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# A Duality for Hopf Algebras and for Subfactors. I

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*Dedicated to Masamichi Takesaki on the occasion of his sixtieth birthday*

**Abstract:** We provide a duality between subfactors with finite index, or finite dimensional semisimple Hopf algebras, and a class of $C^*$-categories of endomorphisms.

## 1. Introduction

The aim of this work is to provide a duality between subfactors with finite index of an infinite factor $M$ or finite-dimensional (semisimple, complex) Hopf algebras and a class of $C^*$-categories.

Hereafter we shall restrict ourselves to the case of concrete $C^*$-categories that are realized by endomorphisms of $M$ [6] and we will provide a general construction of a crossed product algebra. In the sequel of this paper our duality will be formulated in terms of abstract $C^*$-categories.

Our main technique is index theory for infinite factors [13, 7, 14], sector theory in particular, and we rely on the following ideas. Suppose that a subfactor $N \subset M$ has been constructed, then $M$ becomes equipped with a distinguished sector (an endomorphism up to inner automorphisms) $\lambda$, the canonical endomorphism of $M$ into $N$ [17]. The sectors in the irreducible decomposition of $\lambda|_N$ then provide the dual $C^*$-category.

To give insight to this structure let us recall the simple example of a faithful action $\alpha$ of a finite group $G$ on an infinite factor $M$ with irreducible fixed-point subfactor $N$. In this case the sectors in the irreducible decomposition of $\lambda$ furnish the group $G$

$$\lambda \cong \bigoplus_{g \in G} \alpha_g ,$$

while the restriction of $\lambda$ to $N$ corresponds to the dual $\hat{G}$ of $G$

$$\lambda|_N \cong \bigoplus_{\pi \in \hat{G}} d(\pi) \varrho_\pi ,$$

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where the sectors $\varrho_\pi$ of $N$ are naturally associated with the irreducible representations $\pi$ of $G$ (cf. Corollary 3.5 or [9, 11]). In this case $\lambda$ has a meaning of a regular representation.

A subfactor with finite index gives rise to a family of irreducible sectors and we are faced with the reverse problem of deciding when a family of irreducible sectors arises in this way. In other words our problem is to characterize when an endomorphism $\lambda$ of $M$ is canonical with respect to a subfactor. Note that this would symmetrically characterize the situation where $\lambda$ is the restriction of a canonical endomorphism $\tilde{\lambda}: \tilde{M} \to M$ providing a crossed product factor $\tilde{M}$.

Such a characterization is indeed the core of our analysis and will be dealt with at three stages. Starting with an infinite factor $M$ and a finite index endomorphism $\lambda$ of $M$ we shall give necessary and sufficient conditions on two intertwiners between $\lambda$ and $\lambda^2$ to ensure $\lambda$ to be the canonical endomorphism with respect to an irreducible subfactor $N$ of $M$, that we construct canonically. More specifically there should exist an isometry $T \in (\mathrm{id}, \lambda)$ (unique up to a phase for $N$ to be irreducible) and an isometry $S \in (\lambda, \lambda^2)$ such that

$$\text{(b}_1\text{)} \qquad \lambda(S)S = S^2\,, \qquad \lambda(S^*)S = SS^*\,,$$

$$\text{(b}_2\text{)} \qquad S^*\lambda(T) \in \mathbb{C}\backslash\{0\}\,, \qquad T^*S \in \mathbb{C}\backslash\{0\}\,.$$

We arrive therefore at the notion of an irreducible $Q$-system: a triple $(M, \lambda, S)$, where $M$ is an infinite factor, $\lambda$ is an endomorphism of $M$ that contains the identity with multiplicity one (as a sector) and $S \in (\lambda, \lambda^2)$ is an isometry that satisfies the equations ($\mathrm{b}_1$) and ($\mathrm{b}_2$) with $T$ the unique (up to a phase) non-zero intertwiner in $(\mathrm{id}, \lambda)$. By our results, an irreducible $Q$-system corresponds to an irreducible subfactor.

The two basic intertwiners $T$ and $S$ exist because, by results in [15], they correspond to conditional expectations. The final projections of $T$ and $S$ are the Jones projections and the second equations ($\mathrm{b}_2$) are essentially equivalent to the Jones projection relations [13].

Concerning the first line of equations ($\mathrm{b}_1$), they are related to the pentagon relations defining a multiplicative unitary $V$ in the sense of Baaj and Skandalis

$$V_{12}V_{13}V_{23} = V_{23}V_{12}$$

that is associated and determines a Hopf algebra [1].

Indeed equations ($\mathrm{b}_1$) generalize for $T$ and $S$ the notions of fixed and cofixed vector for a multiplicative unitary, reducing to the latter in the special case of the crossed product inclusions by a Hopf algebra. Fixed and cofixed vectors thus appear as more fundamental objects than the multiplicative unitary itself (in the finite-dimensional case) inasmuch as they are present in a widely more general setting where the Hopf structure disappears.

We shall characterize the case where the $Q$-system actually arises by a Hopf algebra action by the distinguished property of the regular representation

$$\lambda^2 \cong d \cdot \lambda\,,$$

namely $\lambda^2$ is a multiple of $\lambda$, a property related to the Ocneanu characterization (see the Appendix), that will provide a duality for Hopf algebras.

A starting point in our analysis is the model of Cuntz for regular actions of Hopf algebras on the $C^*$-algebras $\mathcal{O}_d$ [3]. There is a bijective correspondence between unitaries of $\mathcal{O}_d$ and endomorphisms of $\mathcal{O}_d$. Since a multiplicative unitary $V$ corresponds to a Hopf algebra [1], Cuntz has analyzed the corresponding endomorphism of $\mathscr{O}_d$, more precisely the endomorphism $\lambda_R$ associated with the product $R = VF$ of $V$ with the flip $F$. Our initial results consist in the interpretation of this model in terms of index theory for infinite factors, inspired by the appearance of our formulas for the conditional expectations [14] in this context. In a suitable GNS representation, $\lambda_R$ will become in fact the canonical endomorphism with respect to the fixed-point algebra.

In the case of a compact group, an abstract duality has been obtained by Doplicher and Roberts [6]; their duality and our duality, both based on $C^*$-categories of endomorphisms, rely nevertheless on different viewpoints and methods. Restricted to the common case of a finite group, they provide different descriptions of the dual objects and a direct equivalence between them remains an interesting problem.

## 2. Index of Endomorphisms of $\mathscr{O}_H$

Let $\mathscr{O}_H$ be the Cuntz algebra [4] generated by the Hilbert space $H$ of dimension $d < \infty$. If $\{T_i, i = 1, \ldots, d\}$ is an orthonormal basis of $H$, the $T_i$ are isometries and

$$\varphi(x) = \sum_{i=1}^{d} T_i x T_i^* , \qquad x \in \mathscr{O}_H$$

defines the canonical inner endomorphism of $\mathscr{O}_H$.

If $\lambda$, $\eta \in \mathrm{End}(\mathscr{O}_H)$ we denote by

$$(\lambda, \eta) = \{S \in \mathscr{O}_H,\ \eta(x)S = S\lambda(x),\ x \in \mathscr{O}_H\} \tag{2.1}$$

the linear space of their intertwiners (see [6]).

