Commun. Math. Phys., 157, 429–457 (1993)

Communications in Mathematical Physics

Quasifinite Highest Weight Modules over the Lie Algebra of Differential Operators on the Circle

Victor Kac*, **Andrey Radul****

Mathematics Department, M.I.T., Cambridge, MA 02139, USA

Received: 7 June 1993

Abstract: We classify positive energy representations with finite degeneracies of the Lie algebra $W_{1+\infty}$ and construct them in terms of representation theory of the Lie algebra $\widehat{gl}(\infty, R_m)$ of infinites matrices with finite number of non-zero diagonals over the algebra $R_m = \mathbb{C}[t]/(t^{m+1})$. The unitary ones are classified as well. Similar results are obtained for the sin-algebras.

0. Introduction

0.1. Recent progress in conformal field theory revealed some unusual mathematical objects called the W_n-algebras [Z]. These algebras turned out to be quantizations of the second Gelfand-Dickey structure for Lax equations [FL]. The complicated structure of these algebras is greatly simplified in the limit $n = \infty$, the limiting algebra being the Lie algebra $\widehat{\mathscr{D}}$, the universal central extension of the Lie algebra of differential operators on the circle [KP]. (Physicists denote this Lie algebra by $W_{1+\infty}$ [PSR].) The possibility to get W_n from $\widehat{\mathscr{D}}$ has been studied in [R, RV]. A complete picture for classical W_n was obtained in [KhZ].

The main goal of the present paper is to classify and describe the irreducible quasifinite highest weight representation of the Lie algebra $\widehat{\mathscr{D}}$. The basic technical tool is the analytic completion $\widehat{\mathscr{D}^{\mathcal{O}}}$ of $\widehat{\mathscr{D}}$ and a family of its homomorphisms onto the central extension of the Lie algebra $\widetilde{gl}(\infty, R_m)$ of infinite matrices with finitely many non-zero diagonals over the ring $R_m = \mathbb{C}[t]/(t^{m+1})$.

The Lie algebra $\widehat{\mathscr{D}}$ may be obtained via a general construction (explained in Sect. 1) as a twisted Laurent polynomial algebra over the polynomial algebra $\mathbb{C}[w]$. It is easy to see that the only other Lie algebras obtained by this construction from $\mathbb{C}[w]$ are Lie algebras $\widehat{\mathscr{D}_q}$, the central extension of the Lie algebra of difference

* Supported in part by NSF grant DMS-9103792
** Supported in part by DOE grant DE-F602-88ER25066

operators on the circle. It turns out, however, that a representation theory similar to that of $\widehat{\mathscr{D}}$, may be developed for a larger Lie algebra, the central extension $\widehat{\mathscr{S}_q}$ of the Lie algebra of pseudo-difference operators on the circle (see Sect. 6). The latter Lie algebra has been studied recently by many authors (see [FFZ, GL] and references there).

Being of a very general nature, our methods may be applied to many other examples of infinite-dimensional Lie algebras. Noting that $\mathscr{D}$ (resp. $\mathscr{S}_q$) is a quantization of the Poisson Lie algebra of functions on the cylinder (resp. 2-dimensional tours), one may expect that our approach could be extended to the quantizations of general symplectic manifolds.

0.2. Let us give here the main definitions which will be used for various examples throughout the paper.

Consider a $\mathbb{Z}$-*graded* Lie algebra over $\mathbb{C}$:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}\,.$$

(We do not assume $\mathfrak{g}_i$ to be finite-dimensional.) We let

$$\mathfrak{g}_\pm = \bigoplus_{j>0} \mathfrak{g}_{\pm j}\,.$$

A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called *parabolic* if it contains $\mathfrak{g}_0 + \mathfrak{g}_+$ as a proper subalgebra.

A $\mathfrak{g}$-module V is called graded if

$$V = \bigoplus_j V_j, \quad \mathfrak{g}_i V_j \subset V_{i+j}\,.$$

A graded $\mathfrak{g}$-module V is called *quasifinite* if

$$\dim V_j < \infty \quad \text{for all } j\,.$$

Given $\lambda \in \mathfrak{g}_0^*$, a *highest weight module* is a $\mathbb{Z}$-graded $\mathfrak{g}$-module $V(\mathfrak{g}, \lambda) = \bigoplus_{j \in \mathbb{Z}_+} V_{-j}$ defined by the following properties:

(i) $V_0 = \mathbb{C} v_\lambda$, where v_λ is a non-zero vector,
(ii) $h v_\lambda = \lambda(h) v_\lambda$ for $h \in \mathfrak{g}_0$,
(iii) $\mathfrak{g}_+ v_\lambda = 0$,
(iv) $\mathscr{U}(\mathfrak{g}_-) v_\lambda = V(\mathfrak{g}, \lambda)$.

Here and further $\mathscr{U}(\mathfrak{s})$ stands for the universal enveloping algebra of the Lie algebra $\mathfrak{s}$.

A non-zero vector $v \in V(\mathfrak{g}, \lambda)$ is called *singular* if $\mathfrak{g}_+ v = 0$. The module $V(\mathfrak{g}, \lambda)$ is irreducible if and only if any of its singular vectors is a multiple of v_λ.

The "largest" among the modules $V(\mathfrak{g}, \lambda)$ with a given λ is the *Verma module* $M(\mathfrak{g}, \lambda)$ defined by the property that the map

$$\varphi : \mathscr{U}(\mathfrak{g}_-) \to M(\mathfrak{g}, \lambda)$$

given by $\varphi(u) = u(v_\lambda)$ is a vector space isomorphism.

Any highest weight module $V(\mathfrak{g}, \lambda)$ is a quotient of $M(\mathfrak{g}, \lambda)$. The "smallest" among the $V(\mathfrak{g}, \lambda)$ is the irreducible module $L(\mathfrak{g}, \lambda)$ (which is a quotient of $M(\mathfrak{g}, \lambda)$ by the maximal graded submodule).

We shall write $M(\lambda)$ and $L(\lambda)$ in place of $M(\mathfrak{g},\lambda)$ and $L(\mathfrak{g},\lambda)$ if no ambiguity may arise.

0.3. It is useful to note that the Verma modules can be constructed as follows:

$$M(\mathfrak{g},\lambda)=\mathscr{U}(\mathfrak{g})\otimes_{\mathscr{U}(\mathfrak{g}_0+\mathfrak{g}_+)}\mathbb{C}_\lambda\,,$$

where $\mathbb{C}_\lambda$ is the 1-dimensional $\mathfrak{g}_0+\mathfrak{g}_+$-module given by $h\mapsto\lambda(h)$ if $h\in\mathfrak{g}_0$, $\mathfrak{g}_+\mapsto 0$, and the action of $\mathfrak{g}$ is induced by the left multiplication in $\mathscr{U}(\mathfrak{g})$.

