Commun. Math. Phys. 159, 471–502 (1994)

Communications in
Mathematical
Physics

Representations of Affine Lie Algebras, Elliptic r-Matrix Systems, and Special Functions

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Received: 15 February 1993/in revised form: 29 July 1993

Abstract: The author considers an elliptic analogue of the Knizhnik–Zamolodchikov equations – the consistent system of linear differential equations arising from the elliptic solution of the classical Yang–Baxter equation for the Lie algebra $\mathfrak{sl}_N$. The solutions of this system are interpreted as traces of products of intertwining operators between certain representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_N$. A new differential equation for such traces characterizing their behavior under the variation of the modulus of the underlying elliptic curve is deduced. This equation is consistent with the original system.

It is shown that the system extended by the new equation is modular invariant, and the corresponding monodromy representations of the modular group are defined. Some elementary examples in which the system can be solved explicitly (in terms of elliptic and modular functions) are considered. The monodromy of the system is explicitly computed with the help of the trace interpretation of solutions. Projective representations of the braid group of the torus and representations of the double affine Hecke algebra are obtained.

Introduction

In 1984 Knizhnik and Zamolodchikov [KZ] studied matrix elements of products of intertwining operators between representations of the affinization $\hat{\mathfrak{g}}$ of a finite dimensional simple complex Lie algebra $\mathfrak{g}$ at level k. These matrix elements are analytic functions of several complex variables, and it was found that they satisfy a certain remarkable system of linear differential equations which is now called the Knizhnik–Zamolodchikov (KZ) system:

$$\kappa \frac{\partial \Psi}{\partial z_i} = \sum_{j=1, j \neq i}^{n} \frac{\Omega_{ij}}{z_i - z_j} \Psi . \tag{1}$$

Here $\Psi(z_1, \ldots, z_n)$ is a function of n complex variables with values in the product $W = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ of n representations of $\mathfrak{g}$, κ is a nonzero complex number, and

$$\Omega_{ij} = \sum_p (x_p)_i (x_p)_j ,$$

$$(x_p)_i = \mathrm{Id}_1 \otimes \cdots \otimes \mathrm{Id}_{i-1} \otimes x_p \otimes \mathrm{Id}_{i+1} \otimes \cdots \otimes \mathrm{Id}_n \in \mathrm{End}(W) , \tag{2}$$

where the summation is over an orthonormal base $\{x_p\}$ of $\mathfrak{g}$ with respect to the invariant form, and Id_j denotes the identity operator in V_j.

Solutions of the KZ equations are very interesting special functions which generalize the Gauss hypergeometric function. By now explicit integral representations of these functions have been found [M, SV], and their monodromy has been completely computed [Koh, Dr], [TK] (for $\mathfrak{g} = \mathfrak{sl}_2$), [V]. It turned out that the monodromy of the KZ system expresses in terms of the quantum R-matrix – the quasitriangular structure of the corresponding quantum group $U_q(\mathfrak{g})$, where $q = \exp\left(\frac{2\pi i}{\kappa}\right)$.

Cherednik [Ch1] considered a general consistent system of differential equations of the "factorized" form

$$\kappa \frac{\partial \Psi}{\partial z_i} = \sum_{j=1, j \neq i}^{n} r_{ij}(z_i - z_j) \Psi , \tag{3}$$

where $r(u)$ is a meromorphic function with values in $\mathfrak{g} \otimes \mathfrak{g}$, and $r_{ij}(u)$ denotes the action of $r(u)$ in W: the first factor acts in V_i and the second one in V_j, and the *unitarity* property $r_{ij}(u) = -r_{ji}(-u)$ is assumed (this condition is equivalent to the invariance of solutions of system (3) under the simultaneous translation of all variables z_j by the same constant). It was observed in [Ch1] that system (3) is consistent if and only if the function $r(u)$ is a *classical r-matrix*, i.e. if it satisfies the classical Yang–Baxter equation:

$$\begin{aligned}[r_{ij}(z_i - z_j), r_{ik}(z_j - z_k)] + [r_{ij}(z_j - z_k), r_{jk}(z_k - z_i)] \\ + [r_{ik}(z_k - z_i), r_{jk}(z_i - z_j)] = 0 .\end{aligned} \tag{4}$$

Therefore, a consistent system (3) is called a local r-matrix system.

Clearly, the KZ system is a special case of a local r-matrix system, for a simplest r-matrix $r(u) = \Omega/u$, $\Omega = \sum_p x_p \otimes x_p$, (the summation is over an orthonormal base $\{x_p\}$ of $\mathfrak{g}$). This gives rise to a question: what other r-matrix systems are possible? This question was essentially answered by Belavin and Drinfeld in 1982 [BeDr]. They classified all solutions of (3) satisfying the nondegeneracy condition: $r(u)$ is invertible as a map $\mathfrak{g}^* \to \mathfrak{g}$ for at least one complex number u. This classification states that all such solutions are unitary and, in terms of dependence on u, up to an equivalence relation, there are only three types of functions $r(u)$: rational, trigonometric, and elliptic. For instance, the function $r(u) = \Omega/u$ which is involved in the KZ system is a rational r-matrix.

Trigonometric and elliptic nondegenerate unitary solutions of the classical Yang–Baxter equation are completely classified [BeDr]. On the contrary, a satisfactory classification of rational solutions is unknown.

The Belavin-Drinfeld classification suggests a two-step generalization of the KZ system: KZ equations (rational r-matrix equations) – trigonometric r-matrix equations – elliptic r-matrix equations. One should expect that these local

r-matrix systems should have remarkable properties and provide new interesting special functions as their solutions.

Cherednik [Ch2] found an interpretation of solutions of nondegenerate unitary r-matrix equations in terms of representation theory of affine Lie algebras. It was proved in [Ch2] that the general solution is the coinvariant (or the so-called τ-function) of the Lie algebra of $\mathfrak{g}$-valued rational functions on a rational or elliptic curve with singularities at designated points $z_1, \ldots, z_n$ with respect to a certain representation of this algebra. This interpretation has found many applications.

The properties of the trigonometric r-matrix equations are now fairly well understood. Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a Cartan decomposition of $\mathfrak{g}$, and let $\hat{\Omega}^+, \hat{\Omega}^-, \hat{\Omega}^0$ be the orthogonal projections of Ω to the subspaces $\mathfrak{n}^+ \otimes \mathfrak{n}^-, \mathfrak{n}^- \otimes \mathfrak{n}^+$, and $\mathfrak{h} \otimes \mathfrak{h}$, respectively. Then the simplest trigonometric $\mathfrak{g} \otimes \mathfrak{g}$-valued r-matrix has the form $r(u) = \dfrac{\Omega^+ e^u + \Omega^-}{e^u - 1}$, where $\Omega^\pm = \hat{\Omega}^\pm + \frac{1}{2}\hat{\Omega}_0$. It turns out that there exists a transformation of coordinates which maps the $n+1$-point KZ equations to the n-point trigonometric r-matrix equations with the above $r(u)$ (which are called the trigonometric KZ equations), which allows one to carry over to this case all the results about the KZ equations, including the integral formulas.

Trigonometric r-matrix equations with more general r-matrices were studied by Cherednik [Ch2], who gave an explicit integral formula for the general solution.

Nondegenerate unitary elliptic r-matrices exist only for $\mathfrak{g} = \mathfrak{sl}_N$. They were found in the case $N = 2$ by Sklyanin and in the general case by Belavin [Be] (note that the $N = 2$ elliptic r-matrix is the quasiclassical limit of Baxter's quantum R-matrix which arises in statistical mechanics):

$$\rho(z|\tau) = \Omega\zeta(z) + \sum_{0 \leqq m,n \leqq N-1,\, m^2+n^2>0} (1 \otimes \beta^n \gamma^{-m})(\Omega)\left(\zeta\left(z - \frac{m+n\tau}{N}\middle|\tau\right) - \zeta\left(-\frac{m+n\tau}{N}\middle|\tau\right)\right), \tag{5}$$

where β, γ are two commuting inner automorphisms of $\mathfrak{sl}_N$ of order N with no common invariant vectors, τ lies in the upper half of the complex plane, and $\zeta(z|\tau)$ is the Weierstrass ζ-function. The properties of the corresponding r-matrix equations are not very well understood. For example, integral formulas (or any other explicit representations) for solutions are unknown.

It has been anticipated that some progress in the elliptic case can be achieved by using the intertwining (vertex) operator language which was originally used by Knizhnik and Zamolodchikov [KZ]. Frenkel and Reshetikhin [FR] conjectured that if one takes traces of products of intertwiners rather than matrix elements, one should be able to obtain solutions to the elliptic r-matrix equations. The same idea occurs in the paper of Bernard [Ber] who studied expectation values for the Wess–Zumino–Witten model on an elliptic curve and was led to consider trace expressions of a similar sort. Bernard deduced some differential relations for traces, but they were not a closed system of differential equations since the two commuting inner automorphisms did not enter the game. The idea to consider traces is also suggested by the aforementioned theory developed in [Ch2] which states that solutions of an elliptic r-matrix system should express in terms of the τ-function for the corresponding elliptic curve, which is basically a trace expression involving vertex operators.

This paper is devoted to making this idea work (at the price of a few modifications). More precisely, we represent solutions of the elliptic r-matrix equations in the form

$$F(z_1, \dots, z_n|q) = \mathrm{Tr}|_{M_{\lambda,k}}(\Phi^1(z_1)\dots\Phi^n(z_n)Bq^{-\partial}), \tag{6}$$

where $M_{\lambda,k}$ is a Verma module over (a twisted version of) the affine algebra $\widehat{\mathfrak{sl}}_N$, $\Phi^i(z_i)$: $M_{\lambda_i,k} \to \hat{M}_{\lambda_{i-1},k} \otimes V_i$ are intertwiners for $\widehat{\mathfrak{sl}}_N$ (the action of $\widehat{\mathfrak{sl}}_N$ on V_i: $(a \otimes t^m)v = z^m av, a \in \mathfrak{g}, v \in V_i$; the hat denotes a completion of the Verma module), $q \in \mathbb{C}, |q| < 1, \partial$ is the grading operator in $M_{\lambda,k}$, and B is the map of Verma modules induced by an outer automorphism of $\widehat{\mathfrak{sl}}_N$ of order N (rotation of the affine Dynkin diagram, which is a regular N-gon, through the angle of $2\pi/N$). This representation helps to compute the monodromy of the elliptic r-matrix equations. It is still unclear how to deduce integral formulas for solutions similar to those existing for the rational and trigonometric KZ equations.

Remark. The recent paper [CFW] studies generalized hypergeometric functions on the torus – integrals over twisted cycles of products of powers of elliptic functions. These functions satisfy certain linear differential equations with elliptic coefficients, but it is not clear how these equations are related to the elliptic r-matrix equations.

In Sect. 1 we introduce a realization of an affine Lie algebra twisted by an inner automorphism and twisted versions of Verma modules and evaluation modules. This twisting is necessary to eventually produce solutions of the elliptic r-matrix equations.

In Sect. 2 we define twisted intertwiners and deduce a differential equation for them. Then we define twisted correlation functions and show that they satisfy a twisted trigonometric KZ system. This system, however, can be reduced to the usual trigonometric KZ system by a simple transformation.

In Sect. 3 we study trace expressions of the form (6) and prove that they satisfy an elliptic r-matrix system of differential equations in the variables $\log z_j$. (We call this system the elliptic KZ equations.) We also deduce one more differential equation which expresses the derivative $\frac{\partial F}{\partial q}$ in terms of F. Thus we get a consistent system of $n + 1$ differential equations – the extended elliptic KZ system.

In Sect. 4 we show that the elliptic KZ equations are modular invariant – they are preserved under the action of the congruence subgroup $\Gamma(N)$ of the modular group. The $n + 1^{\text{th}}$ equation involving the derivative by q is "almost" invariant – it undergoes a very minor modification under the action of an element of $\Gamma(N)$. This implies that the fundamental solution of the extended elliptic KZ system changes under the action of the modular group according to a certain representation of this group. In other words, the functions $F(z_1, \dots, z_n|q)$ yield nontrivial examples of vector-valued automorphic (modular) forms.

In Sect. 5 we consider a few simple special cases of the extended elliptic KZ system for $\mathfrak{sl}_2$ in which it can be solved in quadratures. In this case, solutions are expressed in terms of elliptic and modular functions.

In Sect. 6 we compute the monodromy of the elliptic KZ equations. According to the results of [Ch1], this monodromy yields a representation of the generalized braid group of the torus. We compute this representation and show that local monodromies (around the loci $z_i = z_j$) are the same as for the usual KZ equations and can be described in terms of the quantum R-matrix – a known result from the theory of r-matrix systems [Ch1]. Global monodromies (around the τ-cycle of the elliptic curve) are described in terms of ordered products of R-matrices, of the type occurring in the quantum KZ equation (see [FR]). In the special case when for all j V_j is the N-dimensional vector representation of $\mathfrak{sl}_N$, the monodromy representation is a representation of the double affine Hecke algebra recently introduced by Cherednik [Ch3]. As an aside, the examination of monodromy helps to prove that if $\kappa = \frac{1}{NM}$, where M is an integer then the elliptic KZ equations are integrable in elliptic functions.

A generalization of the results of this paper to the case of a quantum affine algebra will be described in a forthcoming paper.

1. A Twisted Realization of Affine Lie Algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ of rank r. Denote by $\langle\,,\rangle$ the standard invariant form on $\mathfrak{g}$ with respect to which the longest root has length $\sqrt{2}$.

Let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$. The form $\langle\,,\rangle$ defines a natural identification $\mathfrak{h}^* \to \mathfrak{h}$: $\lambda \mapsto h_\lambda$ for $\lambda \in \mathfrak{h}^*$. We will use the notation $\langle\,,\rangle$ for the inner product in both $\mathfrak{h}$ and $\mathfrak{h}^*$.

Let Δ^+ be the set of positive roots of $\mathfrak{g}$. For $\alpha \in \Delta^+$, let e_α, f_α, h_α be the standard basis of the $\mathfrak{sl}_2$-subalgebra in $\mathfrak{g}$ associated with α: $[h_\alpha, e_\alpha] = \langle\alpha,\alpha\rangle e_\alpha$, $[h_\alpha, f_\alpha] = -\langle\alpha,\alpha\rangle f_\alpha$, $[e_\alpha, f_\alpha] = h_\alpha$. Let $|\alpha|$ denote the number of summands in the decomposition of α in the sum of simple positive roots.

Let N be the dual Coxeter number of $\mathfrak{g}$. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. The elements ρ and h_ρ satisfy the relations $\rho(h_\alpha) = \alpha(h_\rho) = |\alpha|$.

Let γ be an inner automorphism of $\mathfrak{g}$: $\gamma(a) = \mathrm{Ad}\, C(a)$, $a \in \mathfrak{g}$, where $C = e^{\frac{2\pi i l h\rho}{N}}$, $l \in \mathbb{Z}$, $1 \leqq l < N$, and l, N are coprime[1]. This automorphism is of order N.