In particular we put

$$\mathfrak{M}_n \equiv (\varphi^n, \varphi^n) = (H^n, H^n) \equiv H^n H^{n^*} ,$$

then $\mathfrak{M}_n$ is a $d^n \times d^n$ matrix algebra.

If $u$ is a unitary of $\mathscr{O}_H$

$$\lambda_u(T) = uT , \qquad T \in H$$

determines an endomorphism of $\mathscr{O}_H$ and all endomorphisms of $\mathscr{O}_H$ arise in this way [3].

Let $\tau$ be the unique tracial state of the UHF algebra

$$\mathfrak{M} \equiv \overline{\cup \mathfrak{M}_k}$$

and

$$\omega \equiv \tau \cdot \varepsilon$$

be the state obtained by composition with the conditional expectation of $\mathscr{O}_H$ onto $\mathfrak{M}$,

$$\varepsilon = \int_{\mathbb{T}} \lambda_t \, \mathrm{d}t .$$

We set

$$M \equiv \pi_\omega(\mathscr{O}_H)'' , \qquad M_\omega \equiv \pi_\omega(\mathfrak{M})'' ,$$

where $\pi_\omega$ is the associated faithful GNS representation and we shall omit the symbol $\pi_\omega$ for shortness.

Then $\omega$ is a KMS state for $\lambda_t$, $t \in \mathbb{T}$, $M$ is a $III_{\frac{1}{d}}$ factor and $M_\omega$ is a $II_1$ factor (cf. [2, 4]).

**Lemma 2.1.** *If $u \in \mathfrak{M}$ is a unitary then $\omega \cdot \lambda_u = \omega$.*

*Proof.* $\lambda_u$ and $\lambda_t$, $t \in \mathbb{T}$, commute because $u \in \mathfrak{M}$, therefore $\lambda_u$ commutes with $\varepsilon$. Now

$$\lambda_u(\mathfrak{M}) \subset \mathfrak{M}$$

(see Lemma 2.2) hence $\tau \cdot \lambda_u = \tau$ on $\mathfrak{M}$ by the unicity of the trace and

$$\omega \cdot \lambda_u = \tau \cdot \varepsilon \cdot \lambda_u = \tau \cdot \lambda_u \cdot \varepsilon = \tau \cdot \varepsilon = \omega . \qquad \square$$

**Lemma 2.2.** *If $u \in \mathfrak{M}_n$ is a unitary, then*

$$\lambda_u(\mathfrak{M}_k) \subset \mathfrak{M}_{n+k-1} .$$

*Proof.* By [3], if $x \in \mathfrak{M}_k$ then

$$\lambda_u(x) = u\varphi(u) \ldots \varphi^{k-1}(u) x \varphi^{k-1}(u^*) \ldots u^* . \tag{2.2}$$

Since $\varphi$ acts as a shift on $\mathfrak{M}$, we have $\varphi^k(\mathfrak{M}_n) \subset \mathfrak{M}_{n+k}$, thus all elements in (2.2) belong to $\mathfrak{M}_{n+k-1}$. $\square$

**Lemma 2.3.** *If $u \in \mathfrak{M}$ is a unitary, $\lambda_u$ extends to a normal endomorphism of $M$.*

*Proof.* Since $\omega$ is a KMS state for $\lambda_t$, $\omega$ is faithful on $M$, hence $\lambda_u$ extends to $M$ by its $\omega$-invariance. $\square$

We still denote by $\lambda_u \in \mathrm{End}(M)$ the extension of $\lambda_u$ to $M$ given by Lemma 2.3.

**Propositon 2.4.** *If $u \in \mathfrak{M}_n$ is a unitary, then*

$$d(\lambda_u) \leq d^{n-1} , \tag{2.3}$$

*where $d$ denotes the dimension of $\lambda_u$ (*i.e. $d(\lambda_u) = \mathrm{Ind}(\lambda_u)^{1/2}$ *with* Ind *the minimal index). The bound* (2.3) *is optimal.*

*Proof.* Since $\lambda_u$ commutes with $\lambda_t$, it follows that $\lambda_t$ leaves $\lambda_u(M)$ invariant and by the Takesaki theorem [24] there exists a normal conditional expection $E: M \to \lambda_u(M)$ leaving $\omega$ invariant.

We first show that $d(\lambda_u|_{M_\omega}) \leq d^{n-1}$. Note that $E|_{M_\omega}$ is the $\tau$-preserving conditional expectation of $M_\omega$ onto $\lambda_u(M_\omega)$.

Fix $k \in \mathbb{N}$ and let $E_k$ denote the trace preserving expectation $\mathfrak{M}_{n+k-1}$ onto $\lambda_u(\mathfrak{M}_n)$ (Lemma 2.2).

If $x \in \mathfrak{M}_{n+k-1}$,

$$\|E(x) - x\|_2 \leq \|y - x\|_2 , \qquad \forall y \in \lambda_u(\mathfrak{M})$$

because $E$ is an orthogonal projection in $L^2(N, \tau)$, hence

$$\|E(x) - x\|_2^2 \leq \|E_k(x) - x\|_2^2$$

or

$$\|E(x)\|_2 \geq \|E_k(x)\|_2 .$$

If moreover $x \geq 0$, the Pimsner-Popa inequality [22] yields

$$\|E(x)\|_2 \geq \|E_k(x)\|_2 \geq \mu\|x\|_2\,,$$

where

$$\begin{aligned}\mu^{-1} &= \mathrm{Ind}(\lambda_u(\mathfrak{M}_k), \mathfrak{M}_{n+k-1}) \\ &= \mathrm{Ind}(\mathfrak{M}_k, \mathfrak{M}_{n+k-1}) \\ &= (d^{n-1})^2\end{aligned}$$

thus, since $k$ is arbitrary,

$$d(\lambda_u|_{M_\omega}) \leq d^{n-1}\,.$$

We shall now show that

$$d(\lambda_u) = d(\lambda_u|_{M_\omega})\,. \tag{2.4}$$

Since $\lambda_u(M_\omega) \subset M_\omega$ is the fixed point of $\lambda_u(M) \subset M$ with respect to $\lambda_t$, $t \in \mathbb{T}$, Eq. (2.4) would follow if $\{\lambda_t, t \in \mathbb{T}\}$ restricted to a dominant action on $\lambda_u(M)$ [14]; this is not true because $M_\omega$ is a finite factor, but $\{\lambda_t, t \in \mathbb{T}\}$ has eigen-isometries for all positive integers by definition, hence if $\mathscr{F}$ is type $I_\infty$ factor, $\lambda_t \otimes \mathrm{id}$ defines a dominant action on $\lambda_u(M) \otimes \mathscr{F}$ and

$$d(\lambda_u|_{M_\omega}) = d(\lambda_u \otimes \mathrm{id}\,|_{M_\omega \otimes \mathscr{F}}) = d(\lambda_u \otimes \mathrm{id}) = d(\lambda_u)\,.$$

To see that the bound is optimal, note that $d(\varphi^n) = d^n$ and

$$\varphi^n = \lambda_u \quad \text{with} \quad u = \varphi^n(F)\varphi^{n-1}(F)\ldots F \in \mathfrak{M}_{n+1}\,. \qquad \square$$

*Remark.* It is a natural problem to understand the condition for $\lambda_u$ to admit a conjugate endomorphism within the $C^*$-algebra $\mathscr{O}_H$ and to check when the Watatani index occurs [25]. A related problem is to find a formula for the index

$$\mathrm{Ind}(\lambda_u) = F(u)$$

in terms of an explicit function $F$ for a general unitary $u \in \mathscr{O}_H$. Examples with irrational index are given in [12].