Now, let $\mathfrak{p}=\bigoplus_j\mathfrak{p}_j$ be a parabolic subalgebra of $\mathfrak{g}$, and let $\lambda\in\mathfrak{g}_0^*$ be such that $\lambda\mid_{\mathfrak{g}_0\cap[p,p]}=0$. Then the $\mathfrak{g}_0+\mathfrak{g}_+$-module $\mathbb{C}_\lambda$ extends to $\mathfrak{p}$ by letting $\mathfrak{p}_j\mapsto 0$ for $j<0$, and we may construct the highest weight module

$$M(\mathfrak{g},\mathfrak{p},\lambda)=\mathscr{U}(\mathfrak{g})\otimes_{\mathscr{U}(\mathfrak{p})}\mathbb{C}_\lambda\,.$$

It is called the *generalized Verma module*. It may be characterized by the property that the map φ induces an isomorphism $\mathscr{U}(\mathfrak{g}_-)/\mathscr{U}(\mathfrak{p}\cap\mathfrak{g}_-)\to M(\mathfrak{g},\mathfrak{p},\lambda)$.

Note that if $\dim\mathfrak{g}_j<\infty$ for all j, the $\mathfrak{g}$-module $L(\lambda)$ for any λ is quasifinite. If however $\dim\mathfrak{g}_j=\infty$, which is the case in all of our examples, the classification of quasifinite irreducible highest weight modules becomes a non-trivial problem. The answer to this problem for the Lie algebra $\widehat{\mathscr{D}}$ is given by Theorem 4.2. Moreover, we give an explicit construction of all these modules in terms of irreducible highest weight modules over the Lie algebra $\widehat{gl}(\infty,R_m)$ (Theorems 4.5 and 4.6).

0.4. Recall that an *anti-involution* of a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is an additive map $\omega:\mathfrak{g}\to\mathfrak{g}$ such that

$$\omega(\lambda a)=\bar{\lambda}a,\quad \omega([a,b])=[\omega(b),\omega(a)],\quad \text{for } \lambda\in\mathbb{C},\ a,b\in\mathfrak{g}\,.$$

Given a $\mathfrak{g}$-module V, a Hermitian form h on V is called *contravariant* if for any $a\in\mathfrak{g}$ the operators a and $\omega(a)$ are (formally) adjoint operators on V with respect to h.

Fix an anti-involution ω of the Lie algebra $\mathfrak{g}$ such that $\omega(\mathfrak{g}_j)=\mathfrak{g}_{-j}$. Let $L(\mathfrak{g},\lambda)$ be an irreducible highest weight module over $\mathfrak{g}$ such that $\lambda(h)\in\mathbb{R}$ if $\omega(h)=h$. For $v\in L(\mathfrak{g},\lambda)$ denote by $\langle v\rangle$ the coefficient of v_λ in the decomposition of v with respect to the gradation of $L(\mathfrak{g},\lambda)$. Let

$$h(av_\lambda,bv_\lambda)=\langle\omega(a)bv_\lambda\rangle,\quad a,b\in\mathscr{U}(\mathfrak{g})\,.$$

It is easy to show (see e.g. [K, Chapter 9]) that h is the unique contravariant form on $L(\mathfrak{g},\lambda)$ such that $h(v_\lambda,v_\lambda)=1$; moreover, it is non-degenerate and $h(L(\mathfrak{g},\lambda)_i,L(\mathfrak{g},\lambda)_j)=0$ if $i\neq j$.

The $\mathfrak{g}$-module $L(\mathfrak{g},\lambda)$ is called *unitary* (with respect to ω) if the contravariant form h is positive definite (this is independent of the choice of $v_\lambda\in L(\mathfrak{g},\lambda)_0$).

The classification of unitary quasifinite (irreducible) highest weight modules over $\widehat{\mathscr{D}}$ is given by Theorem 5.2.

Let us note in conclusion that the classification of irreducible quasifinite highest weight $\widehat{\mathscr{D}}$-modules is expressed in terms of Bernoulli polynomials. Is it an indication of a connection to the Riemann-Roch theorems?

0.5. One of the authors wishes to thank D. Lebedev and M. Golenishcheva-Kutuzova for illuminating discussions on the sin Lie algebra.

1. Twisted Laurent Polynomial Algebras and Associated Lie Algebras

1.1. Let A be an associative algebra over a field $\mathbb{F}$ and let σ be an automorphism of A. Define the *twisted Laurent polynomial algebra* $A_\sigma[z,z^{-1}]$ over A in the indeterminate z to be the vector space $\mathbb{F}[z,z^{-1}]\otimes_{\mathbb{F}} A$ over $\mathbb{F}$ of finite sums of the form $\sum_{j\in\mathbb{Z}} z^j\otimes a_j$, $a_j\in A$, with multiplication defined by the rule

$$(z^k\otimes a)(z^m\otimes b)=z^{k+m}\otimes\sigma^m(a)b,\quad a,b\in A,\ \ k,m\in\mathbb{Z}. \tag{1.1.1}$$

Further on we shall often write $z^m a$ in place of $z^m\otimes a$.

Remarks. (a) Replacing z by za^{-1}, where a is an invertible element of A, corresponds to replacing σ by $(Ada)\sigma$, where Ada stands for the inner automorphism:

$$(Ada)b=aba^{-1},\quad b\in A.$$

(b) Applying an automorphism α to A replaces σ by $\alpha^{-1}\sigma\alpha$.

Thus, we obtain the following proposition.

Proposition. *Twisted Laurent polynomial algebras over an associative algebra A are parameterized by the conjugacy classes of the group* $\mathrm{Aut}\,A/AdA$. □

Two automorphisms of A whose images lie in the same conjugacy class of $\mathrm{Aut}\,A/Ad\,A$ are called *equivalent.*

1.2. The algebra $A_\sigma[z,z^{-1}]$ has a canonical $\mathbb{Z}$-gradation, called the *principal* gradation:

$$A_\sigma[z,z^{-1}]=\bigoplus_{j\in\mathbb{Z}}(z^jA). \tag{1.2.1}$$

Let $\mathscr{P}=\bigoplus_{j\in\mathbb{Z}}\mathscr{P}_j$ be a parabolic subalgebra of $A_\sigma[z,z^{-1}]$. It is clear that $\mathscr{P}_{-1}=z^{-1}I$, where I is a (two-sided) ideal of the algebra A. Hence $\mathscr{P}$ contains the following *minimal parabolic subalgebra* $\mathscr{P}(I)$ associated to I:

$$\mathscr{P}(I)=\left(\bigoplus_{j>0}(z^{-1}I)^j\right)\bigoplus\left(\bigoplus_{j\geq 0}(z^jA)\right). \tag{1.2.2}$$