The action of γ on root vectors is as follows: $\gamma(e_\alpha) = \varepsilon^{|\alpha|}e_\alpha$, $\gamma(f_\alpha) = \varepsilon^{-|\alpha|}f_\alpha$, where $\varepsilon = e^{2\pi i l/N}$ is a primitive N^{th} root of unity.

Let x_j, $1 \leqq j \leqq r$, be an orthonormal basis of $\mathfrak{h}$ with respect to the standard invariant form.

Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be the affine Lie algebra associated with $\mathfrak{g}$. The commutation relations in this algebra are

$$[a(t) + \lambda c, b(t) + \mu c] = [a(t), b(t)] + \frac{1}{2\pi i} \oint_{|t|=1} \langle a'(t) b(t)\rangle t^{-1} dt \cdot c \tag{1.1}$$

for any two $\mathfrak{g}$-valued Laurent polynomials $a(t)$, $b(t)$, and complex numbers λ, μ [Ka]. The elements $e_\alpha \otimes t^m$, $f_\alpha \otimes t^m$, $x_i \otimes t^m$, c, for $m \in \mathbb{Z}$, $\alpha \in \Delta^+$, form a basis of $\hat{\mathfrak{g}}$.

Define the subalgebra $\hat{\mathfrak{g}}_\gamma$ of $\hat{\mathfrak{g}}$ consisting of all expressions $a(t) + \lambda c$ with the property $a(\varepsilon t) = \gamma(a(t))$.

Lemma 1.1. (see [PS], p. 36) *The Lie algebra $\hat{\mathfrak{g}}_\gamma$ and $\hat{\mathfrak{g}}$ are isomorphic.*

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[1] Note that throughout the paper we denote the complex number $\mathrm{i} = \sqrt{-1}$ by a roman "i", to distinguish it from the subscript i, which is italic.

Proof. The elements $e_\alpha \otimes t^{|\alpha|+mN}, f_\alpha \otimes t^{-|\alpha|+mN}, x_i \otimes t^{mN}, c$, for $m \in \mathbb{Z}, \alpha \in \Delta^+$, form a basis of $\hat{\mathfrak{g}}_\gamma$. Define a map ϕ: $\hat{\mathfrak{g}}_\gamma \to \hat{\mathfrak{g}}$ by:

$$\phi(e_\alpha \otimes t^{|\alpha|+mN}) = e_\alpha \otimes t^m ,$$

$$\phi(f_\alpha \otimes t^{-|\alpha|+mN}) = f_\alpha \otimes t^m ,$$

$$\phi(x_j \otimes t^{mN}) = x_j \otimes t^m, \quad m \neq 0 ,$$

$$\phi(x_i) = x_i - \frac{1}{N}\rho(x_i)c, \quad \phi(c) = \frac{c}{N} . \tag{1.2}$$

It is easy to check that ϕ is an isomorphism of Lie algebras.

Thus, the twisting of $\hat{\mathfrak{g}}$ by γ does not create a new Lie algebra. However, the twisted realization $\hat{\mathfrak{g}}_\gamma$ of the affinization of $\mathfrak{g}$ will be very convenient in further considerations.

Let us translate some well known results about representations of $\hat{\mathfrak{g}}$ into the "twisted" language.

First of all, define the polarization of $\hat{\mathfrak{g}}_\gamma$: $\hat{\mathfrak{g}}_\gamma = \hat{\mathfrak{g}}_\gamma^+ \oplus \hat{\mathfrak{g}}_\gamma^- \oplus \mathfrak{h} \oplus \mathbb{C}c$. Here $\hat{\mathfrak{g}}_\gamma^+$ is the set of polynomials $a(t)$ vanishing at 0, and $\hat{\mathfrak{g}}_\gamma^-$ is the set of polynomials $a(t)$ vanishing at infinity.

Next, define Verma modules over $\hat{\mathfrak{g}}_\gamma$. This is done exactly in the same way as for the untwisted affine algebra. Let $\lambda \in \mathfrak{h}^*$ be a weight, and let k be a complex number. Define $X_{\lambda,k}$ to be a one dimensional module over $\hat{\mathfrak{g}}_\gamma^+ \oplus \mathfrak{h} \oplus \mathbb{C}c$ spanned by a vector v such that $\hat{\mathfrak{g}}_\gamma^+$ annihilates v, and $cv = kv, hv = \lambda(h)v, h \in \mathfrak{h}$. Define the Verma module

$$M_{\lambda,k} = \mathrm{Ind}_{\hat{\mathfrak{g}}_\gamma^+ \oplus \mathfrak{h} \oplus \mathbb{C}c}^{\hat{\mathfrak{g}}_\gamma} X_{\lambda,k} . \tag{1.3}$$

Now define evaluation representations. Let V be a highest weight module over $\mathfrak{g}$. Define the operator $C \in \mathrm{End}(V)$ by the conditions: $Caw = \gamma(a)Cw$ for any $w \in V, a \in \mathfrak{g}$, and $Cw_0 = w_0$, where w_0 is the highest weight vector of V.

Let $V(z)$ denote the space of V-valued Laurent polynomials in z, and let $V_C(z)$ be the space of those polynomials which satisfy the equivariance condition $w(\varepsilon z) = Cw(z)$.

The natural (pointwise) action of $\hat{\mathfrak{g}}$ on $V(z)$ restricts to an action of $\hat{\mathfrak{g}}_\gamma$ on $V_C(z)$. For this twisted action we have an analogue of Lemma 1.1.

Lemma 1.2. *The isomorphism ϕ transforms the $\hat{\mathfrak{g}}_\gamma$-module $V_C(z)$ into a $\hat{\mathfrak{g}}$-module, isomorphic to $V(z)$.*

Let us introduce the twisted version of currents. Set

$$J_{e_\alpha}(z) = \sum_{m \in \mathbb{Z}} e_\alpha \otimes t^{|\alpha|+mN} \cdot z^{-|\alpha|-mN-1} ,$$

$$J_{f_\alpha}(z) = \sum_{m \in \mathbb{Z}} f_\alpha \otimes t^{-|\alpha|+mN} \cdot z^{|\alpha|-mN-1} ,$$

$$J_h(z) = \sum_{m \in \mathbb{Z}} h \otimes t^{mN} \cdot z^{-mN-1}, \quad h \in \mathfrak{h} . \tag{1.4}$$

Thus by linearity we have defined $J_a(z)$ for any $a \in \mathfrak{g}$.

Define the polarization of currents:

$$J^+_{e_\alpha}(z) = \sum_{m<0} e_\alpha \otimes t^{|\alpha|+mN} \cdot z^{-|\alpha|-mN-1} ,$$

$$J^+_{f_\alpha}(z) = \sum_{m\leqq 0} f_\alpha \otimes t^{-|\alpha|+mN} \cdot z^{|\alpha|-mN-1} ,$$

$$J^+_h(z) = \frac{1}{2} h \otimes 1 \cdot z^{-1} + \sum_{m<0} h \otimes t^{mN} \cdot z^{-mN-1}, \quad h \in \mathfrak{h} . \tag{1.5}$$

This defines $J^+_a(z)$ for all $a \in \mathfrak{g}$. Now set

$$J^-_a(z) = J^+_a(z) - J_a(z) . \tag{1.6}$$

Note that this polarization is not quite the same as the standard polarization of currents for the untwisted $\hat{\mathfrak{g}}$ [Ka], i.e. the isomorphism ϕ does not match up these two polarizations.

The Lie algebra $\hat{\mathfrak{g}}_\gamma$ has a natural $\mathbb{Z}$-grading by powers of t. Thus every Verma module $M_{\lambda,k}$ is naturally $\mathbb{Z}$-graded: the degree of the highest weight vector v is zero, and for every homogeneous vector w $\deg(a \otimes t^{-m} w) = \deg(w) - m$. Let $d \in \operatorname{End}(M_{\lambda,k})$ be the grading operator: if w is a homogeneous vector then $dw = \deg(w) w$. The operator d satisfies the commutation relations $[d, a(t)] = t a'(t)$, $[d, c] = 0$.

Let us find an expression for d in terms of elements of $\hat{\mathfrak{g}}_\gamma$ – a twisted version of the Sugawara construction. We assume that $k \neq -1$.

Proposition 1.3.

$$d = -\frac{1}{k+1} \sum_{m\in\mathbb{Z}} \left(\sum_{\alpha\in\Delta^+} :e_\alpha \otimes t^{|\alpha|+mN} f_\alpha \otimes t^{-|\alpha|-mN}: + \frac{1}{2} \sum_{j=1}^{r} :x_j \otimes t^{mN} x_j \otimes t^{-mN}: \right)$$
$$+ \frac{\langle \lambda, \lambda \rangle}{2(k+1)} , \tag{1.7}$$

where : : is the standard normal ordering:

$$:e_\alpha \otimes t^n f_\alpha \otimes t^{-n}: = \begin{cases} e_\alpha \otimes t^n f_\alpha \otimes t^{-n} & n < 0 \\ f_\alpha \otimes t^{-n} e_\alpha \otimes t^n & n > 0 \end{cases} ,$$

$$:h \otimes t^n h \otimes t^{-n}: = \begin{cases} h \otimes t^n h \otimes t^{-n} & n \leqq 0 \\ h \otimes t^{-n} h \otimes t^n & n > 0 \end{cases} , \quad h \in \mathfrak{h} . \tag{1.8}$$

Proof. Let $\mathcal{M}_{\Lambda,K}$ be the Verma module over $\hat{\mathfrak{g}}$ with highest weight Λ and central charge K. Lemma 1.1 implies that the isomorphism ϕ transforms the module $M_{\lambda,k}$ over $\hat{\mathfrak{g}}_\gamma$ into the module $\mathcal{M}_{\Lambda,K}$ over $\hat{\mathfrak{g}}$, with $K = Nk$, $\Lambda = \lambda + k\rho$. Let D be the grading operator in $\mathcal{M}_{\Lambda,K}$ which is associated with the grading of $\hat{\mathfrak{g}}$ by powers of t. Then, according to Lemma 1.1, $\phi(d) = ND + h_\rho$. Therefore,

$$d = \phi^{-1}(ND + h_\rho) = N\phi^{-1}(D) + h_\rho + \langle \rho, \rho \rangle k . \tag{1.9}$$

For the standard affine algebra $\hat{\mathfrak{g}}$, the operator D is given by the Sugawara formula:

$$D = -\frac{1}{K+N}\sum_{m\in\mathbb{Z}}\left(\sum_{\alpha\in\Delta^+} :e_\alpha\otimes t^m f_\alpha\otimes t^{-m}: + \frac{1}{2}\sum_{j=1}^{r} :x_j\otimes t^m x_j\otimes t^{-m}:\right) + \frac{\langle\lambda,\lambda+2\rho\rangle}{2(K+N)}, \tag{1.10}$$

where the normal ordering is defined by (1.8) and

$$:e_\alpha f_\alpha: = \frac{1}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha). \tag{1.11}$$

Substituting (1.10) into (1.9), after some algebra we get (1.7).

Let us now extend the Lie algebra $\hat{\mathfrak{g}}_\gamma$ by adding a new element ∂ satisfying the relations $[\partial, a(t)] = ta'(t)$, $[\partial, c] = 0$. Denote the obtained Lie algebra by $\tilde{\mathfrak{g}}_\gamma$. Then the action of $\hat{\mathfrak{g}}_\gamma$ in every Verma module $M_{\lambda,k}$ extends to an action of $\tilde{\mathfrak{g}}_\gamma$: one sets $\partial = d - \frac{\langle\lambda,\lambda\rangle}{2(k+1)}$ (to get rid of the free term in (1.7)). The action of ∂ can also be defined in evaluation representations $V_C(z)$: $\partial = z\frac{d}{dz}$. Thus $V_C(z)$ becomes a $\tilde{\mathfrak{g}}_\gamma$-module.

2. Twisted Intertwiners and Knizhnik–Zamolodchikov Equations

We will be interested in $\tilde{\mathfrak{g}}_\gamma$ intertwining operators $\Phi(z)$: $M_{\lambda,k}\to M_{\nu,k}\hat{\otimes} z^\Delta V_C(z)$, where the highest weight of V is μ, $\hat{\otimes}$ denotes the completed tensor product, and Δ is a complex number.

Lemma 1.1 and the results of the untwisted theory imply the following statement:

Proposition 2.1. *Operators Φ are in one-to-one correspondence with vectors in V of weight $\lambda-\nu$. This correspondence is defined by the action of Φ at the vacuum level.*

Let z_0 be a nonzero complex number. Evaluation of the operator $\Phi(z)$ at the point z_0 yields an operator $\Phi(z_0)$: $M_{\lambda,k}\to\hat{M}_{\nu,k}\otimes V$, where $\hat{M}$ denotes the completion of M with respect to the grading.

From now on the notation $\Phi(z)$ will mean the operator Φ evaluated at the point $z\in\mathbb{C}^*$. This will give us an opportunity to regard the operator $\Phi(z)$ as an analytic function of z. This analytic function will be multivalued: $\Phi(z) = z^\Delta\Phi^0(z)$, where Φ^0 is a single-valued function in $\mathbb{C}^*$, and $\Delta = \frac{\langle\nu,\nu\rangle-\langle\lambda,\lambda\rangle}{2(k+1)}$.

Let u belong to the restricted dual module V^*. Introduce the notation $\Phi_u(z) = (1\otimes u)(\Phi(z))$. $\Phi_u(z)$ is an operator: $M_{\lambda,k}\to\hat{M}_{\nu,k}$.

The intertwining property for $\Phi(z)$ can be written in the form

$$[a\otimes t^m, \Phi_u(z)] = z^m\Phi_{au}(z). \tag{2.1}$$

It is convenient to write the intertwining relation in terms of currents.

Lemma 2.2.

$$[J_h^{\pm}(\zeta), \Phi_u(z)] = \frac{1}{2\zeta}\frac{\zeta^N + z^N}{z^N - \zeta^N}\Phi_{hu}(z), \quad h \in \mathfrak{h}\ ;$$

$$[J_{e_\alpha}^{\pm}(\zeta), \Phi_u(z)] = \frac{\zeta^{N-1-|\alpha|}z^{|\alpha|}}{z^N - \zeta^N}\Phi_{e_\alpha u}(z), \quad \alpha \in \Delta_+\ ;$$

$$[J_{f_\alpha}^{\pm}(\zeta), \Phi_u(z)] = \frac{\zeta^{|\alpha|-1}z^{N-|\alpha|}}{z^N - \zeta^N}\Phi_{f_\alpha u}(z), \quad \alpha \in \Delta_+\ . \tag{2.2}$$

The identities marked with $+$ make sense if $|z| > |\zeta|$, and those marked with $-$ make sense if $|z| < |\zeta|$.