We conclude this section with the following proposition.

**Proposition 2.5.** *If* $u \in \mathscr{O}_H$, *then*

$$\lambda_u(M)' \cap M = \{x \in M, \varphi(x) = u^* x u\}\,; \tag{2.5}$$

*if morevoer* $u \in \mathfrak{M}_n$, *then* $\lambda_u(M)' \cap M_\omega \subset \mathfrak{M}_{n-1}$.

*Proof.* If $x \in M$ then $x \in \lambda_u(M)'$ iff

$$x\lambda_u(T) = \lambda_u(T)x\,, \qquad T \in H\,,$$

namely $xuT = uTx = u\varphi(x)T$, that holds iff $xu = u\varphi(x)$, namely iff

$$\psi(x) = x\,, \tag{2.6}$$

where $\psi \equiv \mathrm{ad}(u) \cdot \varphi$, proving (2.5).

Concerning the second statement, let $u \in \mathfrak{M}_n$ and $\Psi$ be the completely positive map

$$\Psi \equiv \Phi \cdot \mathrm{ad}(u^*)\,,$$

where $\Phi$ is the minimal left inverse of $\varphi$,

$$\Phi(x) = \frac{1}{d}\sum_{i=1}^{d} T_i^* x T_i\,, \qquad x \in M\,.$$

Then $\Psi$ is the minimal left inverse of $\psi$ and preserves $\omega$ because $u$ belongs to the centralizer $M_\omega$ of $\omega$ [2] and $\omega \cdot \Phi = \Phi$.

Since $\Phi\colon \mathfrak{M}_k \to \mathfrak{M}_{k-1}$, $k \in \mathbb{N}$, and $u \in \mathfrak{M}_n$,

$$\Psi(\mathfrak{M}_k) = \Phi(\mathfrak{M}_k) \subset \mathfrak{M}_{k-1}\,, \qquad k \geq n\,. \tag{2.7}$$

By the mean ergodic theorem, the weak limit

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \Psi^i(x) \equiv P(x)\,, \qquad x \in M\,,$$

where $P$ is a normal conditional expectation onto the fixed-points $M^\Psi$ of $\Psi$.

By (2.7) $P(\cup\mathfrak{M}_k) \subset \mathfrak{M}_{n-1}$, thus $P(M_\omega) \subset \mathfrak{M}_{n-1}$, and we have

$$\lambda_u(M)' \cap M_\omega \subset M^\Psi \cap M_\omega \subset P(M_\omega) \subset \mathfrak{M}_{n-1}\,,$$

where the first inclusion is checked evaluating $\Psi$ on both sides of (2.6). □

*Remark.* If $E$ (or equivalently $E|_{M_\omega}$ [14]) is minimal, then $\lambda_u(M)' \cap M = \lambda_u(M)' \cap M_\omega$. Indeed in this case $E|_{\lambda_u(M)'\cap M}$ is a trace, thus $\lambda_u(M)' \cap M$ is contained in the fixed-point algebra $M_\omega$ of the modular group $\lambda_t$ of $M$. This case will occur in the next section.

## 3. On the Cuntz Model

Baaj and Skandalis [1] have in particular described a finite-dimensional Hopf algebra in terms of a multiplicative unitary. Let again $H$ be a Hilbert space of dimension $d < \infty$ and $V$ a multiplicative unitary on $H \otimes H$. By definition $V$ satisfies the pentagon equality

$$V_{12}V_{13}V_{23} = V_{23}V_{12}\,.$$

Following J. Cuntz, we consider the unitary $R \equiv VF$, where $F$ is the flip symmetry of $H \otimes H$ and the endomorphism $\lambda_R$ of $\mathcal{O}_H$ associated with $R \in \mathfrak{M}_2$. The pentagon equality is then equivalent to the following property for $\lambda_R$ [3],

$$\lambda_R^2 = \varphi \cdot \lambda_R\,. \tag{3.1}$$

**Proposition 3.1.** $d(\lambda_R) = d$.

*Proof.* By Eq. (3.1) and the multiplicativity of the dimension [8, 16]

$$d(\lambda_R)^2 = d(\varphi)\,d(\lambda_R) = d \cdot d(\lambda_R)\,,$$

and the proposition follows because $d(\lambda_R)$ is finite by Proposition 2.4. □

The fixed point algebra $\mathcal{O}_V$ is defined by

$$\mathcal{O}_V = \{x \in \mathcal{O}_H, \varphi(x) = \lambda_R(x)\}\,.$$

We put $N = \mathcal{O}_V''$. The argument showing $\mathcal{O}_V' \cap \mathcal{O}_H = \mathbb{C}$ [3] can be extended to the following

**Proposition 3.2.** $N' \cap M = \mathbb{C}$.

*Proof.* By Proposition 2.5, since $\lambda_R(M) \subset N$, we have

$$N' \cap M_\omega \subset \lambda_R(M)' \cap M_\omega \subset \mathfrak{M}_1 ;$$

let $S$ be a unit cofixed vector for $V$ [1], i.e. $S \in \mathcal{O}_V \cap H$; if $x \in N' \cap M_\omega$, then $x$ commutes with $S$, hence $x = S^* x S$ and $x \in \mathbb{C}$ because $x \in \mathfrak{M}_1$.

Now $N' \cap M_\omega = \mathbb{C}$ is the fixed-point algebra of $N' \cap M$ with respect to $\{\lambda_t, t \in \mathbb{T}\}$ and $N' \cap M$ is finite-dimensional because $d(\lambda_R) < \infty$, therefore $\{\lambda_t, t \in \mathbb{T}\}$ is inner and ergodic on $N' \cap M$ and this is possible only if $N' \cap M = \mathbb{C}$. $\square$

**Corollary 3.3.** $\lambda_R(N)' \cap M = \mathfrak{M}_1$.

*Proof.* We have $\lambda_R(N)' \cap M = \varphi(N)' \cap M$ that contains $\mathfrak{M}_1$. Let $x \in M$ commute with $\varphi(N)$,

$$x\varphi(y) = \varphi(y)x , \qquad y \in N ,$$

then multiplying by $S_i^*$ on the left and by $S_j$ on the right the above equation we have

$$S_i^* x S_j y = y S_i^* x S_j , \qquad y \in N ,$$

hence $S_i^* x S_j = a_{ij} \in \mathbb{C}$ by Proposition 3.2.

It follows that

$$x = \sum_{i,j} S_i S_i^* x S_j S_j^* = \sum_{i,j} a_{ij} S_i S_j^* \in \mathfrak{M}_1 . \qquad \square$$

Since $H$ is finite-dimensional, $V$ is automatically irreducible up to multiplicity [1]. We now assume that $V$ is irreducible.

