in \mathbf{H} \mid A^T \in H^2,\ E^T \in H^1;\ ndA = 0\}\,. \tag{3.5}$$

The group $\exp(t\mathscr{S})$ *of unitary transformations in* $\mathbf{H}$, *generated by* $\mathscr{S}$, *induces a group of transformations in* $\mathbf{D}$, *which acts continuously with respect to the graph norm*

$$\|(A, E)\|^2_{\mathscr{S}} = \|(A, E)\|^2_{\mathbf{H}} + \|\mathscr{S}(A, E)\|^2_{\mathbf{H}}\,. \tag{3.6}$$

By definition the longitudinal components of $\mathscr{S}(A,E)$ vanish. To make this more explicit we can write

$$\mathscr{S}(A,E) = (0^L + E^T, 0^L - \Delta A^T)\,. \tag{3.7}$$

The operator $\mathscr{S}$ maps to **D** to **H**, and its domain **D** coincides with the phase space for the Yang-Mills equations given by (1.5). Moreover, the corresponding graph norm on **D** is equivalent to the norm

$$\|(A,E)\|^2_{H^2\times H^1} = \|A\|^2_{H^2} + \|E\|^2_{H^1}\,. \tag{3.8}$$

4. Non-Linear Evolution

In terms of the generator $\mathscr{S}$ corresponding to the linearized Eq. (3.1), we can rewrite the full evolution equations (2.8) for curves $\mathbf{x}(t) = (A(t), E(t))$ in **D** as

$$\dot{\mathbf{x}}(t) = \mathscr{S}(\mathbf{x}(t)) + J(\mathbf{x}(t))\,, \tag{4.1}$$

where the non-linear term is given by

$$J(\mathbf{x}) = J(A,E) = \left([\phi, A], \tfrac{1}{2}\,\delta[A\wedge, A] - *[A\wedge, B] - [\phi, E]\right)\,, \tag{4.2}$$

with B given by Eq. (1.4) and ϕ uniquely determined by Eqs. (2.5) and (2.6). In order to apply the general theory on the existence and uniqueness of solutions of non-linear equations of the form (4.1) to the case of Yang-Mills equations we show:

Proposition 2. *The function J, given by Eq.* (4.2), *maps* **D** *to* **D**, *and is of class C^∞ with respect to the graph norm in* **D** *given by Eq.* (3.6).

Since $\exp(t\mathscr{S})$ acts as a group of continuous transformations in **D** endowed with the graph norm $\|\cdot\|_{\mathscr{S}}$, we can rewrite Eq. (4.1) together with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, in the integral form

$$\mathbf{x}(t) = \exp(t\mathscr{S})\mathbf{x}_0 + \int_0^t \exp((t-s)\mathscr{S})j(\mathbf{x}(s))\,ds\,. \tag{4.3}$$

J restricted to **D** is a continuous and smooth function from **D** to **D**, and so we can use the theory of non-linear semigroups [6]: For every $\mathbf{x}_0 \in \mathbf{D}$ there exists $T > 0$ and a unique curve $\mathbf{x}(t)$, defined for $t \in [0,T)$, satisfying the integral equation (4.3) and also the differential equation (4.1) with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. In fact it suffices to show that J is Lipschitz in order to obtain this result [9]. With $\mathscr{S}(\mathbf{x}) = (E^T, -\Delta A^T)$, the following is obvious:

Corollary. *For every $(A_0, E_0) \in$* **D**, *there exists $T > 0$, and a unique continuously differentiable curve $(A(t), E(t))$ in* **D**, *defined for $t \in [0,T)$, satisfying the Yang-Mills evolution equations, given by Eqs.* (2.8), *and the initial condition $(A(0), E(0)) = (A_0, E_0)$.*

5. Constraint Equation

Let $(A(t), E(t))$ be a curve in **D** satisfying the evolution equations. The Lie algebra valued 1-form $A_i dx^i + \phi dt$, where ϕ is determined by Eq. (2.5), define a connection in the pull-back of $\mathscr{P}$ to $M \times [0, T)$. The left-hand side of the constraint equation (1.1) is the covariant, with respect to this connection, co-differential of E, denoted by

$$\delta_A E = \delta E + [A\cdot, E]\,. \tag{5.1}$$

Thus, the constraint equation reads

$$\delta_A E = 0\,. \tag{5.2}$$

One can show by direct computation that the evolution equations (1.2), (1.3) imply the vanishing of the covariant time derivative of $\delta_A E$,

$$\frac{D}{dt}\,\delta_A E = \frac{d}{dt}\,(\delta_A E) + [\phi, \delta_A E] = 0\,. \tag{5.3}$$