Now we are ready to write down the twisted version of the operator Knizhnik–Zamolodchikov (KZ) equations.

Theorem 2.3. *The operator function $\Phi_u(z)$ satisfies the differential equation*

$$\begin{aligned}(k+1)\frac{d}{dz}\Phi_u(z) = &\sum_{\alpha\in\Delta^+}(J_{e_\alpha}^+(z)\Phi_{f_\alpha u}(z) - \Phi_{f_\alpha u}(z)J_{e_\alpha}^-(z))\\ &+ \sum_{\alpha\in\Delta^+}(J_{f_\alpha}^+(z)\Phi_{e_\alpha u}(z) - \Phi_{e_\alpha u}(z)J_{f_\alpha}^-(z))\\ &+ \sum_{j=1}^{r} J_{x_j}^+(z)\Phi_{x_j u}(z) - \Phi_{x_j u}(z)J_{x_j}^-(z))\ .\end{aligned} \tag{2.3}$$

Proof. The logic of deduction is the same as in the untwisted case (see e.g., [FR]). Equation (2.3) is nothing else but the intertwining relation between $\Phi(z)$ and ∂:

$$z\frac{d}{dz}\Phi_u(z) = -[\partial, \Phi_u(z)]\ . \tag{2.4}$$

Substituting the expression for ∂ (Eq. (1.7)) into this relation, after some calculations we obtain (2.4).

Now let us define the twisted correlation functions. Let $V_1, \dots, V_N$ be highest weight representations of $\mathfrak{g}$, and let $\Phi_{u_i}^i(z_i)$: $M_{\lambda_i,k} \to \hat{M}_{\lambda_{i-1},k}$, $1 \leqq i \leqq n$ be interwining operators. Set $\lambda_n = \lambda$ and $\lambda_0 = \nu$. Let v_λ be the highest weight vector of $M_{\lambda,k}$, and let v_ν^* be the lowest weight vector of the restricted dual module to $M_{\nu,k}$. Consider the correlation function

$$\Psi_{u_1, \quad, u_n}(z_1, \dots, z_n) = \langle v_\nu^*, \Phi_{u_1}^1(z_1)\dots\Phi_{u_n}^n(z_n)v_\lambda\rangle, \quad u_i \in V_i\ , \tag{2.5}$$

which makes sense in the region $|z_1| > \dots > |z_n|$. The function Ψ can be regarded as taking values in the space $V_1 \otimes \dots \otimes V_n$.

The function Ψ satisfies a twisted version of the trigonometric KZ equations. Before writing these equations down, let us recall the definition of the trigonometric classical r-matrix [BeDr].

Let $\Omega = \sum_{\alpha\in\Delta^+}(e_\alpha \otimes f_\alpha + f_\alpha \otimes e_\alpha) + \sum_{j=1}^r x_j \otimes x_j$, $\Omega \in \mathfrak{g}\otimes\mathfrak{g}$. For $0 \leqq p \leqq N-1$, let $\mathfrak{g}_p$ be the eigenspace of γ in $\mathfrak{g}$ with the eigenvalue $e^{2\pi i p/N}$. Let Ω_γ^p be the orthogonal projection of Ω to the subspace $\mathfrak{g}_p \otimes \mathfrak{g}_{-p}$ of $\mathfrak{g}\otimes\mathfrak{g}$. The trigonometric r-matrix has the form

$$r(z) = \frac{\Omega_\gamma^0}{2} + \sum_{p=0}^{N-1}\frac{\Omega_\gamma^p z^p}{z^N - 1}\ . \tag{2.6}$$

Introduce the convenient notation.

$$a_i = \mathrm{Id}_{V_1 \otimes \quad \otimes V_{i-1}} \otimes a \otimes 1_{V_{i+1} \otimes \quad \otimes V_n}, \quad a \in U(\mathfrak{g}), \quad 1 \leqq i \leqq n\,;$$

$$\text{if } r = \sum a^s \otimes b^s, \quad a^s, b^s \in U(\mathfrak{g}), \quad \text{then} \quad r_{ij} = \sum (a^s)_i (b^s)_j\,. \tag{2.7}$$

The main properties of the trigonometric r-matrix $r(z)$ are the classical Yang–Baxter equation and unitarity:

$$[r_{ij}(z_i/z_j), r_{ik}(z_i/z_k)] + [r_{ij}(z_i/z_j), r_{jk}(z_j/z_k)] + [r_{ik}(z_i/z_k), r_{jk}(z_j/z_k)] = 0\,,$$
$$r_{ij}(z) = -\, r_{ji}(z^{-1})\,. \tag{2.8}$$

Theorem 2.4. *The function Ψ satisfies the following system of differential equations*:

$$(k+1) z_i \frac{\partial \Psi}{\partial z_i} = \sum_{j \neq 1} r_{ij}(z_i/z_j)\Psi + \frac{1}{2}(h_\lambda + h_\nu)\Psi\,. \tag{2.9}$$

Proof. The proof is analogous to that in the untwisted case (see [FR]). It is based on the direct use of relation (2.3).

Since the twisted intertwining operators are obtained from the usual ones by a simple transformation, we should expect that system (2.9) should reduce to the untwisted trigonometric KZ equations. This turns out to be the case. Indeed, set $\zeta_j = z_j^N$, and $\hat{\Psi}(\zeta_1, \ldots, \zeta_n) = (z_1^{lh_\rho})_1 \ldots (z_n^{lh_\rho})_n \Psi(z_1, \ldots, z_n)$. Then the function $\hat{\Psi}$ satisfies the trigonometric KZ equations:

$$N(k+1)\,\zeta_i \frac{\partial \hat{\Psi}}{\partial \zeta_i} = \sum_{j \neq i} \frac{\Omega_{ij}^+ \zeta_i + \Omega_{ij}^- \zeta_j}{\zeta_i - \zeta_j} \hat{\Psi} + \frac{1}{2}(h_{\hat{\lambda}} + h_{\hat{\nu}} + 2h_\rho)\hat{\Psi}\,, \tag{2.10}$$

where $\Omega^\pm$ are the half Casimir operators defined in the introduction, and $\hat{\lambda} = \lambda + k\rho$, $\hat{\nu} = \nu + k\rho$. Therefore, the standard theory of the KZ equations can be applied to the study of the properties of (2.9).

3. Traces of Intertwiners and Elliptic r-Matrices

From now on the letter $\mathfrak{g}$ will denote the Lie algebra $\mathfrak{sl}_N(\mathbb{C})$ of traceless $N \times N$ matrices with complex entries. The dual Coxeter number of this algebra is N, and the rank is $N-1$. The Cartan subalgebra $\mathfrak{h}$ is the subalgebra of diagonal matrices, and the element C is the matrix $\mathrm{diag}(1, \varepsilon^{-1}, \varepsilon^{-2}, \ldots, \varepsilon^{-N+1})$ (up to a factor).

Let B be the $N \times N$ matrix of zeros and ones corresponding to the cyclic permutation $(12 \ldots N)$. Note that $BC = \varepsilon CB$.

Define a new inner automorphism β of $\mathfrak{g}$: $\beta(a) = BaB^{-1}$, $a \in \mathfrak{g}$. This automorphism has order N and commutes with γ: $\beta \circ \gamma = \gamma \circ \beta$.

The action of the automorphism B naturally extends to $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}_\gamma$. Note that on $\hat{\mathfrak{g}}_\gamma$, unlike $\mathfrak{g}$ and $\hat{\mathfrak{g}}$, B is an outer automorphism: it corresponds to the rotation of the affine Dynkin diagram of $\hat{\mathfrak{g}}_\gamma$ (which is a regular N-gon) through the angle $2\pi/N$.

The action of β in $\hat{\mathfrak{g}}_\gamma$ preserves degree, hence, it preserves the polarization. Therefore, it transforms Verma modules into Verma modules. In other words, we can regard B as an operator B: $M_{\lambda,k} \to M_{\beta(\lambda),k}$, where by convention $\beta(\lambda)(h) = \lambda(\beta^{-1}(h))$. This operator intertwines the usual action of $\hat{\mathfrak{g}}_\gamma$ and the action twisted by β: $\beta(a)Bw = Baw$, $a \in \hat{\mathfrak{g}}_\gamma$, $w \in M_{\lambda,k}$.

Let $\nu = \beta^{-1}(\lambda)$, and let $\Phi^j_{u_j}(z_j)$ be as above (cf. Sect. 2). Let q be a complex number, $0 < |q| < 1$. Following the idea of Frenkel, Reshetikhin ([FR], Remark 2.3) and Bernard [Ber], introduce the following function:

$$F_{u_1, \quad , u_n}(z_1, \ldots, z_n|q) = \mathrm{Tr}|_{M_{\nu,k}}(\Phi^1_{u_1}(z_1) \ldots \Phi^n_{u_n}(z_n)Bq^{-\partial}) . \tag{3.1}$$

This function takes values in $V_1 \hat{\otimes} \ldots \hat{\otimes} V_N$. From now on it will be the main object of our study.

It turns out that the n-point trace $F(z_1, \ldots, z_n|q)$ defined by (3.1) satisfies a remarkable system of differential equations involving elliptic solutions of the classical Yang–Baxter equation for $\mathfrak{sl}_N$. Let us deduce these equations. The idea of the method of deduction is due to Frenkel and Reshetikhin.

Differentiating by z_j, we get

$$\begin{aligned}
&(k+1)\frac{\partial}{\partial z_j}F(z_1, \ldots, z_n|q) \\
&= \mathrm{Tr}\left(\Phi^1_{u_1}(z_1)\cdots\sum_\alpha (J^+_{e_\alpha}(z_j)\Phi^j_{f_\alpha u_j}(z_j) - \Phi^j_{f_\alpha u_j}(z_j)J^-_{e_\alpha}(z_j))\ldots\Phi^n_{u_n}(z_n)Bq^{-\partial}\right) \\
&\quad + \mathrm{Tr}\left(\Phi^1_{u_1}(z_1)\cdots\sum_\alpha (J^+_{f_\alpha}(z_j)\Phi^j_{e_\alpha u_j}(z_j) - \Phi^j_{e_\alpha u_j}(z_j)J^-_{f_\alpha}(z_j))\ldots\Phi^n_{u_n}(z_n)Bq^{-\partial}\right) \\
&\quad + \mathrm{Tr}\left(\Phi^1_{u_1}(z_1)\cdots\sum_{r=1}^{N-1} (J^+_{x_r}(z_j)\Phi^j_{x_r u_j}(z_j) - \Phi^j_{x_r u_j}(z_j)J^-_{x_r}(z_j))\ldots\Phi^n_{u_n}(z_n)Bq^{-\partial}\right).
\end{aligned} \tag{3.2}$$

Now let us pull the currents J^+, J^- around: the currents J^- will move to the right up to the end, then jump at the beginning and continue to move to the right, and so on; the currents J^+ will move to the left up to the beginning, then jump at the end and continue to move to the left, and so on. As we do these permutations, we will need relations (2.2) and also the following identities:

$$q^{-\partial}J^\pm_a(z) = qJ^\pm_a(qz)q^{-\partial} , \tag{3.3}$$

$$BJ^\pm_a(z) = J^\pm_{\beta(a)}(z)B . \tag{3.4}$$

After J^+ and J^- have made M complete circles, Eq. (3.2) will have the form

$$\begin{aligned}
&(k+1)\frac{\partial}{\partial z_j}F(z_1, \ldots, z_n|q) \\
&= \mathrm{Tr}\Big(\Phi^1_{u_1}(z_1)\cdots\sum_\alpha (q^M J^+_{\beta^M(e_\alpha)}(q^M z_j)\Phi^j_{f_\alpha u_j}(z_j) \\
&\quad - \Phi^j_{f_\alpha u_j}(z_j)q^{-M}J^-_{\beta^{-M}(e_\alpha)}(q^{-M}z_j))\ldots\Phi^n_{u_n}(z_n)Bq^{-\partial}\Big) \\
&\quad + \mathrm{Tr}\Big(\Phi^1_{u_1}(z_1)\cdots\sum_\alpha (q^M J^+_{\beta^M(f_\alpha)}(q^M z_j)\Phi^j_{e_\alpha u_j}(z_j) \\
&\quad - \Phi^j_{e_\alpha u_j}(z_j)q^{-M}J^-_{\beta^{-M}(f_\alpha)}(q^{-M}z_j))\ldots\Phi^n_{u_n}(z_n)Bq^{-\partial}\Big)
\end{aligned}$$

$$+ \operatorname{Tr}\left(\Phi^1_{u_1}(z_1) \cdots \sum_{r=1}^{N-1} (q^M J^+_{\beta^M(x_r)}(z_j) \Phi^j_{x_r u_j}(z_j) \right.$$

$$\left. - \Phi^j_{x_r u_j}(z_j) q^{-M} J^-_{\beta^{-M}(x_r)}(q^{-M} z_j)) \ldots \Phi^n_{u_n}(z_n) B q^{-\partial} \right)$$

$$+ \sum_{i=1}^{n} X^M_{ij} F(z_1, \ldots, z_n | q) \,, \tag{3.5}$$

where

$$X^M_{ij} = \begin{cases} \sum_{p=-M}^{M} X^{M,p}_{ij} & i \neq j \\ \sum_{p=-M, p\neq 0}^{M} X^{M,p}_{jj} & i = j \end{cases} \tag{3.6}$$

and

$$X^{M,p}_{ij} = \sum_{\alpha} \frac{q^p (q^p z_i)^{N-1-|\alpha|} z_j^{|\alpha|}}{q^{pN} z_i^N - z_j^N} (\beta^p(e_\alpha))_i (f_\alpha)_j$$

$$+ \sum_{\alpha} \frac{q^p (q^p z_i)^{|\alpha|-1} z_j^{N-|\alpha|}}{q^{pN} z_i^N - z_j^N} (\beta^p(f_\alpha))_i (e_\alpha)_j$$

$$+ \sum_{l} \frac{1}{2z_i} \frac{(q^p z_i)^N + z_j^N}{q^{pN} z_i^N - z_j^N} (\beta^p(x_r))_i (x_r)_j \,. \tag{3.7}$$

