Commun. Math. Phys., 158, 327–339 (1993)

Communications in Mathematical Physics

A Sharp Lower Bound for the Hausdorff Dimension of the Global Attractors of the 2D Navier-Stokes Equations

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Received: 26 June 1992/in revised form: 11 December 1992

Abstract. For a special class of the Navier-Stokes equations on the two-dimensional torus, we give a lower bound in the form $G^{2/3}$ (where G is the Grashof number) for the Hausdorff dimension of its global attractor which is optimal up to a logarithmic term.

1. Preliminaries and Introduction

We continue our previous work [8, 9] on the 2D Navier-Stokes equations for a viscous incompressible fluid with spatially periodic boundary conditions. In [9], we get a lower bound for the Hausdorff dimension of the global attractor in the form $G^{1/3}$ by considering some unstable modes of the associated linear operator for the Navier-Stokes equations. We commented there that one can improve the lower bound by more careful examination. In this paper, following the same technique as in [8, 9], we improve the lower bound to $G^{2/3}$ by considering more unstable eigenmodes. The idea is simple, since the dimension of unstable manifold around a steady state gives a lower bound for the Hausdorff dimension of the global attractor, so we only need to give an estimate for the number of unstable directions around this steady state.

Navier-Stokes equations written in functional form are [4, 15, 16]:

$$\frac{du}{dt} + Au + B(u,u) = f, \tag{1}$$

$$u(0) = u_0\,, \tag{2}$$

Dedicated to the memory of Professor Nicholas D. Kazarinoff in a Hilbert space H, where H consists of those u such that

$$u = \sum_{j=(j_1,j_2)\in Z^2} u_j e^{ij\cdot x}, \quad u_j \in C^2 u_{-j} = \bar{u}_j, \ u_0 = 0, \tag{3}$$

$$j \cdot u_j = 0, \quad \text{for each } j, \tag{4}$$

$$|u| = (2\pi)^2 \sum_{j\in Z^2} |u_j|^2 < \infty, \tag{5}$$

where[1] $j\cdot x = j_1x_1 + j_2x_2$, $u_j = (u_j^1, u_j^2)$, $j\cdot u_j = j_1u_j^1 + j_2u_j^2$.

Let P be the orthogonal projection onto H in $(L^2(\Omega))^2$ (where $\Omega = [0,2\pi]\times[0,2\pi]$), then

$$Au = -P\Delta u,$$
$$B(v,w) = P[(v\cdot\nabla)w].$$

Now as in [8, 9], for $k = (k_1,k_2) \neq (0,0)$, we define

$$W_k = \frac{1}{\sqrt{2}\pi|k|} k' \cos(k\cdot x),$$
$$W_k' = \frac{1}{\sqrt{2}\pi|k|} k' \sin(k\cdot x),$$

where $k' = (k_2, -k_1)$, $|k| = \sqrt{k_1^2 + k_2^2}$, $k\cdot x = k_1x_1 + k_2x_2$.

Let

$$\mathscr{K} = \{k = (k_1,k_2) \mid k_1 > 0 \text{ or } k_1 = 0, \ k_2 > 0\}.$$

We see that W_k, W_k', $k\in\mathscr{K}$ are eigenvectors of A with eigenvalues $|k|^2$; and those W_k, W_k' form an orthonormal basis in H.

It is easy to see for $k\neq(0,0)$, $\gamma = (\alpha,\beta)$,

$$P[\gamma\cos(k\cdot x)] = \frac{\sqrt{2}\pi\gamma\cdot k'}{|k|} W_k \tag{6}$$

$$P[\gamma\sin(k\cdot x)] = \frac{\sqrt{2}\pi\gamma\cdot k'}{|k|} W_k'. \tag{7}$$

As in [9], we consider a special class of the Navier-Stokes equations with external forces $f_0 = \lambda s^2 W_{(0,s)}'$. A corresponding stationary solution is

$$u_0 = \lambda W_{(0,s)}', \tag{8}$$

where $\lambda > 0$ is a parameter.

As in [4, 5, 16], the nondimensional Grashof number is defined by

$$G = \frac{|f|}{\nu^2\lambda_1}. \tag{9}$$

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[1] In this paper, we use row vector (u,v) to denote column vector $\binom{u}{v}$

In here, since the viscosity $\nu = 1$ and $\lambda_1 = 1$, so

$$G = |f_0| = s^2\lambda . \tag{10}$$

An upper bound for the Hausdorff dimension of the global attractor X for the 2D Navier-Stokes equations with periodic boundary conditions has been given (cf. [4, 5, 16]):

$$\dim_H(X) \le cG^{2/3}(1 + \log G)^{1/3} , \tag{11}$$

where c is a nondimensional constant.