Since the scalar product in $\mathfrak{g}$ used in the definition of the Hilbert space structures is ad-invariant, it follows that

$$\frac{d}{dt}\,\|\delta_A E\|^2_{L^2} = 2\left\langle \frac{D}{dt}\,\delta_A E, \delta_A E\right\rangle_{L^2} = 0\,. \tag{5.4}$$

Hence $\|\delta_A E\|^2_{L^2}$ is a constant of motion. In particular, the evolution preserves the constraint equation (1.1).

Appendix

For the proof of the existence and uniqueness result for the Yang-Mills system the knowledge about the ellipticity of a Neumann problem for 1-forms and scalar functions is crucial:

Proposition A.1. *Let M be a smooth Riemannian 3-manifold with smooth boundary.*
(i) *On M the boundary value problem given by*

$$\Delta v = f \quad \textit{with} \quad nv = 0 \quad \textit{and} \quad ndv = 0 \tag{A.1}$$

is elliptic, where v is a 1-form. Especially, for $v \in H^2$, obeying the given boundary conditions, one can estimate

$$\|v\|_{H^2} \le K_1(\|\Delta v\|_{L^2} + \|v\|_{H^1})\,. \tag{A.2}$$

(ii) *For $w \in H^1$ and $nw = 0$ the Neumann problem on M given by*

$$\Delta\psi = -\delta w \quad \textit{with} \quad nd\psi = 0 \tag{A.3}$$

is elliptic and has a unique solution ψ in the space of functions L^2-orthogonal to constants. ψ is of class H^2 and the following estimate holds:

$$\|\psi\|_{H^2} \le K_2\|w\|_{H^1}\,. \tag{A.4}$$

Proof. For (i) we need to show that the boundary value problem (A.1) satisfies the Lopatinskiĭ-Ŝapiro condition [11].[1] Therefore, let $p \in \partial M$ be a boundary point

[1] See also [10] for a more explicit version of that condition

and choose g-orthogonal coordinates (x_1, x_2, x_3) in the tangent space T_pM at this point such that $(0,0,1)$ is the inward pointing normal vector. Fourier-transforming the homogeneous problem corresponding to (A.1) at p with respect to the (x_1, x_2)-coordinates yields

$$(-|\xi|^2 + \partial_{x_3}^2)\tilde{v}(\xi_1, \xi_2, x_3) = 0\,, \tag{A.5}$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2$ and $\tilde{v} = \tilde{v}_\imath dx_\imath$. The boundary conditions turn into

$$\begin{aligned} \tilde{v}_3(\xi_1, \xi_2, 0) &= 0\,, \\ i\xi_2\tilde{v}_3(\xi_1, \xi_2, 0) - \partial_{x_3}\tilde{v}_2(\xi_1, \xi_2, 0) &= 0\,, \\ i\xi_2\tilde{v}_3(\xi_1, \xi_2, 0) - \partial_{x_3}\tilde{v}_1(\xi_1, \xi_2, 0) &= 0\,. \end{aligned} \tag{A.6}$$

As the solution set for the (ordinary) differential equation (A.5) we get

$$\mathscr{U}^+ = \{\tilde{v}\exp((|\xi|x_3) \mid \tilde{v} \in \mathbb{C}^3\} \tag{A.7}$$

so that the linear system corresponding to (A.6) is bijective on $\mathscr{U}^+$. Hence the boundary value problem (A.1) is elliptic and defines a Fredholm operator [11]. Using a general argument about Fredholm operators on Banach space [10, Theorem 12.12] we then get the estimate (A.2).