Now we want to pass to the limit $M \to \infty$. Right now we cannot do so since the limit does not exist. In order to be able to pass to the limit, we should write down Eq. (3.5) for $M = L, L+1, \ldots, L+N-1$, add these N equations together, and divide by N:

$$(k+1) \frac{\partial}{\partial z_j} F(z_1, \ldots, z_n | q)$$

$$= \frac{1}{N} \sum_{M=L}^{L+N-1} \left[\operatorname{Tr}\left(\Phi^1_{u_1}(z_1) \cdots \sum_{\alpha} (q^M J^+_{\beta^M(e_\alpha)}(q^M z_j) \Phi^j_{f_\alpha u_j}(z_j) \right. \right.$$

$$\left. - \Phi^j_{f_\alpha u_j}(z_j) q^{-M} J^-_{\beta^{-M}(e_\alpha)}(q^{-M} z_j)) \ldots \Phi^n_{u_n}(z_n) B q^{-\partial} \right)$$

$$+ \operatorname{Tr}\left(\Phi^1_{u_1}(z_1) \cdots \sum_{\alpha} (q^M J^+_{\beta^M(f_\alpha)}(q^M z_j) \Phi^j_{e_\alpha u_j}(z_j) \right.$$

$$\left. - \Phi^j_{e_\alpha u_j}(z_j) q^{-M} J^-_{\beta^{-M}(f_\alpha)}(q^{-M} z_j)) \ldots \Phi^n_{u_n}(z_n) B q^{-\partial} \right)$$

$$+ \operatorname{Tr}\left(\Phi^1_{u_1}(z_1) \cdots \sum_{r=1}^{N-1} (q^M J^+_{\beta^M(x_r)}(z_j) \Phi^j_{x_r u_j}(z_j) \right.$$

$$\left. - \Phi^j_{x_r u_j}(z_j) q^{-M} J^-_{\beta^{-M}(x_r)}(q^{-M} z_j)) \cdots \Phi^n_{u_n}(z_n) B q^{-\partial} \right)$$

$$+\sum_{i=1}^{n} X_{ij}^{M}\Bigg] F(z_1,\ldots,z_n|q)\,, \tag{3.8}$$

In the obtained equation it will already be possible to take the limit. Moreover, since $\sum_{p=0}^{N-1}\beta^p(h)=0$ for $h\in\mathfrak{h}$, the part of the right-hand side of (3.8) involving currents associated with the Cartan subalgebra elements will disappear as $M\to\infty$. The same will happen to the currents associated with the root elements e_α, f_α because these currents do not contain a term of degree -1 which is the only term that could possibly have given a nonzero limit. Thus, in the limit we get a simple equation:

$$\frac{\partial}{\partial z_j}F(z_1,\ldots,z_n|q)=\left(\sum_{i=1}^{n} X_{ij}^{\infty}\right)F(z_1,\ldots,z_n|q)\,, \tag{3.9}$$

where $X_{ij}^{\infty}=\lim_{L\to\infty}\frac{1}{N}\sum_{M=L}^{L+N-1}X_{ij}^{M}$.

The function X_{ij}^{∞} admits a simple description in terms of elliptic functions. First of all, it is easy to check that $X_{jj}^{M}=0$ for any M.

Next, the function X_{ij}^{M} can be represented in the form $X_{ij}^{M}=\frac{1}{z_i}\rho_{ij}^{M}(z_i/z_j)$, where $\rho^M(z)$ is a rational function with values in $\mathfrak{g}\otimes\mathfrak{g}$:

$$\begin{aligned}\rho^M(z)=\sum_{p=-M}^{M}\Bigg(&\sum_{\alpha\in\Delta^+}\frac{(q^pz)^{N-|\alpha|}}{q^{pN}z^N-1}f_\alpha\otimes\beta^p(e_\alpha)\\&+\sum_{\alpha\in\Delta^+}\frac{(q^pz)^{|\alpha|}}{q^{pN}z^N-1}e_\alpha\otimes\beta^p(f_\alpha)\\&+\sum_{r=1}^{N-1}\frac{q^{pN}z^N+1}{2(q^{pN}z^N-1)}x_r\otimes\beta^p(x_r)\Bigg)\,.\end{aligned} \tag{3.10}$$

Therefore, $X_{ij}^{\infty}=\frac{1}{z_i}\rho_{ij}(z_i/z_j|q)$, where $\rho(z|q)$ is an elliptic function of $\log z$ with values in $\mathfrak{g}\otimes\mathfrak{g}$ (we have used notation (2.7)). We can tell what this function looks like by looking at its residues.

From (3.10) we see that the only poles of $\rho(z|q)$ are at the point $\varepsilon^m q^p$, and all these poles are simple. The residue of $\rho(z|q)$ at $z=\varepsilon^m q^p$ is equal to that of $\rho^M(z)$ for $M\geqq|p|$, i.e. it equals

$$\begin{aligned}\operatorname{Res}|_{z=\varepsilon^m q^p}\rho(z|q)&=\frac{q^p}{N}\Bigg(\sum_{\alpha\in\Delta^+}(\varepsilon^{-m|\alpha|}f_\alpha\otimes\beta^p(e_\alpha)\\&\qquad+\varepsilon^{m|\alpha|}e_\alpha\otimes\beta^p(f_\alpha)+\sum_{r=1}^{N-1}x_r\otimes\beta^p(x_r)\Bigg)\\&=\frac{q^p}{N}(1\otimes\gamma^{-m}\beta^p)(\Omega)\,.\end{aligned} \tag{3.11}$$

Because of the obvious homogeneity property $\sum_i z_i\frac{\partial F}{\partial z_i}=0$, we have the unitarity relation $\rho_{ij}(z)=-\rho_{ji}(-z)$.

Let $q = e^{2\pi i \tau}$. Let

$$\zeta(x|\tau) = \frac{1}{x} + \lim_{M\to\infty} \sum_{-M \leqq m, p \leqq M, m^2+p^2>0} \left[\frac{1}{x - m - p\tau} + \frac{x}{(m+p\tau)^2} \right] \tag{3.12}$$

be the standard Weierstrass function. Equation (3.11) implies that

$$\begin{aligned} \rho(z|q) = {} & \frac{\Omega}{2\pi i N} \zeta\left(\frac{\log z}{2\pi i} \middle| N\tau \right) \\ & + \frac{1}{2\pi i N} \sum_{0 \leqq m, p \leqq N-1, m^2+p^2>0} (1 \otimes \gamma^{-m}\beta^p)(\Omega) \\ & \times \left[\zeta\left(\frac{\log z}{2\pi i} - \frac{m}{N} - p\tau \middle| N\tau \right) - \zeta\left(-\frac{m}{N} - p\tau \middle| N\tau \right) \right]. \end{aligned} \tag{3.13}$$

Thus, we have proved

Theorem 3.1. *The function $F(z_1, \ldots, z_n|q)$ satisfies the system of differential equations*

$$(k+1) z_i \frac{\partial F}{\partial z_i} = \sum_{j \neq i} \rho_{ij}(z_i/z_j|q) F, \quad 1 \leqq i \leqq n. \tag{3.14}$$

The function $\rho(z|q)$ is a solution of the classical Yang–Baxter equation. It is easy to see that this function is nothing else but the elliptic r-matrix for $\mathfrak{sl}_N$ due to A. Belavin [Be]. Thus, we have given a representation-theoretical interpretation of the local system associated with the elliptic solutions of the classical Yang–Baxter equation.

Remark. A Belavin and V. Drinfeld [BeDr] showed that elliptic solutions exist only for the simple Lie algebra $\mathfrak{sl}_N$, and every nondegenerate elliptic solution is equivalent (= conjugate) to const $\cdot \rho(z|q)$ (for a suitable primitive N^{th} root of unity ε).

Observe that Eqs. (3.14) transform into the KZ equations (2.9) as $q \to 0$. This was to be expected since $\lim_{q\to 0} q^{-\frac{\langle\lambda,\lambda\rangle}{2(k+1)}} F(\mathbf{z}|q) = \Psi(\mathbf{z})$. We will call Eqs. (3.14) *the elliptic Knizhnik–Zamolodchikov* (KZ) *equations.*

There is one more equation satisfied by the function F – a differential equation involving the first derivative by q. Differentiating F by q, we obtain

$$-q \frac{\partial F}{\partial q} = \mathrm{Tr}|_{M_{v,k}} (\Phi_{u_1}^1(z_1) \ldots \Phi_{u_n}(z_n) B q^{-\partial} \partial|_{M_{v,k}}). \tag{3.15}$$

Plugging the Sugawara expression for ∂ in (3.15), we see that in order to obtain the differential equation for F, it would suffice to express the traces

$$\begin{aligned} & \mathrm{Tr}|_{M_{v,k}} (\Phi_{u_1}^1(z_1) \ldots \Phi_{u_n}^n(z_n) B q^{-\partial} f_\alpha \otimes t^{-|\alpha|-mN} e_\alpha \otimes t^{|\alpha|+mN}), \\ & \mathrm{Tr}|_{M_{v,k}} (\Phi_{u_1}^1(z_1) \ldots \Phi_{u_n}^n(z_n) B q^{-\partial} e_\alpha \otimes t^{|\alpha|-mN} f_\alpha \otimes t^{-|\alpha|+mN}), \\ & \mathrm{Tr}|_{M_{v,k}} (\Phi_{u_1}^1(z_1) \ldots \Phi_{u_n}^n(z_n) B q^{-\partial} x_r \otimes t^{-mN} x_r \otimes t^{mN}) \end{aligned} \tag{3.16}$$

in terms of the original trace $F(\mathbf{z}|q)$. This can be done as follows.

Take the expression for the first trace in (3.16) and move the factor $e_\alpha \otimes t^{|\alpha|+mN}$ from left to right. When it has made N full circles, we will have

$$\begin{aligned}
&\operatorname{Tr}|_{M_{v,k}}(\Phi^1_{u_1}(z_1)\dots\Phi^n_{u_n}(z_n)Bq^{-\partial}f_\alpha\otimes t^{-|\alpha|-mN}e_\alpha\otimes t^{|\alpha|+mN})\\
&=\sum_{j=1}^{n}\sum_{p=0}^{N-1} z_j^{|\alpha|+mN}q^{-p(|\alpha|+mN)}\operatorname{Tr}|_{M_{v,k}}(\Phi^1_{u_1}(z_1)\dots\Phi^j_{\beta^{-p}(e_\alpha)u_j}(z_j)\\
&\quad\dots\Phi^n_{u_n}(z_n)Bq^{-\partial}f_\alpha\otimes t^{-|\alpha|-mN})\\
&\quad+q^{-N(|\alpha|+mN)}\operatorname{Tr}|_{M_{v,k}}(\Phi^1_{u_1}(z_1)\dots\Phi^n_{u_n}(z_n)\\
&\quad\times Bq^{-\partial}(h_\alpha+(|\alpha|+mN)k))+q^{-N(|\alpha|+mN)}\operatorname{Tr}|_{M_{v,k}}(\Phi^1_{u_1}(z_1)\\
&\quad\dots\Phi^n_{u_n}(z_n)Bq^{-\partial}f_\alpha\otimes t^{-|\alpha|-mN}e_\alpha\otimes t^{|\alpha|+mN})\,. \qquad (3.17)
\end{aligned}$$

It is easy to see that for any $h\in\mathfrak{h}$,

$$\operatorname{Tr}|_{M_{v,k}}(\Phi^1_{u_1}(z_1)\dots\Phi^n_{u_n}(z_n)Bq^{-\partial}h)=-(1-\beta^{-1})^{-1}(h)F(\mathbf{z}|q)\,. \qquad (3.18)$$

Similarly we have

$$\begin{aligned}
&\operatorname{Tr}|_{M_{v,k}}(\Phi^1_{u_1}(z_1)\dots\Phi^n_{u_n}(z_n)Bq^{-\partial}f_\alpha\otimes t^{-|\alpha|-mN})\\
&=-\sum_{j=1}^{n} z_j^{-|\alpha|-mN}(1-\beta^{-1}q^{|\alpha|+mN})^{-1}(f_\alpha)_jF(\mathbf{z}|q)\,. \qquad (3.19)
\end{aligned}$$

Thus, we obtain the following expression for the first trace in (3.16):

$$\begin{aligned}
&\operatorname{Tr}|_{M_{v,k}}(\Phi^1_{u_1}(z_1)\dots\Phi^n_{u_n}(z_n)Bq^{-\partial}f_\alpha\otimes t^{-|\alpha|-mN}e_\alpha\otimes t^{|\alpha|+mN})\\
&=\sum_{i,j=1}^{n} z_i^{-|\alpha|-mN}z_j^{|\alpha|+mN}(1-\beta^{-1}q^{|\alpha|+mN})^{-1}\\
&\quad\times(f_\alpha)_i(1-\beta^{-1}q^{-|\alpha|-mN})^{-1}(e_\alpha)_jF(\mathbf{z}|q)\\
&\quad+(1-q^{-N(|\alpha|+mN)})^{-1}((1-\beta^{-1})^{-1}(-h_\alpha)\\
&\quad+(|\alpha|+mN)k)F(\mathbf{z}|q)\,. \qquad (3.20)
\end{aligned}$$

The second and the third trace in (3.16) are treated quite similarly, and finally after some calculations we obtain:

$$(k+1)q\frac{\partial F}{\partial q}=\sum_{i,j=1}^{n}L_{ij}\left(\frac{z_i}{z_j}\middle|q\right)F\,, \qquad (3.21)$$

where

$$\begin{aligned}
L(z|q)&=\sum_{\alpha\in\Delta^+}\sum_{m\geq 0} z^{-|\alpha|-mN}(1-\beta^{-1}q^{|\alpha|+mN})^{-1}(f_\alpha)\otimes(1-\beta^{-1}q^{-|\alpha|-mN})^{-1}(e_\alpha)\\
&\quad+\sum_{\alpha\in\Delta^+}\sum_{m>0} z^{|\alpha|-mN}(1-\beta^{-1}q^{-|\alpha|+mN})^{-1}(e_\alpha)\otimes(1-\beta^{-1}q^{|\alpha|-mN})^{-1}(f_\alpha)\\
&\quad+\sum_{p=1}^{N-1}\sum_{m>0} z^{-mN}(1-\beta^{-1}q^{mN})^{-1}(x_p)\otimes(1-\beta^{-1}q^{-mN})^{-1}(x_p)\\
&\quad+\frac{1}{2}\sum_{p=1}^{N-1}(1-\beta^{-1})^{-1}(x_p)\otimes(1-\beta^{-1})^{-1}(x_p) \qquad (3.22)
\end{aligned}$$

is a function with values in $\mathfrak{g} \otimes \mathfrak{g}$ (once again, we use notation (2.7)). Note that $L_{ij} = L_{ji}$. If $i = j$, then by L_{ii} we mean $\mu(L)_i$, where μ: $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$ is multiplication: $\mu(a \otimes b) = ab$.

