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Irreducible Representations of Virasoro-Toroidal Lie Algebras

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To A. John Coleman on the occasion of his 75th birthday

Abstract: Toroidal Lie algebras and their vertex operator representations were introduced in [MEY] and a class of indecomposable modules were investigated. In this work, we extend the toroidal algebra by the Virasoro algebra thus constructing a semi-direct product algebra containing the toroidal algebra as an ideal and the Virasoro algebra as a subalgebra. With the use of vertex operators and certain oscillator representations of the Virasoro algebra it is proved that the corresponding Fock space gives rise to a class of irreducible modules for the Virasoro-toroidal algebra.

Introduction

Toroidal algebras $\mathfrak{t}_{[n]}$ are defined for every $n \geqq 1$ and when $n = 1$ they are precisely the untwisted affine algebras. Such an affine algebra $\mathfrak{g}$ can be realized as the universal covering algebra of the loop algebra $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ where $\dot{\mathfrak{g}}$ is a simple finite dimensional Lie algebra over $\mathbb{C}$. It is well known that $\mathfrak{g}$ is a one-dimensional central extension of $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. The toroidal algebras $\mathfrak{t}_{[n]}$ are the universal covering algebras of iterated loop algebras $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}]$ which, for $n \geqq 2$, turn out to be infinite-dimensional central extensions.

Unlike the finite dimensional case, there is a distinguished irreducible highest weight module for any untwisted (or direct) affine Lie algebra. This is the *basic representation.* In 1980 Frenkel and Kac [FK] gave a remarkable construction of the basic representation by using vertex operators $X(\alpha, z)$, where α runs over the root lattice $\dot{Q}$ of $\dot{\mathfrak{g}}$. Already in [FK] it was observed that the Virasoro algebra also operates on the basic representation and in particular the (energy) operator d_0 plays a distinguished role.

A decade later, the vertex operators $X(\alpha, z)$, where α now lies in the affine root lattice, $Q = \dot{Q} \oplus \mathbb{Z}\delta$, were used to produce indecomposable representations of the toroidal algebras $\mathfrak{t}_{[2]}$ [MEY]. Soon after, these results were shown [EM] to extend to arbitrary n.

However these representations are not completely reducible, nor do irreducible representations appear in a natural way in the picture. The objective of this paper is to show how one can greatly improve the situation by enlarging $\mathfrak{t}_{[2]}$. The key point is that our representations, as in the affine case, naturally afford representations of the Virasoro algebra $\mathfrak{Vir}$ too. We thus extend $\mathfrak{t}_{[2]}$ to $\tilde{\mathfrak{t}}_{[2]} := \mathfrak{Vir} \rtimes \mathfrak{t}_{[2]}$.

The vertex representations of $\mathfrak{t}_{[2]}$ constructed in [MEY] arise from a canonical representation of a degenerate Heisenberg algebra $\mathfrak{a}(Q)$ whose centre is infinite dimensional. We will embed Q in a nondegenerate lattice Γ and form the larger Heisenberg (oscillator) algebra $\mathfrak{a}(\Gamma)$. The representations of $\mathfrak{Vir}$ that we will use are the oscillator representations corresponding to $\mathfrak{a}(\Gamma)$. Thus, to any generator d_k, $k \in \mathbb{Z}$, we associate the infinite normally ordered quadratic expression $L_k = \frac{1}{2}\sum_{j\in\mathbb{Z}} \sum_{i=1}^{l+2} : u_i(-j)u_i(j+k)$: where $\{u_i\}_{i=1}^{l+2}$ is an orthonormal basis for $\mathfrak{k} = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$.

The $\tilde{\mathfrak{t}}_{[2]}$-module studied here is the Fock space $V(\Gamma)$ associated with the lattice Γ. It is the tensor product $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$ of a twisted group algebra $\mathbb{C}(\Gamma)$ and the symmetric algebra $S(\mathfrak{a}(\Gamma)_-)$. As a $\mathbb{C}$-space, $V(\Gamma)$ decomposes into a direct sum $\coprod_{m\in\mathbb{Z}} K(m)$. We will show that if $m \neq 0$, $K(m)$ is an irreducible $\tilde{\mathfrak{t}}_{[2]}$-submodule of $V(\Gamma)$ and $K(m) \simeq K(m')$ if and only if $m = m'$. The submodule $K(0)$ is not irreducible. In a forthcoming paper, [F1], the submodule structure of $K(0)$ is investigated.

1. The Heisenberg Algebras $\mathfrak{a}(L)$ and the Canonical Representation

Let $(L, (\cdot|\cdot))$ be a (geometric) lattice, that is, a free $\mathbb{Z}$-module L of finite rank together with a nontrivial symmetric $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$: $L \times L \to \mathbb{Z}$. Let $\mathfrak{l} := \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend $(\cdot|\cdot)$ to a symmetric $\mathbb{C}$-bilinear form (also denoted $(\cdot|\cdot)$) on $\mathfrak{l}$. We call the lattice L nondegenerate if $(\cdot|\cdot)$ is nondegenerate on $\mathfrak{l}$. Let $\mathfrak{l}(n)$ be an isomorphic copy of $\mathfrak{l}$ for every $n \in \mathbb{Z}$ under the correspondence $a(n) \leftrightarrow a$, $a \in \mathfrak{l}$.

Form the Heisenberg algebra $\mathfrak{a}(L) := (\coprod_{n\in\mathbb{Z}} \mathfrak{l}(n)) \oplus \mathbb{C}\cent$, where $\cent$ is some symbol, with multiplication $[\cdot, \cdot]$ on $\mathfrak{a}(L)$ defined by $[a(n), b(m)] := (a|b)n\delta_{n+m,0}\cent$, for all $a, b \in \mathfrak{l}$, $n, m \in \mathbb{Z}$, and $\cent$ is central. $\mathfrak{a}(L)$ is graded with $\deg a(n) := -n$ and by $\cent = 0$. Observe that $\mathfrak{l}(0)$ is an abelian subalgebra of $\mathfrak{a}(L)$ and its complement $\mathring{\mathfrak{a}}(L) := (\coprod_{n\in\mathbb{Z}\setminus\{0\}} \mathfrak{l}(n)) \oplus \mathbb{C}\cent$ is a subalgebra of $\mathfrak{a}(L)$ satisfying $\mathfrak{a}(L) = \mathring{\mathfrak{a}}(L) \times \mathfrak{l}(0)$, where $\times$ denotes the direct product of Lie algebras. One easily proves

Proposition 1. $\text{centre } \mathfrak{a}(L) = \mathfrak{l}(0) \oplus \mathbb{C}\cent \oplus \left(\coprod_{\substack{n\in\mathbb{Z}\setminus\{0\} \\ \gamma\in\text{rad}(\cdot|\cdot)}} \mathbb{C}\gamma(n) \right)$. □

The most famous examples occur when $\dot{Q}$ is a lattice of type ADE, that is, $\dot{Q}$ is of type A_l, D_l or E_l, $(l = 6, 7, 8)$. $\mathfrak{a}(\dot{Q})$ is a Heisenberg algebra with $\dim_{\mathbb{C}}[\text{centre}(\mathfrak{a}(\dot{Q}))] = l + 1$. Another set of examples occurs when $Q = \dot{Q} \oplus \mathbb{Z}\delta$, where $(\dot{Q}|\delta) = 0 = (\delta|\delta)$. Note that Q is a degenerate lattice and the Heisenberg algebra $\mathfrak{a}(Q)$ has centre $[\mathfrak{a}(Q)] = \mathfrak{h}(0) \oplus \mathbb{C}\cent \oplus (\coprod_{n\in\mathbb{Z}\setminus\{0\}} \mathbb{C}\delta(n))$, where $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} Q$. We call $\mathfrak{a}(Q)$ a *degenerate Heisenberg algebra* since the associated skew-symmetric bilinear form ψ: $\mathfrak{a}(Q) \times \mathfrak{a}(Q) \to \mathbb{C}$ given by $\psi(a(k), b(l)) := k\delta_{k+l,0}(a|b)$ has nontrivial radical elements in the homogeneous subspaces of non-zero degree.