We will show in this paper that the above upper bound is optimal up to a logarithmic term. More specifically, for the special class of the Navier-Stokes equations considered here, we will show the following:

$$\dim_H(X) \ge cG^{2/3} , \tag{12}$$

where c is a constant independent of G.

Remark. As remarked in [9], the same result has been stated in [5], but there is a mistake, because in there, the G which appears in (12) is different from the G in (11) which is defined by (9). The relation between the upper bound and lower bound on the dimension of the attractor for the Navier-Stokes equations (both 2 and 3 dimensions) has been studied in [6]. We caution the reader about the differences in parameters and the differences in G. In the two-dimensional case, lower bounds for global attractors given in both [5] and [6] are based on the work of [3] which is concerned with the Navier-Stokes equations on the domain $[0, 2\pi/\alpha] \times [0, 2\pi]$, where $\alpha > 0$ is a small perturbation parameter. Our result cannot be implied by the results in [3, 5, 6]. If one transforms the results of [3, 5, 6] to the domain $[0, 2\pi] \times [0, 2\pi]$, then one can see that the number of unstable modes given in this paper is in the order of square of the number of unstable modes one gets in [3, 5, 6]; the reason is that [3, 5, 6] only consider unstable modes in one space direction, whereas in the present paper we investigate unstable modes in both space directions, see the Remark following Theorem 1 below.

The Navier-Stokes equations linearized around u_0 are

$$\frac{dw}{dt} + L(u_0)w = 0 , \tag{13}$$

where

$$L(u_0)w = Aw + B(w, u_0) + B(u_0, w) . \tag{14}$$

We consider the following eigenvalue problem:

$$L(u_0)V = -\sigma V . \tag{15}$$

We call the eigenvectors corresponding to eigenvalues with negative real part (so $\mathrm{Re}\,\sigma > 0$) unstable modes. It is well known [3, 16] that the number of unstable modes gives a lower bound for the Hausdorff dimension of the global attractor. We want to give an estimate of their numbers.

In Sect. 2, we reduce the eigenvalue problem for $L(u_0)$ to an infinite system (uncoupled) of three term recurrence relations and recall a property of three term recurrence relations. In Sect. 3, we prove Theorem 1. Section 4 gives our main results.

2. Reduction of the Problem and Continued Fractions

The reduction of the problem is the same as in [8, 9]. We use a Fourier expansion, writing the eigenvector in the form:

$$V = W + W', \tag{16}$$

$$W = \sum_{k \in \mathscr{K}} a_k W_k, \tag{17}$$

$$W' = \sum_{k \in \mathscr{K}} a'_k W'_k, \tag{18}$$

where W and W' are even and odd parts of V respectively. We substitute V into (15), and by Lemma 1 below, we get two identical equations for W and W' respectively. So we only need to consider the equation for W:

$$L(u_0)W = -\sigma W. \tag{19}$$

We recall from [8, 9]:

Lemma 1. *For every* $k = (k_1, k_2) \neq (0,0)$ *and* $l = (l_1, l_2) \neq (0,0)$, *we have:*

$$B(W_k, W'_l) + B(W'_l, W_k) = \frac{-k' \cdot l(|k|^2 - |l|^2)}{2\sqrt{2}\pi|k||l|}\left\{\frac{1}{|k+l|}W_{(k+l)} - \frac{1}{|k-l|}W_{(k-l)}\right\},$$

and

$$B(W'_k, W'_l) + B(W'_l, W'_k) = \frac{-k' \cdot l(|k|^2 - |l|^2)}{2\sqrt{2}\pi|k||l|}\left\{\frac{1}{|k+l|}W'_{(k+l)} - \frac{1}{|k-l|}W'_{(k-l)}\right\}.$$