The form of Eq. (3.21) can be simplified. Indeed, the consistency of (3.21) and (3.14) for all k implies that

$$q \frac{\partial \rho(z|q)}{\partial q} = z \frac{\partial}{\partial z} (L(z|q) + L(z^{-1}|q)), \quad \text{i.e.}$$

$$L(z|q) + L(z^{-1}|q) = 2L(1, q) + \int_1^z \frac{q}{z} \frac{\partial \rho(z|q)}{\partial q} dz \,. \tag{3.23}$$

Therefore, we finally get the theorem (in the formulation we use notation (2.7)):

Theorem 3.2. *The function F satisfies the differential equation*

$$(k+1) q \frac{\partial F}{\partial q} = \sum_{i,j=1}^{n} L_{ij}(1|q) F + \sum_{i<j} s\left(\frac{z_i}{z_j}\middle| q\right)_{ij} F \,,$$

$$s(z|q) = \int_1^z \frac{q}{z} \frac{\partial \rho(z|q)}{\partial q} dz \,. \tag{3.24}$$

Corollary.

$$\mathrm{Tr}|_{M_{0,k}} (B q^{-\partial}) = 1 \,. \tag{3.25}$$

Proof. Apply Theorem 3.2 to the case of a single intertwining operator $\Phi(z)$: $M_{0,k} \to M_{0,k} \otimes V_C^0(z)$, where V^0 is the trivial representation of $\mathfrak{g}$. In this case it is obvious that $\Phi(z) = \mathrm{Id}$. Therefore, (3.23) simply follows from the fact that (3.22) acts by zero in V^0.

4. Modular Invariance of the Elliptic KZ Equations

Let us now discuss the modular invariance of the elliptic KZ equations. Introduce new variables $y_j = \frac{N \log z_j}{2\pi i}$, $\tau = \frac{\log q}{2\pi i}$. From now on let us use the notation $\kappa = k + 1$. Rewrite Eqs. (3.14) and (3.24) in the new coordinates:

$$\kappa \frac{\partial F}{\partial y_i} = \sum_{j \neq i} \rho_{ij}^*(y_i - y_j|\tau) F, \quad 1 \leqq i \leqq n \,, \tag{4.1}$$

$$\kappa \frac{\partial F}{\partial \tau} = \sum_{i,j=1}^{n} L_{ij}^*(\tau) F + \sum_{j<i} s_{ij}^*(y_i - y_j|\tau) F \,, \tag{4.2}$$

where

$$\rho^*(y|\tau) = \frac{2\pi i}{N} \rho(z|q) \,,$$

$$s^*(y|\tau) = 2\pi i s(z|q) = \int_0^y \frac{\partial \rho^*(x|\tau)}{\partial \tau} dx \,,$$

$$L^*(\tau) = 2\pi i L(1|q) \,. \tag{4.3}$$

Let $\Gamma(N)$ be the congruence subgroup in $SL_2(\mathbb{Z})$ consisting of the matrices equal to the identity modulo N. We have the following almost obvious property.

Proposition 4.1. *Equations* (4.1) *are invariant with respect to the group* $\Gamma(N)$. *That is, they are preserved under the change of variables* $\hat{y}_i = \frac{y_i}{c\tau + d}$, $\hat{\tau} = \frac{a\tau + b}{c\tau + d}$ *if* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$.

Proof. Partial derivatives with respect to the new coordinates are given by

$$\frac{\partial}{\partial \hat{y}_i} = (c\tau + d)\frac{\partial}{\partial y_i},$$

$$\frac{\partial}{\partial \hat{\tau}} = (c\tau + d)^2 \frac{\partial}{\partial \tau} + \sum_i c(c\tau + d) y_i \frac{\partial}{\partial y_i}. \tag{4.4}$$

Thus we only have to prove that $\rho^*(\hat{y}|\hat{\tau}) = (c\tau + d)\rho^*(y|\tau)$. To do this, it is enough to observe that both sides of this equation

(i) have simple poles at the points of the lattice $m + p\tau$ with the same residues $\frac{1}{N}(1 \otimes \gamma^{-m}\beta^p)(\Omega)$, and

(ii) satisfy the unitarity condition: $\rho_{12}^*(y|\tau) = -\rho_{21}^*(-y|\tau)$.

Now let us study the invariance properties of Eq. (4.2). First of all, it is easy to see that

$$\frac{\partial \rho^*(\hat{y}|\hat{\tau})}{\partial \hat{\tau}} = (c\tau + d)^3 \frac{\partial \rho^*(y|\tau)}{\partial \tau} + c(c\tau + d)^2 \frac{\partial (y\rho^*(y|\tau))}{\partial y}. \tag{4.5}$$

Integrating this equation against dy, we get

$$s^*(\hat{y}, \hat{\tau}) = (c\tau + d)^2 s^*(y|\tau) + c(c\tau + d)\left(y\rho^*(y|\tau) - \lim_{x \to 0} x\rho^*(x|\tau) \right). \tag{4.6}$$

By our definition, $\lim_{x\to 0} x\rho^*(x|\tau) = \frac{\Omega}{N}$. Thus, under the change of variable $\tau \to \hat{\tau}$, $y \to \hat{y}$, Eq. (4.2) transforms into the following equation:

$$\begin{aligned}
&(c\tau + d)^2 \kappa \frac{\partial F}{\partial \tau} + c(c\tau + d) \sum_i \kappa y_i \frac{\partial F}{\partial y_i} \\
&= \sum_{i<j} (c\tau + d)^2 s^*(y_i - y_j|\tau) F \\
&\quad + \sum_{i<j} c(c\tau + d)\left((y_i - y_j)\rho^*(y_i - y_j|\tau) - \frac{\Omega_{ij}}{N} \right) F + \sum_{i,j} L_{ij}(\hat{\tau}) F.
\end{aligned} \tag{4.7}$$

Combining (4.7) with (4.1), we can get rid of the derivatives by y_i and reduce (4.7) to the form:

$$\kappa \frac{\partial F}{\partial \tau} = \sum_{i<j} \left(s_{ij}^*(y|\tau) - \frac{c\Omega_{ij}}{(c\tau + d)N} \right) F + (c\tau + d)^{-2} \sum_{i,j} L_{ij}^*(\hat{\tau}) F. \tag{4.8}$$

Now we need to find the law of transformation of $L^*(\tau)$.

Lemma.

$$L^*(\hat{\tau}) = (c\tau + d)^2 L^*(\tau) + \frac{c(c\tau + d)\Omega}{2N} . \tag{4.9}$$

Proof. Let $C(\tau) = L^*(\hat{\tau}) - (c\tau + d)^2 L^*(\tau) - \frac{c(c\tau + d)\Omega}{2N} \in \mathfrak{g} \otimes \mathfrak{g}$. We know that both Eqs. (4.2) and (4.8) are consistent with (4.1). Therefore, we have

$$\left[C_{11}(\tau) + 2C_{12}(\tau) + C_{22}(\tau) + \frac{c(c\tau + d)(\Omega_{11} + \Omega_{22})}{N}, \rho_{12}^*(y|\tau) \right] = 0 . \tag{4.10}$$

Observe that $\Omega_{11} = \Delta \otimes 1, \Omega_{22} = 1 \otimes \Delta$, where $\Delta \in U(\mathfrak{g})$ is the Casimir element. Since the Casimir element commutes with the Lie algebra action, (4.10) reduces to the relation

$$[C_{11}(\tau) + 2C_{12}(\tau) + C_{22}(\tau), \rho_{12}^*(y|\tau)] = 0 . \tag{4.11}$$

This relation has to hold for all y, which implies that the expression $\tilde{C}(\tau) = C_{11}(\tau) + 2C_{12}(\tau) + C_{22}(\tau)$ commutes with all γ-invariant elements in $\mathfrak{g} \otimes \mathfrak{g}$, i.e. with all elements of the form $x_i \otimes x_j$, $e_\alpha \otimes \beta^p(f_\alpha)$ and $f_\alpha \otimes \beta^p(e_\alpha)$.

We are going to prove that $C(\tau) = 0$. Let

$$C(\tau) = \sum_{i,j} a_{ij}(\tau) x_i \otimes x_j + \sum_{\alpha, p} b_{\alpha p}(\tau)(e_\alpha \otimes \beta^p(f_\alpha) + \beta^p(f_\alpha) \otimes e_\alpha) . \tag{4.12}$$

Pick arbitrary two elements X and Y in $\mathfrak{h}$. Consider the expression

$$\begin{aligned} C_{XY} &= [X \otimes 1, [1 \otimes Y, 2C]] = [X \otimes 1, [1 \otimes Y, \tilde{C}]] \\ &= -2 \sum_{\alpha, p} b_{\alpha p}(\alpha(X)\alpha(\beta^{-p}(Y)) e_\alpha \otimes \beta^p(f_\alpha) \\ &\quad + \alpha(Y)\alpha(\beta^{-p}(X)) f_\alpha \otimes \beta^p(e_\alpha) \end{aligned} \tag{4.13}$$

(for brevity we do not specify explicitly the dependence on τ). This expression has to commute with $x_i \otimes x_j$ for all i, j. This immediately implies that $C_{XY} = 0$ for all X, Y, i.e. $b_{\alpha p} = 0$ for all α, p. Therefore,

$$C = \sum_{i,j} a_{ij} \cdot x_i \otimes x_j . \tag{4.14}$$

Let $X_i = 1 \otimes x_i + x_i \otimes 1$. Then $\tilde{C} = \sum_{i,j} a_{ij} X_i X_j$. Therefore,

$$\begin{aligned} [\tilde{C}, e_\alpha \otimes \beta^p(f_\alpha)] &= \sum_{i,j} \alpha((1 - \beta^{-p})(x_i)) a_{ij}(X_j e_\alpha \otimes \beta^p(f_\alpha) \\ &\quad + e_\alpha \otimes \beta^p(f_\alpha) X_j) . \end{aligned} \tag{4.15}$$

In order for (4.15) to be zero for any α, p, we must have $\sum_{i,j} \alpha((1 - \beta^{-p})(x_i)) a_{ij} x_j = 0$ for all α, p, which can only happen when all a_{ij} are zero (because the roots span the space $\mathfrak{h}^*$). Q.E.D.

The lemma we just proved implies

Proposition 4.2. *Under the change of variables* $(y, \tau) \to (\hat{y}, \hat{\tau})$ *system of equations* (4.1), (4.2) *transforms into a system of equations equivalent to the combination of* (4.1) *and the following equation*:

$$\kappa \frac{\partial F}{\partial \tau} = \sum_{i,j=1}^{n} L_{ij}^{*}(\tau)F + \sum_{j<i} s_{ij}^{*}(y_i - y_j|\hat{\tau})F + \sum_{i=1}^{n} \frac{c\Delta_i}{2N(c\tau + d)} F , \tag{4.16}$$

where Δ_i *is the Casimir operator in the* i^{th} *factor of the tensor product* $V_1 \hat{\otimes} V_2 \hat{\otimes} \dots \hat{\otimes} V_n$.

Thus, the system of equations (4.1), (4.2) is almost invariant under $\Gamma(N)$: the first n equations are unchanged under the action of this group whereas the last equation gets a very simple extra term.

Consider *the fundamental solution* $\mathscr{F}(\mathbf{y}|\tau)$ of the system (4.1), (4.2) – a solution with values in $\operatorname{End}(V_1 \hat{\otimes} \cdots \hat{\otimes} V_n)$ defined by the condition: if $v = v_1 \otimes \cdots \otimes v_n$ is a vector in $V_1 \hat{\otimes} \cdots \hat{\otimes} V_n$, $v_i \in V_i$, $1 \leqq i \leqq n$, and $hv_i = \chi_i(h)v_i$, $\chi_i \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, then

$$\mathscr{F}v = \operatorname{Tr}|_{M_{\lambda_n,k}}(\Phi^{v_1,\lambda_1,\lambda_0}(z_1) \dots \Phi^{v_n,\lambda_n,\lambda_{n-1}}(z_n)Bq^{-\partial}) , \tag{4.17}$$

where λ_j are defined by the formula

$$\lambda_j = (\beta - 1)^{-1}\left(\sum_{i=1}^{n} \chi_i\right) + \sum_{i=1}^{j} \chi_i, \quad 0 \leqq j \leqq n , \tag{4.18}$$

and $\Phi^{v_i,\lambda,\nu}(z)$ is the intertwiner $M_{\lambda,k} \to M_{\nu,k} \hat{\otimes} z^{\Delta} V_i(z) \left(\Delta = \dfrac{\langle \nu, \nu \rangle - \langle \lambda, \lambda \rangle}{2(k+1)}\right)$ such that $\langle v_\nu^*, \Phi^{v_i,\lambda,\nu}(z)v_\lambda \rangle = v_i$.

Let $A \in \Gamma(N)$.

Theorem 4.3. (*on the modular invariance of solutions of the elliptic* KZ *equations*).

$$\mathscr{F}(\hat{y}_1, \dots, \hat{y}_n|\hat{\tau}) = (c\tau + d)^{\frac{1}{2N\kappa}\sum_i \Delta_i} \mathscr{F}(y_1, \dots, y_n|\tau)\chi(A) , \tag{4.19}$$

where $\chi(A)$ *is an operator in* $V_1 \hat{\otimes} V_2 \hat{\otimes} \dots \hat{\otimes} V_n$ *dependent only on the* 2×2 *matrix* A (i.e. *independent of* y_i *and* τ).

Assume that the representations V_j are irreducible. Then Δ_j are simply complex numbers. In this case the function $\chi(A)$ is a projective representation of $\Gamma(N)$:

$$\chi(A_1)\chi(A_2) = \sigma(A_1, A_2)\chi(A_1 A_2) . \tag{4.20}$$

The 2-cocycle $\sigma(A_1, A_2)$ is very easy to describe. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ be called positive if either $c > 0$ or if $c = 0$ but $d > 0$; otherwise, let A be called negative. Then

$$\sigma(A_1, A_2) = \varphi(A_1) + \varphi(A_2) - \varphi(A_1 A_2), \quad \varphi(A) = \begin{cases} 1, & A \text{ is positive} \\ e^{\frac{\pi i \sum_j \Delta_j}{2N\kappa}}, & A \text{ is negative} \end{cases} . \tag{4.21}$$

By definition, this cocycle is a coboundary, so the action of $\Gamma(N)$ in the projectivization of $V_1 \hat{\otimes} V_2 \hat{\otimes} \cdots \hat{\otimes} V_n$ comes from a linear action.

Thus, the theory of the elliptic KZ equations gives us a natural method to assign to every set of irreducible finite dimensional representations $V_1, \ldots, V_n$ of $\mathfrak{sl}_N$ an action of the congruence subgroup $\Gamma(N)$ of $SL_2(\mathbb{Z})$ in the tensor product of these representations, $V_1 \otimes \cdots \otimes V_n$.

In fact this construction allows us to obtain a representation of the entire modular group $SL_2(\mathbb{Z})$. For this we need to assume that V_i are finite dimensional representations of $GL_N(\mathbb{C})$ for all i. For brevity we will also assume that N is odd (this assumption is not very essential, but in the even case one has to be a little bit more careful).