**Proposition 3.4.** $\lambda_R(M)' \cap N = \mathbb{C}$, *provided* $V$ *is irreducible.*

*Proof.* Let $x \in \lambda_R(M)' \cap N$. By Corollary 3.3 $x \in \mathfrak{M}_1$. Now $\varphi(x) = \lambda_R(x)$ because $x \in N$, hence

$$FxF = \varphi(x) = \lambda_R(x) = RxR^* , \tag{3.2}$$

or equivalently

$$FVFx = xFVF , \tag{3.3}$$

namely $x$ commutes with $W \equiv FV^*F$.

On the other hand $\varphi(x) = R^* x R$ by Proposition 2.5, hence

$$FxF = \varphi(x) = R^* x R$$

or

$$xV = Vx . \tag{3.4}$$

Since $V$ is irreducible, (3.3) and (3.4) imply $x \in \mathbb{C}$. $\square$

**Corollary 3.5.** *There is a natural correspondence between subsectors of* $\lambda_R$ (resp. *of* $\lambda_R|_N$) *and subrepresentations of* $W = FV^*F$ (resp. *of* $V$).

*In particular the subsectors of* $\lambda_R$ *have integral dimension and*

$$\lambda_R = \bigoplus_k d(\varrho_k)\,\varrho_k .$$

*Proof.* If $e \in \lambda_R(M)' \cap M$ is a projection, $e$ commutes with $W$ because Eq. (3.3) holds as above and $e \in (H, H)$ by Corollary 3.3, hence $e$ defines a subrepresentation of $W$. Conversely, if a projection $e \in (H, H)$ commutes with $W$, then (3.3) and (3.2) hold, hence $e \in \lambda_R(M)' \cap M$ by Proposition 2.5. The rest is a consequence of the Frobenius reciprocity theorem, see Sect. 6. □

*Remark.* Given a finite-dimensional Hopf algebra $\mathfrak{A}$ and an irreducible representation $\pi_k$ of $\mathfrak{A}$, Corollary 3.5 provides an irreducible sector $\varrho_k$ (hence an irreducible subfactor) associated with $\pi_k$. The corresponding invariants (fusion rules and connection) are the ones inherited from $\pi_k$, in particular the subfactor has finite depth.

We note that the above construction provides a model for prime actions of any finite-dimensional Hopf algebras on a AFD factor, see [20, 23] for the unicity of such actions.

## 4. A Characterization of the Canonical Endomorphism I. First consequences

In this section we give a characterization of the canonical endomorphism that we shall need for our analysis. Recall [17, 18] that if $N \subset M$ is a inclusion of properly infinite von Neumann algebras the canonical endomorphism $\gamma : M \to N$ is defined by

$$\gamma(x) = \Gamma x \Gamma^* , \qquad x \in M ,$$

where $\Gamma = J_N J_M$ is the product of the modular conjugations of $N$ and $M$. Then $\gamma$ maps $M$ into $N$ and it is well defined up to inner automorphisms of $N$.

**Proposition 4.1.** *Let $N \subset M$ be an irreducible inclusion of factors with finite index and $\lambda$ an endomorphism of $M$ with $\lambda(M) \subset N$. Then $\lambda$ is a canonical endomorphism of $M$ into $N$ if and only if*
a) $\lambda(M)' \cap N = \mathbb{C}$,
b) $\lambda \succ \mathrm{id}$.

*Proof.* The only if part was shown in [14, 15]. For the if part, we first assume that $N$ is isomorphic to $M$, namely $N = \varrho(M)$ for some $\varrho \in \mathrm{End}(M)$. Then

$$\eta \equiv \varrho^{-1} \cdot \lambda$$

is an endomorphism of $M$ and it is irreducible because

$$\eta(M)' \cap M = \varrho^{-1}(\lambda(M)' \cap \varrho(M)) = \mathbb{C} .$$

Since $\varrho\eta = \lambda \succ \mathrm{id}$, $\eta$ is a conjugate of $\varrho$, namely $\lambda = \varrho\bar{\varrho}$ and therefore $\lambda$ is a canonical endomorphism of $M$ into $N = \varrho(M)$ [15].

In case $N$ is not isomorphic with $M$ we choose a factor $X$ such that $M_0 \equiv M \otimes X$ is isomorphic to $N_0 \equiv N \otimes X$ [16] then $\lambda \otimes \mathrm{id}$ is the canonical endomorphism of $M_0$ and $N_0$ by the above argument, hence if $\gamma : M \to N$ is a canonical endomorphism, then $\lambda \otimes \mathrm{id}$ is conjugate to $\gamma \otimes \mathrm{id}$ by a unitary in $N_0$, therefore $\lambda$ is conjugate to $\gamma$ by a unitary in $N$ and this entails that $\lambda : M \to N$ is a canonical endomorphism by the Radon-Nikodym property of the canonical endomorphism [18]. □

Note that if condition a) is dropped in Proposition 4.1, then $\lambda$ still contains the canonical endomorphism of $M$ into $N$.

The case of a reducible inclusion of factors in the above proposition can be handled as follows.

**Proposition 4.2.** *Let $N \subset M$ be an inclusion of infinite factors and $\lambda \in \mathrm{End}(M)$ with $\lambda(M) \subset N$.*

*There exist isometries $T \in M$, $S \in N$ with*

$$\lambda(x)T = Tx\,, \qquad x \in M\,, \tag{4.1}$$
$$\lambda(x)S = Sx\,, \qquad x \in N\,, \tag{4.2}$$

*and*

$$T^*S \in \mathbb{C}\backslash\{0\}\,, \qquad S^*\lambda(T) \in \mathbb{C}\backslash\{0\}$$

*if and only if $\lambda$ is a canonical endomorphism of $M$ in $N$ and $N \subset M$ has finite index.*

*Proof.* As before we may assume that $N = \varrho(M)$ for some $\varrho \in \mathrm{End}(M)$. If Eqs. (4.1) and (4.2) hold, setting $\eta = \varrho^{-1} \cdot \lambda$ we have

$$\begin{aligned}\varrho\eta(x)T &= \lambda(x)T = Tx\,, \qquad x \in M\,,\\ \eta\varrho(x)\,\varrho^{-1}(S) &= \varrho^{-1}(\lambda(\varrho(x))\,S)\\ &= \varrho^{-1}(S\lambda(\varrho(x)) = \varrho^{-1}(S)\,\eta\varrho(x)\,,\end{aligned}$$

and $\varrho$ and $\eta$ are conjugate because the conditon in [15, Theorem 5.2] characterize the conjugate sector, see [10]. The converse follows by reversing the argument. □

*Remark.* Proposition 4.2 extends to the case $N$ is a von Neumann subalgebra of $M$ with non-trivial center: the conditions on the intertwiners in the statement still entail that $\lambda$ is the canonical endomorphism of $M$ into $N$, essentially by the same proof [10]; also in this case the index is a scalar.

Coming back to the context of last section with $M = \mathcal{O}_H''$ and $N = \mathcal{O}_V''$ we have:

**Corollary 4.3.** *The extension of $\lambda$ of $\lambda_R$ to $M$ is a canonical endomorphism of $M$ into $N$.*

*Therefore the inclusions*

$$M \supset N \supset \lambda(M) \supset \varphi(N) \supset \ldots$$

*provide a Jones tunnel.*

*Proof.* We may apply Proposition 4.1 because a fixed vector $T \in H$ for $V$ belongs to $(\mathrm{id}, \lambda)$; moreover the irreducibility requirements are fulfilled because of Propositions 3.2 and 3.4. The rest follows because the Jones tunnel and the tunnel associated with the canonical endomorphism coincide, see [14]. □

We now derive some consequences of the above corollary. We keep the notations of the previous Sects. 2 and 3.