As in [8, 9], we substitute W given by (17) into (19), using Lemma 1 with $l = (0, s)$, we get the following recurrence relations for a_k:

$$\begin{aligned}&(k_1^2 + k_2^2 + \sigma)a_{(k_1,k_2)} \\ &\quad + \frac{\lambda k_1(k_1^2 + (k_2 - s)^2 - s^2)}{2\sqrt{2}\pi\sqrt{k_1^2 + (k_2 - s)^2}\sqrt{k_1^2 + k_2^2}}\, a_{(k_1,k_2-s)} \\ &\quad - \frac{\lambda k_1(k_1^2 + (k_2 + s)^2 - s^2)}{2\sqrt{2}\pi\sqrt{k_1^2 + (k_2 + s)^2}\sqrt{k_1^2 + k_2^2}}\, a_{(k_1,k_2+s)} = 0, \quad \text{for} \quad (k_1, k_2) \in \mathscr{K}.\end{aligned} \tag{20}$$

For each fixed $k_1 \geq 0$, the above equation gives a three term recurrence relation among $a_{(k_1,k_2-s)}$, $a_{(k_1,k_2)}$, $a_{(k_1,k_2+s)}$. So the complex problem of solving (19) is reduced to solving (20) for each fixed k_1. It is easy to note that for $k_1 = 0$, if $\operatorname{Re}\sigma > -1$, the only solution of (20) is $a_{(0,k_2)} = 0$, $\forall k_2 > 0$. We assume $k_1 > 0$ in the rest of this paper.

From (20), we get

Lemma 2 (cf. [8, 9]). *For every fixed* $k_1 > 0$,

$$c_{(k_1,k_2)}b_{(k_1,k_2)} + b_{(k_1,k_2-s)} - b_{(k_1,k_2+s)} = 0, \tag{21}$$

where

$$b_{(k_1,k_2)} = \frac{k_1^2 + k_2^2 - s^2}{\sqrt{k_1^2 + k_2^2}}\, a(k_1, k_2)\,, \tag{22}$$

$$c_{(k_1,k_2)} = \frac{2\sqrt{2}\pi(k_1^2 + k_2^2)(k_1^2 + k_2^2 + \sigma)}{\lambda k_1(k_1^2 + k_2^2 - s^2)}\,. \tag{23}$$

Because the eigenvector W belongs to the space H, we want to find nontrivial solutions of (21) such that

$$b_{(k_1,b_2)} \to 0 \quad \text{if} \quad |k_2| \to \infty\,. \tag{24}$$

If the above is true, then from (21), for all $n > 0$,

$$|k_2|^n b_{(k_1,k_2)}) \to 0 \quad \text{if} \quad |k_2| \to \infty\,.$$

So, such a W will be in H.

Now we let

$$d_n = c_{(t,sn+r)} = \frac{2\sqrt{2}\pi[t^2 + (sn + r)^2]}{\lambda t}\,\frac{[t^2 + (sn + r)^2 + \sigma]}{[t^2 + (sn + r)^2 - s^2]}\,, \tag{25}$$

where t is a positive integer, n is an integer and $r = 0, 1, \ldots, s - 1$. And we let

$$e_n = b_{(t,sn+r)} = \frac{t^2 + (sn + r)^2 - s^2}{\sqrt{t^2 + (sn + r)^2}}\, a_{(t,sn+r)}\,. \tag{26}$$

From Lemma 2, for each fixed positive integer t and fixed $r = 0, 1, \ldots, s - 1$, we get the following three term recurrence relations:

$$d_n e_n + e_{n-1} - e_{n+1} = 0, \quad n = 0, \pm 1, \pm 2, \ldots\,. \tag{27}$$

By the trivial solution of (27) we mean the solution $\{e_n\}$, $e_n = 0$ for $\forall n$. We want to find nontrivial solutions of the above Eq. (27) such that

$$\lim_{|n|\to\infty} e_n = 0\,, \tag{28}$$

since they correspond to eigenvectors of (19).

In the following, by the nontrivial solutions of (27) we mean those nontrivial solutions of (27) that also satisfy the condition (28). We have the following:

Theorem 1. *For each integer pair (t, r) (where $t > 0$, $r \geq 0$) satisfies:*

$$s^2 > t^2 + r^2, \quad t^2 + (-s + r)^2 > s^2\,. \tag{29}$$

For any $\lambda > 0$, there is a unique $\sigma = \sigma(\lambda) > -(t^2 + r^2)$ which increases monotonically with λ, such that there is a unique nontrivial solution of (27) (within a constant factor), and

$$\sigma(\lambda) \leq O(\lambda) \quad \text{if} \quad \lambda \to \infty\,.$$

Moreover, if

$$t^2 + r^2 \leq s^2/3\,, \tag{30}$$

then

$$\sigma(\lambda) = O(\lambda) \text{ if } \lambda \to \infty\,.$$

We will give the proof of Theorem 1 in the next section.