We will need to use the Weil representation of the group $SL_2(\mathbb{Z}/N\mathbb{Z})$. This representation is defined as follows. Take the N-dimensional vector space $U = \mathbb{C}^N$ and define an action of the Heisenberg group $H_N = \langle x, y \mid x^N = y^N = 1, xyx^{-1}y^{-1}$ commutes with $x, y \rangle$ in this space by $x \to B, y \to C$. This is the basic irreducible representation of H_N. Now observe that $SL_2(\mathbb{Z}/N\mathbb{Z})$ acts by automorphisms of H_N: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A(x) = x^a y^b, A(y) = x^c y^d$. Moreover, the representation U^A of H_N obtained from U by twisting of U by A is isomorphic to U. Therefore, the group $SL_2(\mathbb{Z}/N\mathbb{Z})$ projectively acts in U in such a way that $Azu = A(z)Au$, $A \in SL_2(\mathbb{Z}/N\mathbb{Z})$, $z \in H_N, u \in PU$, and this action is unique. The space U with the constructed projective action of $SL_2(\mathbb{Z}/N\mathbb{Z})$ is referred to as the Weil representation. This representation defines a homomorphism $W_0 \colon SL_2(\mathbb{Z}/N\mathbb{Z}) \to PGL_N(\mathbb{C})$. Since the group $SL_2(\mathbb{Z}/N\mathbb{Z})$ for odd N does not have non-trivial central extensions, this homomorphism lifts to a map $W\colon SL_2(\mathbb{Z}/N\mathbb{Z}) \to GL_N(\mathbb{C})$.

Proposition 4.4. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, *and let the change of variables* $(y, \tau) \to (\hat{y}, \hat{\tau})$ *be defined as in Proposition* 4.1. *Then*

$$\mathscr{F}(\hat{y}_1, \ldots, \hat{y}_n \mid \hat{\tau}) = (c\tau + d)^{\frac{1}{2N\kappa}\sum_i A_i} \theta(A) \mathscr{F}(y_1, \ldots, y_n \mid \tau) \chi(A) \,, \tag{4.22}$$

where $\chi(A)$ is a projective representation of $SL_2(\mathbb{Z})$ in $V_1 \otimes \cdots \otimes V_N$, and θ is the composition of the three maps:

$$SL_2(\mathbb{Z}) \xrightarrow{\text{mod } N} SL_2(\mathbb{Z}/N\mathbb{Z}) \xrightarrow{W} GL_N(\mathbb{C}) \xrightarrow{\pi_1 \otimes \cdot \otimes \pi_N} \mathrm{Aut}(V_1 \otimes \cdots \otimes V_N) \,, \tag{4.23}$$

where $\pi_j \colon GL_N \to$ Aut(V_j) are the homomorphisms defining the action of the group GL_N in V_i.

The proof of this statement is simple and similar to the proof of Theorem 4.3. The Weil representation of $SL_2(\mathbb{Z}/N\mathbb{Z})$ arises naturally when we consider the transformations of the lattice L of poles of $\rho^*(y \mid \tau)$ which do not preserve the lattice of periods NL.

The function $\chi(A)$ satisfies Eq. (4.20) with the 2-cocycle σ still being defined by (4.21) (now for the entire $SL_2(\mathbb{Z})$). This cocycle is a coboundary, so $\chi(A)$ comes from a linear action of $SL_2(\mathbb{Z})$.

Remark. It is not clear how to compute the representation $\chi(A)$ for any nontrivial example. A good example to start with would be $N = 2, n = 1$, and V_1 is the 4-dimensional irreducible representation of $\mathfrak{sl}_2$. In this case it seems that V will be a direct sum of two irreducible 2-dimensional representations of $\Gamma(2)$, and the solutions of (3.24) will be some nontrivial vector-valued modular functions.

5. Some Examples

Let us consider the special case $N = 2, \mathfrak{g} = sl_2$. In this case β acts as follows: $\beta(e) = f, \beta(f) = e, \beta(h) = -h$. Therefore, we have

$$\rho(z|q) = \frac{1}{2}[a(z|q)(e \otimes e + f \otimes f) + b(z|q)(e \otimes f + f \otimes e) + c(z|q) h \otimes h] , \tag{5.1}$$

where

$$\begin{aligned} a(z|q) &= \frac{1}{2\pi i}\left[\zeta\left(\frac{\log z}{2\pi i} - \frac{1}{2}\middle|2\tau\right) - \zeta\left(\frac{\log z}{2\pi i} - \frac{1}{2} - \tau\middle|2\tau\right)\right] + a_0(q) , \\ b(z|q) &= \frac{1}{2\pi i}\left[\zeta\left(\frac{\log z}{2\pi i}\middle|2\tau\right) - \zeta\left(\frac{\log z}{2\pi i} - \tau\middle|2\tau\right)\right] + b_0(q) , \\ c(z|q) &= \frac{1}{4\pi i}\left[\zeta\left(\frac{\log z}{2\pi i}\middle|2\tau\right) + \zeta\left(\frac{\log z}{2\pi i} - \tau\middle|2\tau\right) - \zeta\left(\frac{\log z}{2\pi i} - \frac{1}{2}\middle|2\tau\right)\right. \\ &\quad \left. - \zeta\left(\frac{\log z}{2\pi i} - \frac{1}{2} - \tau\middle|2\tau\right)\right] + c_0(q) , \end{aligned} \tag{5.2}$$

and the constants a_0, b_0, c_0 are chosen to satisfy the condition $\rho(z|q) = -\rho(-z|q)$. We also get from (3.22),

$$\begin{aligned} 2\pi i L(1|q) = &- \sum_{m \geqq 0} \frac{q^{2m+1}(1 + q^{4m+2})}{(1 - q^{4m+2})^2}(e \otimes e + f \otimes f) \\ &- \sum_{m \geqq 0} \frac{2q^{4m+2}}{(1 - q^{4m+2})^2}(e \otimes f + f \otimes e) \\ &+ \frac{1}{2}\left(\frac{1}{8} + \sum_{m > 0} \frac{q^{2m}}{(1 + q^{2m})^2}\right) h \otimes h . \end{aligned} \tag{5.3}$$

Now let us calculate some nontrivial 1-point traces.

Example 1. Let $\Phi(z)$: $M_{-\lambda,k} \to \hat{M}_{\lambda,k} \otimes V_C^1(z)$ be an intertwining operator, where V^1 is the two-dimensional irreducible representation of $\mathfrak{sl}_2$. Then we must have $\lambda = \pm\frac{1}{2}$. Denote the corresponding operators by $\Phi^{\pm}$, and introduce the notation: $T_{\pm}(q) = \mathrm{Tr}(\Phi^{\pm} B q^{-\partial})$. It is clear that these traces do not depend on z. Let us compute them.

We have proved that $T_{\pm}(q)$ satisfy the equation

$$\kappa q \frac{\partial F}{\partial q} = L(1|q)_{11} F . \tag{5.4}$$

In the case the matrix L we are considering turns out to be a scalar 2×2 matrix:

$$2\pi i L(1|q)|_{V^1} = \frac{1}{16} + \frac{1}{2}\sum_{m > 0} \frac{q^{2m}}{(1 + q^{2m})^2} - \sum_{m \geqq 0} \frac{2q^{4m+2}}{(1 - q^{4m+2})^2} . \tag{5.5}$$

Therefore, we can explicitly integrate Eq. (5.4), which gives us the following answer.

Let $v_\pm$ be the basis of V^1, such that $hv_\pm = \pm v_\pm$, $ev_- = v_+$, $fv_+ = v_-$, $ev_+ = fv_- = 0$. Let $\eta(q)$ be the Dedekind function:

$$\eta(q) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) . \tag{5.6}$$

Then

$$T_\pm(q) = \eta(q^2)^{\frac{3}{4\kappa}} v_\pm . \tag{5.7}$$

Since the Dedekind function is a modular form of weight 1/2, the function (5.7) is a modular function of weight $3/8\kappa$, which is by no means a surprise in view of formula (4.16).

Example 2. Let $\Phi(z)$: $M_{-\lambda,k} \to \hat{M}_{\lambda,k} \otimes V_C^2(z)$ be an intertwining operator, where V^2 is the three-dimensional irreducible representation of $\mathfrak{sl}_2$. Then we must have $\lambda = 0$ or ± 1. Denote the corresponding operators by Φ^0, $\Phi^\pm$, and introduce the notation: $T_0(q) = \mathrm{Tr}(\Phi^0 B q^{-\partial})$, $T_\pm(q) = \mathrm{Tr}(\Phi^\pm B q^{-\partial})$. These traces are computed similarly to Example 1. The matrix $L(1|q)$ is now no longer scalar, but it is a diagonal matrix, so Eq. (5.4) is still easy to integrate. Here is the answer:

Let v_+, v_0, v_- be the basis of V^2, such that $hv_+ = 2v_+$, $hv_- = -2v_-$, $hv_0 = 0$, $ev_- = fv_+ = v_0$. Then:

$$T_0(q) = \eta(q^2)^{\frac{4}{\kappa}} \eta(q^4)^{-\frac{2}{\kappa}} v_0 ,$$

$$T_\pm(q) = \frac{1}{2} \eta(q^4)^{\frac{1}{\kappa}} \eta(q^2)^{\frac{1}{\kappa}} \left[\left(\frac{\eta(q)}{\eta(-q)} \right)^{\frac{1}{\kappa}} (v_+ + v_-) \pm \left(\frac{\eta(-q)}{\eta(q)} \right)^{\frac{1}{\kappa}} (v_+ - v_-) \right] . \tag{5.8}$$

Let us now calculate a simplest 2-point trace (a part of this computation is due to A. Kirillov, Jr.).

Example 3. Consider intertwining operators: $\Phi^\pm(z)$: $M_{\lambda,k} \to \hat{M}_{\lambda\pm 1,k} \otimes V_C^1(z)$. We can combine four traces out of these operators:

$$T_{\pm\pm}(z|q) = \mathrm{Tr}(\Phi^\pm(z_1)\Phi^\pm(z_2) B q^{-\partial}), \quad z = \frac{z_1}{z_2} . \tag{5.9}$$

It is possible to calculate these traces explicitly using the elliptic KZ equations.

We have proved that the traces (5.9) satisfy the equation

$$\kappa z \frac{\partial F}{\partial z} = \rho(z|q) F . \tag{5.10}$$

In the case under consideration, the traces take values in the four dimensional space $V_C^1 \otimes V_C^1$, and this equation can be explicitly solved. Indeed, let us seek the solution in the form

$$\begin{aligned} g(z|q) = {} & g_{++}(z|q) v_+ \otimes v_+ + g_{+-}(z|q) v_+ \otimes v_- \\ & + g_{-+}(z|q) v_- \otimes v_+ + g_{--}(z|q) v_- \otimes v_- . \end{aligned} \tag{5.11}$$

Consider the functions

$$h_1(z|q) = \frac{1}{2}(g_{++}(z|q) + g_{--}(z, q)),$$

$$h_2(z|q) = \frac{1}{2}(g_{++}(z|q) - g_{--}(z, q)),$$

$$h_3(z|q) = \frac{1}{2}(g_{+-}(z|q) + g_{-+}(z, q)),$$

$$h_4(z|q) = \frac{1}{2}(g_{+-}(z|q) - g_{-+}(z, q)), \tag{5.12}$$

We have

$$g(z|q) = h_1(z|q)(v_{++} - v_{--}) + h_2(z|q)(v_{++} + v_{--}) + h_3(z|q)(v_{+-} - v_{-+}) \\ + h_4(z|q)(v_{+-} + v_{-+}). \tag{5.13}$$

Equation (5.10) yields a separate first order linear differential equation for each of the functions h_1, h_2, h_3, h_4:

$$\kappa z\frac{\partial h_1}{\partial z} = \frac{1}{2}(-a + c)h_1, \quad \kappa z\frac{\partial h_2}{\partial z} = \frac{1}{2}(a + c)h_2,$$

$$\kappa z\frac{\partial h_3}{\partial z} = \frac{1}{2}(-b - c)h_3, \quad \kappa z\frac{\partial h_4}{\partial z} = \frac{1}{2}(b - c)h_4. \tag{5.14}$$

These equations are easily solved. To write down the solutions, it is convenient to use the function

$$E(z|q) = -\frac{1}{2}\wp'\left(\frac{\log z}{2\pi i}\middle|2\tau\right),$$

$$-\frac{1}{2}\wp'(y|2\tau) = \sum_{m,\, p\in\mathbb{Z}} (y - m - 2p\tau)^{-3}. \tag{5.15}$$

In terms of this function, the solutions of (5.14) can be written in the form:

$$h_1(z|q) = C_1(q)E(-z|q)^{\frac{1}{4\kappa}},$$

$$h_2(z|q) = C_2(q)E(-qz|q)^{\frac{1}{4\kappa}},$$

$$h_3(z|q) = C_3(q)E(z|q)^{\frac{1}{4\kappa}},$$

$$h_4(z|q) = C_4(q)E(qz|q)^{\frac{1}{4\kappa}}. \tag{5.16}$$

The coefficients $C_j(q)$ are easily found from Eq. (3.24):

$$C_1(q) = C_1 \eta(q^4)^{\frac{2}{\kappa}} \eta(q^2)^{-\frac{1}{\kappa}} \left(\frac{\eta(-q)}{\eta(q)} \right)^{\frac{1}{\kappa}},$$

$$C_2(q) = C_2 \eta(q^4)^{\frac{2}{\kappa}} \eta(q^2)^{-\frac{1}{\kappa}} \left(\frac{\eta(q)}{\eta(-q)} \right)^{\frac{1}{\kappa}},$$

$$C_3(q) = C_3 ,$$

$$C_4(q) = C_4 \eta(q^2)^{\frac{4}{\kappa}} \eta(q^4)^{-\frac{2}{\kappa}} , \tag{5.17}$$

It remains to say what values of constants C_j correspond to the traces (5.9). This information is given below:

$$T_{++}: \; C_1 = C_2 = 1, C_3 = C_4 = 0 ;$$

$$T_{--}: \; C_1 = -1, C_2 = 1, C_3 = C_4 = 0 ;$$

$$T_{-+}: \; C_1 = C_2 = 0, C_3 = -1, C_4 = 1 ;$$

$$T_{+-}: \; C_1 = C_2 = 0, C_3 = C_4 = 1 . \tag{5.18}$$

In general, solutions of the elliptic KZ equations cannot be expressed in terms of classical elliptic and modular functions. Their components are more complicated special functions associated with an elliptic curve.

