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Representations of Quantum $so(8)$ and Related Quantum Algebras

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Abstract: We study irreducible representations of the quantum group $U_\varepsilon(so(8))$ when $\varepsilon \in \mathbb{C}^*$ is a primitive l^{th} root of unity. By a theorem of De Concini and Kac, there is a finite number of such representations associated to each point of a complex algebraic variety of dimension 28 and the generic representation has dimension l^{12}.

We give explicit constructions of essentially all the irreducible representations whose dimension is divisible by l^8. In addition, we construct all cyclic representations of minimal dimension. This minimal dimension is l^5, in accordance with a conjecture of De Concini, Kac and Procesi.

1. Introduction

If $\underline{g}$ is finite-dimensional complex simple Lie algebra, there is a well-known family $\{U_q(\underline{g});\ q \in \mathbb{C}^\times\}$ of Hopf algebras over $\mathbb{C}$ which "tend" in a precise sense, to the universal enveloping algebra of $\underline{g}$ as q tends to 1. The algebra $U_q(\underline{g})$ is generated by elements $e_i, f_i, k_i^{\pm 1}$, $i = 1, \ldots, n = rk(\underline{g})$, satisfying certain relations which may be found in Sect. 2.

If q is not a root of unity, the representation theory of $U_q(\underline{g})$ is the "same" as that of $\underline{g}$ [8]. On the other hand, if $q = \varepsilon$ is an l^{th} root of unity, where we assume that l is odd and greater than 1, there are finitely many finite-dimensional irreducible $U_\varepsilon(\underline{g})$-modules associated to every point of a certain complex algebraic veriety of dimension $m = \dim(\underline{g})$ [5]. All such representations have dimension at most $l^{(m-n)/2}$. Although the results of [5] give an adequate parametrization of the set of irreducible representations of $U_\varepsilon(\underline{g})$, they do not give any explicit description of the representations themselves (except in the sl_2 case). It is of interest to give such descriptions, partly to test certain conjectures made in [5 and 6], and also because of certain analogies between the representation theory of $U_\varepsilon(\underline{g})$ and that of $\underline{g}$ over

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a field of finite characteristic. Although there are several deep general results concerning the latter theory, there seem to be almost no explicit constructions of the representations in the literature.

The generators e_i, f_i act injectively on the generic $U_\varepsilon(g)$-module: such modules are called cyclic. It was shown in [2 and 3] that, if g is of type A_n, B_n or C_n, the minimal dimension of a cyclic module is l^n, and all minimal cyclic modules were described explicitly. In the remaining cases, the minimal dimension is divisible by, and strictly greater than l^n, and the minimal cyclic modules appear to be much harder to construct.

In this paper, we study the prototype of the remaining cases, namely $g = so(8, \mathbb{C})$. We reduce the study of cyclic $U_\varepsilon(so(8))$-modules to that of a certain auxiliary algebra $\mathscr{A}_\varepsilon$. More precisely, we construct a homomorphism from $U_{\varepsilon^2}(so(8))$ to the tensor product of $\mathscr{A}_\varepsilon$ and a Laurent quasi-polynomial algebra on 8 generators. It is well-known (and easy to prove) that every irreducible representation of the latter algebra has dimension l^4 and depends on 8 parameters. Pulling back a tensor product of irreducible representations of $\mathscr{A}_\varepsilon$ and of the quasi-polynomial algebra gives an irreducible representation of $U_{\varepsilon^2}(so(8))$, and all cyclic $U_{\varepsilon^2}(so(8))$-modules arise in this way (certain noncyclic representations can also be obtained). In fact, this reduction to an auxiliary algebra can be carried out for arbitrary g. To illustrate the technique, we start with the simpler case $g = sl_3$, where we recover very easily certain results of Arnaudon [1]. In the so(8) case, we show that the minimal dimension of representations of $\mathscr{A}_\varepsilon$ is l, so that the minimal dimension of cyclic representations of $U_{\varepsilon^2}(so(8))$ is l^5. We construct all such representations, as well as representations of dimension dl^5 for $1 \leqq d \leqq l$. As a further illustration of the method, we construct representations of $\mathscr{A}_\varepsilon$ of dimension dl^4 for $1 \leqq d \leqq l$, by reducing to a second auxiliary algebra. Since several of our results depend on straightforward, but very tedious, computations, we have omitted many of the details.

2. Notation and Preliminaries

In this section, we recall certain basic facts about quantum groups and their representations. See [5] for further details. We also introduce some closely related quantum algebras and study their representations.

2.1. Let q be an indeterminate. For $n, r \in \mathbb{N}$, let

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$[n]_q! = [n]_q[n-1]_q \ldots [3]_q[2]_q[1]_q,$$

$$[n;r]_q = \frac{[n]_q!}{[r]_q![n-r]_q!}.$$

It is known [8] that these are all elements of $Z[q, q^{-1}]$, and hence can be specialized by letting $q = \lambda$ for any non-zero complex number λ. The resulting complex numbers are denoted $[n]_\lambda$, etc.

2.2. Let $\underline{g}$ be a complex simple Lie algebra of rank n, and let (a_{ij}) be the Cartan matrix of $\underline{g}$. In this paper, we shall only be concerned with cases when the Cartan matrix is symmetric. Let ε be a primitive l^{th} root of unity, with l odd, greater than 1 and coprime to the determinant of (a_{ij}). Then $U_\varepsilon(\underline{g})$ is the associative algebra over $\mathbb{C}$ with generators $e_i, f_i, k_i^{\pm 1}$, $i = 1, \ldots, n$, and the relations:

$$k_i k_i^{-1} = 1 = k_i^{-1} k_i ,$$

$$k_i k_j = k_j k_i ,$$

$$k_i e_j k_i^{-1} = \varepsilon^{a_{ij}} e_j ,$$

$$k_i f_j k_i^{-1} = \varepsilon^{-a_{ij}} f_j ,$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{\varepsilon - \varepsilon^{-1}} ,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r [1 - a_{ij}; r]_\varepsilon e_i^r e_j e_i^{1-a_{ij}-r} = 0, \quad i \neq j ,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r [1 - a_{ij}; r]_\varepsilon f_i^r f_j f_i^{1-a_{ij}-r} = 0, \quad i \neq j .$$

It is well-known that $U_\varepsilon(\underline{g})$ has a Hopf algebra structure, but we shall make no use of it in this paper.

2.3. Let U_ε^+ be the subalgebra of $U_\varepsilon(\underline{g})$ generated by the e_i, $i = 1, \ldots, n$, and define U_ε^- similarly. Let U_ε^0 be the subalgebra of $U_\varepsilon(\underline{g})$ generated by the $k_i^{\pm 1}$. Multiplication induces an isomorphism of vector spaces,

$$U_\varepsilon(\underline{g}) \cong U_\varepsilon^- \otimes U_\varepsilon^0 \otimes U_\varepsilon^+ .$$

Let ω be the conjugate-linear anti-automorphism of $U_\varepsilon(\underline{g})$ defined by

$$\omega(e_i) = f_i, \quad \omega(f_i) = e_i, \quad \omega(k_i) = k_i^{-1} .$$

This is called the Cartan involution of $U_\varepsilon(\underline{g})$.

2.4. Let $\mathrm{Br}_{\underline{g}}$ be the braid group associated to $\underline{g}$. Thus, $\mathrm{Br}_{\underline{g}}$ is the abstract group with generators $T_i, i = 1, \ldots, n$, and the following defining relations:

$$T_i T_j = T_j T_i \quad \text{if } a_{ij} = 0 ,$$

$$T_i T_j T_i = T_j T_i T_j \quad \text{if } a_{ij} a_{ji} = 1 .$$

2.5. Lusztig showed in [6] that $\mathrm{Br}_{\underline{g}}$ acts as a group of automorphisms of $U_\varepsilon(\underline{g})$. In fact,

$$T_i e_i = -f_i k_i, \quad T_i k_j = k_j k_i^{-a_{ij}} ,$$

$$T_i e_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} \varepsilon_i^{-s} e_i^{(-a_{ij}-s)} e_j e_i^{(s)}, \quad i \neq j ,$$

where $e_i^{(s)} = e_i^s / [s]_{\varepsilon_i}!$, and the action of $\mathrm{Br}_{\underline{g}}$ on f_i is determined by

$$T_i \omega = \omega T_i .$$

Note, in particular, that if $a_{ij} = -1$,

$$T_i e_j = -\varepsilon^{-\frac{1}{2}}[e_i, e_j]_{\varepsilon^{\frac{1}{2}}},$$

where

$$[x, y]_\lambda = \lambda xy - \lambda^{-1} yx, \quad \lambda \in \mathbb{C}^\times, \ x, y \in U_\varepsilon(\underline{g}).$$

2.6. The braid group allows us to define (non-canonically) root vectors e_α, f_α in $U_\varepsilon(\underline{g})$ corresponding to every positive root α. Let $s_1, \ldots, s_n$ be the fundamental reflections in the Weyl group W of $\underline{g}$. Let

$$w_0 = s_{i_1} s_{i_2} \ldots s_{i_N}$$

be a reduced expression for the longest element w_0 of W. Then the positive root vectors are

$$e_{i_1}, T_{i_1} e_{i_2}, T_{i_1} T_{i_2} e_{i_3}, \ldots, T_{i_1} \ldots T_{i_{N-1}} e_{i_N},$$

and $f_\alpha = \omega(e_\alpha)$. For any choice of w_0, the positive root vectors are in U_ε^+, and the negative root vectors in U_ε^-.