We recall the canonical Fock space representation of $\mathfrak{a}(L)$. Let $\mathfrak{a}(L)_- := \coprod_{n<0} \mathfrak{l}(n)$ and let $S(\mathfrak{a}(L)_-)$ be the corresponding symmetric algebra.

Define an action of $\mathring{\mathfrak{a}}(L)$ on $S(\mathfrak{a}(L)_-)$: for $n, m > 0$, $a, b \in \mathfrak{l}$, and $f \in S(\mathfrak{a}(L)_-)$

$$\begin{cases} \mathfrak{c} \cdot f = f \\ a(-n) \cdot f = L_{a(-n)} f \\ a(n) \cdot f = \partial_{a(n)} f, \end{cases} \tag{1}$$

where $L_{a(-n)} f = a(-n) f$ is the left multiplication operator and $\partial_{a(n)}$ is the unique derivation of $S(\mathfrak{a}(L)_-)$ satisfying

$$\partial_{a(n)}(b(-m)) = n\delta_{m,n}(a|b) .$$

Proposition 2. *$S(\mathfrak{a}(L)_-)$ is an $\mathring{\mathfrak{a}}(L)$-module and the following are equivalent:*
(i) *$S(\mathfrak{a}(L)_-)$ is an irreducible $\mathring{\mathfrak{a}}(L)$-module.*
(ii) *L is nondegenerate.*
(iii) *$S(\mathfrak{a}(L)_-)$ is a faithful $\mathring{\mathfrak{a}}(L)$-module.*

□

Let M be any nondegenerate lattice containing L. One may choose $M = L$ if L is already nondegenerate. Put $\mathfrak{m} := \mathbb{C} \otimes_{\mathbb{Z}} M$ and fix $\lambda \in \mathfrak{m}$. Let $\mathbb{C}e^{\lambda}$ be the one-dimensional space spanned by the symbol e^{λ}. Consider the $\mathbb{C}$-space

$$V_L(\lambda) := \mathbb{C}e^{\lambda} \otimes_{\mathbb{C}} S(\mathfrak{a}(L)_-) . \tag{2}$$

Of course, as $\mathbb{C}$-spaces, we have $V_L(\lambda) \cong S(\mathfrak{a}(L)_-)$. We make $V_L(\lambda)$ into an $\mathfrak{a}(L)$-module by extending (1) as follows:

$$\begin{cases} \mathfrak{c} \cdot (e^{\lambda} \otimes f) = e^{\lambda} \otimes \mathfrak{c} \cdot f = e^{\lambda} \otimes f, \\ a(-n) \cdot (e^{\lambda} \otimes f) = e^{\lambda} \otimes L_{a(-n)} f, \\ a(n) \cdot (e^{\lambda} \otimes f) = e^{\lambda} \otimes \partial_{a(n)} f, \\ a(0) \cdot (e^{\lambda} \otimes f) = (a|\lambda)(e^{\lambda} \otimes f) . \end{cases} \tag{3}$$

Note that $V_L(\lambda)$ is an irreducible $\mathfrak{a}(L)$-module if and only if L is a nondegenerate lattice but that $V_L(\lambda)$ is never a faithful $\mathfrak{a}(L)$-module.

2. Toroidal Algebras

Let $\dot{\mathfrak{g}}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$. Let A be any commutative algebra with unity over $\mathbb{C}$. Consider the Lie algebra $\mathfrak{g}_A := \dot{\mathfrak{g}} \otimes_{\mathbb{C}} A$ with bracket $[x \otimes a, y \otimes b] = [x, y] \otimes ab$, $x, y \in \dot{\mathfrak{g}}$ and $a, b \in A$. The structure of the universal covering algebra of $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} A$ has been worked out in [Ka].

Let Ω_A be the A-module of differentials and $d: A \to \Omega_A$ the differential map. Thus d is linear and satisfies $d(ab) = a \cdot db + b \cdot da$. Let $\overline{} : \Omega_A \to \Omega_A/dA$ be the canonical map. Then for $a, b \in A$ we have $\overline{d(ab)} = 0$.

Theorem [Ka, and Kac, Ex. 7.9]. *The Lie algebra $\mathfrak{g} := (\dot{\mathfrak{g}} \otimes_{\mathbb{C}} A) \oplus \Omega_A/dA$ with multiplication defined by*

$$\begin{cases} [x \otimes a, y \otimes b] := [x, y] \otimes ab + (x|y)\overline{(da)b} \\ \Omega_A/dA \ \ \text{central} \end{cases} \tag{4}$$

is the universal covering algebra of $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} A$. (Here $(\cdot|\cdot)$ *denotes the Killing form on* $\dot{\mathfrak{g}}$.)

□

When $A = \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}]$ we denote the algebra $\mathfrak{g}$ simply by $\mathfrak{t}_{[n]}$ and we call it the *toroidal algebra*. Consider the case $n = 2$ so that $A = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$. Then it is easy to check that a $\mathbb{C}$-basis for Ω_A/dA is given (see [MEY]) by

$$\begin{cases} a(p, q) := \overline{s^{p-1}t^q dt}\ , & (p, q) \in \mathbb{Z} \times \mathbb{Z} \backslash \{0\}\ , \\ a(p, 0) := \overline{s^p t^{-1} dt}\ , & p \in \mathbb{Z}\ , \\ a(0, 0) := \overline{s^{-1} ds}\ . \end{cases} \tag{5}$$

Next, let $\dot{\mathfrak{h}}$ be a fixed Cartan subalgebra of $\dot{\mathfrak{g}}$ and consider the subalgebra $\mathfrak{b}$ of $\mathfrak{t}_{[2]}$ generated by the subspace $\dot{\mathfrak{h}} \otimes_{\mathbb{C}} \mathbb{C}[s, s^{-1}]$. Using (4), we have for $h, h' \in \dot{\mathfrak{h}}$ and $n, m \in \mathbb{Z}$, $[h \otimes s^m,\ h' \otimes s^n] = [h, h'] \otimes s^{n+m} + (h|h')\overline{(ds^m)s^n} = (h|h')m\delta_{m+n,0}\overline{s^{-1}ds}$, and hence $\mathfrak{b}$ can be identified as the Heisenberg algebra $\mathfrak{a}(\dot{Q})$ under the correspondences $h \otimes s^n \leftrightarrow h(n)$ and $\overline{s^{-1}ds} \leftrightarrow c\!\!\!/$.