Remark. Given two ideals I and J of A, we have the following graded subalgebra of $A_\sigma[z,z^{-1}]$:

$$\mathscr{P}(I,J)=\left(\bigoplus_{j>0}(z^{-1}I)^j\right)\oplus A\oplus\left(\bigoplus_{j>0}(zJ)^j\right). \tag{1.2.3}$$

1.3. We denote $\widetilde{A}_\sigma$ the algebra $A_\sigma[z,z^{-1}]$ viewed as a Lie algebra with respect to the usual bracket:

$$[f,g]'=fg-gf.$$

Fix a trace on the algebra A, i.e., a linear map $\mathrm{tr}:A\to V$, where V is a vector space over $\mathbb{F}$, such that $\mathrm{tr}(ab)=\mathrm{tr}(ba)$. Then we may construct a remarkable central extension $\widehat{A}_{\sigma,\mathrm{tr}}$ of $\widetilde{A}_\sigma$ by a central subalgebra V:

$$0\to V\to\widehat{A}_{\sigma,\mathrm{tr}}\to A_\sigma\to 0$$

as follows. It is straightforward to check that the formula

$$\begin{aligned}\Psi_{\sigma,\mathrm{tr}}(z^r a, z^s b) &= -\Psi_{\sigma,\mathrm{tr}}(z^s b, z^r a)\\ &= \begin{cases} \mathrm{tr}((1+\sigma+\cdots+\sigma^{r-1})(\sigma^{-r}(f)g)) & \text{if } r=-s>0,\\ 0 & \text{if } r+s\neq 0 \text{ or } r=s=0\end{cases}\end{aligned} \tag{1.3.1}$$

defines a 2-cocycle on $\widetilde{A}_\sigma$ with values in V. Then $\widehat{A}_{\sigma,\mathrm{tr}} = \widetilde{A} + V$ with V central and the bracket of two elements $f, g \in \widetilde{A} \subset \widehat{A}_{\sigma,\mathrm{tr}}$ is given by the usual formula:

$$[f,g] = [f,g]' + \Psi_{\sigma,\mathrm{tr}}(f,g)\,.$$

Remarks. (a) Replacing z by za^{-1} corresponds to replacing $\Psi_{\sigma,\mathrm{tr}}$ by $\Psi_{(Ada)\sigma,\mathrm{tr}}$.
(b) Applying an autormorphism α to A replaces $\Psi_{\sigma,\mathrm{tr}}$ by $\Psi_{\alpha^{-1}\sigma\alpha,\mathrm{tr}\circ\alpha}$.
(c) Since $\Psi_{\sigma,\mathrm{tr}}(z^r, z^s) = \mathrm{tr}(1)r\delta_{r,-s}$, the cocycle $\Psi_{\sigma,\mathrm{tr}}$ is nontrivial if $\mathrm{tr}(1)\neq 0$.
(d) Suppose that the map $\sigma - 1 : A \to A$ is surjective and that tr vanishes on its kernel. Then we have an isomorphism $\sigma - 1 : A/\mathrm{Ker}(\sigma-1) \xrightarrow{\sim} A$, and $\varphi : \mathrm{tr}\circ(1-\sigma)^{-1} : A \to V$ is a well-defined map. We have:

$$\Psi_{\sigma,\mathrm{tr}}(z^r f, z^{-r} g) = \varphi([z^r f, z^{-r} g])\,,$$

hence in this case the cocycle $\Psi_{\sigma,\mathrm{tr}}$ is trivial.
(e) Suppose that $\mathrm{tr}(\sigma(a)) = \mathrm{tr}\,a$, $a\in A$. Then tr extends to a trace of the algebra $A_\sigma[z,z^{-1}]$ by letting $\mathrm{tr}(z^k a) = \delta_{k,0}\,\mathrm{tr}\,a$.

Example. Let $A = \mathrm{Mat}_n\,\mathbb{F}$; then any automorphism of A is equivalent to $\sigma = 1$ (by Remark 1(a)). Take the usual trace $\mathrm{tr} : A \to \mathbb{F}$, then $\widehat{A}_{\sigma,\mathrm{tr}}$ is isomorphic to the usual affine algebra $gl_n(\mathbb{F})^\wedge$.

We have the corresponding to (1.2.1) $\mathbb{Z}$-gradation:

$$\widehat{A}_{\sigma,\mathrm{tr}} = \bigoplus_j \widehat{A}_j, \quad \text{where } \widehat{A}_j = z^j A \text{ if } j\neq 0, \text{ and } \widehat{A}_0 = A + V\,. \tag{1.3.2}$$

For each (two-sided) non-zero ideal I of A we have the associated parabolic subalgebra of $\widehat{A}_{\sigma,\mathrm{tr}}$,

$$\mathfrak{p}(I) = \bigoplus_{j>0} (z^{-1}I)^j \bigoplus \left(A \bigoplus V\right) \bigoplus \left(\bigoplus_{j>0} z^j A\right)\,. \tag{1.3.3}$$

1.4. We turn now to the main examples of the Lie algebras $\widetilde{A}_\sigma$ and $\widehat{A}_\sigma$, those associated to the polynomial algebra $A = \mathbb{C}[w]$ in the indeterminate w. We show that the Lie algebras $\widetilde{A}_\sigma$ are isomorphic to the Lie algebras of all regular differential (resp. difference) operators on the punctured complex plane $\mathbb{C}^\times = \mathbb{C}\setminus\{0\}$, and that the Lie algebras $\widehat{A}_{\sigma,\mathrm{tr}}$ are their well known central extensions.

For $q\in\mathbb{C}^\times$ define the following operator on $\mathbb{C}[z,z^{-1}]$:

$$D_q f(z) = \begin{cases} \dfrac{f(qz)-f(z)}{q-1} & \text{if } q\neq 1\,,\\ z\partial_z f(z) & \text{if } q=1\,.\end{cases}$$

Denote by $\mathscr{D}_q^a$ the associative algebra of all operators on $\mathbb{C}[z,z^{-1}]$ of the form

$$E = e_k(z)D_q^k + e_{k-1}(z)D_q^{k-1} + \cdots + e_0(z)\,, \quad \text{where } e_i(z)\in\mathbb{C}[z,z^{-1}]$$

(the superscript a stand for "associative") and let $\mathscr{D}_q$ denote the corresponding Lie algebra.