Let $T \in H$ and $S \in H$ be a fixed and cofixed vector for $V$ respectively. Then $T$ and $S$ satisfies (4.1) and (4.2). Because of corollary 4.3 we may rely on the analysis made in [15]. In particular $T$ and $S$ are the unqiue isometries (up to a phase factor) obeying the relations (4.1) and (4.2) and we may choose the phase so that (compare with Proposition 4.2) we have the relations

$$T^*S = \frac{1}{d}\,, \qquad S^*\lambda(T) = \frac{1}{d} \tag{4.3}$$

because $d$ is the dimension $d(\lambda)$ of $\lambda$. The Jones projections for the tunnel in Corollary 4.3 are

$$e = TT^*\,, \qquad f = SS^*$$

and $\lambda(e), \lambda(f), \lambda^2(e)\dots$ and so on. The Jones relation $efe = \frac{1}{d^2}\, e$ is

$$TT^*SS^*TT^* = \frac{1}{d^2}\, TT^* \tag{4.4}$$

that follows by (4.3), but

$$T^*SS^*T = |\langle S, T\rangle|^2\,, \tag{4.5}$$

where $\langle S, T\rangle$ is the scalar product in $H$, hence (4.4) and (4.5) gives $|\langle S, T\rangle| = \frac{1}{d}$.

**Corollary 4.4.** *Let $V$ be an irreducible multiplicative unitary on a Hilbert space $H\otimes H$ with* $\dim(H) = d < \infty$.

*There exist a unique (up to a phase) fixed unit vector $T$ and a unique cofixed unit vector $S$. Their scalar product satisfies*

$$|\langle S, T\rangle| = \frac{1}{d}\,. \tag{4.6}$$

*Proof.* We have just seen that (4.6) holds if $S$ is a cofixed unit vector and $T$ is a fixed vector. On the other hand (co-)fixed vectors form a linear space that must be one-dimensional for (4.6) to be satisfied for all norm one elements.

Alternatively, the unicity of the (co-)fixed vectors follows from the multiplicity one in [15, Theorem 4.1]. □

**Corollary 4.5.** *$M$ is the crossed product of $N$ by a finite-dimensional Hopf algebra.*

*Proof.* By a result of Ocneanu [21] (see the appendix for its extension to the infinite factor case), the statement is equivalent to the fact that $N \subset M$ has depth 2, namely $\lambda(N)' \cap M$ is a factor. But $\lambda(N)' \cap M = \mathfrak{M}_1$, by Corollary 3.3. □

**Lemma 4.6.** $\mathscr{O}_H \cap N = \mathscr{O}_V$.

*Proof.* The inclusion $\mathscr{O}_V \subset \mathscr{O}_H \cap N$ is obvious. To check the reverse inclusion let $x \in \mathscr{O}_H \cap N$. The conditional expectation $E$ of $M$ onto $N$

$$E = S^*\lambda(x)\,S\,,$$

maps $\mathscr{O}_H$ onto $\mathscr{O}_V$ (compare the formulas for the conditional expectations in [14, 15] and in [3]), hence $x = E(x) \in \mathscr{O}_V$. □

Denote now

$$M_{2n} = \lambda^n(M)\,, \qquad M_{2n+1} = \lambda^n(N)\,.$$

**Proposition 4.7.** *$A_n \equiv M_n' \cap M \subset \mathscr{O}_H$ and $B_n \equiv M_n' \cap N \subset \mathscr{O}_V$.*

*Proof.* We have

$$\begin{aligned} A_{2n+1} &= \lambda^n(N) \cap M = \varphi^n(N)' \cap M = \mathfrak{M}_n \\ &= (H^n, H^n) \subset \mathscr{O}_H\,, \end{aligned} \tag{4.7}$$

where $\varphi^n(N)' \cap M = \mathfrak{M}_n$ as in Corollary 3.3.

On the other hand $B_n \subset A_{n+1}$ hence

$$B_n \subset \mathscr{O}_H \cap N = \mathscr{O}_V$$

by the above lemma. □

Let $A = \cup A_n^-$ (uniform closure). Then by (4.7)

$$A = \cup \mathfrak{M}_n^- = \mathfrak{M}$$

and $\mathcal{O}_H$ is generated by $A$ and $T$, more precisely $\mathcal{O}_H$ is the crossed product of $\mathfrak{M}$ by the shift $\varphi$, see [4].

If we denote

$$B = \cup B_n^- ,$$

then

$$B = A \cap \mathcal{O}_V = \mathfrak{M} \cap \mathcal{O}_V$$

because the conditonal expectations $\varepsilon = \int \lambda_t$ of $\mathcal{O}_H$ onto $\mathfrak{M}$ and $E = S^* \lambda(\cdot) S$ of $\mathcal{O}_H$ onto $\mathcal{O}_V$ commute.

Moreover $\mathcal{O}_V$ is generated by $B$ and $S$, for example because any $x \in \mathcal{O}_V$ can be written as

$$x = \frac{1}{d} \hat{E}(xS^*) S$$

by [15], where $\hat{E}$ is the dual expectation of $\mathcal{O}_H$ onto $\lambda(\mathcal{O}_H)$, or by [3, Proposition 3.4].

The following corollary extends a result in [5] concerning the case of a group action. Our proof was inspired by a conversation with Izumi on his model [12], where this isomorphism is realized from the start.

**Corollary 4.8.** *$\mathcal{O}_V$ is isomorphic to $\mathcal{O}_H$.*

*Proof.* With the above premises, it is sufficient to show that $B$ is a UHF algebra of type $d^\infty$ and $S$ implements the shift on $B$.

We shall show that $B_{2n}$ is isomorphic to $\mathfrak{M}_{n-1}$; since $S$ implements $\lambda$ and $\lambda$ acts as a shift on $\mathfrak{M}$, the proof will be complete.

Let $\Gamma = J_N J_M$ implement $\lambda$ by Corollary 4.3. We have

$$\begin{aligned} B_{2n} &= \lambda^n(M)' \cap N \\ &= \Gamma^n M' \Gamma^{-n} \cap N \\ &\cong M' \cap \Gamma^{-n} N \Gamma^n \\ &\cong J_M M' J_M \cap J_M \Gamma^{-n} J_N N' J_N \Gamma^n J_M \\ &= M \cap \Gamma^{n-1} N' \Gamma^{n-1} = \lambda^{n-1}(N)' \cap M = \mathfrak{M}_{n-1} , \end{aligned}$$

where the first $\cong$ means isomorphic, the second $\cong$ means anti-isomorphic and we made use of the relation $J_M \Gamma^{-n} J_N = \Gamma^{n-1}$. $\square$

It follows that $(\mathcal{O}_V, \lambda_R|_{\mathcal{O}_V})$ is isomorphic with $(\mathcal{O}_H, \lambda_{R^*})$.