The subalgebra $\mathfrak{e} := \mathfrak{b} \oplus (\coprod_{p \in \mathbb{Z}\backslash\{0\}} \mathbb{C}a(p, 0))$ of $\mathfrak{t}_{[2]}$ can be identified as the Heisenberg algebra $\mathfrak{a}(Q)$, where $Q = \dot{Q} \oplus \mathbb{Z}\delta$ as in Sect. 1 under the above correspondences together with $a(p, 0) \leftrightarrow \delta(p)$, $p \in \mathbb{Z}$. The Heisenberg algebras $\mathfrak{a}(\dot{Q})$ and $\mathfrak{a}(Q)$ and their representations will play a central role in the sequel.

3. Vertex Representations of Toroidal Algebras

Let $\dot{Q}$ be of type ADE, and let $\Gamma = \dot{Q} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\mu = Q \oplus \mathbb{Z}\mu$, where $(\dot{Q}|\mu) = 0 = (\mu|\mu)$ and $(\delta|\mu) = 1$. Following [EMY] we let $\varepsilon: Q \times Q \to \{\pm 1\}$ be a bimultiplicative map satisfying

$$\begin{cases} \textbf{CC(i)} & \varepsilon(\alpha, \alpha) = (-1)^{(\alpha|\alpha)/2}\ , \\ \textbf{CC(ii)} & \varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)}\ , \\ \textbf{CC(iii)} & \varepsilon(\alpha, \delta) = 1\ , \end{cases} \tag{6}$$

where $\alpha, \beta \in Q$. Extend ε to a bimultiplicative map $\varepsilon: Q \times \Gamma \to \{\pm 1\}$. For $\gamma \in \Gamma$ let e^γ be a symbol and form the vector space $\mathbb{C}[\Gamma]$ with $\mathbb{C}$-basis $\{e^\gamma : \gamma \in \Gamma\}$. Then $\mathbb{C}[\Gamma]$ contains the subspace $\mathbb{C}[Q] := \coprod_{\gamma \in Q} \mathbb{C}e^\gamma$. We give $\mathbb{C}[Q]$ a twisted group algebra structure by defining $e^\alpha e^\beta := \varepsilon(\alpha, \beta)e^{\alpha+\beta}$, $\alpha, \beta \in Q$. Then $\mathbb{C}[\Gamma]$ becomes a $\mathbb{C}[Q]$-module in such a way that $e^\alpha e^\gamma := \varepsilon(\alpha, \gamma)e^{\alpha+\gamma}$, $\alpha \in Q$, $\gamma \in \Gamma$. Now form the *full Fock space*

$$V(\Gamma) := \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)\ . \tag{7}$$

Note that, as $\mathbb{C}$-spaces, $V(\Gamma) = \coprod_{\lambda \in \Gamma} V_\Gamma(\lambda)$, where $V_\Gamma(\lambda) := \mathbb{C}e^\lambda \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$.

Let z be a complex variable and $\alpha \in Q$. Define

$$T_{\pm}(\alpha, z) := -\sum_{n \gtrless 0} \frac{1}{n}\alpha(n)z^{-n}\ . \tag{8}$$

Then the *vertex operator*, $X(\alpha, z)$, for α on $V(\Gamma)$ is defined by

$$\begin{cases} X(\alpha, z) := z^{(\alpha|\alpha)/2} \exp T(\alpha, z), \text{ where} \\ \exp T(\alpha, z) := \exp T_-(\alpha, z) e^\alpha z^{\alpha(0)} \exp T_+(\alpha, z), \text{ and} \\ z^{\alpha(0)}(e^\lambda \otimes f) := z^{(\alpha|\lambda)}(e^\lambda \otimes f), \quad f \in S(\mathfrak{a}(\Gamma)_-)\ . \end{cases} \tag{9}$$

The $X(\alpha, z)$ can be formally expanded in powers of z to give

$$X(\alpha, z) = \sum_{n \in \mathbb{Z}} X_n(\alpha) z^{-n}$$

and the coefficients $X_n(\alpha)$ are called *moments*. The $X_n(\alpha)$ are operators on $V(\Gamma)$ and for any $f \in S(\mathfrak{a}(\Gamma)_-)$ and $\lambda \in \Gamma$, one has $X_n(\alpha)(e^\lambda \otimes f) = e^{\lambda+\alpha} \otimes f'$, where $f' \in S(\mathfrak{a}(\Gamma)_-)$. Thus, in the decomposition of the full Fock space $V(\Gamma) = \coprod_{\lambda \in \Gamma} V_\Gamma(\lambda)$ one can view the moments $X_n(\alpha)$ as operators which move an element in the "λ-stalk" $V_\Gamma(\lambda)$ to an element in the "$(\lambda + \alpha)$-stalk" $V_\Gamma(\lambda + \alpha)$.

The determination of the commutation relations of the moments can be made by standard techniques of contour integration [GO, MP] and yields

CR.0 $[\alpha(k), X_n(\beta)] = (\alpha|\beta) X_{n+k}(\beta)$.
CR.1 $[X_m(\alpha), X_n(\beta)] = 0, \ (\alpha|\beta) \geqq 0$.
CR.2 $[X_m(\alpha), X_n(\beta)] = \varepsilon(\alpha, \beta) X_{n+m}(\alpha + \beta), \ (\alpha|\beta) = -1$.
CR.3 If $(\alpha|\alpha) = (\beta|\beta) = -(\alpha|\beta) = 2$, then

$$[X_m(\alpha), X_n(\beta)] = \varepsilon(\alpha, \beta)\{m X_{n+m}(\alpha + \beta) + \sum_{k \in \mathbb{Z}} : \alpha(k) X_{m+n-k}(\alpha + \beta):\} ,$$

$$\text{where}: \alpha(k) X_{m+n-k}(\beta) ::= \begin{cases} \alpha(k) X_{m+n-k}(\beta) & \text{if } k \leqq m+n-k \\ X_{m+n-k}(\beta)\alpha(k) & \text{if } k > m+n-k . \end{cases}$$

Next we will state a result from [MEY] which gives vertex representations of the toroidal Lie algebra $\mathfrak{t}_{[2]}$. We fix a simple finite dimensional Lie algebra $\dot{\mathfrak{g}}$ of type ADE with Cartan subalgebra $\dot{\mathfrak{h}}$, root lattice $\dot{Q}$, root system $\dot{\Delta}$ and basis of simple roots $\{\alpha_1, \ldots, \alpha_l\}$. We assume that $\{e_{\pm\alpha_i}\}$, $\{\alpha_i\}$ is a Chevalley basis of $\dot{\mathfrak{g}}$ (we identify $\dot{\mathfrak{h}}$ with $\dot{\mathfrak{h}}^*$ by the Killing form as usual) so that $e_{\pm\alpha_i} \in \dot{\mathfrak{g}}^{\pm\alpha_i}$ and $[e_{\alpha_i}, e_{-\alpha_i}] = -\alpha_i$.

Now in $\mathfrak{t}_{[2]}$ we can identify an affine algebra $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[s, s^{-1}] \oplus \mathbb{C}\not{c}$. Its root system is denoted Δ, its root lattice Q and its set of real roots Δ^{re}.