2.7. The following formula is useful and follows immediately from the definition of the braid group action:

$$T_i T_j e_i = e_j \quad \text{if } a_{ij} = -1.$$

If w is any element of the Weyl group and

$$w = s_{j_1} \ldots s_{j_r}$$

is a reduced expression for it, then the element

$$T_w = T_{j_1} \ldots T_{j_r}$$

of $\mathrm{Br}_{\underline{g}}$ depends only on w, and not on the choice of reduced expression. In particular, there is a well-defined elelment $T_0 \in \mathrm{Br}_{\underline{g}}$ associated to $w_0 \in W$. If α_i and α_j are simple roots and $w(\alpha_i) = \alpha_j$, then $T_w(e_i) = e_j$, cf. [4].

2.8. Let $\mathrm{Rep}(U_\varepsilon(\underline{g}))$ be the set of isomorphism classes of finite-dimensional irreducible representations of $U_\varepsilon(\underline{g})$. Let Z_0 be the subalgebra of $U_\varepsilon(\underline{g})$ generated by the elements e_α^l, f_α^l for all positive roots α of $\underline{g}$, and by the $k_i^{\pm l}, i = 1, \ldots, n$.

Proposition (cf. [6]). *Z_0 is a Hopf subalgebra of $U_\varepsilon(\underline{g})$ and is contained in the centre of $U_\varepsilon(\underline{g})$. Assigning to an element of* $\mathrm{Rep}(U_\varepsilon(\underline{g}))$ *its Z_0-character is a finite-to-one surjective map* $\mathrm{Rep}(U_\varepsilon(\underline{g})) \to \mathrm{Spec}(Z_0)$.

Note that if we define $Z_0^\pm = Z_0 \cap U_\varepsilon^\pm$, $Z_0^0 = Z_0 \cap U_\varepsilon^0$, then

$$Z_0 \cong Z_0^- \otimes Z_0^0 \otimes Z_0^+ .$$

2.9. Let G be the adjoint group of $\underline{g}$, and define maps

$$X\colon \mathrm{Spec}(Z_0^+) \to G, \quad Y\colon \mathrm{Spec}(Z_0^-) \to G, \quad K\colon \mathrm{Spec}(Z_0^0) \to G,$$

as follows. Let $e_{\beta_1}, \ldots, e_{\beta_N}$ be the positive root vectors of $U_\varepsilon(\underline{g})$ in the order in which they appear in (2.6), and let $x_\beta = e_\beta^l$. Let E_β be the root vectors in $\underline{g}$ obtained from simple root vectors by the same procedure as in (2.6) (note that $\mathrm{Br}_{\underline{g}}$ acts as a group of automorphisms of $\underline{g}$). Define $y_\beta = f_\beta^l$ and F_β similarly. Then,

$$Y = \exp((\varepsilon^2 - \varepsilon^{-2})^l y_{\beta_N} F_{\beta_N}) \exp((\varepsilon^2 - \varepsilon^{-2})^l y_{\beta_{N-1}} F_{\beta_{N-1}}) \ldots \exp((\varepsilon^2 - \varepsilon^{-2})^l y_{\beta_1} F_{\beta_1}) ,$$

and $X = T_0(Y)$, the action of T_0 on Y being

$$T_0(\ldots \exp((\varepsilon^2 - \varepsilon^{-2})^l y_\beta F_\beta) \ldots) = \ldots \exp((\varepsilon^2 - \varepsilon^{-2})^l T_0(y_\beta) T_0(F_\beta)) \ldots .$$

Finally, identify $\mathrm{Spec}(Z_0^0)$ with the Cartan subgroup H of G in the natural way; if $h \in \underline{h}$, the Lie algebra of H, then

$$\exp(2\pi\sqrt{-1}h)(k_i^l) = \exp(\alpha_i(h)) .$$

Define $K(t) = t^2$, $t \in H$.

Proposition. *The map YKX: $\mathrm{Spec}(Z_0) \to G$ exhibits $\mathrm{Spec}(Z_0)$ as an (unramified) covering with 2^n sheets of the big cell $G_0 \subset G$.*

De Concini, Kac and Procesi make the following important conjecture in [6]:

Conjecture. Let V be an irreducible representation of $U_\varepsilon(\underline{g})$, and let g_V be the image under the map YKX of the Z_0-character of V. Let $2d_V$ be the dimension of the conjugacy class of g_V in G. Then, $\dim(V)$ is divisible by l^{d_V}.

2.10.

Definition. *A quasi-polynomial algebra is an associative algebra over $\mathbb{C}$ with generators $x_i, i = 1, \ldots, r$, and relations:*

$$x_i x_j = \lambda_{ij} x_j x_i, \quad i < j ,$$

for some scalars $\lambda_{ij} \in \mathbb{C}^\times$ (cf. [7]).

Denote by $\mathbb{C}_\varepsilon[x, z]$ the quasi-polynomial algebra with generators x, z, and the relation:

$$[x, z]_\varepsilon = 0 ,$$

where the ε-bracket is defined by

$$[x, z]_\varepsilon = \varepsilon x z - \varepsilon^{-1} z x .$$

The Laurent quasi-polynomial algebra $\mathbb{C}_\varepsilon[x, z, x^{-1}, z^{-1}]$ is now defined in the obvious way.

To describe the irreducible representations of the algebra $\mathbb{C}_\varepsilon[x, z]$ (resp. $\mathbb{C}_\varepsilon[x, z, x^{-1}, z^{-1}]$), let $\{v_0, \ldots, v_{l-1}\}$ be the standard basis of $\mathbb{C}^l$ and, for each λ, $\mu \in \mathbb{C}^\times$, define operators $X_\lambda, Z_\mu \in \mathrm{End}(\mathbb{C}^l)$ as follows:

$$Z_\mu v_i = \mu \varepsilon^{2i} v_i ,$$

$$X_\lambda v_i = v_{i+1}, \quad i < l - 1 ,$$

$$X_\lambda v_{l-1} = \lambda v_0 .$$

Proposition.

(i) *The elements x^l, z^l are in the centre of $\mathbb{C}_\varepsilon[x, z, x^{-1}, z^{-1}]$ and hence act as scalars on any irreducible representation of both $\mathbb{C}_\varepsilon[x, z]$ and $\mathbb{C}_\varepsilon[x, z, x^{-1}, z^{-1}]$.*

(ii) *Any irreducible representation of* $\mathbb{C}_\varepsilon[x,z]$ *on which* x^l *and* z^l *act as non-zero scalars is of dimension* l. *The action of* x *and* z *on* $\mathbb{C}$ *is defined by*:

$$x \to X_\lambda, \quad z \to Z_\mu$$

for some $\lambda, \mu, \in \mathbb{C}^\times$. *Further, these representations extend naturally to representations of* $\mathbb{C}_\varepsilon[x, z, x^{-1}, z^{-1}]$ *and exhaust all the irreducible representations of* $\mathbb{C}_\varepsilon[x, z, x^{-1}, z^{-1}]$.
(iii) *Any finite-dimensional irreducible representation of* $\mathbb{C}_\varepsilon[x,z]$ *on which either* x^l *or* z^l *is zero is one-dimensional.*
(iv) *Let* W *be any finite-dimensional representation of* $\mathbb{C}_\varepsilon[x,z]$ *on which* $x^l = \lambda$ *and* $z^l = \mu$ *for some* $\lambda, \mu \in \mathbb{C}^\times$. *Then*

$$W \simeq \mathbb{C}^l \otimes W'$$

for some vector space W'. *The action of* x (resp. z), *is given by the operator* $X_\lambda \otimes 1$ (resp. $Z_\mu \otimes 1$).

Proof. The proof of parts (i) to (iii) is straightforward (see also [7]). For part (iv), note that the action of z on W is diagonalizable and that the only possible eigenvalues of z are among the l^{th} roots of μ. Let W' be any eigenspace of z. Since $[x,z]_\varepsilon = 0$ it follows that $x^i W'$ is an eigenspace of z for all i and so,

$$W = \bigoplus_i x^i W' .$$

Pick a basis $\{w_1, \ldots, w_r\}$ of W'. Since x is injective on W and x^l is a scalar on W the elements $\{x^i w_1, \ldots, x^i w_r\}$ form a basis of $x^i W'$. Thus, we can write

$$W \simeq \mathbb{C}^l \otimes W' .$$

That the action of x and z is as given is now clear.

For brevity of notation we shall often omit the parameters λ, μ from the operators X, Z, but it should be kept in mind that X then defines a one-parameter family of operators, and similarly for Z.