Now, any automorphism of $\mathbb{C}[w]$ is equivalent to $\sigma_q, q \in \mathbb{C}^\times$, defined by

$$\sigma_q(w) = qw + 1\,.$$

Note that

$$\sigma_q^n(w) = q^n w + [n]\,, \tag{1.4.1}$$

where, as usual, for $n \in \mathbb{Z}$:

$$[n] = \frac{q^n - 1}{q - 1} \quad \text{if } q \neq 1 \text{ and } = n \text{ if } q = 1\,.$$

Proposition. (a) *The linear map* $\mathbb{C}[w]_{\sigma_q}[z, z^{-1}] \longrightarrow \mathscr{D}_q^a$ *defined by* $z^k f(w) \mapsto z^k f(D_q)$ *is an isomorphism of associative algebras.*
(b) *Let* tr : $\mathbb{C}[w] \longrightarrow \mathbb{C}$ *be the evaluation map at* $w = 0$. *Then the 2-cocycle* $\Psi_{\sigma,\mathrm{tr}}$ *on the Lie algebra* $\mathbb{C}[w]_{\sigma_q}[z, z^{-1}]$ *induces, via the above isomorphism, the following 2-cocycle on the Lie algebra* $\mathscr{D}_q$:

$$\Psi(z^m f(D_q), z^n g(D_q)) = \begin{cases} \sum\limits_{-m \leq j \leq -1} f([j])g([j+m]) & \text{if } m = -n \geq 0\,, \\ 0 & \text{if } m + n \neq 0\,. \end{cases} \tag{1.4.2}$$

Proof. This is straightforward using (1.4.1). □

We shall denote by

$$\widehat{\mathscr{D}_q} = \mathscr{D}_q + \mathbb{C}C$$

the central extension of $\mathscr{D}_q$ corresponding to the cocycle (1.4.2) so that the bracket of two elements from the subspace $\mathscr{D}_q$ is given by

$$[E_1, E_2] = E_1 E_2 - E_2 E_1 + \Psi(E_1, E_2)C\,.$$

1.5. Let $\mathscr{D}^a = \mathscr{D}_1^a$, $\mathscr{D} = \mathscr{D}_1$, $\widehat{\mathscr{D}} = \widehat{\mathscr{D}}_{1,\mathrm{tr}}$, $D = D_1 (= z\partial_z)$. As we have seen $\mathscr{D}^a$ is the associative algebra of all regular differential operators on the punctured complex plane $\mathbb{C}^\times$, i.e., operators of the form

$$E = e_k(z)\partial_z^k + e_{k-1}(z)\partial_z^{k-1} + \cdots + e_0(z)\,, \quad \text{where } e_i(z) \in \mathbb{C}[z, z^{-1}]\,. \tag{1.5.1}$$

It is not difficult to see that the cocycle Ψ given by (1.4.2) is given by the following formula:

$$\Psi(f\partial_z^m, g\partial_z^n) = \frac{m!\,n!}{(m+n+1)!}\,\mathrm{Res}_{z=0}\, dz\, f^{(n+1)}(z) g^{(m)}(z)\,, \tag{1.5.2}$$

where as usual $f^{(n)}$ stands for $\partial_z^n f$. This cocycle appeared (probably for the first time) in [KP]. It has been shown independently by several authors ([Li] and [F] among them) that $\widehat{\mathscr{D}}$ is the unique, up to isomorphism, non-trivial central extension of the Lie algebra $\mathscr{D}$ by a one-dimensional algebra.

It is, however, more convenient to write the differential operators as linear combinations of elements of the form $z^k f(D)$, where f is a polynomial in D, since it is easier to compute their product [cf. (1.1.1)]:

$$(z^m f(D))(z^k g(D)) = z^{m+k} f(D + k) g(D)\,. \tag{1.5.3}$$

The bracket in $\widehat{\mathscr{D}}$ is then given by

$$[z^r f(D), z^s q(D)] = z^{r+s}(F(D+s)g(D) - f(D)g(D+r) \\ + \Psi(z^r f(D), z^s g(D)))C\,, \tag{1.5.4}$$

where

$$\Psi(z^r f(D), z^s g(D)) = \begin{cases} \sum\limits_{-r \le j \le -1} f(j)g(j+r) & \text{if } r = -s > 0 \\ 0 & \text{if } r+s \neq 0 \text{ or } r = s = 0\,. \end{cases} \tag{1.5.5}$$

1.6. Consider now the associative algebra $\mathscr{D}_q^a$, the corresponding Lie algebra $\mathscr{D}_q$ and its central extension $\widehat{\mathscr{D}_q}$ in the case $q \neq 1$. Introduce the following basis of $\mathscr{D}_q^a$:

$$T_{m,n} = q^{\frac{1}{2}(m+1)n} z^m((q-1)D_q + 1)^n\,, \quad m \in \mathbb{Z},\ n \in \mathbb{Z}_+\,.$$

Then we have

$$T_{m,n} T_{m',n'} = q^{\frac{1}{2}(m'n - mn')} T_{m+m',n+n'}\,. \tag{1.6.1}$$

The cocycle (1.4.2) on the Lie algebra $\mathscr{D}_q$ becomes:

$$\Psi(T_{m,n} T_{m',n'}) = \delta_{m,-m'} \frac{\sinh(\hbar m(n+n'))}{\sinh(\hbar(n+n'))}\,, \tag{1.6.2}$$

where we let $q = e^{2\hbar}$. Consequently, the commutation relations of the Lie algebra $\widehat{\mathscr{D}_q}$ become:

$$[T_{m,n}, T_{m',n'}] = 2\sinh(\hbar(m'n - mn'))T_{m+m',n+n'} \\ + \delta_{m,-m'} \frac{\sinh(\hbar m(n+n'))}{\sinh(\hbar(n+n'))} C\,. \tag{1.6.3}$$

Remarks. (a) Commutation relations (1.6.3) correspond to the automorphism σ_q' of $\mathbb{C}[w]$ given by $\sigma_q'(w) = qw$ (which is equivalent to σ_q), and to the trace being an evaluation map at $w = 1$. The evaluation map at $w = 0$ gives the cocycle $\Psi_0(T_{m,n}, T_{m',n'}) = m\delta_{m,-m'}\delta_{n,-n'}$, which is equivalent to Ψ due to the argument of Remark 1.3(d).
(b) If we take $A = \mathbb{C}[w, w^{-1}]$, $\sigma(x) = qx$, where $q = e^{2\hbar} \neq 1$, and $\mathrm{tr}\left(\sum a_i w^i\right) = a_0$, then in the basis $T_{m,n} = q^{\frac{1}{2}mn} z^m w^m$ $(m, n \in \mathbb{Z})$ we obtain the commutation relations of the trigonometric Sin-Lie algebra:

$$[T_{m,n}, T_{m',n'}] = 2\sinh(\hbar(m'n - mn'))T_{m+m',n+n'} + m\delta_{m,-m'}\delta_{n,-n'} C\,.$$