Considering (ii) the ellipticity of the Neumann problem (A.3) can be shown in the same way as for (A.1). The existence of a unique solution under the given conditions as a well established fact [10]. Hence one concludes from ellipticity that ψ is in H^2 for $w \in H^1$. Therefore one then obtains

$$\|\psi\|_{H^2} \leq K_3\|\delta w\|_{L^2}\,, \tag{A.8}$$

which yields (A.4) by using the continuity of the operator $\delta: H^1 \to L^2$. $\square$

In order to study the linearized Yang-Mills equations, given in Sect. 3, we need:

Lemma A.2. *The operator $\Delta = \delta d$ on $\mathscr{E}$ with domain*

$$\mathbf{V} = \{v \in \mathscr{E} \cap H^2 \mid ndv = 0\} \tag{A.9}$$

is self-adjoint with respect to the L^2-product.

Proof Clearly Δ maps $\mathbf{V}$ to $\mathscr{E}$, and is symmetric on $\mathbf{V}$, since by Stoke's theorem

$$\int\limits_M (w \cdot \delta dv)dV = \int\limits_M (*dw \cdot *dv)dV + \int\limits_{\partial M} tw \wedge *ndv\,, \tag{A.10}$$

and the boundary term vanishes for $v \in \mathbf{v}$. Furthermore $nv = 0$ by definition of $\mathbf{V}$. In order to prove that Δ is self-adjoint in $\mathscr{E}$ one has to show [12, Theorem 5.19] that range $(\mathbf{I} + \Delta) = \mathscr{E}$. Therefore let $\mathbf{W} = \mathscr{E} \cap H^1$, and $a(v, w)$ be the bilinear form on $\mathbf{W}$ given by

$$a(v, w) = \langle v, w\rangle_{L^2} + \langle *dv, *dw\rangle_{L^2}\,. \tag{A.11}$$

Clearly $a(v, w)$ is continuous and symmetric on $\mathbf{W}$. Since for all $w \in H^1$, which are L^2-orthogonal to the harmonic fields, there is a constant $C > 0$ such that Friedrichs inequality

$$(\|*dw\|_{L^2}^2 + \|\delta w\|_{L^2}^2) \geq C\|w\|_{H^1}^2 \tag{A.12}$$

[7, Theorem 7.7.9] holds, this implies that $a(v, v) \geq C\|v\|_{H^1}^2$ for $v \in \mathbf{W}$. Hence the Lax-Milgram lemma [13] implies the existence of an isomorphism $\tilde{\mathbf{A}}: \mathbf{W} \to \mathbf{W}^*$ such that $a(v,w) = \langle \mathbf{A}v \,|w\rangle$ for all $v, w \in \mathbf{W}$. Given $g \in \mathbf{W}^*$, the equation $\mathbf{A}v = f$ means that

$$\int_M (w \cdot v)dV + \int_M (*dw - *dv)dV = \langle f \mid w\rangle \tag{A.13}$$

for each $w \in \mathbf{W}$. By applying formally Stokes' theorem (A.10) this yields $(\mathbf{I}+\Delta)v = f$ in the sense of distributions on M and $ndv = 0$ weakly on ∂M.

It remains to show that for each $f \in \mathscr{E} \subset \mathbf{W}^*$, the equation $\mathbf{A}v = f$ implies that $v \in H^2$, and $ndv = 0$ strongly on ∂M. The boundary value problem

$$(\mathbf{I} + \Delta)v = f \quad \text{with} \quad nv = 0 \quad \text{and} \quad ndv = 0 \tag{A.14}$$

is elliptic by Proposition A.1. Thus the corresponding regularity result[2] in [14] guarantees that any solution of (A.14) is in H^2 for $f \in L^2$. This proves that Δ is self-adjoint in $\mathscr{E}$. □

In Sect. 3 the Yang-Mills equations have been formulated on the Hilbert space

$$\mathbf{H}: = \{(A,E) \in H^1 \times L^2 \,|\, A^L \in H^2,\ E^L \in H^1,\ nA^L = nE^L = 0\}, \tag{A.15}$$

endowed with a scalar product

$$\begin{aligned}\langle (A,E),(\tilde{A},\tilde{E})\rangle_{\mathbf{H}} = \langle A^L, \tilde{A}^L\rangle_{H^2} + \langle *dA^T, *d\tilde{A}^T\rangle_{L^2} \\ + \langle E^L, \tilde{E}^L\rangle_{H^1} + \langle E^T, \tilde{E}^T\rangle_{L^2}.\end{aligned} \tag{A.16}$$