## 5. A Characterization of the Canonical Endomorphism II

In this section we shall give a characterization of the canonical endomorphism of an infinite factor $M$ without reference to the subfactor $N$, that will appear as an output of our construction.

Note that, if $N$ is isomorphic to $M$ (one can always reduce to this case by a tensoring trick), our result may be reformulated as a condition for an endomorphism $\lambda$ to admit a "square root"

$$\lambda = \varrho \bar{\varrho} .$$

**Theorem 5.1.** *Let $M$ be an infinite factor and $\lambda \in \mathrm{End}(M)$ an endomorphism of $M$ with finite index. The following are equivalent:*
(i) *There exists an irreducible subfactor $N \subset M$ such that $\lambda$ is the canonical endomorphism of $M$ into $N$.*
(ii) a) $\lambda \succ \mathrm{id}$ *with multiplicity* 1, i.e. *there exists a unique isometry $T \in (\mathrm{id}, \lambda)$ (up to a phase).*
b) *There exists an isometry $S \in (\lambda, \lambda^2)$ such that*

$$(\mathrm{b}_1) \qquad \lambda(S)S = S^2\,, \qquad \lambda(S^*)S = SS^*\,,$$

$$(\mathrm{b}_2) \qquad S^*\lambda(T) \in \mathbb{C}\backslash\{0\}\,, \qquad T^*S \in \mathbb{C}\backslash\{0\}\,.$$

*Moreover if* (ii) *holds then the subfactor $N$ is canonically constructed from $\lambda$ and $S$. In fact $N$ is the unique subfactor of $M$ such that Eqs.* (4.1) *and* (4.2) *hold.*

The implication (i) $\Rightarrow$ (ii) has been shown in Proposition 4.2. We now assume that (ii) hold. To obtain (i) we need a few lemmas. Let us define

$$E \equiv S^*\lambda(\cdot)S\,,$$

then $E$ is a completely positive normal unital map of $M$ into $M$.

**Lemma 5.2.** *$E$ is a faithful normal conditional expectation of $M$ onto a von Neumann subalgebra $N$ of $M$.*

*Proof.* We have to show the relation

$$E(E(x)y) = E(x)E(y)\,, \qquad x, y \in M\,, \tag{5.1}$$

in fact this implies $E^2 = E$ and that the range of $M$

$$N \equiv E(M)$$

is an algebra, hence a von Neumann subalgebra because $E$ is involutive and normal.
To check (5.1) we use the relations $(\mathrm{b}_1)$ as follows:

$$\begin{aligned} E(E(x)y) &= S^*\lambda(S^*\lambda(x)Sy)S \\ &= S^*\lambda(S^*)\lambda^2(x)\lambda(S)\lambda(y)S \\ &= S^*S^*\lambda^2(x)\lambda(S)\lambda(y)S \\ &= S^*\lambda(x)S^*\lambda(S)\lambda(y)S \\ &= S^*\lambda(x)SS^*\lambda(y)S = E(x)E(y)\,. \end{aligned}$$

It remains to show that $E$ is faithful. But if $x \in M$ and $E(x^*x) = 0$ then

$$\begin{aligned} S^*\lambda(x^*x)S = E(x^*x) &= 0 \\ \Leftrightarrow \lambda(x)S &= 0 \\ \Rightarrow T^*\lambda(x)S &= xT^*S = 0\,, \end{aligned}$$

thus $x = 0$ because $T^*S \in \mathbb{C}\backslash\{0\}$. $\square$

Notice that $\lambda(M) \subset N$ because

$$E(\lambda(x)) = S^*\lambda_2(x)S = \lambda(x)\,, \qquad x \in M\,,$$

hence $\lambda$ restricts to an endomorphism of $N$.

**Lemma 5.3.** $S \in (\lambda|_N, \mathrm{id}|_N)$.

*Proof.* First of all $S \in N$ because

$$E(S) = S^* \lambda(S) S = S^* S^2 = S,$$

moreover if $y \in N$ we may write $y = E(x)$ for some $x \in M$, therefore

$$\begin{aligned} \lambda(y) S &= \lambda(E(x)) S \\ &= \lambda(S^* \lambda(x) S) S = \lambda(S^*) \lambda^2(x) \lambda(S) S \\ &= \lambda(S^*) \lambda^2(x) S S = \lambda(S^*) S \lambda(x) S \\ &= S S^* \lambda(x) S = S E(x) = S y, \end{aligned}$$

and the proof is complete. □

**Lemma 5.4.** *$\lambda$ is the canonical endomorphism of $M$ into $N$.*

*Proof.* The condition ($b_2$) implies the statement by the Proposition 4.2 and the previous lemmas. □

**Lemma 5.5.** *$N$ is an irreducible subfactor of $M$.*

*Proof.* Since $\lambda \succ \mathrm{id}$ with multiplicity 1, there exists a unique normal faithful conditional expectation of $N'$ of onto $M'$ [15, Proposition 4.3]. Then $N' \cap M \subset Z(M) = \mathbb{C}$ [2], namely $N$ is an irreducible subfactor of $M$. □

*Proof of Theorem 5.1.* The proof now follows by Proposition 5.5 and Lemma 5.4. □

*Remark.* The multiplicity 1 assumption for $\lambda \succ \mathrm{id}$ in Theorem 5.1 is only needed for $N$ to be an irreducible factor. If we drop it, the theorem remains true with $N$ a von Neumann subalgebra with finite index and $T$ a given isometry in $(\mathrm{id}, \lambda)$.

## 6. *Q*-systems, Crossed Product and Hopf Algebras

We begin with a reformulation of the results in the previous section. We define a (irreducible) *$Q$-system* to be a set $(M, \lambda, S)$, where $M$ is an infinite factor, $\lambda \in \mathrm{End}(M)$ contains the identity (with multiplicity one), and $S \in (\lambda, \lambda^2)$ obeys the assumptions ($b_1$) and ($b_2$) of Theorem 5.1 (in the reducible case we should also fix $T$, see last remark).

There is an obvious notion of *isomorphism* between $Q$-systems $(M_1, \lambda_1, S_1)$ and $(M_2, \lambda_2, S_2)$; in the irreducible case it is an isomorphism of $M_1$ with $M_2$ that interchanges $\lambda_1$ and $\lambda_2$ and maps $S_1$ to $S_2$; there is also a notion of *cocycle equivalence*: $(M_1, \lambda_1, S_1)$ is cocycle equivalent to $(M_2, \lambda_2, S_2)$ if there is a unitary $u \in M_2$ such that $(M_1, \lambda_1, S_1)$ is isomorphic with $(M_2, \mathrm{ad}(u) \cdot \lambda_2, u\lambda(u) S_2 u^*)$.