Example 4. (T. Kojima, private communication). Consider the elliptic KZ equations with coefficients in $V^1 \otimes V^3$. Let v_1, v_{-1} be the basis of V^1 introduced in Example 1 (earlier we used the notation v_+, v_- for this basis), and let w_{-3}, w_{-1}, w_1, w_3 be a basis of V^3 in which $fw_3 = w_1, fw_1 = 2w_{-1}, fw_{-1} = 3w_{-3}, fw_{-3} = 0$, $ew_{-3} = w_{-1}$, $ew_{-1} = 2w_1$, $ew_1 = 3w_3$, $ew_3 = 0$. Let us look for solutions of the elliptic KZ of the form:

$$F(z|q) = h(z|q)(v_1 \otimes w_1 + v_{-1} \otimes w_{-1}) + f(z|q)(v_1 \otimes w_{-3} + v_{-1} \otimes w_3) . \tag{5.19}$$

The functions h and f can be found from the following 2×2 linear system of differential equations:

$$\kappa z \frac{\partial h}{\partial z} = \left(\frac{1}{2} c + a \right) h + \frac{1}{2} bf,$$

$$\kappa z \frac{\partial f}{\partial z} = \frac{3}{2} bh - \frac{3}{2} cf ; \tag{5.20}$$

System (5.20) is defined on the elliptic curve $\mathbb{C}/\langle 1, 2\tau \rangle$ and has four singular points – the points of order $\leqq 2$. Therefore, if we make a change of variable $w = \wp\left(\frac{\log z}{2\pi i} \middle| 2\tau \right)$ then system (5.20) will become a system with rational coefficients and four regular singularities at $E_1 = \wp(1/2|2\tau)$, $E_2 = \wp(\tau|2\tau)$, $E_3 = \wp(\tau + 1/2|2\tau)$, and ∞. Using Eqs. (5.20) and (5.2), we find that the leading coefficients of the right-hand side of (5.20) are equal to κ^{-1} times the following four matrices:

$$\begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix}, \begin{pmatrix} 1/4 & 1/2 \\ 3/2 & -3/4 \end{pmatrix}, \begin{pmatrix} -5/4 & 0 \\ 0 & 3/4 \end{pmatrix}, \begin{pmatrix} 1/4 & -1/2 \\ -3/2 & -3/4 \end{pmatrix}, \tag{5.21}$$

at E_1, E_2, E_3, ∞, respectively. We see that the E_1 coefficient is a scalar matrix, which means that this singular point is removable, so our system reduces to a system with three singularities, and therefore its solutions can be expressed in terms of the Gauss hypergeometric function. It has been shown by T. Kojima that the same is true for all solutions of the elliptic r-matrix equations with values in $V_1 \otimes V_3$.

6. Monodromy of the Elliptic KZ Equations

In this section we will study the monodromy of the elliptic KZ equations with respect to the lattice of periods, and compute the monodromy matrices. Although it is difficult to compute the solutions, the calculation of monodromy is fairly straightforward.

Let us first describe how to interchange the order of intertwining operators.

Let $\Phi^{w,\lambda,\nu}(z)$: $M_{\lambda,k} \to M_{\nu,k} \hat{\otimes} z^{\Delta} V_C(z)$ be the intertwining operator such that $\langle v_\nu^*, \Phi^{w,\lambda,\nu}(z)v_\lambda\rangle = z^{\Delta}w$, $w \in V^{\lambda-\nu}$. Suppose that z_1, z_2 are nonzero complex numbers, and we have a product $\Phi^{w_1,\lambda_1,\lambda_0}(z_1)\Phi^{w_2,\lambda_2,\lambda_1}(z_2)$: $M_{\lambda_2,k} \to \hat{M}_{\lambda_0,k} \otimes V_1 \otimes V_2$, where V_1 and V_2 are finite dimensional representations of $\mathfrak{g}$. The question is: can this product be expressed in terms of products of the form $\Phi(z_2)\Phi(z_1)$? Of course, we can only talk about such an expression after analytic continuation, since the former is defined for $|z_1| > |z_2|$, and the latter for $|z_1| < |z_2|$. However, if we apply analytic continuation, the answer to the question is positive, and given by the following theorem.

Theorem 6.1 (see [FR, TK]). *Let $x_{i\nu}$ be a basis of $V_1^{\nu-\lambda_0}$, and let $y_{i\nu}$ be a basis of $V_2^{\lambda_2-\nu}$. Then*

$$\Phi^{x_{r\lambda_1},\lambda_1,\lambda_0}(z_1)\Phi^{y_{s\lambda_1},\lambda_2,\lambda_1}(z_2)$$
$$= A^{\pm} \sum_{\nu,i,j} \check{R}^{\pm}_{ijrs\lambda_1\nu}(\lambda_2,\lambda_0)^{V_1V_2}\sigma\Phi^{x_{j\nu},\nu,\lambda_0}(z_2)\Phi^{y_{i\nu},\lambda_2,\nu}(z_1), \tag{6.1}$$

where $A^{\pm}$ is the analytic continuation along a path in which z_1 passes z_2 from the right (for plus) and from the left (for minus), respectively, $\check{R}(\lambda,\mu)^{V_1V_2}$ is a matrix, and σ is the permutation: $V_1 \otimes V_2 \to V_2 \otimes V_1$.

Clearly, the matrix $\check{R}^{\pm}(\lambda,\mu)^{V_1V_2}$ represents a linear operator $(V_1 \otimes V_2)^{\lambda-\mu} \to (V_2 \otimes V_1)^{\lambda-\mu}$. Therefore, if we define

$$\check{R}^{\pm}(\lambda)^{V_1V_2} = \bigoplus_{\mu} \check{R}^{\pm}(\lambda,\mu)^{V_1V_2}, \tag{6.2}$$

then $\check{R}^{\pm}(\lambda)^{V_1V_2}$ will correspond to an operator: $V_1 \otimes V_2 \to V_2 \otimes V_1$. This operator has the property: $\check{R}^{+}(\lambda)^{V_1V_2}\check{R}^{-}(\lambda)^{V_2V_1} = \mathrm{Id}$.

The operator $\check{R}^{\pm}(\lambda)$ has been computed [Koh, Dr, TK, SV], and it turned out to be proportional to the product of the quantum R-matrix of the quantum group $U_{\hat{q}}(\mathfrak{g})$ and the permutation σ (the order of this product depends on the sign, plus or minus), where $\hat{q} = e^{2\pi i/N\kappa}$. Hereafter we will assume that the matrix $\check{R}^{\pm}(\lambda)^{V_1V_2}$ is known.

Let us now compute the monodromy of solutions of the elliptic KZ equations. Again, we will need to assume that $V_1, \ldots, V_n$ are finite dimensional representations of GL_N.

The elliptic KZ system is a local system with singularities, $\mathscr{L}$, on the space E^n, where E is the elliptic curve $\mathbb{C}^*/\Gamma$, and Γ is the multiplicative subgroup in $\mathbb{C}^*$ generated by q^N. The fiber of this local system is $V_1 \otimes \cdots \otimes V_n$. Such an interpretation, however, is not very convenient for computation of monodromy since $\mathscr{L}$ has too many singularities: they occur whenever $z_i/z_j = \varepsilon^m q^p$. It would be more natural to regard the elliptic KZ system as a local system on the n^{th} power of a smaller elliptic curve $\hat{E} = \mathbb{C}^*/\hat{\Gamma}$, where $\hat{\Gamma}$ is generated by q and ε. In this case, the singularities would occur only on the loci $z_i = z_j$. But unfortunately, the elliptic KZ system is not a local system with singularities on $\hat{E}^n$: its right-hand side is not q, ε-periodic. Therefore we would like to produce a local system on $\hat{E}^n$ starting with $\mathscr{L}$. For this purpose we will use the fact that $\mathscr{L}$ has a finite group of symmetries.

Recall the notation of Sect. 4: H_N denotes the Heisenberg group of order N^3. Let H_N^n denote the n^{th} Cartesian power of H_N. Since the group H_N is naturally embedded in GL_N (it is generated by B and C), we have a natural representation of H_N in V_i and hence a representation of H_N^n in $V_1 \otimes \cdots \otimes V_n$. On the other hand, the group H_N operates on E: $Bz = qz$, $Cz = \varepsilon^{-1}z$, so the group H_N^n naturally operates on E^n. These two actions can be combined into an action of H_N^n in the trivial bundle over E_N with fiber $V_1 \otimes \cdots \otimes V_n$. This action has the following property.

Lemma. *The group H_N^n preserves the local system $\mathscr{L}$.*

Proof. The lemma follows from the definition of the elliptic r-matrix (formula (3.13)).

Now we can create a new "local system" $\mathscr{S} = \mathscr{L}/H_N^n$. The fiber of this local system is no longer the space $V_1 \otimes \cdots \otimes V_n$ but rather the quotient of this space by the action of the center of H_N^n (which is, of course, not a vector space). In this section we will describe the monodromy of this local system. This monodromy will be a linear representation of a suitable central extension of the fundamental group of $\hat{E}^n \backslash \{\text{diagonals}\}$, and it obviously contains all the information about the monodromy of the elliptic KZ equations.

As before, we will use the fundamental solution $\mathscr{F}(z_1, \ldots, z_n|q)$ defined by (4.17) (now we prefer to use the $\mathbf{z}, q$ variables rather than $\mathbf{y}, \tau$ variables). This solution takes values in the spaces $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$.

Below we study the monodromy of the fundamental solution. This is equivalent to studying the monodromy of the local system $\mathscr{S}$.

We will use the representation of $\hat{E}$ as a parallelogram on the complex plane: $\frac{\log z}{2\pi i} = \frac{x}{N} + y\tau$, $0 \leqq x, y < 1$, and write $z = (x, y)$. We choose a base point $(z_1^0, \ldots, z_n^0) = (x_1^0, y_1^0, \ldots, x_n^0, y_n^0)$ on $\hat{E}_n$ such that $1 > x_1^0 > \cdots > x_n^0 > 0$, $1 > y_1^0 > \cdots > y_n^0 > 0$.

It follows from the definition of $\mathscr{F}(z_1, \ldots, z_n|q)$ that it can be represented in the form $\mathscr{F}_0(z_1, \ldots, z_n|q) z_1^{D_1} z_2^{D_2} \ldots z_n^{D_n}$, where $\mathscr{F}_0$ is a single-valued function, and $D_1, \ldots, D_n$ are operators in $V_1 \otimes \cdots \otimes V_n$ defined as follows. Let vectors $v_j \in V_j$

satisfy the condition $hv_j = \chi_j(h)v_j$, $h \in \mathfrak{h}$, $\chi_j \in \mathfrak{h}^*$. Let

$$\lambda_j = (\beta - 1)^{-1}\left(\sum_{i=1}^{n} \chi_i\right) + \sum_{i=1}^{j} \chi_i, \quad 0 \leqq j \leqq n .$$

Then

$$D_j(v_1 \otimes \cdots \otimes v_n) = \frac{\langle \lambda_{j-1}, \lambda_{j-1}\rangle - \langle \lambda_j, \lambda_j\rangle}{2\kappa} \cdot v_1 \otimes \cdots \otimes v_n \tag{6.3}$$

(from this definition of D_j it immediately follows that $[D_i, D_j] = 0$ for any i, j).

The fundamental solution has a defining property: if u is a vector in $V_1 \otimes \cdots \otimes V_n$ then $\mathscr{F}_0 \cdot u$ is the solution of the system (3.14), (3.24) which tends to u as $q \to 0$ and $z_i/z_{i+1} \to \infty$.

This observation helps us find the monodromy of the function $\mathscr{F}$ around the ε-cycles on $\hat{E}^n$. It z_j is rotated around the origin anticlockwise, through the angle $2\pi i l/N$, with the rest of the variables fixed, the function $\mathscr{F}$ multiplies by the matrix $e^{2\pi i l D_j/N}$ from the right, and undergoes a conjugation by C_j – the action of C in V_j. We denote these monodromy operators by E_j:

$$E_j(\mathscr{F}) = C_j^{-1}\mathscr{F} C_j e^{2\pi i l D_j/N} . \tag{6.4}$$

The monodromy of the fundamental solution in the neighborhood of the locus $z_j = z_{j+1}$ can be found with the help of Theorem 6.1. Using this theorem and the representation of solutions as traces of products of intertwiners, we immediately find that the monodromy of the function $\mathscr{F}$ around the locus $z_j = z_{j+1}$ (in the anticlockwise direction) is

$$b_{j,j+1}(\mathscr{F}) = \mathscr{F} S_{j,j+1} , \tag{6.5}$$

where

$$S_{j,j+1} = \check{R}_j^-(V_1, \ldots, V_j, V_{j+1}, \ldots, V_n)\check{R}_j^-(V_1, \ldots, V_{j+1}, V_j, \ldots, V_n) ,$$

and the linear operators

$$\begin{aligned}\check{R}_j^{\pm}(V_1, \ldots, V_n)\colon V_1 \otimes \cdots \otimes V_j \otimes V_{j+1} \otimes \cdots \otimes V_n \\ \to V_1 \otimes \cdots \otimes V_{j+1} \otimes V_j \otimes \cdots \otimes V_n\end{aligned}$$

are defined as follows: if $v_i \in V_i$, $1 \leqq i \leqq n$, and $hv_i = \chi_i(h)v_i$, $\chi_i \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, then

$$\begin{aligned}&\check{R}_j^{\pm}(V_1, \ldots, V_n)(v_1 \otimes \cdots \otimes v_j \otimes v_{j+1} \otimes \cdots \otimes v_n) \\ &= v_1 \otimes \cdots \otimes \check{R}^{\pm}(\lambda_{j+1})^{V_j V_{j+1}}(v_j \otimes v_{j+1}) \otimes \cdots \otimes v_n ,\end{aligned}$$

where λ_j are defined by (4.18).

The monodromy is obviously the same as for the usual (trigonometric) KZ equations, whose solutions are given by matrix elements of intertwiners rather than traces (see Sect. 2) – a known fact which was first observed by I. Cherednik [Ch2].

Let us now find the monodromy of the function $\mathscr{F}$ around the q-cycles. Consider the cycle on $\hat{E}^n$ in which z_j passes $z_{j+1}, \ldots, z_n$ from the right, hits the circle $|z| = |q|$ (which is identified with the circle $|z| = 1$ through the map $z \mapsto qz$), jumps over to the circle $|z| = 1$, and then passes $z_1, \ldots, z_{j-1}$ from the left, returning to its initial position (this corresponds to the path $y_j(t) = y_j^0 - t \bmod 1$, with the rest of x_i and y_i remaining unchanged). Using Theorem 6.1 and the expression of $\mathcal{F}$ in terms of traces, we find the monodromy matrices Q_j for the described q-cycles. Indeed, we have to interchange $\Phi(z_j)$ in the trace expression with $\Phi(z_{j+1}), \ldots, \Phi(z_n)$, then with $q^{-\partial}$ and B, and then with $\Phi(z_1), \ldots, \Phi(z_{j-1})$. This results in an expression of Q_j as a product of R-matrices. To write this expression down, introduce the following notation: if $s \in S_n$ is a permutation of n items then set $\check{R}_j^{\pm}(s) = \check{R}_j^{\pm}(V_{s(1)}, \ldots, V_{s(n)})$. Let t_j be the elementary transpositions $(j, j+1)$, and let $s_{jm} = t_{m-1} \ldots t_{j+1} t_j$, $j < m \leqq n$, $s_{jm} = t_m \ldots t_{j-2} t_{j-1}$, $1 \leqq m < j$ (we make a convention that for two permutations σ_1, σ_2 $\sigma_1 \sigma_2(j) = \sigma_1(\sigma_2(j))$, $1 \leqq j \leqq n$, i.e. the factors in a product of permutations are applied from right to left). Also let $R_j^{\pm}(s) = s^{-1} t_j \check{R}_j^{\pm}(s) s$. Then $R_j^{\pm}$ is a linear operator in $V_1 \otimes \cdots \otimes V_n$.