Furthermore we had the subspace

$$\mathbf{D} = \{(A,E) \in \mathbf{H} \,|\, A^T \in H^2,\ E^T \in H^1;\ ndA = 0\}, \tag{A.17}$$

equipped with the graph norm $\|\cdot\|_{\mathscr{S}}$ of the operator $\mathscr{S}$, which can be written as

$$\|(A,E)\|^2_{\mathscr{S}} = \|(A,E)\|^2_{\mathbf{H}} + \|\Delta A^T\|^2_{L^2} + \| * dE^T\|^2_{L^2}. \tag{A.18}$$

Lemma A.3. (i) *The scalar product* (A.16) *defines on* $\mathbf{H}$ *a norm* $\|(A,E)\|_{\mathbf{H}}$, *equivalent to the norm*

$$|||(A,E)|||^2 = \|A^L\|^2_{H^2} + \|A^T\|^2_{H^1} + \|E^L\|^2_{H^1} + \|E^T\|^2_{L^2}. \tag{A.19}$$

(ii) *The graph norm* $\|(A,E)\|_{\mathscr{S}}$ *on* $\mathbf{D}$ *is equivalent to the norm on* $H^2 \times H^1$, *given by*

$$\|(A,E)\|^2_{H^2\times H^1} = \|A\|^2_{H^2} + \|E\|^2_{H^1}. \tag{A.20}$$

Proof To show (i) we need to estimate

$$C_1\|*dA^T\|_{L^2} \leq \|A^T\|_{H^1} \leq C_2\|*dA^T\|_{L^2}. \tag{A.21}$$

The left-hand side is obvious by the continuity of $*d: H^1 \to L^2$. For the right-hand side we observe that $A^T \in H^1 \cap \mathscr{E}$ is L^2-orthogonal to the space $\mathscr{H}$ and $\delta A^T = 0$, such that Friedrichs' inequality (A.12) yields the required estimate for $\|A^T\|_{H^1}$ in terms of $\|*dA^T\|^2_{L^2}$.

For (ii) e use the same argument for the term $\|*dE^T\|^2_{L^2}$. It remains to show that

$$C_3\|A^T\|^2_{H^2} \geq (\|\Delta A^T\|^2_{L^2} + \|A^T\|^2_{H^1}) \geq C_4\|A^T\|^2_{H^2}. \tag{A.22}$$

[2] For scalar functions this can also be found in [10]

Since Δ, considered as a mapping from H^2 to L^2 is continuous, the left-hand estimate is obvious. On the other side we are led to consider for $A^T \in \mathscr{C} \cap H^2$, the boundary value problem

$$\Delta A^T = f \quad \text{with} \quad nA^T = 0 \quad \text{and} \quad ndA^T = 0\,. \tag{A.23}$$

As shown in Proposition A.1 this problem is elliptic, so that we can read off the required estimate from Eq. (A.2). $\square$

Writing the linearized evolution equations (3.1) for the Yang-Mills fields in the terms of the operator $\mathscr{S}$, given by (3.4) as

$$(\dot{A}^L + \dot{A}^T, \dot{E}^L + \dot{E}^T) = \mathscr{S}(A,E) = (0^L + E^T, 0^L - \Delta A^T)\,, \tag{A.24}$$

Proposition 1 can be reformulated as follows.

Proposition A.4. The operator $\mathscr{S}$, given by Eq. (A.24), with domain $\mathbf{D}$ given by Eq. $(A.17)$, generates a group $\exp(t\mathscr{S})$ of unitary transformations in the Hilbert space $\mathbf{H}$. This induces a group of transformations in $\mathbf{D}$, which are continuous with respect to the graph norm

$$\|(A,E)\|^2_{\mathscr{S}} = \|(A,E)\|^2_{\mathbf{H}} + \|\mathscr{S}(A,E)\|^2_{\mathbf{H}}\,. \tag{A.25}$$