Proposition 3. *Let $\mathfrak{s}$ be the Lie algebra of operators on $V(\Gamma)$ generated by the moments $X_m(\alpha)$, $\alpha \in \Delta^{\text{re}}$, $m \in \mathbb{Z}$. Then $\mathfrak{s}$ is isomorphic to $\mathfrak{t}_{[2]}$ under the assignment*

$$e_{\pm\alpha_i} \otimes \pm s^m t^n \mapsto X_m(\pm \alpha_i + n\delta), \quad n, m \in \mathbb{Z}, \ 1 \leqq i \leqq l . \tag{10}$$

□

Now for $\alpha \in \Gamma$ define the *elementary Schur polynomials* $S_r(\alpha)$, $r \in \mathbb{Z}$ by the expressions

$$\begin{cases} \exp T_-(\alpha, z) =: \sum_{r=0}^{\infty} S_r(\alpha) z^r \\ S_r(\alpha) := 0, \quad r < 0 , \end{cases} \tag{11}$$

where $T_-(\alpha, z)$ is defined in (8). As in [MEY], we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} X_k(\varphi) z^{-k} (e^\lambda \otimes 1) &= X(\varphi, z)(e^\lambda \otimes 1) \\ &= z^{(\varphi|\varphi)/2} \exp T_-(\varphi, z) e^\varphi z^{\varphi(0)} (e^\lambda \otimes 1) \\ &= \sum_{r=0}^{\infty} (e^{\lambda+\varphi} \otimes S_r(\varphi)) z^{r + (\varphi|\lambda) + \frac{(\varphi|\varphi)}{2}} . \end{aligned}$$

Matching powers of z we get

$$X_k(\varphi)(e^\lambda \otimes 1) = \varepsilon(\varphi, \lambda) e^{\lambda+\varphi} \otimes S_{-k-(\varphi|\lambda+\frac{\varphi}{2})}(\varphi) . \tag{12}$$

4. The Virasoro–Heisenberg and Virasoro-Toroidal Algebras

In this section we introduce the Virasoro–Heisenberg and Virasoro-toroidal algebras. We first define a representation by derivations of the Virasoro algebra on both the Heisenberg algebra $\mathfrak{a}(Q)$ and on the toroidal algebra $\mathfrak{t}_{[2]}$ and then form the corresponding semi-direct product Lie algebras. The representations used here can be identified as certain copies of the well-known module of tensor fields [FF, K].

Recall that the Virasoro algebra, denoted $\mathfrak{Vir}$, is the infinite dimensional Lie algebra with generators $\{d_k, \ k \in \mathbb{Z}\}$ and relations $[d_k, d_l] = (k-l)d_{k+l} + \frac{1}{12}\delta_{k+l,0}(k^3 - k)z$, where z is a central symbol. Define an action of $\mathfrak{Vir}$ on $\mathfrak{a}(Q)$ in such a way that z acts trivially and for $k \in \mathbb{Z}$ and d_k is the unique derivation satisfying

$$d_k \cdot a(n) = -na(n+k) . \tag{13}$$

Define an action of $\mathfrak{Vir}$ on $\mathfrak{t}_{[2]}$ in such a way that z acts trivially and, for $k \in \mathbb{Z}$, d_k acts as the unique derivation satisfying

$$d_k \cdot (e_{\pm\alpha_i} \otimes s^m t^n) := \left\{\frac{k}{2}(\alpha_i|\alpha_i) - (m+k)\right\}(e_{\pm\alpha_i} \otimes s^{m+k} t^n) , \tag{14}$$

where $n, m \in \mathbb{Z}$ and $1 \leqq i \leqq l$.

One can verify directly that (13) and (14) do determine representations of $\mathfrak{Vir}$ on $\mathfrak{a}(Q)$ and $\mathfrak{t}_{[2]}$ respectively. As $\mathbb{C}$-spaces, define

$$\tilde{\mathfrak{a}} := \mathfrak{Vir} \oplus \mathfrak{a}(Q) \quad \text{and} \quad \tilde{\mathfrak{t}}_{[2]} := \mathfrak{Vir} \oplus \mathfrak{t}_{[2]} .$$

We make $\tilde{\mathfrak{a}}$ (resp. $\tilde{\mathfrak{t}}_{[2]}$) into a Lie algebra in such a way that $\mathfrak{Vir}$ is a subalgebra and $\mathfrak{a}(Q)$ (resp. $\mathfrak{t}_{[2]}$) is an ideal via (13) (resp. (14)):

$$\begin{cases} [d_k, a(n)] := d_k \cdot a(n) \\ [d_k, e_{\pm\alpha_i} \otimes s^m t^n] := d_k \cdot (e_{\pm\alpha_i} \otimes s^m t^n) . \end{cases}$$

We call $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{t}}_{[2]}$ the *Virasoro–Heisenberg* and *Virasoro-toroidal algebras* respectively.

5. Oscillator Representations of the Virasoro Algebra

A very interesting class of representations of $\mathfrak{Vir}$ are the so-called *oscillator representations*. The operators used arise from the Fourier components of the energy-momentum tensor in quantum field theory and can be expressed in terms of the canonical representation of a corresponding representation of the Heisenberg (oscillator) algebra $\mathfrak{a}(L)$.

Let L be an arbitrary nondegenerate lattice of rank l, $\mathfrak{l} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and $\{a_i\}_{i=1}^{l}$ an orthonormal basis for $\mathfrak{l}$. Consider the normally ordered sums

$$L_k := \frac{1}{2} \sum_{i=1}^{l} \sum_{j \in \mathbb{Z}} : a_i(-j) a_i(j+k) : , \tag{15}$$

where $k \in \mathbb{Z}$ and for all $n \in \mathbb{Z}$, $a_i(n) \in \mathfrak{a}(L)$. *The normal ordering* defined by

$$: a_i(r) a_i(s) : := \begin{cases} a_i(r) a_i(s) , & r \leqq s \\ a_i(s) a_i(r) , & r > s \end{cases}$$

ensures that only a finite number of the terms in L_k act nontrivially and hence the L_k make sense as operators on $V_L(\lambda)$. A proof along the same lines as in [KR] gives

Proposition 4. *The assignment $d_k \mapsto L_k$, $k \in \mathbb{Z}$ and $z \mapsto lI$ defines a representation of $\mathfrak{Vir}$ on $V_L(\lambda)$.*

□

Let $\dot{Q}$ be of type ADE and $\Gamma = \dot{Q} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\mu$ as before. Let $\{u_i\}_{i=1}^{l}$ be an orthonormal basis for $\dot{\mathfrak{h}} := \mathbb{C} \otimes_{\mathbb{Z}} \dot{Q}$. Let $u_{l+1} := \frac{\delta}{2} + \mu$ and $u_{l+2} := \sqrt{-1}\left(\frac{\delta}{2} - \mu\right)$. Then $\{u_i\}_{i=1}^{l+2}$ is an orthonormal basis for $\mathfrak{k} := \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$. Applying Proposition 4 with $\Gamma = L$ we obtain a representation of $\mathfrak{Vir}$ on $V_\Gamma(\lambda)$, $\lambda \in \Gamma$, with the centre z acting as multiplication by $l + 2$. Since $V(\Gamma) = \coprod_{\lambda \in \Gamma} V_\Gamma(\lambda)$ we can at once extend the representation of $\mathfrak{Vir}$ to all of $V(\Gamma)$.