Corollary. *Let* W *be any finite-dimensional representation of* $\bigotimes_{i=1}^r \mathbb{C}_\varepsilon[x_i, z_i]$, *on which* x_i^l *and* z_i^l *act as non-zero scalars for all* $i = 1, \ldots, r$. *Then,*

$$W \simeq (\mathbb{C}^l)^{\otimes r} \otimes W' ,$$

where W' *is a common eigenspace of the* z_i. *The action of* x_i (resp. z_i) *is given by the operator* X_i (resp. Z_i). *Here* X_i *etc. means the operator* X *in the* i^{th} *place and* 1 *elsewhere.*

2.11. The *quantum Heisenberg algebra* $\mathscr{H}_\varepsilon$ is the associative algebra over $\mathbb{C}$ with generators a, b and the single relation

$$[a,b]_\varepsilon = \varepsilon - \varepsilon^{-1} .$$

Before discussing the representation theory of $\mathscr{H}_\varepsilon$, we prove:

Proposition.
(i) *The elements* $\{b^r a^s : r, s \geqq 0\}$ *span* $\mathscr{H}_\varepsilon$.
(ii) *The elements* a^l *and* b^l *lie in the centre of* $\mathscr{H}_\varepsilon$.
(iii) *Let* $c = ab - 1$. *Then* $c^l = a^l b^l - 1$.

Proof. This is immediate from the next lemma, which is easily proved by induction.

Lemma. *In $\mathscr{H}_\varepsilon$, we have, for $m \geqq r \geqq 1$,*
(i) $[a, b^r]_{\varepsilon^r} = (\varepsilon^r - \varepsilon^{-r})b^{r-1}$,
(ii) $[a^r, b]_{\varepsilon^r} = (\varepsilon^r - \varepsilon^{-r})a^{r-1}$,
(iii) $(ab - 1)^m = \sum_{r=0}^{m}(-1)^{m-r}\varepsilon^{(m-1)r}[m;r]_\varepsilon a^r b^r$.

Remark. Note that $[a, c]_\varepsilon = 0 = [c, b]_\varepsilon$.

2.12.
Proposition. *Let V be a finite-dimensional irreducible representation of $\mathscr{H}_\varepsilon$. Let $a^l = \lambda$, $b^l = \mu$ on V, where λ, $\mu \in \mathbb{C}$.*

(i) $\dim(V) \leqq l$.
(ii) *If* $\lambda\mu \neq 1$, *then* $\dim(V) = l$.
(iii) *If* $\lambda\mu = 1$, *then* $\dim(V) = 1$.

In each case, V is determined uniquely (up to isomorphism) by λ and μ.

Proof. Let $v \in V$ be any eigenvector of a. By Proposition 2.11(i), (ii) the elements $\{v, bv, \ldots, b^{l-1}v\}$ span V, thus proving (i).

Next suppose that $\lambda \neq 0$ and $\lambda\mu \neq 1$. By Proposition (2.11)(iii), $c^l \neq 0$. The result follows by applying Proposition 2.10(ii) to the subalgebra of $\mathscr{H}_\varepsilon$ generated by a and c.

If $\lambda\mu = 1$ and $\lambda \neq 0$, then $c^l = 0$. Since a preserves $\ker(c)$, we can choose a common eigenvector v of a and c. But, as a is invertible on V, v is then also a common eigenvector of a and b, proving that $\dim(V) = 1$.

If $\lambda = 0$, let $0 \neq v \in \ker(a)$. By Proposition 2.11(i),(ii), the vectors $b^r v$, $0 \leqq r < l$ span V. Suppose that there is a linear relation

$$b^r v = \sum_{p=0}^{r-1} \beta_p b^p v ,$$

where $\beta_p \in \mathbb{C}$ and $\{v, bv, \ldots, b^{r-1}v\}$ are linearly independent with $r < l$. Applying a to both sides and using Lemma 2.11(i) gives a contradiction. Thus, $\{v, bv, \ldots, b^{l-1}v\}$ is a basis of V, and $\dim(V) = l$.

The uniqueness statement is clear from the above constructions.

Remark. The elements a^{l-1}, b^{l-1} act as non-zero operators on any finite-dimensional representation of $\mathscr{H}_\varepsilon$. The proof for an irreducible representation is clear from the preceding proposition, and since any finite-dimensional representation contains an irreducible representation, the statement follows in general.

3. Construction of Cyclic Representations: Reduction to an Auxiliary Algebra

We begin this section with the representation theory of $U_{\varepsilon^2}(sl_3)$, which serves as a simple example of the methods used for the $so(8)$ case and which can be also used in general. The sl_3 theory is due to Arnaudon [1], but we state and prove the results in a slightly different way.

3.1. Let $\mathscr{I}$ be the subcategory of the category of finite-dimensional representations of $U_{\varepsilon^2}(sl_3)$ on which the l^{th} powers of e_1, f_2, k_1, k_2 act as non-zero scalars.

Let $V \in \mathscr{I}(sl_3)$. Regarded as a module for the quasi-polynomial algebra generated by k_1, k_2, e_1, f_2, one proves as in Proposition 2.10(iv) that V is isomorphic to $\mathbb{C}^l \otimes \mathbb{C}^l \otimes W$, for some auxiliary vector space W, the action of these generators on the tensor product being

$$e_1 = X_1, \quad f_2 = X_2 , \tag{1}$$

$$k_1 = Z_1^2 Z_2, \quad k_2 = (Z_1 Z_2^2)^{-1} , \tag{2}$$

where as usual X_1 etc. means the operator $X \otimes 1 \otimes 1$ normalized so that $X^l = e_1^l$.

To find the action of the remaining generators, say f_1 (and similarly for e_2), we write f_1 as a polynomial with coefficients in $\mathrm{End}(W)$ in the noncommuting variables X_i, Z_i. The relations $k_i f_1 k_i^{-1} = \varepsilon^{2a_{i1}} f_1$, $[e_1, f_1] = \dfrac{k_1 - k_1^{-1}}{\varepsilon^2 - \varepsilon^{-2}}$ and the Serre relation $[f_2, [f_1, f_2]_\varepsilon]_\varepsilon = 0$, imply that:

$$f_1 = -\frac{X_1^{-1}}{(\varepsilon^2 - \varepsilon^{-2})^2}(\{Z_1^2 Z_2 \varepsilon^{-2}\} + a_1 Z_2 + b_1 Z_2^{-1}) , \tag{3}$$

$$e_2 = -\frac{X_2^{-1}}{(\varepsilon^2 - \varepsilon^{-2})^2}(\{Z_2^2 Z_1 \varepsilon^{-2}\} + a_2 Z_1 + b_2 Z_1^{-1}) , \tag{4}$$

for some $a_i, b_i \in \mathrm{End}(W)$, $i = 1, 2$. Here and elsewhere we use the following notation: for any invertible operator A on a vector space V, $A + A^{-1} = \{A\}$.

Imposing the relation $[e_2, f_1] = 0$ and the remaining two Serre relations, we find that the operators a_i, b_i, $i = 1, 2$, must satisfy:

$$[a_i, b_i]_{\varepsilon^2} = \varepsilon^2 - \varepsilon^{-2} , \tag{5}$$

$$[b_j, a_i]_{\varepsilon^2} = \varepsilon^2 - \varepsilon^{-2} , \quad i \neq j \tag{6}$$

$$[a_1, a_2] = (\varepsilon^2 - \varepsilon^{-2})(b_2 - b_1) , \tag{7}$$

$$[b_1, b_2] = (\varepsilon^2 - \varepsilon^{-2})(a_1 - a_2) . \tag{8}$$

Notice that these relations are completely independent of the auxiliary space W. So, if we define $\mathscr{A}_\varepsilon(sl_3)$ as the associative algebra over $\mathbb{C}$ with generators a_i, b_i subject to the relations (5)–(8), then W is a representation of $\mathscr{A}_\varepsilon(sl_3)$. Further, if V' is any proper $U_{\varepsilon^2}(sl_3)$ sub-representation of V then this argument also proves that W must have a proper sub-representation. Conversely, given any representation W of $\mathscr{A}_\varepsilon(sl_3)$ we can define a representation of $U_{\varepsilon^2}(sl_3)$ on $V = \mathbb{C}^l \otimes \mathbb{C}^l \otimes W$ by the formulas (1)–(4). Clearly $V \in \mathscr{I}$.

Thus we have shown that V is an irreducible representation of $U_{\varepsilon^2}(sl_3)$ if and only if W is an irreducible representation of $\mathscr{A}_\varepsilon(sl_3)$. The above results are summarized in the following proposition.

Proposition
(i) *The map*

$$\pi : U_{\varepsilon^2}(sl_3) \to \mathbb{C}_{\varepsilon^2}[x_1, z_1, x_1^{-1}, z_1^{-1}] \otimes \mathbb{C}_{\varepsilon^2}[x_2, z_2, x_2^{-1}, z_2^{-1}] \otimes \mathscr{A}_\varepsilon(sl_3) ,$$

given by the formulas (1)–(4) (*with* X, Z *replaced by* x, z) *defines a homomorphism of algebras.*

(ii) *Let W be an irreducible representation of $\mathscr{A}_\varepsilon(sl_3)$. Tensoring W with an irreducible representation $\mathbb{C}^l \otimes \mathbb{C}^l$* of $\bigotimes_{i=1}^2 \mathbb{C}_{\varepsilon^2}[x_i, z_i, x_i^{-1}, z_i^{-1}]$ *and pulling back through π gives an irreducible representation of $U_{\varepsilon^2}(sl_3)$. All irreducible representations of $U_{\varepsilon^2}(sl_3)$* in $\mathscr{I}(sl_3)$ *arise in this way.*

3.2. Arnaudon [1] constructs representations of $\mathscr{A}_\varepsilon(sl_3)$ directly, but it is simpler to observe that $\mathscr{A}_\varepsilon(sl_3)$ is essentially $U_{\varepsilon^2}(sl_2)$:

Proposition. *There is a one-parameter family of homomorphism of algebras π_λ: $\mathscr{A}_\varepsilon(sl_3) \to U_{\varepsilon^2}(sl_2)$ given by:*

$$a_1 \to \lambda^{-1} k^{-1}(1 + e) ,$$

$$a_2 \to \lambda^{-1}(k^{-1} - (\varepsilon^2 - \varepsilon^{-2})^2 \varepsilon^2 (1 + \varepsilon^2 \lambda^3 k) f ,$$

$$b_1 \to \lambda k ,$$

with b_2 determined by (7), *and* $\lambda \in \mathbb{C}^\times$.