2. Lie Algebras $\widehat{\mathscr{D}}$ and $\widehat{\mathscr{D}^{\mathscr{O}}}$.

2.1. Let as before $D = z\partial_z$ and let

$$L_k^n = z^k D^n \in \mathscr{D} \subset \widehat{\mathscr{D}} \quad (k \in \mathbb{Z}, n \in \mathbb{Z}_+).$$

Define the *order* and the *weight* by

$$\operatorname{ord} L_R^n = n, \quad \operatorname{wt} L_k^n = k, \quad \operatorname{ord} C = \operatorname{wt} C = 0.$$

It is clear from (1.5.4) and (1.6.1) that the order defines a filtration of $\widehat{\mathscr{D}}$:

$$\widehat{\mathscr{D}}^0 \subset \widehat{\mathscr{D}}^1 \subset \widehat{\mathscr{D}}^2 \subset \cdots, \tag{2.1.1}$$

and the weight defines the principal $\mathbb{Z}$-gradation of $\widehat{\mathscr{D}}$:

$$\widehat{\mathscr{D}} = \bigoplus_{j\in\mathbb{Z}} \widehat{\mathscr{D}}_j. \tag{2.1.2}$$

Note that we have:

$$\Psi(L_r^0, L_s^0) = \delta_{r,-s} r, \tag{2.1.3}$$

$$\Psi(L_r^1, L_s^1) = -\delta_{r,-s}\frac{r^3 - r}{6}, \tag{2.1.4}$$

$$\Psi(L_r^0, L_s^1) = \Psi(L_r^1, L_s^0) = \delta_{r,-s}\frac{r(r-1)}{2} \quad \text{if } r \geq 0. \tag{2.1.5}$$

It follows that $\widehat{\mathscr{D}}^0$ is isomorphic to the oscillator Lie algebra:

$$[L_r^0, L_s^0] = \delta_{r,-s} r C. \tag{2.1.6}$$

Furthermore, $\widehat{\mathscr{D}}^1$ contains a 1-parameter family of Virasoro algebras $\mathrm{Vir}(\beta)$, $\beta \in \mathbb{C}$, ("complementary" to $\widehat{\mathscr{D}}^0$) defined by

$$L_k(\beta) = -(L_k^1 + \beta(k+1)L_k^0), \tag{2.1.7}$$

so that

$$[L_r(\beta), L_s(\beta)] = (r-s)L_{r+s}(\beta) + \delta_{r,-s}\frac{r^3 - r}{12} C_{\mathrm{Vir}(\beta)}, \tag{2.1.8}$$

where

$$C_{\mathrm{Vir}(\beta)} = (12\beta^2 - 2)C. \tag{2.1.9}$$

Remark. $z^{n+s}\partial_z^n = z^s D(D-1)\cdots(D-n+1)$.

2.2. Let $\mathscr{O}$ be the algebra of all holomorphic functions on $\mathbb{C}$ with topology of uniform convergence on compact sets. We define a completion $\mathscr{D}^{a\mathscr{O}}$ of the (associative) algebra of differential operators on $\mathbb{C}^\times$ by considering differential operators of infinite order of the form $z^k f(D)$, where $f \in \mathscr{O}$. The usual product of differential operators extends to $\mathscr{D}^{a\mathscr{O}}$:

$$(z^r f(D))(z^s g(D)) = z^{r+s} f(D+s) g(D), \tag{2.2.1}$$

where by $f(D+s)$ we mean the power series expansion in D. The principal gradation extends as well: $\mathscr{D}^{a\mathscr{O}} = \bigoplus_{j\in\mathbb{Z}} \mathscr{D}_k^{a\mathscr{O}}$, where $\mathscr{D}_k^{a\mathscr{O}} = \{z^k f(D) | f(w) \in \mathscr{O}\}$. Identifying

$\mathscr{D}_k^{a\mathscr{O}}$ with $\mathscr{O}$ and $\mathscr{D}^{a\mathscr{O}}$ with the direct sum of $\mathscr{D}_k^{a\mathscr{O}}$ as topological vector spaces, we make $\mathscr{D}^{a\mathscr{O}}$ a topological associative algebra. It is a completion of the subalgebra $\mathscr{D}^{a}$.

We denote by $\mathscr{D}^{\mathscr{O}}$ the corresponding (topological) Lie algebra. Then the cocycle Ψ extends by continuity from $\mathscr{D}$ to a 2-cocycle on $\mathscr{D}^{\mathscr{O}}$ by formula (1.5.5). We let

$$\widehat{\mathscr{D}^{\mathscr{O}}} = \mathscr{D}^{\mathscr{O}} \oplus \mathbb{C}C$$

be the corresponding central extension. Note that for elements $z^r e^{\lambda D}$ ($r \in \mathbb{Z}, \lambda \in \mathbb{C}$) the commutator in $\widehat{\mathscr{D}^{\mathscr{O}}}$ is especially simple:

$$[z^r e^{\lambda D}, z^s e^{\mu D}] = (e^{\lambda s} - e^{\mu r}) z^{r+s} e^{(\lambda+\mu)D} + \delta_{r,-s} \frac{e^{-\lambda r} - e^{-\mu s}}{1 - e^{\lambda+\mu}} C\,. \tag{2.2.2}$$

Remarks. (a) One may consider $z^k e^{\lambda D}$ as a generating series for the L_k^n and derive (1.5.4) and (1.5.5) by taking derivatives of (2.2.2).
(b) Of course, $\mathscr{D}^{\mathscr{O}}$ is isomorphic to $\widetilde{\mathscr{O}}_{\sigma_1}$, and $\widehat{\mathscr{D}^{\mathscr{O}}}$ to $\widehat{\mathscr{O}}_{\sigma_1,\mathrm{tr}}$.
(c) Consider the following traces on $\mathscr{O}$:

$$\mathrm{tr}_{a,b}\, f(w) = f(a) - f(b), \qquad \text{where } a, b, \in \mathbb{C},$$
$$\mathrm{tr}_s^{[m]}\, f(w) = f^{(m)}(s), \qquad \text{where } s \in \mathbb{C},\ m \in \mathbb{N}\,.$$