Proposition 5. (i) $[L_k, a(n)] = -na(n+k)$, $a \in Q$, $n, k \in \mathbb{Z}$, *and hence $V_\Gamma(\lambda)$ is an $\tilde{\mathfrak{a}}$-module.*
(ii) $[L_k, X_m(\alpha)] = \left\{\frac{k}{2}(\alpha|\alpha) - (m+k)\right\} X_{m+k}(\alpha)$, $m, k \in \mathbb{Z}$, $\alpha \in Q$, *and hence $V(\Gamma)$ is a $\tilde{\mathfrak{t}}_{[2]}$-module.*

Proof. (i) follows by a standard calculation [KR] and (ii) follows easily from the well-known commutation relation

CR.4 $$[L_k, X(\alpha, z)] = z^k \left\{\frac{k}{2}(\alpha|\alpha) + z\frac{d}{dz}\right\} X(\alpha, z)$$

whose proof can be found in [KF] or [GO]. □

6. Representations of the Virasoro–Heisenberg Algebra

The objective of this section is to study the structure of the $\tilde{\mathfrak{a}}$-module $V_\Gamma(\lambda)$ which we simply denote by $V(\lambda)$. We begin this section by pointing out that the Lie algebra $\tilde{\mathfrak{a}}$ admits a triangular decomposition in the sense of [MP]. Indeed, define $\tilde{\mathfrak{a}}^n$ to be $\mathbb{C}d_n + \mathfrak{h}(n)$, $n \neq 0$, and $\tilde{\mathfrak{a}}_0 := \tilde{\mathfrak{a}}^0$ to be the linear span of $\{d_0, \mathfrak{h}(0), \mathfrak{c}, z\}$. Define $\tilde{\mathfrak{a}}_\pm := \coprod_{n \gtrless 0} \tilde{\mathfrak{a}}^n$. Then $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_- \oplus \tilde{\mathfrak{a}}_0 \oplus \tilde{\mathfrak{a}}_+$ provides a triangular decomposition with root spaces $\tilde{\mathfrak{a}}^n$ determined by the eigenfunctions $n\phi$: $\tilde{\mathfrak{a}}_0 \to \mathbb{C}$ with $\langle \phi, d_0 \rangle = -1$ and $\phi|_{\mathfrak{h}(0) \oplus \mathbb{C}\mathfrak{c} \oplus \mathbb{C}z} = 0$ and with anti-linear anti-involution $\tilde{\sigma}: \tilde{\mathfrak{a}} \to \tilde{\mathfrak{a}}$ defined by $\tilde{\sigma}(d_n) := d_{-n}$, $\tilde{\sigma}(a(n)) := a(-n)$, $\tilde{\sigma}(z) := z$ and $\tilde{\sigma}(\mathfrak{c}) := \mathfrak{c}$ where $n \in \mathbb{Z}$ and $a \in \mathfrak{h}$.

Moreover, we note that centre $[\tilde{\mathfrak{a}}] = \mathfrak{h}(0) \oplus C\mathfrak{c} \oplus \mathbb{C}z$ and hence $\dim_{\mathbb{C}}(\text{centre}\,[\tilde{\mathfrak{a}}]) = l + 3$ where $l = \operatorname{rank} \dot{Q}$.

Now let $\mathfrak{Vir}_+ := \coprod_{n>0} \mathbb{C}d_n$ and introduce the subalgebra

$$\hat{\mathfrak{a}} := \langle d_k, a(n) \colon a \in Q, n \in \mathbb{Z}, k > 0 \rangle \subset \tilde{\mathfrak{a}} ,$$

where the angular brackets denote the subalgebra generated by the enclosed symbols. Observe that $\hat{\mathfrak{a}} = \mathfrak{Vir}_+ \ltimes \mathfrak{a}(Q)$ and we have the inclusion of algebras $\mathfrak{a} \subset \hat{\mathfrak{a}} \subset \tilde{\mathfrak{a}}$, where $\mathfrak{a} = \mathfrak{a}(Q)$.

Proposition 6. *Fix $\lambda \in \Gamma$ arbitrarily. Then $V_Q(\lambda) := \mathbb{C}e^\lambda \otimes S(\mathfrak{a}(Q)_-)$ is an $\hat{\mathfrak{a}}$-invariant subspace of $V(\lambda)$.*

Proof. Clearly $V_Q(\lambda)$ is $\mathfrak{a}(Q)$-invariant. Now using (13), extend the action of $\mathfrak{Vir}$ on $\mathfrak{a}(Q)$ to one on $S(\mathfrak{a}(Q))$ uniquely so that each d_k becomes a derivation and z acts trivially. Then, for any homogeneous polynomial $f \in S(\mathfrak{a}(Q)_-)$, it is clear that $d_0(f) = (\deg f)f$. To prove the proposition, it suffices to show that for any $n \geqq 0$,

$$d_n \cdot (e^\lambda \otimes f) = \left(\delta_{n,0} \frac{(\lambda|\lambda)}{2} f + d_n(f)\right) \cdot (e^\lambda \otimes 1)\,, \tag{16}$$

since the right side of this equation belongs to $V_Q(\lambda)$. Consider $n = 0$ first:

$$\begin{aligned} d_0 \cdot (e^\lambda \otimes f) &= \frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{i=1}^{l+2} :u_i(-j)u_i(j):(e^\lambda \otimes f) \\ &= \frac{1}{2} \sum_{i=1}^{l+2} u_i(0)u_i(0)(e^\lambda \otimes f) + \sum_{j>0} \sum_{i=1}^{l+2} u_i(-j)u_i(j)(e^\lambda \otimes f) \\ &= \left(\frac{(\lambda|\lambda)}{2} + \deg f\right)(e^\lambda \otimes f) \\ &= \left(\frac{(\lambda|\lambda)}{2} f + d_0(f)\right)(e^\lambda \otimes 1). \end{aligned}$$

As for $n > 0$, first note that $d_n \cdot (e^\lambda \otimes 1) = 0$ since every summand $:u_i(-j)u_i(j+n):$ appearing in the definition of L_n can be written $u_i(p)u_i(q)$, where $p \leqq 0$, $q > 0$ (after removing the normal ordering) and each of these terms kills $e^\lambda \otimes 1$. Thus

$$\begin{aligned} d_n \cdot (e^\lambda \otimes f) &= d_n \cdot f \cdot (e^\lambda \otimes 1) \\ &= f \cdot d_n \cdot (e^\lambda \otimes 1) + [d_n, f] \cdot (e^\lambda \otimes 1) \\ &= d_n(f) \cdot (e^\lambda \otimes 1)\,, \end{aligned}$$

as required. □

Before proceeding to the main result of this section we will need a preliminary definition. For $k > 0$ let $\mathscr{P}_k$ be the set of all partitions of k. We know that there is a one-to-one correspondence between the elements $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}_+^r$ with $m_1 \geqq m_2 \geqq \cdots \geqq m_r$, $\sum m_i = k$ and the monomials $\delta(-\mathbf{m}) := \delta(-m_1) \cdots \delta(-m_r)$ of degree k. Now we give $\mathscr{P}_k$ the lexicographical ordering as follows. For $\mathbf{m} = (m_1, \ldots, m_r)$, $\mathbf{n} = (n_1, \ldots, n_s) \in \mathscr{P}_k$, r, $s > 0$, we say $\mathbf{m} < \mathbf{n}$ if $m_i < n_i$ for the first i such that $m_i \neq n_i$. Clearly then the partition $(k) \in \mathscr{P}_k$ of length 1 is the unique maximal element with respect to this ordering and the partition $(1, \ldots, 1) \in \mathscr{P}_k$ of length k is the unique minimal element. In the next three results, we assume that all given tuples $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}_+^r$, $r > 0$, are ordered in such a way that $m_1 \geqq m_2 \geqq \cdots \geqq m_r$.