**Theorem 6.1.** *Given an infinite factor $M$, there is a natural bijective correspondence between irreducible subfactors of $M$ with finite index and irreducible $Q$-systems based on $M$. Conjugate subfactors correspond to cocycle equivalent $Q$-systems.*

*Proof.* Beside Theorem 5.1 we have only to observe that conjugate subfactors correspond to cocycle equivalent $Q$-systems, that is an elementary consequence of the Radon-Nikodym property for the canonical endomorphism [15]. □

Let $(M, \lambda, S)$ be a $Q$-system and $N \subset M$ the corresponding subfactor of $M$. Then $\lambda$ is the associated canonical endomorphism, namely

$$\lambda(x) = \Gamma x \Gamma^*, \qquad x \in M,$$

where $\Gamma = J_N J_M$. We define

$$\tilde{M} \equiv \Gamma^{-1} M \Gamma \,, \tag{6.1}$$

the *crossed product* of $M$ by $\lambda, S$. By Proposition 6.1, $\tilde{M}$ is well defined up to isomorphisms of $M \subset \tilde{M}$, moreover the *dual* $Q$-system $(\tilde{M}, \tilde{\lambda}, \tilde{S})$, where $\tilde{\lambda} \equiv \mathrm{ad}(\Gamma)$ on $\tilde{M}$ and $\tilde{S} \equiv T$ is the isometry in $(\mathrm{id}, \lambda)$ with $T^* S = \frac{1}{d} \in \mathbb{R}^+$, is also well defined within cocycle equivalence.

We also remark that Takesaki duality is immediate in this context and states that the bi-dual $Q$-system $(\tilde{\tilde{M}}, \tilde{\tilde{\lambda}}, \tilde{\tilde{S}})$ is isomorphic to $(M, \lambda, S)$; the isomorphism is realized by $\tilde{\tilde{\lambda}}$.

In the following we shall characterize the $Q$-systems arising from Hopf algebra actions.

**Theorem 6.2.** *Let $(M, \lambda, S)$ be an irreducible Q-system. The following are equivalent:*
(i) $\lambda^2 \cong d \cdot \lambda$, *namely $\lambda^2$ is equivalent to $\lambda \oplus \lambda \oplus \ldots \oplus \lambda$ for some $d \in \mathbb{N}$.*
(ii) *there exists a Hopf algebra $\mathfrak{A}$, such that $\tilde{M}$ is the crossed product of $M$ by a (faithful) action of $\mathfrak{A}$ on $M$.*

*Moreover if* (i) *holds, then $\mathfrak{A}$ is unique up to isomorphism.*

*Proof.* It is possible to give a direct proof of this theorem following the analysis made in the previous sections. However these arguments also provide a proof of Ocneanu's characterization (in our setting) that we isolate in the Appendix. By Theorem 5.1, we may then reduce to this case because of Lemma 6.3. □

**Lemma 6.3.** *Let $N \subset M$ be an irreducible inclusion of infinite factors with finite index and $\lambda : M \to N$ the canonical enodmorphism. The following are equivalent:*
(i) *$N \subset M$ has depth 2.*
(ii) *$\lambda^2 \cong d \cdot \lambda$ for some $d \in \mathbb{N}$.*
(iii) *Let $\lambda = \oplus d_i \varrho_i$ be the irreducible decomposition of $\lambda$. Then $d(\varrho_i) = d_i$, in particular $d(\varrho_i)$ is an integer, and the $^*$-semiring generated by $\{\varrho_i\}$ does not contain further irreducible sectors.*

*Proof.* (i)$\Leftrightarrow$(ii): We may assume $N = \varrho(M)$ for some $\varrho \in \mathrm{End}(M)$ [16]. Then condition (i) means that

$$\varrho\bar{\varrho}\varrho \cong d \cdot \varrho \tag{6.2}$$

and condition (ii) that

$$(\varrho\bar{\varrho})^2 \cong d \cdot \varrho\bar{\varrho} \,, \tag{6.3}$$

where $d = d(\varrho)^2$ by the multiplicativity of the dimension either by (6.2) or by (6.3).

Clearly by multiplying on the right by $\bar{\varrho}$ Eq. (6.2) implies Eq. (6.3). Conversely if (6.3) holds then

$$\varrho\bar{\varrho}\varrho\bar{\varrho} \succ d \cdot \mathrm{id} \,. \tag{6.4}$$

By Frobenius reciprocity (Proposition 6.4), Eq. (6.4) implies

$$\varrho\bar{\varrho}\varrho \succ d \cdot \varrho \,. \tag{6.5}$$

Moreover the dimension of both sides of (6.5) is $d(\varrho)^3$, hence

$$\varrho\bar{\varrho}\varrho = d \cdot \varrho \,.$$

(ii)$\Leftrightarrow$(iii): Clearly the completeness requirement on the family $\{\varrho_i\}$ is equivalent to the fact that $\lambda^2$ is quasi-equivalent to $\lambda$, since $\lambda$ is selfconjugate, and we may therefore assume this property. Thus we have to show that (ii) is equivalent to the equality $d(\varrho_i) = d_i$.

It follows by Proposition 6.4 that $\lambda' \equiv \sum d(\varrho_i) \cdot \varrho_i$ satisfies the equation (ii) and therefore

$$\sum d(\varrho_i)^2 = d\,. \tag{6.6}$$

Moreover, evaluating the dimension on $\lambda = \bigoplus_i d_i \varrho_i$, we have

$$\sum d_i d(\varrho_i) = d\,. \tag{6.7}$$

By Proposition 6.4, $\lambda = \varrho\bar{\varrho} \succ d_i \cdot \varrho_i$ is equivalent to $\bar{\varrho}_i \varrho \succ d_i \cdot \varrho$. Now (ii) holds iff (6.2) holds, namely iff $\sum d_i \cdot \varrho_i \varrho = d \cdot \varrho$, thus only if

$$\sum d_i^2 = d\,, \tag{6.8}$$

but since (6.6) and (6.7) hold, Eq. (6.8) is possible iff $d(\varrho_i) = d_i$. □

The following proposition expresses the Frobenius reciprocity. In this form it is implicit in [8]; cf. [6] for the case of $C^*$-categories with permutation symmetry and [20, 23] for the $II_1$-factor case.

**Proposition 6.4.** *Let $M$ be an infinite factor and $\varrho, \eta \in \mathrm{Sect}(M)$ irreducible sectors with finite index. If $\alpha, \beta$ are a sum of finite index sectors then*

$$\alpha\varrho\beta \succ \eta \Leftrightarrow \bar{\alpha}\eta\bar{\beta} \succ \varrho\,,$$

*and the multiplicites of the containment are equal.*

*Proof.* By considering irreducible subsectors of $\alpha$ and $\beta$ we may assume that $\alpha$ and $\beta$ are irreducible. Then

$$\begin{aligned} \alpha\varrho\beta \succ \eta &\Leftrightarrow \alpha\varrho\beta\bar{\eta} \succ \mathrm{id} \\ &\Leftrightarrow \varrho\beta\bar{\eta} \succ \bar{\alpha} \\ &\Leftrightarrow \varrho\beta\bar{\eta}\alpha \succ \mathrm{id} \\ &\Leftrightarrow \beta\bar{\eta}\alpha \succ \bar{\varrho} \\ &\Leftrightarrow \bar{\alpha}\eta\bar{\beta} \succ \varrho \end{aligned}$$

by a repeated use of the characterization of the conjugate sector. The multiplicity is preserved in all these equivalences. □

We conclude this section with a brief categorical formulation of our duality for Hopf algebras.