Here and further $f^{(m)}$ stands for the m^{th} derivative of $f(w)$. We denote the corresponding cocycles by $\Psi_{a,b} := \Psi_{\sigma_1,\mathrm{tr}_{a,b}}$ and $\Psi_s^{(m)} := \Psi_{\sigma_1,\mathrm{tr}_s^{[m]}}$. On $\mathscr{D}^{\mathscr{O}}$ these cocycles are nontrivial (in continuous cohomology). But, due to Remark 1.3(d) when restricted to $\mathscr{D}$ they become trivial. Since

$$(\sigma_1 - 1)\frac{e^{xw} - 1}{e^x - 1} = e^{xw} \tag{2.2.3}$$

using Remark 1.3(d), we obtain the following explicit formulas for these trivial cocycles on $\mathscr{D}$:

$$\Psi_{a,b}(z^k f(D), z^r g(D)) = \delta_{k,-r} \Lambda_{a,b}([z^k f(D), z^{-k} g(D)])\,, \tag{2.2.4}$$
$$\Psi_s^{[m]}(z^k f(D), z^r g(D)) = \delta_{k,-r} \Lambda_s^{[m]}([z^k f(D), z^{-k} g(D)])\,, \tag{2.2.5}$$

where $\Lambda_{a,b}$ and $\Lambda_s^{[m]}$ are the linear functions on $\mathbb{C}[w]$ defined by the following generating series in x:

$$\Lambda_{a,b}(e^{xw}) = \frac{e^{ax} - e^{bx}}{e^x - 1}, \quad \Lambda_x^{(m)}(e^{xw}) = \frac{x^m e^{sx}}{e^x - 1}\,. \tag{2.2.6}$$

2.3. The following theorem describes closed ideals of $\mathscr{D}^{\mathscr{O}}$ (resp. $\widehat{\mathscr{D}^{\mathscr{O}}}$).

Theorem. (a) *The center Z of $\mathscr{D}^{\mathscr{O}}$ consists of elements of the form $f(D)$, where $f(w) \in \mathscr{O}$ is a 1-periodic function* (i.e., $f(w+1) = f(w)$). *The center of $\widehat{\mathscr{D}^{\mathscr{O}}}$ is $\widehat{Z} = Z \oplus \mathbb{C}C$.*
(b) *Let I be an ideal of $\mathscr{O}$ which is invariant under the translation $w \mapsto w+1$, and let $I' = \{f(D) - f(D+1) | f(w) \in I\}$* (resp. $\widehat{I'} = \{f(D) - f(D+1) + f(0)C | f(w) \in I\}$).

Let Y be a subspace of Z (resp. $\widehat{Z}$). *Let $I^{(k)} = z^k I \subset \mathscr{D}^{\mathscr{O}}$ for $k \neq 0$ and let $I^{(0)} = I' + Y$* (resp. $= \widehat{I}' + Y$). *Then*

$$J(I,Y) = \bigoplus_{k \in \mathbb{Z}} I^{(k)}$$

is a closed ideal of $\mathscr{D}^{\mathscr{O}}$ (resp .$\widehat{\mathscr{D}^{\mathscr{O}}}$).

(c) *Every closed ideal of $\mathscr{D}^{\mathscr{O}}$* (resp. *of $\widehat{\mathscr{D}^{\mathscr{O}}}$*) *is one of the $J(I,Y)$.*

Proof. The statements (a) and (b) are clear. We shall prove (c) for $\mathscr{D}^{\mathscr{O}}$, the proof for $\widehat{\mathscr{D}^{\mathscr{O}}}$ being the same. Let J be a closed ideal of the Lie algebra $\mathscr{D}^{\mathscr{O}}$. Since J is $\operatorname{ad} D$-stable, it follows that J is graded ideal:

$$J = \bigoplus_{k \in \mathbb{Z}} z^k I_k \,,$$

where I_k is a closed subspace of $\mathscr{O}$. We have

$$\left[D^2, z^k f(D)\right] = 2kz^k Df(D) + k^2 z^k f(D)\,.$$

It follows that $wI_k \subset I_k$ if $k \neq 0$, i.e., that I_k for $k \neq 0$ is an ideal of $\mathscr{O}$. Furthermore, we let for $k \in \mathbb{Z}$ and an ideal I of $\mathscr{O}$:

$$I[k] = \{f \in \mathscr{O} \mid f(w+k) \in I\}\,.$$

We claim that

$$I_k[\pm 1] + I_k \subset I_{k\pm 1} \quad \text{if } k \neq 0 \text{ and } k \pm 1 \neq 0\,. \tag{2.3.1$_\pm$}$$

Indeed, since

$$\left[z^{\pm 1}, z^k f(D)\right] = z^{k\pm 1}(f(D) - f(D \pm 1))\,,$$

we see that if $f(w) \in I_k$ then $f(w) - f(w \pm 1) \in I_{k\pm 1}$. Since I_k and $I_{k\pm 1}$ are ideals, $wf(w) \in I_k$ and $wf(w) - (w \pm 1)f(w \pm 1) \in I_{k\pm 1}$. Thus, $f(w)$, $f(w \pm 1) \in I_{k\pm 1}$, completing the proof of (2.3.1)$_\pm$. We conclude, in particular, that

$$I_1 = I_2 = \cdots \quad \text{and} \quad I_{-1} = I_{-2} = \cdots\,. \tag{2.3.2}$$

Next, we prove that

$$I_k = I_k[n] \quad \text{for all } n \in \mathbb{Z}, \text{ provided that } k \neq 0\,. \tag{2.3.3}$$

Indeed, due to (2.3.2) we may assume that $|k| \geq 2$, so that both numbers $k+1$ or $k-1$ are non-zero. Applying (2.3.1)$_+$ to I_k and (2.3.1)$_-$ to I_{k-1}, we get

$$I_k[1] + I_k \subset I_{k+1} \quad \text{and} \quad I_{k+1}[-1] + I_{k+1} \subset I_k\,.$$

It follows that $I_k = (I_k[1])[-1] \subset I_{k+1}[-1] \subset I_k$, and $I_k = (I_k[-1])[1] \subset I_k[1] \subset I_{k+1} = I_k$. Hence $I_k = I_k[\pm 1]$ proving (2.3.3), which means that each I_k is invariant under the translation $w \mapsto w+1$.

In order to complete the proof of the theorem, it remains to show that

$$I_1 = I_{-1}\,, \tag{2.3.4}$$

$$I_0 \supset I'\,, \tag{2.3.5}$$

$$I_0 \subset I' + Z\,. \tag{2.3.6}$$

First, we prove that $I_{-1} \subset I_1$; the reverse inclusion is proved similarly. Let $f(w) \in I_{-1}$; we have:

$$[z, [z, z^{-1} f(D)]] = z(f(D) - 2f(D+1) + f(D+2)).$$

Hence $f(w) - 2f(w+1) + f(w+2) \in I_1$. Considering $wf(w) \in I_{-1}$, we conclude that $f(w+2) - f(w+1) \in I_1$ and $f(w+1) - f(w) \in I_1$. Considering $wf(w)$ once more, we conclude that $f(w) \in I_1$, proving (2.3.4).