Proof. Direct verification.

Pulling back irreducible representations of $U_{\varepsilon^2}(sl_2)$ by π_λ gives rise to a one-parameter family of representations of $\mathscr{A}_\varepsilon$ of dimension d, with $1 \leqq d < l$ and a four parameter family of representations of dimension l. These representations are irreducible for generic values of λ and are the representations that Arnaudon constructs.

3.3. We now turn to the general simply-laced case. Choose a partition of $\{1, \ldots, n\}$ into disjoint sets I, J, such that:

$$a_{rs} = 0, \quad \text{if } r, s \in I \text{ or } r, s \in J, \quad r \neq s .$$

Let $\mathscr{I}(g)$ be the subcategory of representations of $U_{\varepsilon^2}(g)$ such that the l^{th} powers of the elements, k_i, e_r, f_s act as non-zero scalars on any representation of $\mathscr{I}(g)$ for all $i = 1, \ldots, n$, $r \in I$ and $s \in J$. Let $V \in \mathscr{I}(g)$. Regarding V as a representation of the quasi-polynomial subalgebra generated by k_i, e_r and f_s, where $i = 1, \ldots, n$, $r \in I$ and $s \in J$ we can write V as:

$$V \simeq (\mathbb{C}^l)^{\otimes n} \otimes W ,$$

with the action of the generators given by:

$$k_i \to \prod_{r \in I} Z_r^{a_{ri}} \prod_{s \in J} Z_s^{-a_{si}} ,$$

and

$$e_r \to X_r, \quad r \in I, \quad f_s \to X_s, \quad s \in J .$$

To determine the action of the remaining Chevalley generators, we proceed as in the case of $g = sl_3$, and find that their action on V is of the form

$$-\frac{X_i^{-1}}{(\varepsilon^2 - \varepsilon^{-2})^2} \left(\left\{ \prod_{r \in I} Z_r^{a_{ri}} \prod_{s \in J} Z_s^{-a_{si}} \varepsilon^{-2} \right\} + \text{linear polynomial in } \{Z_j^{\pm 1} \colon a_{ij} = -1\} \right),$$

where the polynomial has coefficients in End(W). Again, the relations between the coefficients of the polynomials is independent of W (this essentially follows from the sl_3 case) and so we can associate an auxiliary algebra $\mathscr{A}_\varepsilon(\underline{g})$ to $U_{\varepsilon^2}(\underline{g})$, so that any representation of $U_{\varepsilon^2}(\underline{g})$ from $\mathscr{I}(\underline{g})$ arises from a representation of $\mathbb{C}_\varepsilon[x_i, z_i, x_i^{-1}, z_i^{-1}: i = 1, \ldots, n] \otimes \mathscr{A}_\varepsilon(\underline{g})$.

We do not write down the defining relations of $\mathscr{A}_\varepsilon(\underline{g})$ in the general case. In the next section, we do so for $\underline{g} = so(8)$.

4. Quantum $so(8)$

In this section, we identify the algebra $\mathscr{A}_\varepsilon(so(8))$.

4.1. The nodes of the Dynkin diagram of $so(8)$ are numbered $1, 2, 3, 4$, with 4 being the middle node.

Definition. *$\mathscr{A}_\varepsilon$ is the associative algebra over $\mathbb{C}$ with generators $a_i, b_i, c_i, i = 1, 2, 3$ and the following relations*:

$$[a_i, a_j] = [b_i, b_j] = [c_i, c_j] = 0\ , \tag{9}$$

$$[a_i, b_i]_{\varepsilon^2} = \varepsilon^2 - \varepsilon^{-2} = [c_i, a_i]_{\varepsilon^2}\ , \tag{10}$$

$$[a_i, b_j] = [a_j, b_i], \quad i \neq j\ , \tag{11}$$

$$[b_i, c_i] = [b_j, c_j]\ , \tag{12}$$

$$[b_i, c_j]_{\varepsilon^2} = 0, \quad i \neq j\ , \tag{13}$$

$$[b_1, [a_2, c_3]]_{\varepsilon^2} = (\varepsilon^2 - \varepsilon^{-2})^2\ . \tag{14}$$

Remark 1. Relation (14) implies, together with the other relations, that

$$[b_i, [a_j, c_k]]_{\varepsilon^2} = (\varepsilon^2 - \varepsilon^{-2})^2$$

whenever i, j, k are distinct.

2. For any $\sigma_1, \sigma_2, \sigma_3 \in \{\pm 1\}$,

$$a_i \to \sigma_i a_i, \quad b_i \to \sigma_i b_i, \quad c_i \to \sigma_i c_i\ , \tag{15}$$

is an automorphism of $\mathscr{A}_\varepsilon$.

Introduce the following elements of $\mathscr{A}_\varepsilon$:

$$c_0 = \frac{1}{\varepsilon^2 - \varepsilon^{-2}} [c_i, b_i]\ , \tag{16}$$

$$d_k = \frac{1}{\varepsilon^2 - \varepsilon^{-2}} [a_j, c_i], \quad i, j, k \text{ distinct}\ , \tag{17}$$

$$d_0 = \frac{1}{\varepsilon^2 - \varepsilon^{-2}} [a_i, d_i]\ . \tag{18}$$

It follows from relation (12) that c_0 is well defined. That d_k is well-defined follows by taking the bracket of both sides of Eq. (19) below with a_j and using (9) and (11). For d_0, one uses (9) and the Jacobi identity.

Proposition. *The following relations hold in* $\mathscr{A}_\varepsilon$:

$$[c_0, a_i] = (\varepsilon^2 - \varepsilon^{-2})(b_i - c_i) \, , \tag{19}$$

$$[b_i, d_0] = (\varepsilon^2 - \varepsilon^{-2})(a_i - d_i) \, , \tag{20}$$

$$[d_j, b_i] = (\varepsilon^2 - \varepsilon^{-2})c_k, \; i,j,k \in \{1,2,3\} \text{ distinct} \, , \tag{21}$$

$$[b_i, c_0]_{\varepsilon^2} = 0 \, , \tag{22}$$

$$[b_i, d_i]_{\varepsilon^2} = \varepsilon^2 - \varepsilon^{-2} \, , \tag{23}$$

$$[c_0, c_i]_{\varepsilon^2} = 0 \, , \tag{24}$$

$$[d_i, d_0]_{\varepsilon^2} = 0 \, , \tag{25}$$

$$[c_i, d_j]_{\varepsilon^2} = 0, \quad i \neq j \in \{1,2,3\} \, , \tag{26}$$

$$[d_i, a_j]_{\varepsilon^2} = 0, \quad i \neq j \in \{1,2,3\} \, , \tag{27}$$

$$[d_0, a_j]_{\varepsilon^2} = 0 \, , \tag{28}$$

$$[c_0, d_i]_{\varepsilon^2} + [c_j, c_k]_{\varepsilon^2} = 0, \quad i,j,k \text{ distinct} \, , \tag{29}$$

$$[c_i, d_0]_{\varepsilon^2} + [d_j, d_k]_{\varepsilon^2} = 0, \quad i,j,k \text{ distinct} \, , \tag{30}$$

$$[c_0, d_0]_{\varepsilon^2} + [c_j, d_j]_{\varepsilon^2} + [c_k, d_k]_{\varepsilon^2} + [d_i, c_i]_{\varepsilon^2} = \varepsilon^2 - \varepsilon^{-2}, \; i,j,k \text{ distinct} \, . \tag{31}$$

Corollary. *The* l^{th} *powers of the elements* a_i, b_i, c_i, d_i, $i = 1,2,3$, c_0 *and* d_0 *are all in the centre of* $\mathscr{A}_\varepsilon$.

4.2. The analogue of Proposition 3.1 for $so(8)$ is:

Theorem.
(i) *Let* $V \in \mathscr{I}(so(8))$. *Then,*

$$V \simeq (\mathbb{C}^l)^{\otimes 4} \otimes W \, ,$$

for some W, *and the action of the Chevalley generators is given by:*

$$k_i \to Z_i^2 Z_4, \quad e_i \to X_i \, ,$$

$$k_4 \to (Z_1 Z_2 Z_3 Z_4^2)^{-1}, \quad f_4 \to X_4 \, ,$$

$$f_i \to -\frac{X_i^{-1}}{(\varepsilon^2 - \varepsilon^{-2})^2} (\{Z_i^2 Z_4 \varepsilon^{-2}\} + a_i Z_4 + b_i Z_4^{-1}) \, ,$$

$$e_4 \to -\frac{X_4^{-1}}{(\varepsilon^2 - \varepsilon^{-2})^2} \Bigg(\{Z_1 Z_2 Z_3 Z_4^2 \varepsilon^{-2}\} + Z_1 Z_2 Z_3 \Big(c_0 + \sum_i c_i Z_i^{-2} \Big) + (Z_1 Z_2 Z_3)^{-1} \Big(d_0 + \sum_i d_i Z_i^2 \Big) \Bigg) \, ,$$

where $i = 1,2,3$. *The operators* a_i, b_i, c_i, d_i, $i = 1,2,3$, *and* c_0, d_0, *satisfy relations* (9)–(14) *and* (19)–(31).
(ii) $\mathscr{A}_\varepsilon \simeq \mathscr{A}_\varepsilon(so(8))$.