Proposition 7. *Fix $\lambda \in \Gamma \backslash Q$ and suppose $\mathbf{m} \in \mathbb{Z}_+^r$, $\mathbf{n} \in \mathbb{Z}_+^s$, where $\mathbf{n}, \mathbf{m} \in \mathscr{P}_k$ and $\mathbf{n} < \mathbf{m}$ in the lexicographical ordering. Consider the elements $e^\lambda \otimes \delta(-\mathbf{m})$, $e^\lambda \otimes \delta(-\mathbf{n}) \in V_Q(\lambda)$. Then $d_{\mathbf{m}} \cdot (e^\lambda \otimes \delta(-\mathbf{m})) \in \mathbb{C}^\times (e^\lambda \otimes 1)$ and $d_{\mathbf{m}} \cdot (e^\lambda \otimes \delta(-\mathbf{n})) = 0$, where $d_{\mathbf{m}} := d_{m_r} \cdots d_{m_1} \in \mathfrak{U}(\mathfrak{Vir}_+)$.*

Proof.

$$\begin{aligned} d_{\mathbf{m}}\cdot(e^{\lambda}\otimes\delta(-\mathbf{m})) &= d_{m_r}\cdots d_{m_1}\cdot(e^{\lambda}\otimes\delta(-m_1)\cdots\delta(-m_r)) \\ &= d_{q_t}^{p_t}\cdots d_{q_1}^{p_1}\cdot(e^{\lambda}\otimes\delta(-q_1)^{p_1}\cdots\delta(-q_t)^{p_t}), \\ &\quad \text{where } q_i > q_j \text{ for } i<j \text{ and } p_k>0,\ 1\leqq k\leqq t \\ &= (p_1!(m_{q_1})^{p_1}(\lambda|\delta)^{p_1})d_{q_t}^{p_t}\cdots d_{q_2}^{p_2}\cdot(e^{\lambda}\otimes\delta(-q_2)^{p_2}\cdots \\ &\quad \delta(-q_t)^{p_t}), \\ &\quad \text{by (16) and (13)} \\ &= \cdots = \left((p_1!\cdots p_t!)\prod_{i=1}^{t} m_{q_i}^{p_i}(\lambda|\delta)^{p_1+\cdots+p_t}\right)(e^{\lambda}\otimes 1) \\ &\in \mathbb{C}^{\times}(e^{\lambda}\otimes 1), \\ &\quad \text{since } (\lambda|\delta)\neq 0. \end{aligned}$$

Similarly, $d_{\mathbf{m}}\cdot(e^{\lambda}\otimes\delta(-\mathbf{n}))=0$. □

Proposition 8. *Let* $\lambda\in\Gamma\backslash Q$. *Then* $V_Q(\lambda)=\mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}S(\mathfrak{a}(Q)_-)$ *is an irreducible* $\hat{\mathfrak{a}}$-*module.*

Proof. First note that $\lambda\in\Gamma\backslash Q$ is equivalent to the condition $(\lambda|\delta)\neq 0$.

Let W be a submodule of $V_Q(\lambda)$ and let $0\neq x\in W$. Write $x=e^{\lambda}\otimes\sum_{i=1}^{m}g_ih_i$, where $g_i\in S(\mathfrak{a}(\dot{Q})_-)$ are linearly independent and $h_i\in D=S(\sum_{m>0}\mathbb{C}\delta(-m))$. By Proposition 2, $S(\mathfrak{a}(\dot{Q})_-)$ is an irreducible $\mathfrak{a}(\dot{Q})$-module and hence by the Jacobson density theorem we can eliminate $g_2,\ldots,g_m$ and reduce g_1 to 1 with some operator from $\mathfrak{U}(\mathfrak{a}(\dot{Q}))$. Thus we can assume without loss of generality that $x=e^{\lambda}\otimes h$, where $h=\sum_{i=1}^{r}\alpha_i\delta(-\mathbf{m}_i)$, $\alpha_i\in\mathbb{C}^{\times}$, where $\mathbf{m}_1>\mathbf{m}_2>\cdots>\mathbf{m}_r$ in the lexicographical ordering. Finally, from Proposition 7, $d_{\mathbf{m}_1}\cdot(e^{\lambda}\otimes h)\in\mathbb{C}^{\times}(e^{\lambda}\otimes 1)\in W$. It follows that $W=V_Q(\lambda)$. □

Proposition 9. *Let* $\lambda\in\Gamma\backslash Q$. *Then* $V(\lambda)=\mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}S(\mathfrak{a}(\Gamma)_-)$ *is an irreducible* $\tilde{\mathfrak{a}}$-*module.*

Proof. Since $\Gamma=Q\oplus\mathbb{Z}\mu$ we can write $S(\mathfrak{a}(\Gamma)_-)=S(\mathfrak{a}(Q)_-)M$, where $M:=S(\coprod_{n>0}\mathbb{C}\mu(-n))$. Let W be a non-trivial $\tilde{\mathfrak{a}}$-submodule of $V(\lambda)$. Let $0\neq f\in W$ be arbitrary and write $f=\sum_{\mathbf{n}}e^{\lambda}\otimes f_{\mathbf{n}}\mu(-\mathbf{n})$, where $\mu(-\mathbf{n})=\mu(-n_1)\mu(-n_2)\cdots$ and $n_1\geqq n_2\geqq\cdots$ and $f_{\mathbf{n}}\in S(\mathfrak{a}(Q)_-)$. Then we can use the $\delta(n)$, $n>0$, to eliminate all terms but one and reduce to the case where $f=e^{\lambda}\otimes h\in W$, where $h\in S(\mathfrak{a}(Q)_-)$. But by Proposition 8, $V_Q(\lambda)$ is an irreducible $\hat{\mathfrak{a}}$-module. Thus $e^{\lambda}\otimes 1\in W$. Now write $M=\coprod_{n\geqq 0}M_n$, where M_n denotes the subspace of M spanned by elements of degree n. It suffices to show that for all n,

$$R:=\mathfrak{U}(\tilde{\mathfrak{a}})\cdot(e^{\lambda}\otimes 1)\supset\mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}S(\mathfrak{a}(Q)_-)M_n\,. \tag{17}$$