Remark. If one transforms the results of [3, 5, 6] to the domain $[0, 2\pi] \times [0, 2\pi]$, then one can see that in there, the unstable modes are corresponding to $r = 0$ here, so the total number of unstable modes is at most $2s$ instead of the order of s^2 we get here, see Lemma 4 below.

Remark. Same result is true with the conditions (29) and (30) replaced by

$$s^2 < t^2 + r^2, \quad t^2 + (-s + r)^2 < s^2,$$

and

$$t^2 + (-s + r)^2 \leq s^2/3,$$

respectively.

Now, we recall a result [11, 17, 7, 8] on three term recurrence relations which is crucial for the proof of Theorem 1. For other property of continued fractions, we refer to [7]. We consider the three term recurrence relations:

$$d_n e_n + e_{n-1} - e_{n+1} = 0, \tag{31}$$

where d_n, e_n are complex numbers and $n = 0, \pm 1, \pm 2, \ldots$. We have

Theorem 2 (cf. [11, 17, 7, 8]). *Assume*

$$\operatorname{Re} d_n > 0, \quad \textit{for} \quad \forall n \neq 0, 1, \tag{32}$$

$$\lim_{|n| \to \infty} \operatorname{Re} d_n = \infty. \tag{33}$$

Then the following two conditions are equivalent:
(A) *There exists a non-trivial solution* $\{e_n\}$ *of* (31) *such that*

$$\lim_{|n| \to \infty} e_n = 0. \tag{34}$$

(B) *The following equation is true.*

$$d_0 + \cfrac{1}{d_{-1} + \cfrac{1}{d_{-2} + \cdots}} = \cfrac{-1}{d_1 + \cfrac{1}{d_2 + \cdots}}. \tag{35}$$

Moreover, the solution which satisfies condition (A) *is unique within a constant factor; and property* (A) *implies*

$$e_n \neq 0, \quad \textit{for} \quad \forall n.$$

3. The Proof of Theorem 1

The proof below is similar to the proofs given in [8, 9]. We only consider real σ. We restrict σ,

$$\sigma > -t^2 - r^2.$$

Step 1. For fixed integer pair (t, r) $(t \geq 1,\ r \geq 0)$, we first recall

$$d_n = c_{(t, sn+r)} = \frac{2\sqrt{2}\pi(t^2 + (sn + r)^2)}{\lambda t} \frac{[t^2 + (sn + r)^2 + \sigma]}{[t^2 + (sn + r)^2 - s^2]}. \tag{36}$$

If (t, r) satisfies the conditions of Theorem 1, by the restriction on σ, we see that

$$d_0 < 0,\ d_n > 0, \quad \text{for} \quad \forall n \neq 0\,, \tag{37}$$

$$\lim_{|n|\to\infty} d_n = \infty\,, \tag{38}$$

hence the conditions of Theorem 2 are satisfied. Instead of solving (27), we only need to solve (35). We rewrite (35) in the following form:

$$-d_0 = \cfrac{1}{d_{-1} + \cfrac{1}{d_{-2} + \cdots}} + \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cdots}}\,. \tag{39}$$

We define $f(\sigma)$ be the left-hand side of (39) and $g(\sigma)$ be the right-hand side of (39). We want to find σ satisfies:

$$f(\sigma) = g(0)\,. \tag{40}$$

Step 2. By the definition of f,

$$f(\sigma) = \frac{2\sqrt{2}\pi(t^2 + r^2)}{\lambda t} \frac{[t^2 + r^2 + \sigma]}{[s^2 - (t^2 + r^2)]}\,, \tag{41}$$

so

$$\lim_{\sigma\to\infty} f(\sigma) = \infty\,, \tag{42}$$

$$f(-t^2 - r^2) = 0\,. \tag{43}$$

Since $d_n > 0$, $\forall n \neq 0$, by the definition of g and a property of continued fraction, we obtain:

$$\begin{aligned} g(\sigma) &< \frac{1}{d_{-1}} + \frac{1}{d_1}\,, \\ &= \frac{\lambda t}{2\sqrt{2}\pi[t^2 + (-s + r)^2]} \frac{[t^2 + (-s + r)^2 - s^2]}{[t^2 + (-s + r)^2 + \sigma]} \\ &\quad + \frac{\lambda t}{2\sqrt{2}\pi[t^2 + (s + r)^2]} \frac{[t^2 + (s + r)^2 - s^2]}{[t^2 + (s + r)^2 + \sigma]}\,, \end{aligned} \tag{44}$$

hence

$$\lim_{\sigma\to\infty} g(\sigma) = 0. \tag{45}$$

By the definition of g, it is trivial to see:

$$g(-t^2 - r^2) > 0\,. \tag{46}$$

Now we compare (42), (43) with (45) and (46), by the intermediate theorem, we obtain that for every $\lambda > 0$, there is a $\sigma > -t^2 - r^2$ satisfies (40).

Step 3. We show the σ obtained in Step 2 is unique. From (40), we get

$$\begin{aligned} \frac{2\sqrt{2}\pi(t^2 + r^2)}{\lambda t[s^2 - (t^2 + r^2)]} &= \cfrac{1}{(t^2 + r^2 + \sigma)d_{-1} + \cfrac{1}{(t^2 + r^2 + \sigma)^{-1}d_{-2} + \cdots}} \\ &\quad + \cfrac{1}{(t^2 + r^2 + \sigma)d_1 + \cfrac{1}{(t^2 + r^2 + \sigma)^{-1}d_2 + \cdots}}\,, \end{aligned} \tag{47}$$

so σ is unique, otherwise the left-hand side of (47) is a constant but the right-hand side of (47) decreases monotonically with σ, hence, a contradiction.

Step 4. We show the unique $\sigma = \sigma(\lambda)$ increases monotonically with $\lambda > 0$. From (47), we have

$$\frac{2\sqrt{2}\pi(t^2+r^2)}{t[s^2-(t^2+r^2)]} = \cfrac{1}{\lambda^{-1}(t^2+r^2+\sigma)d_{-1}+\cfrac{1}{\lambda(t^2+r^2+\sigma)^{-1}d_{-2}+\cdots}} + \cfrac{1}{\lambda^{-1}(t^2+r^2+\sigma)d_{1}+\cfrac{1}{\lambda(t^2+r^2+\sigma)^{-1}d_{2}+\cdots}}. \tag{48}$$

From the above equation, we get our results; otherwise the right- hand side increases monotonically with λ, which is a contradiction.

Step 5. Now we estimate the growth rate of $\sigma(\lambda)$. From (40), by a property of continued fraction, we have

$$\cfrac{1}{d_{-1}+\cfrac{1}{d_2}} + \cfrac{1}{d_1+\cfrac{1}{d_2}} < f(\sigma) < \frac{1}{d_{-1}} + \frac{1}{d_1}. \tag{49}$$

From the second inequality of (49),

$$\frac{2\sqrt{2}\pi(t^2+r^2)}{\lambda t}\frac{[t^2+r^2+\sigma]}{[s^2-(t^2+r^2)]} < \frac{\lambda t}{2\sqrt{2}\pi[t^2+(-s+r)^2]}\frac{[t^2+(-s+r)^2-s^2]}{[t^2+(-s+r)^2+\sigma]} + \frac{\lambda t}{2\sqrt{2}\pi[t^2+(s+r)^2]}\frac{[t^2+(s+r)^2-s^2]}{[t^2+(s+r)^2+\sigma]},$$

so, we obtain

$$(\sigma+s^2)(\sigma+t^2+r^2) < \frac{[s^2-(t^2+r^2)](\lambda t)^2}{(2\sqrt{2}\pi)^2(t^2+r^2)}\left\{\frac{t^2+(-s+r)^2-s^2}{t^2+(-s+r)^2} + \frac{t^2+(s+r)^2-s^2}{t^2+(s+r)^2}\right\}, \tag{50}$$