Sketch of Proof. The fact that the image of the generators has the general form above follows from the discussion in Sect. 3.3. The relations

$$[a_i, a_j] = 0 = [b_i, b_j], \quad [a_i, b_j] = [a_j, b_i] ,$$

ensure that the copies of $U_{\varepsilon^2}(sl_2)$ corresponding to nodes 1, 2 and 3 of the Dynkin diagram of $so(8)$ commute. The other relations follow from the $U_{\varepsilon^2}(sl_3)$ relations between nodes i and 4, $i = 1, 2, 3$.

Part (ii) is now clear from the definitions of $\mathscr{A}_\varepsilon$ and $\mathscr{A}_\varepsilon(so(8))$ and Proposition 4.1.

The proof that any irreducible representation of $U_{\varepsilon^2}(so(8))$ on which the e_i^l and f_4^l act as non-zero scalars is equivalent to one of these pull-back representations is as in [3] and Sect. 3.1.

4.3. It is interesting to compute the Z_0-characters of the representations of $U_{\varepsilon^2}(so(8))$ described in Theorem 4.3. For this, we must first choose a set of root vectors. We take the following reduced expression of the longest element w_0 of the Weyl group of $so(8)$:

$$w_0 = s_2 s_4 s_2 s_3 s_4 s_2 s_1 s_4 s_3 s_2 s_4 s_1 .$$

We then find that the positive root vectors, in the order described in (2.6), are

$$e_2, T_2(e_4), e_4, T_2T_4(e_3), T_4(e_3), e_3 ,$$

$$T_4T_2T_3T_4(e_1), T_2T_3T_4(e_1), T_3T_4(e_1), T_2T_4(e_1), T_4(e_1), e_1 .$$

The negative root vectors are obtained by replacing e's by f's.

The action of the non-simple root vectors in the representations described in Theorem 4.2 is as follows:

$$\begin{aligned}
T_4 e_i &= \frac{1}{(\varepsilon^2 - \varepsilon^{-2})} X_4^{-1} X_i Z_i ((c_0 + \varepsilon^{-2} Z_4^2) Z_j Z_2 + c_j Z_j^{-1} Z_2 \\
&\quad + c_2 Z_j Z_2^{-1} + d_i Z_j^{-1} Z_2^{-1}) , \\
T_2 e_4 &= \frac{\varepsilon^{-2}}{(\varepsilon^2 - \varepsilon^{-2})} X_4^{-1} X_2 Z_2^{-1} ((d_0 + \varepsilon^2 Z_4^{-2}) Z_1^{-1} Z_3^{-1} \\
&\quad + d_3 Z_1^{-1} Z_3 + d_1 Z_3^{-1} Z_1) + c_2 Z_1 Z_3) , \\
T_2 T_4 e_i &= -\varepsilon^{-2} X_4^{-1} X_2 X_i Z_2^{-1} Z_i (c_2 Z_j + d_i Z_j^{-1}) , \\
T_3 T_4 e_1 &= -\varepsilon^{-2} X_4^{-1} X_1 X_3 Z_1 Z_3^{-1} (c_3 Z_2 + d_1 Z_2^{-1}) , \\
T_2 T_3 T_4 e_1 &= \varepsilon^{-4} (\varepsilon^2 - \varepsilon^{-2}) X_4^{-1} X_1 X_2 X_3 Z_1 Z_2^{-1} Z_3^{-1} d_1 , \\
T_4 T_2 T_3 T_4 e_1 &= \frac{\varepsilon^{-4}}{\varepsilon^2 - \varepsilon^{-2}} X_4^{-2} X_1 X_2 X_3 Z_1^2 (\varepsilon^{-2} Z_4^2 d_1 + \varepsilon^2 [c_0, d_1]_{\varepsilon^4} \\
&\quad + [c_1, d_1]_{\varepsilon^2} Z_1^{-2}) ,
\end{aligned}$$

where $\{i, j\} = \{1, 3\}$.

For the negative root vectors we have,

$$T_4 f_i = -\frac{1}{(\varepsilon^2 - \varepsilon^{-2})} X_i^{-1} X_4 (b_i + \varepsilon^2 Z_i^{-2}) Z_4^{-1} ,$$

$$T_2 f_4 = -\frac{1}{(\varepsilon^2 - \varepsilon^{-2})} X_2^{-1} X_4 (a_2 + \varepsilon^{-2} Z_2^2) Z_4 ,$$

$$T_2 T_4 f_i = \frac{\varepsilon^2}{(\varepsilon^2 - \varepsilon^{-2})^2} (X_i X_2)^{-1} X_4 \left(\frac{[a_2, b_i]_{\varepsilon^2}}{\varepsilon^2 - \varepsilon^{-2}} + Z_i^{-2} Z_2^2 + \varepsilon^2 a_2 Z_i^{-2} + \varepsilon^{-2} b_i Z_2^2 \right) ,$$

$$T_3 T_4 f_1 = \frac{\varepsilon^2}{(\varepsilon^2 - \varepsilon^{-2})^2} (X_1 X_3)^{-1} X_4 \left(\frac{[a_3, b_1]_{\varepsilon^2}}{\varepsilon^2 - \varepsilon^{-2}} + Z_1^{-2} Z_3^2 + \varepsilon^2 a_3 Z_1^{-2} + \varepsilon^{-2} b_1 Z_3^2 \right) ,$$

$$\begin{aligned} T_2 T_3 T_4 f_1 = -\frac{\varepsilon^4}{(\varepsilon^2 - \varepsilon^{-2})^4} (X_1 X_2 X_3)^{-1} X_4 \Bigg(& \frac{[[a_3, a_2, b_1]_{\varepsilon^2}]_{\varepsilon^2}}{\varepsilon^2 - \varepsilon^{-2}} Z_4 \\ & + \frac{\varepsilon^2 [b_3, [a_2, b_1]_{\varepsilon^2}]}{\varepsilon^2 - \varepsilon^{-2}} Z_4^{-1} + \varepsilon^{-2} [a_2, b_1]_{\varepsilon^2} Z_3^2 Z_4 \\ & + \varepsilon^{-2} [a_3, b_1]_{\varepsilon^2} Z_2^2 Z_4 + \varepsilon^2 (\varepsilon^2 - \varepsilon^{-2}) a_2 a_3 Z_1^{-2} Z_4 \\ & + (\varepsilon^2 - \varepsilon^{-2}) a_2 Z_1^{-2} Z_3^2 Z_4 + (\varepsilon^2 - \varepsilon^{-2}) a_3 Z_1^{-2} Z_2^2 Z_4 \\ & + \varepsilon^{-4} (\varepsilon^2 - \varepsilon^{-2}) b_1 Z_2^2 Z_3^2 Z_4 + \varepsilon^{-2} (\varepsilon^2 - \varepsilon^{-2}) Z_1^{-2} Z_2^2 Z_3^2 Z_4 \Bigg) , \end{aligned}$$

$$T_4 T_2 T_3 T_4 f_1 = -\frac{\varepsilon^2}{(\varepsilon^4 - \varepsilon^{-2})^3} (X_1 X_2 X_3)^{-1} X_4^2 \left(\frac{[b_3, [a_2, b_1]_{\varepsilon^2}]}{\varepsilon^2 - \varepsilon^{-2}} + \varepsilon^2 [b_3, a_2] Z_1^{-2} \right) Z_4^{-1} .$$

To compute the action of the l^{th} powers of the root vectors, and hence the Z_0-characters of the representations, we must, of course, construct some representations of $\mathscr{A}_\varepsilon$. The next two sections are devoted to this problem.

5. Small Representations of $\mathscr{A}_\varepsilon$

5.1. From the results of [5], we know that every finite-dimensional irreducible representation of $U_{\varepsilon^2}(so(8))$ has dimension at most l^{12}. It follows from Theorem 4.9 that any finite-dimensional irreducible representation of $\mathscr{A}_\varepsilon$ has dimension at most l^8. As to the minimal possible dimension of representations of $\mathscr{A}_\varepsilon$ we have:

Proposition. *Every irreducible representation of $\mathscr{A}_\varepsilon$ has dimension at least l.*

Proof. Let W be a finite-dimensional irreducible representation of $\mathscr{A}_\varepsilon$. If some b_i^l (resp. c_i^l) is zero on W, the result follows by applying Proposition 2.12 to the quantum Heisenberg subalgebra generated by a_i and b_i (resp. c_i and a_i). Thus, we are reduced to the case when $b_i^l, c_i^l \neq 0$, $i \in \{1, 2, 3\}$. Since, $[b_1, c_2]_{\varepsilon^2} = 0$ the result follows from Proposition 2.10.