We will establish (17) by induction on the degree n. Since $M_0=\mathbb{C}$ and $M_1=\mathbb{C}\mu(-1)$, we can write $M=\mathbb{C}\oplus\mathbb{C}\mu(-1)\oplus(\coprod_{n=2}^{\infty}M_n)$. Since $\mathfrak{U}(\tilde{\mathfrak{a}})\cdot(e^{\lambda}\otimes 1)\supset\mathfrak{U}(\tilde{\mathfrak{a}}_-)\cdot(e^{\lambda}\otimes 1)\supset\mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}S(\mathfrak{a}(Q)_-)$, (17) is clear when $n=0$. For

$n = 1$ consider $d_{-1} \cdot (e^\lambda \otimes 1) = L_{-1}(e^\lambda \otimes 1) \in R$. By definition we have

$$\begin{aligned} L_{-1} \cdot (e^\lambda \otimes 1) &= \sum_{i=1}^{l+2} u_i(-1)u_i(0) \cdot (e^\lambda \otimes 1) \\ &= \sum_{i=1}^{l+2} u_i(-1)(\lambda|u_i) \cdot (e^\lambda \otimes 1) \\ &= e^\lambda \otimes \left(\sum_{i=1}^{l+2} (\lambda|u_i)u_i(-1) \right) \\ &= e^\lambda \otimes \lambda(-1)\,. \end{aligned}$$

Now writing $\lambda = \alpha + a\mu$, $\alpha \in Q$, $a \in \mathbb{C}^\times$, we have $L_{-1}(e^\lambda \otimes 1) = e^\lambda \otimes \alpha(-1) + a(e^\lambda \otimes \mu(-1)) \in R$. But $e^\lambda \otimes \alpha(-1) \in R$ from the case $n = 0$. Thus we have $e^\lambda \otimes \mu(-1) \in R$ and this shows (17) when $n = 1$.

Suppose then that (17) holds for all $0 \leqq n \leqq k - 1$. We call this the first induction hypothesis. We need to show $R \supset \mathbb{C}e^\lambda \otimes_{\mathbb{C}} M_k S(\mathfrak{a}(Q)_-)$. We prove this by induction on the lexicographical ordering defined on $\mathscr{P}_k$. We "anchor" at the top with the partition (k). That is, we will first show $e^\lambda \otimes \mu(-k) \in R$.

Recall that for $k > 0$,

$$\begin{aligned} L_{-k} &= \frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{i=1}^{l+2} : u_i(-j)u_i(j-k): \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} \left\{ \sum_{i=1}^{l} : u_i(-j)u_i(j-k): + :\delta(-j)\mu(j-k): \right. \\ &\quad \left. + :\mu(-j)\delta(j-k): \right\}. \end{aligned}$$

Note that in the expansion of $L_{-k}(e^\lambda \otimes 1) \in R$ the only $j \in \mathbb{Z}$ which contribute are $j = k, k-1, \ldots, \left[\frac{k+1}{2}\right]$, where $[x]$ denotes the largest integer less than or equal to x. We compute

$$\begin{aligned} L_{-k}(e^\lambda \otimes 1) &= \left(\sum_{i=1}^{l+2} u_i(-k)u_i(0) \right)(e^\lambda \otimes 1) \\ &\quad + \left\{ \left(\sum_{i=1}^{l} u_i(-k+1)u_i(-1) + \delta(-k+1)\mu(-1) \right. \right. \\ &\quad \left. + \mu(-k+1)\delta(-1) \right) + \cdots \\ &\quad + c \sum_{i=1}^{l} u_i\left(-\left[\frac{k+1}{2}\right]\right) u_i\left(\left[\frac{k+1}{2}\right] - k\right) \\ &\quad + c\delta\left(-\left[\frac{k+1}{2}\right]\right) \mu\left(\left[\frac{k+1}{2}\right] - k\right) \\ &\quad \left. + c\mu\left(-\left[\frac{k+1}{2}\right]\right) \delta\left(\left[\frac{k+1}{2}\right] - k\right) \right\} (e^\lambda \otimes 1) \\ &= (e^\lambda \otimes \lambda(-k)) + y\,, \end{aligned}$$

where $c = \frac{1}{2}$ or 1 depending on the parity of k, $y \in \mathbb{C}e^{\lambda} \otimes_{\mathbb{C}} (\coprod_{n<k} M_n S(\mathfrak{a}(Q)_-))$, and $y \in R$ by the first induction hypothesis. Thus $e^{\lambda} \otimes \mu(-k) \in R$ by writing $\lambda = \alpha + a\mu$ and arguing as we did earlier in the case $n = 1$.

Next we fix $\mathbf{m} = (m_1, \ldots, m_r) \in \mathscr{P}_k$ and assume that for every $\mathbf{n} \in \mathscr{P}_k$ satisfying $\mathbf{n} > \mathbf{m}$ we have $e^{\lambda} \otimes \mu(-\mathbf{n}) \in R$. We call this the second induction hypothesis. We need to show that $e^{\lambda} \otimes \mu(-\mathbf{m}) \in R$. Since $\sum_{i=2}^{r} m_i < k$, by the first induction hypothesis $e^{\lambda} \otimes \mu(-m_2) \cdots \mu(-m_r) \in R$. But then R also contains the element $x := L_{-m_1}(e^{\lambda} \otimes \mu(-m_2) \cdots \mu(-m_r))$. Now, in the sum defining L_{-m_1}, the only $j \in \mathbb{Z}$ which contribute in the calculation of x are $j \in \{m_1\} \cup \left\{m_1 - 1, \ldots, \left[\frac{m_1+1}{2}\right]\right\} \cup \{m_1 + m_2, \ldots, m_1 + m_r\}$. We calculate

$$\begin{aligned}
&L_{-m_1}(e^{\lambda} \otimes \mu(-m_2) \cdots \mu(-m_r)) \\
&= (e^{\lambda} \otimes \lambda(-m_1)\mu(-m_2) \cdots \mu(-m_r)) \\
&\quad + \left\{ \sum_{i=1}^{l} u_i(-m_1+1)u_i(-1) + \delta(-m_1+1)\mu(-1) + \mu(-m_1+1)\delta(-1) + \cdots \right. \\
&\quad + c\sum_{i=1}^{l} u_i\left(-\left[\frac{m_1+1}{2}\right]\right)u_i\left(\left[\frac{m_1+1}{2}\right] - m_1\right) \\
&\quad + c\delta\left(-\left[\frac{m_1+1}{2}\right]\right)\mu\left(\left[\frac{m_1+1}{2}\right] - m_1\right) \\
&\quad \left. + c\mu\left(-\left[\frac{m_1+1}{2}\right]\right)\delta\left(\left[\frac{m_1+1}{2}\right] - m_1\right)\right\}\cdot(e^{\lambda} \otimes 1) \\
&\quad + \sum_{i=2}^{r} m_i (e^{\lambda} \otimes \mu(-m_1 - m_i)\,\mu(-m_2) \cdots \overline{\mu(-m_i)} \cdots \mu(-m_r)),
\end{aligned}$$

where the overbar denotes omission and $c = \frac{1}{2}$ or 1, as before.