so

$$\sigma(\lambda) \le O(\lambda), \text{ if } \lambda \to \infty. \tag{51}$$

From the first inequality of (49), we get

$$\begin{aligned}&(\sigma+t^2+r^2)[\sigma+t^2+(-s+r)^2]\\ &\quad\times\frac{(2\sqrt{2}\pi)^2(t^2+r^2)}{(\lambda t)^2[s^2-(t^2+r^2)]}\frac{[t^2+(-s+r)^2]}{[t^2+(-s+r)^2-s^2]}\\ &\quad+(\sigma+t^2+r^2)[\sigma+t^2+(s+r)^2]\\ &\quad\times\frac{(2\sqrt{2}\pi)^2(t^2+r^2)}{(\lambda t)^2[s^2-(t^2+r^2)]}\frac{[t^2+(s+r)^2]}{[t^2+(s+r)^2-s^2]}\\ &\quad+\frac{[\sigma+t^2+r^2]}{[\sigma+t^2+(-2s+r)^2]}\frac{[t^2+r^2]}{[s^2-(t^2+r^2)]}\frac{[t^2+(-2s+r)^2-s^2]}{[t^2+(-2s+r)^2]}\\ &\quad+\frac{[\sigma+t^2+r^2]}{[\sigma+t^2+(2s+r)^2]}\frac{[t^2+r^2]}{[s^2-(t^2+r^2)]}\frac{[t^2+(2s+r)^2-s^2]}{[t^2+(2s+r)^2]} > 1.\end{aligned} \tag{52}$$

We denote I to be the sum of the first two terms of left-hand side of the above inequality, and II to be the sum of the last two terms of the left-hand side of the above inequality. Since

$$\begin{aligned} \mathrm{II} &< \frac{[t^2+r^2]}{[s^2-(t^2+r^2)]}\frac{[t^2+(-2s+r)^2-s^2]}{[t^2+(-2s+r)^2]} \\ &+ \frac{[t^2+r^2]}{[s^2-(t^2+r^2)]}\frac{[t^2+(2s+r)^2-s^2]}{[t^2+(2s+r)^2]} \\ &< 2\frac{t^2+r^2}{s^2-(t^2+r^2)}, \end{aligned} \tag{53}$$

so, if $t^2+r^2 \le s^2/3$, then by the above inequality (53), we obtain

$$\mathrm{II} < 1. \tag{54}$$

Hence, from (52) and (54), we imply

$$\sigma(\lambda) \ge O(\lambda), \text{ if } \lambda \to \infty. \tag{55}$$

Combining (51) and (55), we get

$$\sigma(\lambda) = O(\lambda), \text{ if } \lambda \to \infty. \tag{56}$$

We have proved Theorem 1. □

4. Main Results

From Theorem 1, and considering both even and odd parts of V in (16), we easily get

Theorem 3. *For each integer pair (t,r) (where $t>0$, $r\ge 0$) satisfies:*

$$s^2 > t^2+r^2, \quad t^2+(-s+r)^2 > s^2. \tag{57}$$

For any $\lambda > 0$, there is an eigenvalue $-\sigma(\lambda) < (t^2+r^2)$ of $L(u_0)$ which decreases monotonically with λ; the corresponding eigenspace is at least two.

And

$$\sigma(\lambda) \le O(\lambda) \text{ if } \lambda \to \infty.$$

Moreover, if $t^2+r^2 \le s^2/3$, then

$$\sigma(\lambda) = O(\lambda) \text{ if } \lambda \to \infty.$$

In the following, we require the integer pair (t,r) satisfying the conditions of Theorem 3. For each such pair, we denote $\lambda(t,r)$ to be the critical value of λ such that

$$\sigma(\lambda(t,r)) = 0. \tag{58}$$

So, if $\lambda > \lambda(t,r)$, by Theorem 3, there are unstable modes corresponding to $-\sigma(\lambda) < 0$.

Remark. It can be shown, when restricted to odd functions, the eigenvalue 0 is simple, so global bifurcations occur [12]. These and other related issues will be pursued elsewhere.

We want to give an estimate for $\lambda(t,r)$. For simplicity below, we abbreviate $\lambda(t,r)$ as λ_0. Taking $\sigma = 0$ in (50), we obtain

$$\lambda_0 > \frac{s}{t}\,\frac{2\sqrt{2}\pi(t^2+r^2)}{\sqrt{s^2-(t^2+r^2)}}\left\{\frac{t^2+(-s+r)^2-s^2}{t^2+(-s+r)^2}+\frac{t^2+(s+r)^2-s^2}{t^2+(s+r)^2}\right\}^{-1/2},$$

so

$$\lambda_0 > \frac{2\pi(t^2+r^2)}{\sqrt{s^2-(t^2+r^2)}}\,\frac{s}{t}\,. \tag{59}$$