The inclusion (2.3.5) follows from the inclusions $[z, I_{-1}] \subset I_0$ and $[z^{-1}, I_1] \subset I_0$. Finally, in order to prove (2.3.6), note that the map $\varphi := (\operatorname{ad} z)^2 : z^{-1} I_{-1} \to z I_1$ is surjective (this follows from the proof of (2.3.4)). Let now $f \in I_0$ and let $g \in z^{-1} I_{-1}$ be a pre-image of $[z, f(D)]$ under the map φ. Then $[z, g] \in I'$ and $[f - [z, g], z] = 0$, hence $f - [z, g] \in Z$, proving (2.3.6). $\square$

We have the following corollary of the proof:

Corollary. *The Lie algebra $\mathscr{D}/\mathbb{C}$ is simple.*

2.4. Consider a parabolic subalgebra $\mathfrak{p}$ of $\widehat{\mathscr{D}}$:

$$\mathfrak{p} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{p}_j, \quad \text{where } \mathfrak{p}_j = \widehat{\mathscr{D}}_j \text{ for } j \geq 0 \text{ and } \mathfrak{p}_j \neq 0 \text{ for some } j < 0.$$

For each positive integer k we have: $\mathfrak{p}_{-k} = z^{-k} I_{-k}$, where I_{-k} is a subspace of $A = \mathbb{C}[w]$. Since

$$[f(D), z^{-k} P(D)] = z^{-k}(f(D-k) - f(D)) P(D),$$

we see that I_{-k} is an ideal of the polynomial algebra A. It is clear that $I_{-k} \neq 0$ for all $k = 1, 2, \ldots$. Let $b_k(w)$ be the monic (i.e., with the leading coefficient equal to 1) polynomial which is a generator of the ideal I_{-k}. Thus, to a parabolic subalgebra $\mathfrak{p}$ we have associated a sequence of monic polynomials $b_1 = b_1(w)$, $b_2 = b_2(w)$, $\ldots$. The polynomials $b_k, k = 1, 2, \ldots$ are called the *characteristic* polynomials of $\mathfrak{p}$.

Lemma. *Let $\{b_k\}_{k \in \mathbb{N}}$ be the sequence of characteristic polynomials of a parabolic subalgebra $\mathfrak{p}$ of the Lie algebra $\widehat{\mathscr{D}}$. Then*

(a) *$b_k(w)$ divides $b_{k+1}(w)$ and $b_{k+1}(w+1)$ for all $k \in \mathbb{N}$.*
(b) *$b_{k+l}(w)$ divides $b_k(w-l) b_l(w)$ for all $k, l \in \mathbb{N}$.*

Proof. Since $[z, z^{-k-1} b_{k+1}(D)] = z^{-k}(b_{k+1}(D) - b_{k+1}(D+1))$, wee see that $b_k(w)$ divides $b_{k+1}(w) - b_{k+1}(w+1)$. Since $[z, z^{-k-1} D b_{k+1}(D)] = z^{-k}(D b_{k+1}(D) - (D+1) b_{k+1}(D+1))$, we see that $b_k(w)$ divides $w(b_{k+1}(w) - b_{k+1}(w+1)) + b_{k+1}(w+1)$, proving (a).

The proof of (b) is similar by computing the commutators $[z^{-k} b_k(D), z^{-l} b_l(D)]$ and $[z^{-k} b_k(D), z^{-l} D b_l(D)]$. $\square$

Given a monic polynomial $b(w)$, we let

$$b_k^{\min}(w) = b(w) b(w-1) \ldots b(w-k+1),$$
$$b_k^{\max}(w) = lcm\{b(w), b(w-1), \ldots, b(w-k+1)\}.$$

It is easy to see that there exist (unique) parabolic subalgebras, which we denote by $\mathfrak{p}_{\min}(b)$ and $\mathfrak{p}_{\max}(b)$, for which the characteristic polynomials are $\{b_k^{\min}(w)\}$ and $\{b_k^{\max}(w)\}$ respectively. We clearly have

$$\dim \widehat{\mathscr{D}}_{-k} / \mathfrak{p}_{\min}(b)_{-k} = k \deg b. \tag{2.4.1}$$

Lemma 2.4 implies the following.

Proposition. *Let b be a monic polynomial and let $\mathfrak{p}$ be a parabolic subalgebra such that $b_1(w) = b$. Then*

$$\mathfrak{p}_{\min}(b) \subset \mathfrak{p} \subset \mathfrak{p}_{\max}(b).$$

In particular, if the difference of any two distinct roots of b is not an integer, then

$$\mathfrak{p} = \mathfrak{p}_{\min}(b) = \mathfrak{p}_{\max}(b). \qquad \square$$

Remark. $\mathfrak{p}_{\min}(b) = \mathfrak{p}((b))$ (cf. (1.3.3)).

2.5. Given a monic polynomial $b = b(w)$, consider the following subspace of $\widehat{\mathscr{D}_0}$:

$$\widehat{\mathscr{D}_0^b} = \{b(D)g(D) - b(D+1)g(D+1) + b(0)g(0)C | g(w) \in \mathbb{C}[w]\}.$$

In order to study modules over $\widehat{\mathscr{D}}$ induced from its parabolic subalgebras, we need the following proposition.

Proposition. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\widehat{\mathscr{D}}$ and let $b = b(w)$ be its first characteristic polynomial. Then*

$$[\mathfrak{p}, \mathfrak{p}] = \left(\bigoplus_{k\neq 0} \mathfrak{p}_{-k}\right) \bigoplus \widehat{\mathscr{D}_0^b}.$$

In particular,

$$\dim \mathfrak{p}/[\mathfrak{p}, \mathfrak{p}] = \dim \mathfrak{p}_0/[\mathfrak{p}, \mathfrak{p}]_0 = \deg b(w). \tag{2.5.1}$$

Proof. Note that $[\mathfrak{p}, \mathfrak{p}]_0 = [\mathfrak{p}_1, \mathfrak{p}_{-1}]$ and that $[zf(D+1), z^{-1}b(D)g(D)] = b(D)f(D)g(D) - b(D+1)f(D+1)g(D+1) + b(0)f(0)g(0)C$. The rest is straightforward. $\square$

2.6. Let $\mathfrak{g}$ be a finite-dimensional semi simple Lie algebra over $\mathbb{C}$ and let $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$ be its root space decomposition with respect to a Cartan subalgebra $\mathfrak{g}_0$. An embedding $\mathfrak{g} \subset \mathscr{D}^{\mathscr{O}}$ is called *graded* if $\mathfrak{g}_\alpha \subset \mathscr{D}^{\mathscr{O}}_{k(\alpha)}$ for all α.