Let x_1 denote the sum in the brace brackets and x_2 the sum with the overbar. By the first induction hypothesis $x_1 \cdot (e^{\lambda} \otimes 1) \in R$ and since $(m_1 + m_i, m_2, \ldots, \overline{m_i}, \ldots, m_r) > (m_2, m_3, \ldots, m_r)$ for each $2 \leqq i \leqq r$, the second induction hypothesis implies $x_2 \in R$. Finally since the left side belongs to R we conclude that $e^{\lambda} \otimes \lambda(-m_1)\mu(-m_2) \cdots \mu(-m_r) \in R$. Expressing $\lambda = \alpha + a\mu$, $\alpha \in Q$, $a \in \mathbb{C}^{\times}$, the first induction hypothesis gives $e^{\lambda} \otimes \mu(-m_1) \cdots \mu(-m_r) \in R$ as required. This completes the proof of Proposition 9. □

Finally, we indicate how to identify $V(\lambda)$, $\lambda \in \Gamma \backslash Q$, as an irreducible highest weight module. Indeed, recall that $\tilde{\mathfrak{a}}$ admits a triangular decomposition $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_- \oplus \tilde{\mathfrak{a}}_0 \oplus \tilde{\mathfrak{a}}_+$. Let $\alpha \in (\tilde{\mathfrak{a}}_0)^*$ be defined by $\alpha(a(0)) = (\lambda | a)$ for all $a \in \mathfrak{h}$, $\alpha(\not{c}) = 1$, $\alpha(d_0) = \frac{(\lambda|\lambda)}{2}$, $\alpha(z) = l + 2$. Consider the Verma module $M(\alpha) = \mathfrak{U}(\tilde{\mathfrak{a}}) \otimes_{\mathfrak{U}(\mathfrak{b})} \mathbb{C}_{\alpha}$, where $\mathfrak{b} = \tilde{\mathfrak{a}}_0 \oplus \tilde{\mathfrak{a}}_+$ with unique irreducible quotient $L(\alpha)$.

Proposition 10. (i) $V(\lambda) \cong L(\alpha)$.
(ii) *If* $\lambda, \lambda' \in \Gamma \backslash Q$, *then* $V(\lambda) \cong V(\lambda')$ *if and only if* $\lambda \equiv \lambda'$.

Proof. (i) Since $e^{\lambda} \otimes 1$ is a highest weight vector for $\tilde{\mathfrak{a}}$ with weight α and since it generates the irreducible module $V(\lambda)$, $V(\lambda) \cong L(\alpha)$.

(ii) By [MP] Proposition 2.3.4, $L(\alpha)$ is uniquely determined by α and clearly $\lambda, \lambda' \in \Gamma \backslash Q$ determines the same α if and only if $\lambda = \lambda'$. □

7. Irreducible Representations of the Virasoro-Toroidal Algebras

In this section we show that the full Fock space $V(\Gamma) = \mathbb{C}(\Gamma) \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$ decomposes into a sum of subspaces $K(m)$, $m \in \mathbb{Z}$, and for $m \neq 0$, $K(m)$ is an irreducible $\tilde{\mathfrak{t}}_{[2]}$-submodule with $K(m) \simeq K(m')$ if and only if $m = m'$.

Note that, as $\mathbb{C}$-spaces, $V(\Gamma)$ is the direct sum of the $\mathbb{C}$-spaces $K(m) := \mathbb{C}[m\mu + Q] \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$. It is clear that each $K(m)$, $m \in \mathbb{Z}$, is a $\tilde{\mathfrak{t}}_{[2]}$-module. Suppose that $m \neq 0$, and hence $m\mu + Q \subset \Gamma \backslash Q$. We will need the following formula which is a special case of (12) in Sect. 3:

$$X_{-(\gamma|\lambda + \frac{\gamma}{2})}(\gamma)(e^{\lambda} \otimes 1) = \varepsilon(\gamma, \lambda)(e^{\lambda+\gamma} \otimes 1), \quad \gamma \in Q,\ \lambda \in \Gamma. \tag{18}$$

Proposition 11. *For $m \neq 0$, $K(m)$ is an irreducible $\tilde{\mathfrak{t}}_{[2]}$-module.*

Proof. It suffices to show

(a) $K(m) = \mathfrak{U}(\tilde{\mathfrak{t}}_{[2]}) \cdot (e^{m\mu} \otimes 1)$ and,
(b) every nonzero submodule R of $K(m)$ contains $e^{m\mu} \otimes 1$.

For (a), note that $K(m) = \coprod (\mathbb{C}e^{m\mu+\alpha} \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)) = \coprod_{\alpha} V_\Gamma(m\mu + \alpha)$, where α runs through Q. By (18), $\mathfrak{U}(\tilde{\mathfrak{t}}_{[2]}) \cdot (e^{m\mu} \otimes 1)$ contains $e^{m\mu+\alpha} \otimes 1$ for every $\alpha \in Q$ and since $m\mu + \alpha \in \Gamma \backslash Q$ $(m \neq 0)$, Proposition 9 implies $\mathfrak{U}(\tilde{\mathfrak{a}}) \cdot (e^{m\mu} \otimes 1) \supset \mathbb{C}e^{m\mu+\alpha} \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$, $\forall \alpha \in Q$. This establishes (a).

To prove (b), we note that as an $\hat{\mathfrak{a}}$-module $K(m)$ is a direct sum of non-isomorphic modules $V_\Gamma(m\mu + \alpha)$, and hence so too is R. Thus

$$e^{m\mu+\beta} \otimes 1 \in V_\Gamma(m\mu + \beta) \subset R$$

for some $\beta \in Q$. Now by (18), $e^{m\mu} \otimes 1 \in R$ and we are done. □

Proposition 12. *$K(m) \cong K(m')$ if and only if $m = m'$.*

Proof. $K(0)$ is not irreducible [F1]. Consider $m \neq 0$. Define

$$\mathbf{Vac}(K(m), \tilde{\mathfrak{a}}) := \{x \in K(m) \colon \tilde{\mathfrak{a}}_+ \cdot x = 0\}.$$

Note that since $V(m\mu + \alpha)$ is irreducible over $\tilde{\mathfrak{a}}$ we have $\mathbf{Vac}(V(m\mu + \alpha), \tilde{\mathfrak{a}}) = \mathbb{C}e^{m\mu+\alpha} \otimes 1$. Moreover, since $K(m) = \coprod_{\alpha \in Q} V(m\mu + \alpha)$, $\mathbf{Vac}(K(m), \tilde{\mathfrak{a}}) = \coprod_{\alpha \in Q} \mathbb{C}e^{m\mu+\alpha} \otimes 1$. Now for $\alpha \in Q$, $\delta(0) \cdot (e^{m\mu+\alpha} \otimes 1) = (m\mu + \alpha|\delta)e^{m\mu+\alpha} \otimes 1 = m(e^{m\mu+\alpha} \otimes 1)$. Thus $\delta(0)$ acts as m on $\mathbf{Vac}(K(m), \tilde{\mathfrak{a}})$. Finally, if $K(m) \cong K(m')$, where $m, m' \neq 0$, then $\mathbf{Vac}(K(m), \tilde{\mathfrak{a}}) \cong \mathbf{Vac}(K(m'), \tilde{\mathfrak{a}})$, as $\mathbb{C}\delta(0)$-modules and hence $m = m'$. □

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