By taking $\sigma = 0$, from (52) and (53), we get

$$\begin{aligned}&\frac{(2\sqrt{2}\pi)^2(t^2+r^2)^2}{(\lambda_0 t)^2[s^2-(t^2+r^2)]}\left\{\frac{[t^2+(-s+r)^2]^2}{t^2+(-s+r)^2-s^2}+\frac{[t^2+(s+r)^2]^2}{t^2+(s+r)^2-s^2}\right\}\\&> 1-\frac{2(t^2+r^2)}{s^2-(t^2+r^2)}\,.\end{aligned} \tag{60}$$

Hence if $\dfrac{2(t^2+r^2)}{s^2-(t^2+r^2)} < 1$, then

$$\begin{aligned}\lambda_0 &< \frac{2\sqrt{2}\pi(t^2+r^2)}{t\sqrt{s^2-(t^2+r^2)}}\left\{1-\frac{2(t^2+r^2)}{s^2-(t^2+r^2)}\right\}^{-1/2}\\&\quad\times\left\{\frac{[t^2+(-s+r)^2]^2}{t^2+(-s+r)^2-s^2}+\frac{[t^2+(s+r)^2]^2}{t^2+(s+r)^2-s^2}\right\}^{1/2}.\end{aligned} \tag{61}$$

If $t^2+r^2 \le s^2/4$, then (61) implies

$$\begin{aligned}\lambda_0 &< \frac{\sqrt{2}\pi s}{t}\left\{\frac{[t^2+(-s+r)^2]^2}{t^2+(-s+r)^2-s^2}+\frac{[t^2+(s+r)^2]^2}{t^2+(s+r)^2-s^2}\right\}^{1/2}\\&< \sqrt{2}\pi s\left[1+\left(\frac{s+r}{t}\right)^2\right]\sqrt{\frac{1}{1-\frac{s^2}{t^2}+\frac{(s-r)^2}{t^2}}+1}\,.\end{aligned} \tag{62}$$

So, if $s^2/t^2-(s-r)^2/t^2 \le 1/2$ and $t \ge \delta s$ (where δ is a fixed positive small number), we have

$$\lambda_0 < \sqrt{6}\pi\left(1+\frac{4}{\delta^2}\right)s\,. \tag{63}$$

Also if $t^2+r^2 \le s^2/4$ and $t \ge \delta s$, from (59), we get

$$\lambda_0 > \frac{4\pi}{\sqrt{3}}\frac{t^2+r^2}{t} \ge \frac{4\pi}{\sqrt{3}}\delta s\,.$$

We just proved the following Lemma 3.

We define $\mathscr{F}$ as

$$\mathscr{F} = \left\{(t,r)\,\middle\|\,\begin{matrix}t,r \text{ are integers}, & r\ge 0\\ t^2+r^2\le s^2/4, & s^2\le t^2/2+(-s+r)^2\\ t\ge\delta s & \end{matrix}\right\}.$$

It is obvious that any $(t,r)\in\mathscr{F}$ satisfies the conditions of Theorem 3.

Lemma 3. *For any* $(t,r) \in \mathscr{F}$,

$$\frac{4\pi}{\sqrt{3}}\delta s < \lambda(t,r) < \sqrt{6}\pi\left(1+\frac{4}{\delta^2}\right)s\,. \tag{64}$$

We denote $n(\mathscr{F})$ the total number of elements in $\mathscr{F}$. Now we take

$$\lambda = \sqrt{6}\pi\left(1+\frac{4}{\delta^2}\right)s\,,$$

and let the external force be

$$f_0 = \lambda s^2 W'_{(0,s)} = \sqrt{6}\pi(1+4/\delta^2)s^3 W'_{(0,s)}\,. \tag{65}$$

By (10), the Grashof number becomes

$$G = \sqrt{6}\pi\left(1+\frac{4}{\delta^2}\right)s^3\,. \tag{66}$$

From Theorem 3 and Lemma 3, we see that for the choice of external force (65), the numbers of unstable modes is at least $2n(\mathscr{F})$, so [3, 16].

$$\dim_H(X) \geq 2n(\mathscr{F})\,. \tag{67}$$

We want to show

Lemma 4.