Proof. In order to prove that $\mathscr{S}$ with domain $\mathbf{D}$ generates a 1-parameter group of unitary transformations in $\mathbf{H}$ we have to show that $\mathscr{S}$ is skew adjoint, that is $\mathscr{S}$ is skew-symmetric and range $(\mathbf{I} + \mathscr{S}) = \mathbf{H}$, cf. [12]. If (A,E) and $(\tilde{A},\tilde{E})$ belong to $\mathbf{D}$, then A^T and $\tilde{A}^T$ belong to $\mathbf{V}$, the domain of Δ given by Eq. (A.9). The skew-symmetry of $\mathscr{S}$ is proven by using (A.10):

$$\begin{aligned}\langle (A,E), \mathscr{S}(\tilde{A},\tilde{E})\rangle_{\mathbf{H}} &= \langle *dA^T, *d\tilde{E}^T\rangle_{L^2} - \langle E^T, \Delta\tilde{A}^T\rangle_{L^2}\\ &= \langle \Delta A^T, \tilde{E}^T\rangle_{L^2} - \langle *dE^T, *d\tilde{A}^T\rangle_{L^2}\\ &= -\langle \mathscr{S}(A,E), (\tilde{A},\tilde{E})\rangle_{\mathbf{H}}\,.\end{aligned} \tag{A.26}$$

To show that range $(\mathbf{I} + \mathscr{S}) = \mathbf{H}$, consider the corresponding system of equations given by

$$A^L = f\,, \quad (A^T + E^T) = g\,, \quad E^L = h\,, \quad \text{and} \quad (E^T - \Delta A^T) = k\,, \tag{A.27}$$

with $(f,g,h,k) \in ((\mathscr{D} \oplus \mathscr{H}) \cap H^2) \times (\mathscr{C} \cap H^1) \times ((\mathscr{D} \oplus \mathscr{H}) \cap H^1) \times \mathscr{C}$ arbitrary. The solvability of the longitudinal equations is obvious. Eliminating E^T we get

$$(\mathbf{I} + \Delta)A^T = (g - k) \quad \text{with} \quad (g-k) \in \mathscr{C} \text{ arbitrary}\,. \tag{A.28}$$

This equation is always solvable in H^2 as shown in Lemma A.2. For $E^T = (g - A^T) \in \mathscr{C} \cap H^1$ we end up with a pair $(A,E) \in \mathbf{D}$ satisfying Eq. (A.27).

The 1-parameter group $\exp(t\mathscr{S})$ of unitary transformations in $\mathbf{H}$ generated by $\mathscr{S}$ commutes with $\mathscr{S}$ and preserves the domain $\mathbf{D}$, [15, p. 239]. Hence one gets for every (A,E) in $\mathbf{D}$ and every $t \in \mathbb{R}$,

$$\begin{aligned}\|\exp(t\mathscr{S})(A,E)\|^2_{\mathscr{S}} &= \|\exp(t\mathscr{S})(A,E)\|^2_{\mathbf{H}} + \|\exp(t\mathscr{S})\mathscr{S}(A,E)\|^2_{\mathbf{H}}\\ &= \|(A,E)\|^2_{\mathbf{H}} + \|\mathscr{S}(A,E)\|^2_{\mathbf{H}} = \|(A,E)\|^2_{\mathscr{S}}\,,\end{aligned} \tag{A.29}$$

which shows that operators $\exp(t\mathscr{S})$ are bounded on $\mathbf{D}$. The group property of $\exp(t\mathscr{S})$ in $\mathbf{D}$ follow from its group property in $\mathbf{H}$. Moreover, the same arguments, as used above, yield

$$\begin{aligned}\|\exp(t\mathscr{S})(A,E) - (A,E)\|_{\mathscr{S}} &= \|\exp(t\mathscr{S})(A,E) - (A,E)\|_{\mathbf{H}}\\ &+ \|\exp(t\mathscr{S})\mathscr{S}(A,E) - \mathscr{S}(A,E)\|\mathbf{H} \underset{t\to 0}{\longrightarrow} 0\,,\end{aligned} \tag{A.30}$$

which implies that $\exp(t\mathscr{S})$ is a continuous group of operators in $\mathbf{D}$. $\square$

To estimate the non-linear terms of the Yang-Mills evolution we need further:

Proposition A.5. *Let M be a compact 3-manifold with boundary, then the following holds for 1-forms and functions of the indicated Sobolev classes, respectively:*

$$v, w \in H^2 \Rightarrow v \cdot w \in H^2 \quad \text{and} \quad \|v \cdot w\|_{H^2} \leq K_1 \|v\|_{H^2} \|w\|_{H^2}, \tag{A.31}$$

$$v \in H^2, w \in H^1 \Rightarrow v \cdot w \in H^1 \quad \text{and} \quad \|v \cdot w\|_{H^1} \leq K_2 \|v\|_{H^2} \|w\|_{H^1}. \tag{A.32}$$

If v and w are 1-forms the dot-product stands for any algebraic product.

Proof. For functions (A.31) reflects the Banach algebra property of the space H^2 over the 3-manifold M, cf. [16]; for 1-forms the estimate follows from that property.

To show (A.32) we observe from the Sobolev embedding theorem that u and w are in L^4 if they are in H^1. Hence we get from that theorem and Hölder's inequality

$$\|u \cdot w\|^2_{L^2} \leq \|u\|^2_{L^4} \|w\|^2_{L^4} \leq C_1 \|u\|^2_{H^1} \|w\|^2_{H^1}. \tag{A.33}$$

On the compact 3-manifold M, any $v \in H^2$ is continuous, hence $\sup|v| \leq C_2 \|v\|_{H^2}$ and

$$\|v \cdot u\|^2_{L^2} \leq (\sup|v|)^2 \int_M |u|^2 dV \leq C_3 \|v\|^2_{H^2} \|u\|^2_{L^2}. \tag{A.34}$$

Using these two estimates we see from

$$\begin{aligned} \tfrac{1}{2}\|v \cdot w\|^2_{H^1} &\leq \|Dv \cdot w\|^2_{L^2} + \|v \cdot Dw\|^2_{L^2} + \|v \cdot w\|^2_{L^2} \\ &\leq C_1 \|Dv\|^2_{H^1} \|w\|^2_{H^1} + C_3 \|v\|^2_{H^2} \|Dw\|^2_{L^2} \\ &\quad + C_1 \|v\|^2_{H^1} \|w\|^2_{H^1}, \end{aligned} \tag{A.35}$$

that $(v \cdot w)$ is of class H^1 and obeys the estimate (A.32), stated above. $\square$

Using this proposition, we now can show that the non-linear term $J(\mathbf{x})$ in the Yang-Mills evolution equations is of class C^∞.

Proposition A.6. *Let the Hilbert spaces $\mathbf{H}$ and $\mathbf{D}$ be given by Eqs.* (3.2) *and*

$$\mathbf{D} = \{(A, E) \in H^2 \times H^1 \,|\, nA = nE = 0,\, ndA = 0\}, \tag{A.36}$$

respectively. The non-linear operator

$$\begin{aligned} &J : \mathbf{D} \to \mathbf{H}, \\ &(A, E) \mapsto ([\phi, A], \tfrac{1}{2}\,\delta[A\wedge, A] - *[A\wedge, B] - [\phi, E]), \end{aligned} \tag{A.37}$$

where ϕ is a solution of the Neumann problem

$$\Delta\phi = -\delta E \quad \text{and} \quad nd\phi = 0, \tag{A.38}$$

and $B = *dA + \frac{1}{2} *[A\wedge, A]$, has its range in $\mathbf{D}$ and $J.\mathbf{D} \to \mathbf{D}$ is bounded, continuous and smooth with respect to the graph norm $\|\cdot\|_{\mathscr{S}}$ of $\mathbf{D}$.

Proof. For the range of J we observe from Proposition A.1 that ϕ is of Sobolev class H^2 such that $[\phi, A]$ also is in H^2 by (A.31). Similarly we find $*[A\wedge, A]$ in H^2, hence B and $*d[A\wedge, A]$ are of class H^1, and from (A.32) we get $*[A\wedge, B]$ in H^1.