An irreducible $Q$-system may be described in this case as a tensor (or monoidal) $C^*$-category of endomorphisms in the sense of [6], stable under composition (the monoidal operation), equivalence of objects, sub-objects, with finitely many inequivalent irreducible objects $\{\varrho_1 = \mathrm{id}, \varrho_2, \ldots, \varrho_n\}$, with $d(\varrho_i) \in \mathbb{N}$ and a distinguished intertwiner $S \in (\lambda, \lambda^2)$, where $\lambda \cong \bigoplus_i d(\varrho_i)\varrho_i$, satisfying the basic equations ($b_1$) and ($b_2$). We shall say that two $C^*$-categories of this kind are equivalent if there is a monoidal linear invertible $^*$-functor between them interchanging the distinguished object and intertwiner. Note that this notion does not require the $Q$-systems to be isomorphic. It is not difficult show the following corollary that we shall consider in more generality somewhere else.

**Corollary 6.5.** *There is a bijective correspondence between finite-dimensional Hopf algebras, up to isomorphism, and $C^*$-categories of endomorphisms as above, up to equivalence.*

## Appendix. A Proof of the Ocneanu Characterization

The purpose of this appendix is to give a proof of Ocneanu's characterization [21] of inclusions of factors arising from a crossed product by Hopf algebra, that is suitable for our analysis.

Our proof, based on sector analysis and the arguments in this paper, works also for inclusions of infinite factors. The same proof has been noticed on this basis by Izumi and is implicit in [12].

Let $N \subset M$ be an irreducible inclusion of infinite factors with finite index. We take as a definition for $N \subset M$ to have depth 2 that $\lambda(N)' \cap M$ is a factor, where $\lambda: M \to N$ is a canonical endomorphism.

**Theorem A.1.** *Let $N \subset M$ be an irreducible inclusion of factors with finite index. The following are equivalent:*
(i) *$M$ is the crossed product of $N$ by a Hopf algebra.*
(ii) *$N \subset M$ has depth 2.*

*Proof.* The implication (i) $\Rightarrow$ (ii) is known by duality, cf. [19].

To show that (ii) $\Rightarrow$ (i) we may assume that $N$ and $M$ are infinite (tensoring by a type $I_\infty$ factor).

Let $\lambda: M \to N$ be the canonical endomorphism of $M$ into $T$. There exists a unique isometry (up to a phase) $T$ with

$$\lambda(x)T = Tx\,, \qquad x \in M\,.$$

Let

$$H = \{L \in M, \lambda(x)L = Lx \ \ \forall x \in N\}\,.$$

Then $H$ will be a Hilbert space of isometries of $M$, provided we show that the left support $G$ of $H$ is 1. But $G \in \mathfrak{M}_1 \equiv \lambda(N)' \cap M$ and $uGu^* = G$ for all unitaries $u$ of $\mathfrak{M}_1$ because $uH = H$, hence $G$ belongs to the center of $\mathfrak{M}_1$ and $G = 1$ since $\mathfrak{M}_1$ is a factor.

Clearly

$$\lambda(x) = \varphi(x) \qquad x \in N\,,$$

where $\varphi$ is the inner endomorphism of $M$ implemented by $H$.

In particular

$$\lambda^2(x) = \varphi \cdot \lambda(x) \qquad x \in M\,, \tag{A.1}$$

and by the multiplicativity of the dimension this implies $d(\lambda) = d(\varphi) = d$, where $d = \dim(H)$. By (A.1) we have (cf. the argument in Corollary 3.3)

$$\lambda(M)' \cap M \subset \lambda(N)' \cap M = \varphi(N)' \cap M = \varphi(M)' \cap M\,,$$

that is to say $(\lambda, \lambda) \subset (\varphi, \varphi)$. Let $S \equiv d \cdot E(T)$, where $E$ is the expectation of $M$ onto $N$; by [15] $S \in N$ is an isometry and

$$\lambda(x)S = Sx\,, \qquad x \in N$$

and

$$S^*T = \frac{1}{d}, \qquad S^*\lambda(T) = \frac{1}{d}. \tag{A.2}$$

The projection $f \equiv SS^* \in \lambda(N)' \cap N \subset \mathfrak{M}_1$ is a Jones projection, hence a minimal projection of $\mathfrak{M}_1$.

Choose an isometry $\hat{T} \in H$ with $\hat{T}\hat{T}^* = f$, then $v = \hat{T}^*S$ is a unitary in $N' \cap M = \mathbb{C}$, hence $S = \hat{T}v \in H$, in particular since $H = \mathfrak{M}_1 T$, we have

$$S \in (\lambda(N)' \cap M)T. \tag{A.3}$$

Now $\lambda(M) \subset N$ has still depth 2 (see Lemma 6.3) and formula (A.3) applied to $\lambda(M) \subset N$ gives, with $\hat{E}$ the expectation of $N$ onto $\lambda(M)$,

$$d\hat{E}(S) \in \lambda^2(M)' \cap N \subset \lambda^2(N)' \cap M = \varphi^2(N)' \cap M = (H^2, H^2)S,$$

but $\lambda(T) = d\hat{E}(S)$ because both intertwine $\lambda|_{\lambda(M)}$ and $\mathrm{id}|_{\lambda(M)}$, and by formulas (A.2), hence

$$\lambda(T) \in (H^2, H^2)S.$$

It follows that

$$R \equiv \sum_{i=1}^{d} \lambda(T_i)T_i^* \in (H^2, H^2),$$

where $\{T_i, i = 1, \ldots, d\}$ is an orthonormal basis of $H$.

Then $\lambda$ restricts to the endomorphism $\lambda_R$ of $\mathcal{O}_H$ and since (A.1) holds $R = VF$, where $V$ is a multiplicative unitary of $H \otimes H$ and $F$ is the flip on $H \otimes H$ [3]. Moreover

$$\mathfrak{A} \equiv \varphi(H^*)\lambda(H) \subset \mathfrak{M}_1$$

is a Hopf algebra (see [3]). The coaction $\delta$ of $\mathfrak{A}$ on $M$ is given by $\lambda$ that maps $M$ into $\varphi(M) \cdot \mathfrak{A} \cong M \otimes \mathfrak{A}$ because

$$\lambda(M) = \lambda(\{N, T\}'') = \{\varphi(N), \lambda(T)\}'' \subset \{\varphi(M), \lambda(T)\}''$$

and $\lambda(T) = \sum_i \varphi(T_i)\varphi(T_i^*)\lambda(T) \in \varphi(M) \cdot \mathfrak{A}$.

The fixed point algebra for $\delta$ is by definition

$$M^\delta = \{x \in M, \delta(x) = x \otimes 1\} = \{x \in M, \lambda(x) = \varphi(x)\}.$$

Clearly $N \subset M^\delta$. The expectation $E: M \to N$ is given by the formula [15]

$$E = S^*\lambda(\cdot)S,$$

hence $E(x) = x$ if $x \in M^\delta$, i.e. $N = M^\delta$. □

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**Note added in proof.** Concerning the problem of computing the index (remark before Proposition 2.5), let $F = p - (1-p)$ be the spectral resolution of the flip and $F_\mu = p - \mu(1-p)$. In a recent manuscript *On a family of almost commuting endomorphisms* V. Jones shows in particular that $\mathrm{Ind}(\lambda_{F_\mu}) = 1$ for all $\mu \in \mathbb{T}$ except $\mu = 1$.

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