$$n(\mathscr{F}) \geq c_2 s^2, \quad \textit{for } s \textit{ large},$$

where c_2 *is given by* (76) *below.*

Proof. It is obvious that: $(t,r) \in n(\mathscr{F})$ if and only if

$$\sqrt{2(2rs-r^2)} \leq t \leq \sqrt{\frac{s^2}{4}-r^2}\,, \tag{68}$$

$$\delta s \leq t\,. \tag{69}$$

Let $r = cs$, c is a nonnegative number, we have

$$\begin{aligned}\sqrt{s^2/4-r^2}-\sqrt{2(2rs-r^2)} &= \frac{s^2/4+r^2-4rs}{\sqrt{s^2/4-r^2}+\sqrt{2(2rs-r^2)}}\\ &= \frac{(1/4+c^2-4c)s}{\sqrt{1/4-c^2}+\sqrt{2(2c-c^2)}}\\ &= \frac{[(c-2)^2-15/4]s}{\sqrt{1/4-c^2}+\sqrt{2(2c-c^2)}}\,.\end{aligned} \tag{70}$$

We want (70) to be positive, so c has to satisfy:

$$1/4-c^2>0, \quad 2c-c^2 \geq 0\,,$$
$$(c-2)^2-15/4>0\,,$$

solving them, we get

$$0 \leq c < \frac{4-\sqrt{15}}{2}\,.$$

Let $c_0 = \dfrac{4-\sqrt{15}}{4} > 0$, considering each integer r satisfies:

$$0 \le r \le c_0 s\,. \tag{71}$$

From (70), we see for such an integer r

$$\sqrt{\frac{s^2}{4} - r^2} - \sqrt{2(2rs - r^2)} \ge c_1 s\,, \tag{72}$$

where

$$c_1 = \frac{c_0^2 - 4c_0 + 1/4}{\sqrt{1/4} + \sqrt{2(2c_0 - c_0^2)}} > 0\,.$$

Now for each r satisfies (71), let's count the numbers of t which satisfy (68) and (69).

If $\delta s \le \sqrt{2(2rs - r^2)}$, then from (72), the numbers of integers t that satisfy (68) and (69) are at least

$$c_1 s\,. \tag{73}$$

If $\delta s > \sqrt{2(2rs - r^s)}$, the numbers of integers t that satisfy (68) and (69) are at least

$$\left(\sqrt{1/4 - c_0^2} - \delta\right)s = \frac{\sqrt{1/4 - c_0^2}}{2}s\,, \tag{74}$$

where we have chosen

$$\delta = \frac{\sqrt{1/4 - c_0^2}}{2} > 0\,. \tag{75}$$

From (71), (73), and (74), we get that

$$n(\mathscr{F}) \ge c_0 \min\left(c_1, \frac{\sqrt{1/4 - c_0^2}}{2}\right) s^2\,,$$

which gives Lemma 4 with

$$c_2 = c_0 \min\left(c_1, \frac{\sqrt{1/4 - c_0^2}}{2}\right)\,. \tag{76}$$

□

From Lemma 4 and (66) and (67), we finally obtain

Theorem 4. *For the choices of external forces given by* (65), *we have*

$$\dim_H(X) \ge c_3 G^{2/3}\,, \tag{77}$$

where

$$c_3 = 2c_2 \left[\sqrt{6}\pi \left(1 + \frac{16}{1/4 - c_0^2}\right)\right]^{-2/3}\,,$$

c_2 *is given by* (76) *and* $c_0 = \dfrac{4-\sqrt{15}}{4}$.

Remark. The above estimate gives a lower bound for the global attractor of the Navier-Stokes equations on the 2D torus. For the three-dimensional situation, if we define the nondimensional Grashof number by

$$G = \frac{|f|}{\nu^2 \lambda_1^{3/4}},$$

then $G^{2/3}$ also is a lower bound for the attractor of the 3D system. Moreover, one can show a lower bound for the attractor in the form G.

Note added in proof. In my previous paper [9], I stated that the results of [9] give a positive answer to a problem of Arnold [1, 2]: *Is it true that the minimum of the Hausdorff dimension of minimal attractors of the Navier-Stokes equation (on, say, the two-dimensional torus) grows as the Reynolds number increases?* Recently, Arnold informed me that his problem means: *whether it is true that the dimension of* **any** *attracting set is growing with the Reynolds number. For instance, do there exist stable periodic orbits for any Reynolds number or are they all unstable for sufficiently high Reynolds number?* Evidently, his problems are still open.

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Communicated by J. L. Lebowitz