5.2. We first look for l-dimensional representations of $\mathscr{A}_\varepsilon$.

Proposition. *Let $\rho: \mathscr{A}_\varepsilon \to \mathrm{End}(W)$ be an l-dimensional representation of $\mathscr{A}_\varepsilon$. Then, possibly after composing ρ with one of the automorphisms* (15) *of $\mathscr{A}_\varepsilon$, ρ factors through the quotient of $\mathscr{A}_\varepsilon$ by the relations*:

$$a_i = a_j, \quad b_i = b_j, \quad c_i = c_j\ , \tag{32}$$

for $i, j \in \{1, 2, 3\}$.

Proof. By Proposition 5.1, W is necessarily an irreducible representation of $\mathscr{A}_\varepsilon$. Assume first that $b_1^l \neq 0$ on W. By Remark 2.12, $c_2^{l-1} \neq 0$ on W. We can therefore choose an eigenvector w of b_1 such that $c_2^{l-1} w \neq 0$. Thus, the elements $\{w, c_2 w, \ldots, c_2^{l-1} w\}$ are non-zero and form a basis of W since they belong to distinct eigenspaces of b_1. As $[b_1, b_3] = 0$, w is also an eigenvector for b_3, and since $[b_3, c_2]_{\varepsilon^2} = 0$ it follows that b_3 acts as a multiple of b_1 on all of W. One proves similarly that b_2 is a multiple of b_1.

We are therefore reduced to the case $b_i^l = 0$, for all $i = 1, 2, 3$. Choose $0 \neq w$ such that $b_i \cdot w = 0$, $i = 1, 2, 3$. By applying Proposition 2.12 to the quantum Heisenberg algebra generated by a_1 and b_1, we find that the kernel of b_1 on W is one-dimensional, and that the elements $\{w, a_1 w, \ldots, a_1^{l-1} w\}$ form a basis of W. Since $[[a_1, b_i], b_1]_{\varepsilon^2} = 0$ the operator $[a_1, b_i]$ preserves the kernel of b_1. So, there exists scalars v_i, $i \in \{2, 3\}$, such that $b_i a_1 \cdot \mathrm{w} = v_i w$. Lemma 2.11 now shows that

$$b_1 \cdot a_1^r w = -\varepsilon^{2r}(\varepsilon^2 - \varepsilon^{-2})[r-1]_{\varepsilon^2} a_1^{r-1} w\ ,$$

$$b_i \cdot a_1^r w = v_i \varepsilon^{2r-2}[r-1]_{\varepsilon^2} a_1^{r-1} w\ .$$

Thus b_2 and b_3 are scalar multiples of b_1. Note that in both the cases considered above, the multiples must be non-zero since b_i cannot be the zero operator on W.

We have thus shown that $b_i = \mu_i b$ for some non-zero scalars μ_i and some non-zero operator b on W. Similarly one can show that

$$a_i = \lambda_i a, \quad c_i = \mu_i c\ ,$$

for some scalars λ_i, μ_i and operators a, c on W. The relations in $\mathscr{A}_\varepsilon$, imply that for some λ, μ, v and $\sigma_i \in \{-1, 1\}$, $i \in \{1, 2, 3\}$, we have

$$\lambda_i = \sigma_i \lambda, \quad \mu_i = \sigma_i \mu, \quad v_i = \sigma_i v\ .$$

This completes the proof.

5.3. Denote by $\mathscr{A}_\varepsilon^{\mathrm{sym}}$ the quotient of $\mathscr{A}_\varepsilon$ by the relations (32). Note that $\mathscr{A}_\varepsilon^{\mathrm{sym}}$ is generated by elements a, b, c with the following relations:

$$[a, b]_{\varepsilon^2} = \varepsilon^2 - \varepsilon^{-2} = [c, a]_{\varepsilon^2}\ , \tag{33}$$

$$[b, c]_{\varepsilon^2} = 0\ , \tag{34}$$

$$[b, [a, c]]_{\varepsilon^2} = (\varepsilon^2 - \varepsilon^{-2})^2\ . \tag{35}$$

Proposition. *The following formulae define irreducible representations of* $\mathscr{A}_\varepsilon^{\mathrm{sym}}$ on $\mathbb{C}^l$ *on which the* l^{th} *powers of b and c act as non-zero scalars*:

$$a = \varepsilon^2 X^{-1} Z^{-1} + Z^{-1} + X^{-1} ,$$

$$b = X ,$$

$$c = Z ,$$

where X and Z are the operators defined in Sect. 2.10, *with arbitrary normalizations, such that* $ZX = \varepsilon^4 XZ$.

Conversely, every finite-dimensional irreducible representation of $\mathscr{A}_\varepsilon^{\mathrm{sym}}$ *on which* b^l *and* c^l *are non-zero is equivalent to a representation of this type.*

Proof. The fact that the above formulae do define a representation of $\mathscr{A}_\varepsilon^{\mathrm{sym}}$ is an easy verification, with irreducibility following from the fact that the operators X and Z generate $\mathrm{End}(\mathbb{C}^l)$ as an algebra.

Let W be any irreducible representation of $\mathscr{A}_\varepsilon^{\mathrm{sym}}$ on which $b^l \neq 0$ and $c^l \neq 0$. By Proposition 2.10(iv) we can write

$$W \simeq \mathbb{C}^l \otimes W' ,$$

where W' is an eigenspace of c. The action of b is $X \otimes 1$ and that of c is $Z \otimes 1$ for suitably normalized operators. The action of a is determined by writing a as a polynomial in the non-commuting variables X, Z. Using the relations in $\mathscr{A}_\varepsilon^{\mathrm{sym}}$, it is not hard to show that:

$$a = \varepsilon^2 X^{-1} Z^{-1} + X^{-1} + Z^{-1} .$$

In particular a has scalar coefficients. Thus, the irreducibility of W forces $\dim(W') = 1$.

This completes the proof of the proposition.

5.4. To complete the study of representations of $\mathscr{A}_\varepsilon^{\mathrm{sym}}$ we must study representations on which either b^l or c^l is zero. Let W be an irreducible representation on which $b^l = 0$. Since c preserves $\ker(b)$ we can choose $w \in W$ such that:

$$b \cdot w = 0, \quad c \cdot w = vw ,$$

for some $v \in \mathbb{C}$. Using relation (33), we find that

$$[a, b] \cdot w = \varepsilon^2(\varepsilon^2 - \varepsilon^{-2}) .$$

The relations (34) and (35) now force

$$v = -\varepsilon^{-2} .$$

Set $w_i = a^i \cdot w$. Then, using Lemma 2.11(ii), we find:

$$a \cdot w_i = w_{i+1},\ i \neq l-1 , \tag{36}$$

$$a \cdot w_{l-1} = \lambda w_0 , \tag{37}$$

$$b \cdot w_i = -\varepsilon^{2i}(\varepsilon^{2i} - \varepsilon^{-2i}) w_{i-1} , \tag{38}$$

$$c \cdot w_i = -\varepsilon^{-4i-2} w_i + \varepsilon^{-2i}(\varepsilon^{2i} - \varepsilon^{-2i}) w_{i-1} \tag{39}$$

for some $\lambda \in \mathbb{C}$ (we set $w_{-1} = 0$). Thus, since W is irreducible, the elements w_i, $0 \leqq i \leqq l-1$, span W. Equation (39) also implies that the w_i are linearly independent. It is easy to verify that the formulas (36)–(39) do in fact define a representation of $\mathscr{A}_\varepsilon^{\mathrm{sym}}$. The case when $c^l = 0$ can be dealt with similarly.

Remark. The discussion in the preceding two sections shows that any irreducible representation of $\mathscr{A}_\varepsilon^{\mathrm{sym}}$ is l-dimensional, and that the space of irreducible representations is parametrized by $\mathbb{C}^2\setminus\{(0,0)\}$.

5.5. In the preceding subsections, we constructed all l-dimensional irreducible representations of $\mathscr{A}_\varepsilon$, and showed that they factor through the quotient $\mathscr{A}_\varepsilon^{\mathrm{sym}}$. We shall now show that irreducible representations of $\mathscr{A}_\varepsilon$ of dimension dl, for each $d = 1, \ldots, l$ can be constructed by passing to a larger quotient $\tilde{\mathscr{A}}_\varepsilon$ of $\mathscr{A}_\varepsilon$ by the relations

$$a_1 = a_3, \quad b_1 = b_3, \quad c_1 = c_3 .$$

Thus $\tilde{\mathscr{A}}_\varepsilon$ has generators a_i, b_i, c_i, $i = 1, 2$, and relations

$$[a_i, a_j] = [b_i, b_j] = [c_i, c_j] = 0 ,$$

$$[b_i, c_1]_{\varepsilon^2} = 0 ,$$

$$[b_1, c_1]_{\varepsilon^2} = [b_1, c_2\}_{\varepsilon^2} = [b_2, c_1]_{\varepsilon^2} = 0 ,$$

$$[a_i, b_i]_{\varepsilon^2} = \varepsilon^2 - \varepsilon^{-2} ,$$

$$[c_i, a_i]_{\varepsilon^2} = \varepsilon^2 - \varepsilon^{-2} ,$$

$$[b_1, [a_2, c_1]]_{\varepsilon^2} = (\varepsilon^2 - \varepsilon^{-2})^2 ,$$

$$[b_1, c_1] = [b_2, c_2] ,$$

$$[a_1, b_2] = [a_2, b_1] .$$

Proposition. *There exists a homomorphism of algebras* $\tilde{\mathscr{A}}_\varepsilon \to \mathbb{C}_{\varepsilon^2}[x, z, x^{-1}, z^{-1}] \otimes U_{\varepsilon^2}(sl_2)$ *given by*

$$a_1 \to (x^{-1} + z^{-1}) \otimes 1 + \varepsilon^2 x^{-1} z^{-1} \otimes k^{-1}(1 - \varepsilon^{-2}(\varepsilon^2 - \varepsilon^{-2})^2 f) ,$$

$$a_2 \to \varepsilon^{-2} \otimes ke + x^{-1} \otimes (k^{-1} + e) + z^{-1} \otimes k + \varepsilon^2 x^{-1} z^{-1} \otimes 1 ,$$

$$b_1 \to x \otimes 1 ,$$

$$b_2 \to x \otimes k ,$$

$$c_1 \to z \otimes 1 ,$$

$$c_2 \to z \otimes (k^{-1} + e) .$$

Proof. Direct verification.