With respect to the boundary conditions we immediately see from (A.36) and (A.38) that

$$
\begin{aligned}
n[\phi, A] &= [\phi, nA] = 0\,, \\
nd[\phi, A] &= n[d\phi\wedge, A] + [\phi, ndA] = 0\,, \\
n\delta[A\wedge, A] &= \delta(n[A\wedge, A] = 0\,, \\
n(*[A\wedge, B] - [\phi, E]) &= *[tA\wedge, tB] - [\phi, nE] = 0\,.
\end{aligned}
\tag{A.39}
$$

Hence the operator J maps **D** to **D**. In order check continuity it suffices by Lemma A.3 to estimate

$$
\begin{aligned}
&\mathscr{E}_A^2 = \|[\phi, A] - [\tilde{\phi}, \tilde{A}\|_{H^2}^2 + \tfrac{1}{4}\,\|\delta[A\wedge, A] - \delta[\tilde{A}\wedge, \tilde{A}]\|_{H^1}^2\,, \\
&\mathscr{E}_B^2 = \| *[A\wedge, B] - *[\tilde{A}\wedge, \tilde{B}]\|_{H^1}^2 \quad \text{and} \\
&\mathscr{E}_E^2 = \|[\phi, E] - [\tilde{\phi}, \tilde{E}]\|_{H^1}^2\,,
\end{aligned}
\tag{A.40}
$$

for $\|A - \tilde{A}\|_{H^2}$ and for $\|E - \tilde{E}\|_{H^1}$ sufficiently small. For the first expression we get from (A.31), the elliptic estimate (A.4) for $\|\phi\|_{H^2}$, and the continuity of the operator $\delta\colon H^2 \to H^1$,

$$
\begin{aligned}
\mathscr{E}_A^2 \le C_1(&\|E\|_{H^1}^2\|A - \tilde{A}\|_{H^2}^2 + (1 + \|A\|_{H^2}^2)\|E - \tilde{E}\|_{H^1}^2 \\
&+ (1 + \|A\|_{H^2}^2)\,\|A - \tilde{A}\|_{H^2}^2)\,.
\end{aligned}
\tag{A.41}
$$

Considering $\mathscr{E}_B$ we obtain by using the same arguments,

$$\|B\|_{H^1} \le C_2(\|A\|_{H^2} + \|A\|_{H^2}^2)$$

and (A.42)

$$\|B - \tilde{B}\|_{H^1} \le C_3\|A - \tilde{A}\|_{H^2}(1 + \|A\|_{H^2})\,.$$

From these estimates and the argument (A.32) we obtain

$$\mathscr{E}_B^2 \le C_4\|A - \tilde{A}\|_{H^2}^2(1 + \|A\|_{H^2})^4\,. \tag{A.43}$$

Finally we see from (A.32) and (A.4) that

$$\mathscr{E}_H^2 \le C_5\|E - \tilde{E}\|_{H^1}^2(1 + \|E\|_{H^1})^2\,. \tag{A.44}$$

Putting these terms together we end up with the required estimate

$$\|J(A, E) - J(\tilde{A}, \tilde{E})\|_{\mathscr{Y}}^2 \le C_6(1 + \|E\|_{H^1} + \|A\|_{H^2})^4\|(A, E) - (\tilde{A}, \tilde{E})\|_{\mathscr{Y}}^2\,, \tag{A.45}$$

which proves the continuity of $J\colon \mathbf{D} \to \mathbf{D}$. With respect to the differentiability of J write $\mathbf{y} = (a, e)$ for an arbitrary element $\mathbf{y} \in \mathbf{D}$. Then

$$
\begin{aligned}
DJ(A, E)\,(a, e) = ([\psi, A] + [\phi, a],\ &\delta[A\wedge, a] - *[a\wedge, B] \\
&- *[A\wedge, b] - [\psi, E] - [\phi, e])\,, \\
&\text{where } \Delta\psi = -\delta e,\ nd\psi = 0 \text{ and } b = *da + *[A\wedge, a]\,.
\end{aligned}
\tag{A.46}
$$

Observing that a, e, ψ and b are of the same Sobolev classes as A, E, ϕ and B, all the estimates used to prove continuity of J also can be applied here. Hence one shows, literally as above, that $dJ(A, E)\,(a, e)$ is continuous. Analogous arguments also hold for the higher derivatives of J. Actually, derivatives of J of order ≥ 4 vanish identically. □

This result proves Proposition 2, and completes the proof of the Main Theorem.

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