Now the operators Q_j are expressed as follows:

$$Q_j(\mathcal{F}) = B_j \mathcal{F} R_j^-(s_{jj+1}) R_{j+1}^-(s_{jj+2}) \ldots R_{n-1}^-(s_{jn}) B_j^{-1} R_1^+(s_{j2}) \times \ldots R_{j-2}^+(s_{jj-1}) R_{j-1}^+(\mathrm{Id}), \tag{6.6}$$

where B_j denotes the action of B in V_j.

Remarks. 1. Expressions similar to (6.6) (ordered products of R-matrices) occur in the theory of correlation functions for quantum affine algebras. Such correlation functions satisfy a quantum analogue of the Knizhnik–Zamolodchikov equations – a system of difference equations discovered by Frenkel and Reshetikhin [FR]. The structure of the right-hand side of this system resembles (6.6). It is not clear if it is merely a coincidence or not.

2. It is seen from the definitions of E_j and Q_j that $[E_i, E_j] = [Q_i, Q_j] = 0$ for any i, j.

As we have already remarked, the monodromy of the local system $\mathcal{S}$ defines a representation of a central extension of the fundamental group of the complement of the diagonals $z_i = z_j$ in $\hat{E}_n$ – the pure braid group of the torus. To describe this representation in more detail, let us assume that $V_i = V$ for all i, where V is some finite-dimensional GL_N-module. This does not cause any loss of generality since we can always set $V = V_1 \oplus \cdots \oplus V_n$ to include the previously considered case. But now the elliptic KZ system (and the local system $\mathcal{S}$) has an additional symmetry – the symmetry under the simultaneous permutation of the variables z_i and the factors in the product $V \otimes V \otimes \cdots \otimes V$. Therefore, the monodromy representation can in fact be regarded as a representation of a certain (in general, not central) extension of the full braid group of the torus.

The braid group of the torus, BT_n, is generated by the elements T_i, $1 \leqq i \leqq n-1$, X_1, Y_1, satisfying the defining relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}; \quad T_i T_j = T_j T_i, \quad j > i+1;$$

$$(T_1 X_1)^2 = (X_1 T_1)^2; \quad (T_1 Y_1^{-1})^2 = (Y_1^{-1} T_1)^2;$$

$$T_j X_1 = X_1 T_j, \quad T_j Y_1 = Y_1 T_j, \quad j > 1;$$

$$X_2 Y_1^{-1} X_2^{-1} Y_1 = T_1^2, \quad \text{where} \quad X_{i+1} = T_i X_i T_i;$$

$$X_0 Y_1 = Y_1 X_0, \quad \text{where} \quad X_0 = X_1 X_2 \ldots X_n. \tag{6.7}$$

It is also convenient to define the elements $Y_{j+1} = T_j^{-1} Y_j T_j^{-1}$ and $Y_0 = Y_1 Y_2 \ldots Y_n$.

To picture the braid group of the torus geometrically, one should imagine n "beetles" crawling on the surface of the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ starting from some fixed positions $z_1, \ldots, z_n$ ($z_j = N^{-1}x_j + \tau y_j$, $x_1 > x_2 > \ldots > x_n$, $y_1 > y_2 > \ldots > y_n$) so that at no time two beetles can be at the same point and after some period of time (say 1) the "beetles" return to their original positions, possibly with some permutation. Then the beetles will trace out some collection of curves in Torus $\times [0, 1]$ – a braid diagram. Such diagrams can be composed by attaching the bottom of one of them to the top of the other. Under this composition law, braid diagrams form a group – the braid group of the torus BT_n. The element T_j corresponds to the intertwining of the j^{th} and $j + 1^{\text{th}}$ braids (the j^{th} and $j + 1^{\text{th}}$ "beetles" switch, the j^{th} "beetle" passing the $j + 1^{\text{th}}$ one from the right), and the elements X_j and Y_j arise when the j^{th} "beetle" crawls around the x-cycle and y-cycle of the torus, respectively, in the negative direction of the x-axis (respectively, y-axis), with the rest of the "beetles" unmoved.

Now we are in a position to formulate the result about the monodromy of the local system $\mathcal{S}$.

Theorem 6.2. *The monodromy representation of an extension of BT_n associated to the local system $\mathcal{S}$ is defined as follows:*

$$X_j \mapsto e^{-2\pi i D_j/N} C_j^{-1} ,$$

$$Y_j \mapsto R_j^- \ldots R_{n-1}^- B_j^{-1} R_1^+ \ldots R_{j-2}^+ R_{j-1}^+ ,$$

$$T_j \mapsto \check{R}_j^- . \tag{6.8}$$

Remark. Since all the spaces V_j are the same, we have dropped the permutations labeling the R-matrices.

Proof. The theorem follows from Theorem 6.1 and formulas (6.4)–(6.6).

The extension of BT_n involved in Theorem 6.2 is very easy to describe. Consider the quotient of the group BT_n by the relations $T_i^2 = 1$. Then we will obtain the group $S_n \ltimes (\mathbb{Z}^n \oplus \mathbb{Z}^n)$, where S_n acts on both copies of $\mathbb{Z}^n$ by permutations of components. Let H be the Heisenberg group – the central extension of $\mathbb{Z} \oplus \mathbb{Z}$ by $\mathbb{Z}$ by means of the 2-cocycle $\omega(\mathbf{x}, \mathbf{y}) = x_1 y_2 - x_2 y_1$. Then we can construct the group $S_n \ltimes H^n$ (H^n is the n^{th} Cartesian power of H) which is a rank n abelian extension of $S_n \ltimes (\mathbb{Z}^n \oplus \mathbb{Z}^n)$. Let us denote the pullback of this abelian extension to BT_n by $\widehat{BT_n}$. The monodromy representation of $\mathcal{S}$ is thus a representation of $\widehat{BT_n}$ in $V \otimes \cdots \otimes V$.

Now assume that V is irreducible. Then instead of the group $\widehat{BT_n}$ we will have to deal with a rank 1 central extension $\widetilde{BT_n}$ of BT_n which is constructed as follows. Take the group $S_n \ltimes (\mathbb{Z}^n \oplus \mathbb{Z}^n)$, construct a rank 1 central extension of this group by adjoining a new central element c satisfying the relations $X_i Y_i = Y_i X_i c$, $X_i Y_j = Y_j X_i$, $i \neq j$, and then pull this extension back to BT_n. It follows then that the monodromy representation of $\mathcal{S}$ is a projective representation of BT_n which comes from a linear representation of $\widetilde{BT_n}$.

The group $\widetilde{BT_n}$ is closely related to the double affine braid group of type A_{n-1} defined by I. Cherednik in his recent paper [Ch3] – the group generated by the elements T_i, X_i, Y_i and a central element δ satisfying modified relations (6.7):

the relation $X_2 Y_1^{-1} X_2^{-1} Y_1 = T_1^2$ has to be replaced by $X_2 Y_1^{-1} X_2^{-1} Y_1 = \delta T_1^2$. Let us describe the connection between them.

Observe that the elliptic KZ system commutes with the diagonal action of the Heisenberg group H_N^{diag} generated by X_0 and Y_0 in $V \otimes \cdots \otimes V$. Decompose $V \otimes \cdots \otimes V$ into a sum of irreducible representations of H_N^{diag}: $V \otimes \cdots \otimes V = \bigoplus_i P_i \otimes W_i$, where P_i are distinct irreducible representations of H_N, and W_i are multiplicity spaces. Then each summand $P_i \otimes W_i$ is a subrepresentation of $\widetilde{BT_n}$. Let $\phi_i : \widetilde{BT_n} \to \operatorname{End}(P_i \otimes W_i)$ be the corresponding homomorphism.

Assume that n and N are coprime. Then we can define a homomorphism $\xi : \widetilde{BT_n} \to H_N$ by $\xi(X_i) = C^{1/n}$, $\xi(Y_i) = B^{1/n}$, $\xi(T_i) = 1$. (Here $1/n$ is regarded as an element of the ring $\mathbb{Z}/N\mathbb{Z}$.) Composing this homomorphism with the diagonal action of H_N in $V \otimes \cdots \otimes V$, we get a projective representation $\psi_i(\cdot)$ of BT_n in $\operatorname{End}(P_i \otimes W_i)$. Notice that $\psi_i(g) = \phi_i(g)$ if $g \in H_N^{\text{diag}}$. Let us write ϕ_i as $\phi_i(g) = \psi_i(g) \cdot \psi_i(g)^{-1} \phi_i(g)$. Because the elliptic KZ system commutes with H_N^{diag}, $\psi_i(g)$ commutes with $\psi_i(g)^{-1}\phi_i(g)$, which implies that $\psi_i(g)^{-1}\phi_i(g) = \operatorname{Id} \otimes \chi_i(g)$, where $\chi_i(\cdot)$ is some projective action of BT_n in W_i. We also have $\psi_i(g) = \tilde{\psi}_i(g) \otimes 1$, where $\tilde{\psi}_i$ is an action of the group in P_i. Therefore, we have $\phi_i = \tilde{\psi}_i \otimes \chi_i$. Therefore, we can easily compute the 2-cocycle on BT_n corresponding to χ_i as the difference of the 2-cocycles for ϕ_i and ψ. This cocycle is the pullback from $S_n \ltimes (\mathbb{Z}^n \oplus \mathbb{Z}^n)$ of the 2-cocycle given by:

$$\omega((s_1, \mathbf{x}^1, \mathbf{y}^1), (s_2, \mathbf{x}^2, \mathbf{y}^2)) = \sum_{i=1}^{n} (x_i^1 y_i^2 - x_i^2 y_i^1) - \frac{1}{n} \sum_{i,j=1}^{n} (x_i^1 y_j^2 - x_i^2 y_j^1) ,$$
$$s_{1,2} \in S_n, \quad \mathbf{x}^{1,2}, \mathbf{y}^{1,2} \in \mathbb{Z}^n . \tag{6.9}$$

The extension of BT_n by means of this cocycle is exactly the double affine braid group. We denote this group by $\overline{BT_n}$.

The special case when V is the N-dimensional vector representation of GL_N is especially interesting. In this case, we are getting a representation of the *double affine Hecke algebra* $\mathfrak{h}_n^{\hat{q}}$. This algebra was recently defined by I. Cherednik [Ch3] as the quotient of the group algebra $\mathbb{C}[\overline{BT_n}]$ by the relations $(T_j - \hat{q})(T_j + \hat{q}^{-1}) = 0$. Indeed, the matrix $\check{R}_j^-$ is diagonalizable and has the eigenvalues $\hat{q}^{1-\frac{1}{2N}}$ and $-\hat{q}^{-1-\frac{1}{2N}}$, (recall $\hat{q} = e^{2\pi i/N\kappa}$). This statement easily follows from the fact that this matrix is a monodromy matrix of the elliptic KZ equations. Therefore, the matrix $\tilde{R}_j^- = \hat{q}^{\frac{1}{2N}} \check{R}_j^-$ satisfies the equation

$$(\tilde{R}_j^- - \hat{q})(\tilde{R}_j^- + \hat{q}^{-1}) = 0 . \tag{6.10}$$

Thus, the correspondence

$$X_1 \mapsto e^{-2\pi i l D_1/N} C_1^{-1}, \quad Y_1 \mapsto \hat{R}_1^- \hat{R}_2^- \ldots \hat{R}_{n-1}^- B_1^{-1}, \quad T_j \mapsto \tilde{R}_j^- , \tag{6.11}$$

where $\hat{R}_j^- = \hat{q}^{\frac{1}{2N}} R_j^-$, defines a projective representation of BT_n in the space $V \otimes V \otimes \cdots \otimes V$ which can be written as $P \otimes W$, where P is the N-dimensional irreducible projective representation of BT_n obtained by composing the homomorphism ξ defined above with the standard action of H_N in $\mathbb{C}^N$, and W is a representation of the double affine Hecke algebra $\mathfrak{H}_n^{\hat{q}}$.

The element δ acts in the representation W by multiplication by $\hat{q}^{-1/N} \varepsilon^{-1/n}$.

Remark. In fact, the monodromy of the elliptic KZ system (3.14) extended by Eq. (3.24) yields a representation of the semidirect product $SL_2(\mathbb{Z}) \ltimes \widehat{BT}_n$ in the space $\mathrm{End}(V \otimes \cdots \otimes V)$. As we have already remarked in Sect. 4, it is not clear how to compute the modular part of this monodromy, i.e. the action of $SL_2(\mathbb{Z})$. This computation, at least for one nontrivial example, is a very interesting and challenging problem.

As a conclusion, let us note that the study of monodromy helps us to find out for what special values of κ the elliptic KZ equations are integrable in elliptic functions.

Proposition 6.3. *If $\kappa = 1/MN$, where M is an integer, then the matrix elements of the fundamental solution $\mathscr{F}(z_1, \ldots, z_n|q)$ are finite products of rational powers of theta functions of expressions $q^m \varepsilon^p z_i/z_j$, $0 \leqq m, n \leqq N-1$.*

Idea of Proof. If $\kappa = 1/NM$ then $\hat{q} = 1$. Therefore, $\check{R}^+$ is a scalar matrix times the permutation of factors, so $S_{j,j+1} = \alpha \mathrm{Id}$, and $b_{j,j+1}(\mathscr{F}) = \alpha\mathscr{F}$. Let $\alpha = e^{2\pi i s}$ (s is rational). Then we have

$$\mathscr{F}(z_1, \ldots, z_n|q) = \prod_{i<j} \sum_{0 \leqq m, p \leqq N-1} \Theta(q^m \varepsilon^p z_i/z_j | q^N)^s \exp\left(\int \omega\right), \tag{6.12}$$

where ω is a matrix-valued elliptic differential form on E^n and

$$\Theta(z|q) = \prod_{m \geqq 0} (1 - q^m z)(1 - q^{m+1} z^{-1})(1 - q^{m+1}). \tag{6.13}$$

Remark. A similar result holds for the trigonometric KZ equations (1): if $k + h^\vee = 1/M$, where M is an integer then solutions of the KZ equations are algebraic functions.

Acknowledgements I would like to thank my adviser I. Frenkel for stimulating my work by systematic, inspiring discussions. I am grateful to I Cherednik, W. Feit, H Garland, M Kashiwara, D. Kazhdan, and A Varchenko for many useful suggestions, and to A. Kirillov, Jr and T. Kojima for sharing with me the results of their work on the subject and for illuminating remarks

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Communicated by N Yu Reshetikhin