Pulling back the tensor product of an irreducible representation of $\mathbb{C}_{\varepsilon^2}[x, z, x^{-1}, z^{-1}]$ and a d-dimensional irreducible representation W of $U_{\varepsilon^2}(sl_2)$, for $1 \leqq d \leqq l$, gives a dl-dimensional representation $V = \mathbb{C}^l \otimes W$ of $\tilde{\mathscr{A}}_\varepsilon$. Since b_1^l and c_1^l act as non-zero scalars on V, the irreducibility of V follows by arguments similar to the ones in Sect. 3. We omit the details.

5.6. We now consider the Z_0-characters of the representations of $U_{\varepsilon^2}(so(8))$ constructed in Theorem 4.2 and Proposition 5.5. We shall not write down these characters for arbitrary values of the parameters. We shall assume that $X_i^l = 1$,

$i = 1, 2, 3, 4$, $Z_1^l = Z_3^l = \lambda$ (say), $Z_2^l = Z_4^l = 1$, $k^l = \lambda$, $x^l = -\lambda^{-2}$, $f^l = 0$. Let $Z^l = \mu$, $e^l = \nu$.

Proposition. *On the representations of $U_{\varepsilon^2}(so(8))$ obtained by imposing the above restrictions, the l^{th} powers of all root vectors act as 0 except the following*:

$$f_4^l = 1, \quad e_i^l = 1, \quad i = 1, 2, 3 ,$$

$$(T_2(e_4))^l = -\frac{(\lambda^2 - \lambda^{-2})}{(\varepsilon^2 - \varepsilon^{-2})^l},$$

$$(T_2 T_4(e_i))^l = \lambda^2, \quad i = 1, 3 ,$$

$$(T_3 T_4(e_1))^l = \lambda^{-2}(\mu\nu + 1) ,$$

$$(T_2 T_3 T_4(e_1))^l = -(\varepsilon^2 - \varepsilon^{-2})^l(\mu + \lambda^{-2}(\mu\nu + 1)) ,$$

$$(T_4 T_2 T_3 T_4(e_1))^l = -(\mu\nu + 1) .$$

In addition, we have $k_1^l = k_3^l = \lambda^2$, $k_2^l = 1$, $k_4^l = -\lambda^{-2}$.

Proof. A direct calculation: [see [4], Sect. 4.7, for the method].

Remark. The Z_0-characters of the representations depend only on λ, μ and ν. In particular, if in Proposition 5.5 one uses a restricted representation of $U_\varepsilon(sl_2)$, i.e. one for which $k^l = 1$, $e^l = f^l = 0$, the Z_0-character of the resulting representation of $U_{\varepsilon^2}(so(8))$ is the same as that obtained by using the trivial representation of $U_\varepsilon(sl_2)$. In particular, we see that it is possible to have irreducible representations of $U_\varepsilon(\underline{g})$ of different dimensions with the same Z_0-character.

5.7. It is interesting to test Conjecture 2.9 with the representations of $U_{\varepsilon^2}(so(8))$ that we have constructed.

Proposition. *Let V be one of the representations of $U_{\varepsilon^2}(so(8))$ whose Z_0-character is computed in Proposition* 5.6. *Let g_V be the element of $SO(8)$ associated to V as in Sect.* 2.9.

(a) *If $\lambda = 1$ and $\nu = 0$, the conjugacy class of g_V has dimension* 10.
(b) *If $\lambda \neq 1$, the conjugacy class of g_V has dimension* 12.

Proof. The element g_V can be computed from the data in Proposition 5.6. The dimension of the conjugacy class of g_V was computed by first finding the Jordan canonical form of g_V, then finding a simple element of $SO(8)$ with the same Jordan canonical form, and then finding the centralizer of the latter element. The calculations were verified using Mathematica.

This result is exactly in accordance with Conjecture 2.9, for the representation V has dimension dl^5, $1 \leqq d < l$ in case (a), and dimension l^6 in case (b).

6. Large Representations of $\mathscr{A}_\varepsilon$

In this section, we give a procedure for constructing essentially all irreducible representations of $\mathscr{A}_\varepsilon$ with dimension divisible by l^4. Using the results of Sect. 4, this gives representations of $U_{\varepsilon^2}(so(8))$ with dimension divisible by l^8.

6.1. In the same way as the representation theory of $U_{\varepsilon^2}(so(8))$ was reduced to that of $\mathscr{A}_\varepsilon$, in Sect. 4, we shall reduce the representation theory of $\mathscr{A}_\varepsilon$ to that of the algebra $\mathscr{B}_\varepsilon$ defined as follows.

Definition. *$\mathscr{B}_\varepsilon$ is the associative algebra over $\mathbb{C}$ with generators u_i, v_i, $i = 1, 2, 3$, and the following defining relations:*

$$\begin{aligned} [u_i, u_j] &= 0 = [v_i, v_j] , \\ [u_i, [v_j, u_i]_\varepsilon]_\varepsilon &= v_j , \\ [v_i, [u_j, v_i]_\varepsilon]_\varepsilon &= u_j , \\ [v_i, u_i]_\varepsilon &= [v_j, u_j]_\varepsilon , \\ [u_i, v_j]_\varepsilon &= [u_j, v_i]_\varepsilon, \quad i \neq j . \end{aligned}$$

Note that we can replace ε by an arbitrary non-zero complex number in this definition. It is not difficult to see that $\mathscr{B}_1$ is isomorphic to the tensor product of four copies of $U(sl_2)$. However, it is clear that $\mathscr{B}_\varepsilon$ has $2^4 + 1$ one-dimensional representations, and so is not isomorphic to the tensor product of four copies of quantum sl_2.

6.2. Define the following elements of $\mathscr{A}_\varepsilon$:

$$\begin{aligned} \gamma_i &= c_i b_i - \varepsilon^2 c_0, \quad i = 1, 2, 3 , \\ \gamma_0 &= c_0 , \\ b_0 &= \gamma_0[[a_1, b_2], b_3] - \varepsilon^6(\varepsilon^2 - \varepsilon^{-2})^2 b_1 b_2 b_3 . \end{aligned}$$

Note that, by Eqs. (9) and (11), b_0 is well-defined.

Proposition. (a) *The elements b_i, γ_i, $i = 0, 1, 2, 3$, generate a quasi-polynomial subalgebra of $\mathscr{A}_\varepsilon$:*

$$\begin{aligned} b_i b_j = b_j b_i, &\quad \gamma_i \gamma_j = \gamma_j \gamma_i , \\ b_i \gamma_i = \gamma_i b_i, &\quad i = 1, 2, 3 , \\ b_i \gamma_j = \varepsilon^4 \gamma_j b_i \quad i = 1, 2, 3, &\quad j = 0, 1, 2, 3, \quad i \neq j , \\ b_0 \gamma_i = \varepsilon^8 \gamma_i b_0, &\quad i = 0, 1, 2, 3 . \end{aligned}$$

(b) *The l^{th} powers of the elements b_i, γ_i, $i = 0, 1, 2, 3$, lie in the centre of $\mathscr{A}_\varepsilon$.*

6.3. **Theorem.** (i) *There is a homomorphism of algebras*

$$\sigma: \mathscr{A}_\varepsilon \to \bigotimes_{i=0}^{3} \mathbb{C}_\varepsilon[x_i^{1/2}, x_i^{-1/2}, z_i, z_i^{-1}] \otimes \mathscr{B}_\varepsilon \quad \textit{such that}$$

$$\begin{aligned} \sigma(a_i) = {} & x_0^{-1} x_i (1 + \varepsilon^{-2} z_0^2) z_j^2 z_k^2 \\ & + x_j^{-1}(\{\varepsilon^{-2} z_j^2\} + (\varepsilon^2 - \varepsilon^{-2}) u_j) z_k^{-2} + x_k^{-1}(\{\varepsilon^{-2} z_k^2\} + (\varepsilon^2 - \varepsilon^{-2}) u_k) z_j^{-2} \\ & + x_0 x_j^{-1} x_k^{-1}(\{\varepsilon^{-2} z_j^2\} + (\varepsilon^2 - \varepsilon^{-2}) u_j)(\{\varepsilon^{-2} z_k^2\} + (\varepsilon^2 - \varepsilon^{-2}) u_k) \\ & + x_0^{-1}(1 + \varepsilon^2 z_0^{-2}) z_j^{-2} z_k^{-2} + z_0^2 z_j^2 z_k^2 \\ & + (\varepsilon^2 - \varepsilon^{-2})^2 (x_i^{-1} x_j x_k)^{1/2} (\varepsilon^{-1} z_j^2 z_k^2 [v_i, u_j]_\varepsilon + (z_j^2 v_k + z_k^2 v_j) \\ & + \varepsilon [u_j, v_k]_\varepsilon) , \end{aligned}$$

$$\sigma(b_i) = (z_0 z_j z_k)^{-2} ,$$

$$\sigma(c_i) = (x_i + \varepsilon^2 x_0)(z_0 z_j z_k)^2 ,$$

where $\{i, j, k\} = \{1, 2, 3\}$.
(ii) *If* V_i *is an irreducible representation of* $\mathbb{C}_\varepsilon[x_i^{1/2}, x_i^{-1/2}, z_i, z_i^{-1}]$, $i = 0, 1, 2, 3$, *and* V *is an irreducible representation of* $\mathscr{B}_\varepsilon$, *the pull-back* $\sigma^*(\bigotimes_{i=0}^{3} V_i \otimes V)$ *is an irreducible representation of* $\mathscr{A}_\varepsilon$. *Moreover,* $\sigma^*(\bigotimes_{i=0}^{3} V_i \otimes V) \cong \sigma^*(\bigotimes_{i=0}^{3} V_i' \otimes V')$ iff $V_i \cong V_i'$ *and* $V \cong V'$.
(iii) *The irreducible representations which arise in* (ii) *are precisely those on which the* l^{th} *powers of the elements* b_i, γ_i, $i = 0, 1, 2, 3$, *of* $\mathscr{A}_\varepsilon$ *act as non-zero scalars.*

Proof. The first part of the theorem is a direct verification. The other parts are proved by the methods used in Sects. 3 and 4. We omit the details.

Remark. It is not difficult to see that the conditions that the b_i^l and γ_i^l are non-zero scalars corresponds to a Zariski open subset of $\mathrm{Spec}(Z_0)$. In conjunction with Theorem 4.2, this theorem therefore reduces the construction of the irreducible representations of $U_{\varepsilon^2}(so(8))$ corresponding to a (smaller) Zariski open subset of $\mathrm{Spec}(Z_0)$ to the problem of finding the irreducible representations of $\mathscr{B}_\varepsilon$.

6.4. Since the generic representation of $U_{\varepsilon^2}(so(8))$ has dimension $l^{1/2}$ and depends on 28 (complex) parameters, it follows from Theorems 4.2 and 6.3 that the generic representation of $\mathscr{B}_\varepsilon$ has dimension l^4 and depends on 12 parameters. (It is interesting to note that the same is true for the representations of the tensor product of four copies of $U_\varepsilon(sl_2)$, although as we remarked in Sect. 6.2, $\mathscr{B}_\varepsilon$ is not isomorphic to such a tensor product.) We do not know how to describe the most general representation of $\mathscr{B}_\varepsilon$. We restrict ourselves here to the discussion of the "symmetric" representations of $\mathscr{B}_\varepsilon$, i.e. those which factor through the quotient $\mathscr{B}_\varepsilon^{\mathrm{sym}}$ of $\mathscr{B}_\varepsilon$ by the relations:

$$u_i = u_j\ (= u), \quad v_i = v_j\ (= v) ,$$

$i, j = 1, 2, 3$.
Before discussing the representation theory of $\mathscr{B}_\varepsilon^{\mathrm{sym}}$, we make a few remarks on its structure.

Remark.
(i) If we define $w = [u, v]_\varepsilon$, the defining relations of $\mathscr{B}_\varepsilon^{\mathrm{sym}}$ take the symmetric form

$$[u, v]_\varepsilon = w, \quad [v, w]_\varepsilon = u, \quad [w, u]_\varepsilon = v .$$

This makes it clear that the cyclic permutations of u, v, w define algebra automorphisms of $\mathscr{B}_\varepsilon^{\mathrm{sym}}$.
(ii) If ψ is an indeterminate, let

$$P(\psi) = \prod_{k=0}^{l-1} (\psi + \{\varepsilon^k\}) .$$

Then, $P(u)$, $P(v)$ and $P(w)$ are in the centre of $\mathscr{B}_\varepsilon^{\mathrm{sym}}$.
(iii) The element

$$\Omega = \varepsilon^{-2}u^2 + \varepsilon^2 v^2 + \varepsilon^{-2}w^2 - \varepsilon^{-1}(\varepsilon^2 - \varepsilon^{-2})wvu$$

is invariant under the automorphisms in (i), and is in the centre of $\mathscr{B}_\varepsilon^{\mathrm{sym}}$.

We conjecture that the elements $P(u)$, $P(v)$, $P(w)$ and Ω generate the centre of $\mathscr{B}_\varepsilon^{\mathrm{sym}}$ as an algebra.

6.5. We now describe a family of l-dimensional irreducible representations of $\mathscr{B}_\varepsilon^{\mathrm{sym}}$.

Proposition. *There is a one-parameter family of homomorphisms of algebras* ρ_μ: $\mathscr{B}_\varepsilon^{\mathrm{sym}} \to \mathbb{C}_\varepsilon[x, z, x^{-1}, z^{-1}]$, $\mu \in \mathbb{C}$, *such that*

$$\rho_\mu(u) = \frac{z + z^{-1}}{\varepsilon^2 - \varepsilon^{-2}},$$

$$\rho_\mu(v) = \frac{x + x^{-1}}{\varepsilon^2 - \varepsilon^{-2}} + \mu x \prod_{i=1}^{l-2} (\varepsilon^{-2i} z - \varepsilon^{2i} z^{-1}).$$

If V is an irreducible representation of $\mathbb{C}_\varepsilon[x, z, x^{-1}, z^{-1}]$, the pull-back $\rho_\mu^*(V)$ is an irreducible representation of $\mathscr{B}_\varepsilon$, and $\rho_\mu^*(V) \cong \rho_{\mu'}^*(V')$ iff *$\mu = \mu'$ and $V \cong V'$.*

Thus, $\mathscr{B}_\varepsilon^{\mathrm{sym}}$ has a 3-parameter family of irreducible representations of dimension l. Combining this result with Theorems 4.2 and 6.3, we obtain a 19-parameter family of l^9-dimensional representations of $U_{\varepsilon^2}(so(8))$.

6.6. There are also a finite number of irreducible representations of $\mathscr{B}_\varepsilon^{\mathrm{sym}}$ of dimension $< l$, but these are not easily described in terms of quasi-polynomial algebras.

Proposition. *$\mathscr{B}_\varepsilon^{\mathrm{sym}}$ has five irreducible representations of dimension d for $1 \leqq d < l$; the action of u, v, with respect to a suitable basis, is described as follows*:

(a)

$$u = \operatorname{diag}(a_0, \ldots, a_{d-1}),$$
$$v_{i,i+1} = m_i, \quad i = 0, 1, \ldots, d-2,$$
$$v_{i+1,i} = 1, \quad i = 0, 1, \ldots, d-2,$$
$$v_{ij} = 0 \quad \text{if } j \neq i-1, i+1,$$

where,

$$a_k = \frac{\{\varepsilon^{2k-d+1}\}}{\varepsilon^2 - \varepsilon^{-2}},$$

and

$$m_k = \frac{1}{(\varepsilon^2 - \varepsilon^{-2})^2}\left(1 - \frac{[d+1]_\varepsilon [d-1]_\varepsilon}{[2k-d+3]_\varepsilon [2k-d+1]_\varepsilon}\right).$$

(b)

$$u = \operatorname{diag}(a_0, \ldots, a_{d-1}),$$
$$v_{i,i+1} = m_i, \quad i = 0, 1, \ldots, d-2,$$
$$v_{i+1,i} = 1, \quad i = 1, 2, \ldots, d-2,$$
$$v_{00} = m,$$
$$v_{ij} = 0 \quad \text{if } j \neq i-1, i+1, (i,j) \neq (0,0),$$

where

$$a_k = \pm \frac{\{\varepsilon^{2k+1}\}}{\varepsilon^2 - \varepsilon^{-2}},$$

$$m_k = \frac{1}{(\varepsilon^2 - \varepsilon^{-2})^2}\left(1 - \frac{[2d-1]_\varepsilon [2d+1]_\varepsilon}{[2k+3]_\varepsilon [2k+1]_\varepsilon}\right),$$

and

$$m^2 = \frac{1}{(\varepsilon - \varepsilon^{-1})^2} - \frac{1}{(\varepsilon - \varepsilon^{-1})(\varepsilon^2 - \varepsilon^{-2})^2}\left(1 - \frac{[2d+1]_\varepsilon [2d-1]_\varepsilon}{[3]_\varepsilon}\right).$$

We conjecture that the representations described in Propositions 6.5 and 6.6 exhaust the irreducible representations of $\mathscr{B}_\varepsilon^{\mathrm{sym}}$ (up to isomorphism). This is true under the assumption that u (say) is semisimple with distinct eigenvalues.

We hope to take up the more general problem of the representation theory of $\mathscr{B}_\varepsilon$ in a subsequent paper.

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