Proposition. (a) *The graded embeddings in $\mathscr{D}^{\mathscr{O}}$ of the Lie algebra $sl_2(\mathbb{C})$ with the standard basis E, H, F are parameterized by $k \in \mathbb{Z}\setminus\{0\}$ and by k-periodic functions $f, g \in \mathscr{O}$ as follows:*

$$H = \frac{2}{k}D + f(D), \quad E = z^k x(D), \quad F = z^{-k}y(D),$$

where

$$x(D)y(D+k) = -\left(\frac{D}{k}\right)^2 + \frac{D}{k}(f(D)+1) + g(D).$$

(b) *The only graded embeddings of $sl_2(\mathbb{C})$ in $\mathscr{D}$ are as follows* ($k \in \mathbb{Z}\setminus\{0\}$, $\lambda, \mu, \in \mathbb{C}$):

$$H = \frac{2}{k}D + \lambda, \quad E = z^k x(D), \quad F = z^{-k}y(D),$$

with the following four possibilities for $x(D)$ and $y(D)$:

(i) $x(D) = \frac{1}{k}D^2 + \frac{1}{k}(\lambda+1)D + \mu$, $y(D) = -\frac{1}{k}$

(ii) $x(D) = \frac{1}{k}D - \mu, \quad y(D) = -\frac{1}{k}D - \lambda - \mu - 1$;
(iii) *and* (iv) *are obtained from* (i) *and* (ii) *by the substitution* $x'(D) = y(D-k)$, $y'(D) = x(D+k)$.

(c) *A semi simple Lie algebra of rank* ≥ 2 *has no graded embeddings in* $\mathscr{D}^{\mathscr{O}}$.

Proof. Note that the equation

$$[h(D), z^k x(D)] = \lambda z^k x(D)$$

implies that

$$h(D+k) - h(D) = \lambda \, .$$

All solutions of the latter equation are $h(0) = \frac{\lambda}{k}D + f(D)$, where $f(D+k) = f(D)$. Now (a) easily follows. (b) follows from (a). If rank $\mathfrak{g} \geq 2$, we always can find an element $h \in \mathfrak{g}_0$ such that $\alpha(h)/\beta(h)$ is an irrational number for two distinct roots α and β. Hence (a) implies (c). $\square$

Remarks. (a) Let L_n denote the subalgebra of operators of $\mathscr{D}^{\mathscr{O}}$ leaving invariant the subspace $\sum_{k=0}^{n} \mathbb{C}z^k$ of $\mathbb{C}[z, z^{-1}]$, and let I_n denote the ideal of L_n of operators acting on this subspace trivially. By Jacobson's density theorem we have an exact sequence of associative algebras:

$$0 \to I_n \to L_n \to \mathrm{Mat}_n(\mathbb{C}) \to 0 \, .$$

Proposition 2.6(b) shows that this is a non-split exact sequence.
(b) It follows from the proof of the proposition that $\mathrm{ad}\, \mathscr{D}_0^{\mathscr{O}}$ is not diagonalizable on $\mathscr{D}^{\mathscr{O}}$.

3. Interplay between $\widehat{\mathscr{D}}$ and $\widehat{gl}(\infty)[m]$

3.1. Let R be an associative algebra over $\mathbb{C}$. Denote by R^∞ a free R-module with a fixed basis $\{v_j\}_{j\in\mathbb{Z}}$. As usual, define the operators E_{ij} by

$$E_{ij} v_k = \delta_{jk} v_i \, . \tag{3.1.1}$$

Denote by $\widetilde{M}(\infty, R)$ the associative subalgebra of $\mathrm{End}\, R^\infty$ consisting of all operators $\sum_{i,j\in\mathbb{Z}} a_{ij}E_{ij}$ whose matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ have a finite number of non-zero diagonals. Letting $\deg E_{ij} = j - i$ defines the principal $\mathbb{Z}$-gradation:

$$\widetilde{M}(\infty, R) = \bigoplus_{j\in\mathbb{Z}} \widetilde{M}(\infty, R)_j \, . \tag{3.2.1}$$

Fix $s \in \mathbb{C}$ and a nilpotent element $t \in R$. Consider the free R-module $R[z, z^{-1}]z^s$ and identify it with R^∞ by choosing the basis $v_j = z^{-j+s}$, $j \in \mathbb{Z}$. By associating to an element $z^k f(D) \in \mathscr{D}^a$ the operator $z^k f(D+t)$ on $R[z, z^{-1}]z^s$, we obtain an embedding $\varphi_{s,t} : \mathscr{D}^a \hookrightarrow \widetilde{M}(\infty, R)$ of associative algebras over $\mathbb{C}$, which is compatible with the principal gradations. Explicitly:

$$\varphi_{s,t}(z^k f(D)) = \sum_{j\in\mathbb{Z}} f(-j+s+t) E_{j-k,j} \, . \tag{3.1.3}$$

The homomorphism $\varphi_{s,t} : \mathscr{D}^a \to \widetilde{M}(\infty, R)$ extends via (3.1.3) to a homomorphism

$$\varphi_{s,t} : \mathscr{D}^{a\mathscr{O}} \to \widetilde{M}(\infty, R).$$

3.2. Given a non-negative integer m, consider the algebra of truncated polynomials $R_m = \mathbb{C}[t]/(t^{m+1})$, and let $\widetilde{M}(\infty)[m] = \widetilde{M}(\infty, R_m)$. We denote the homomorphism $\varphi_{s,t} : \mathscr{D}^{a\mathscr{O}} \to \widetilde{M}(\infty)[m]$ given by (3.1.3) by $\varphi_s^{[m]}$. By Taylor's formula we have:

$$\varphi_s^{[m]}(z^k f(D)) = \sum_{i=0}^{m} \sum_{j\in\mathbb{Z}} \frac{f^{(i)}(-j+s)}{i!} t^i E_{j-k,j}\,. \tag{3.2.1}$$

Let

$$I_s^{[m]} = \left\{ f \in \mathscr{O} | f^{(i)}(n+s) = 0 \text{ for all } n \in \mathbb{Z} \text{ and all } i = 0, \ldots, m \right\},$$

and let $J_s^{[m]} = \bigoplus_{k\in\mathbb{Z}} z^k I_s^{[m]} \in \mathscr{D}^{a\mathscr{O}}$. We clearly have:

$$\operatorname{Ker} \varphi_s^{[m]} = J_s^{[m]}\,. \tag{3.2.2}$$

Fix now $\vec{s} = (s_1, \ldots, s_N) \in \mathbb{C}^N$ such that $s_i - s_j \notin \mathbb{Z}$ if $i \neq j$, and fix $\vec{m} = (m_1, \ldots, m_N) \in \mathbb{Z}_+^N$. Let $\widetilde{M}(\infty)[\vec{m}] = \bigoplus_{i=1}^{M} \widetilde{M}(\infty)[m_i]$. Consider the homomorphism

$$\varphi_{\vec{s}}^{[\vec{m}]} = \bigoplus_i \varphi_{s_i}^{[m_i]} : \mathscr{D}^{a\mathscr{O}} \to \widetilde{M}(\infty)[\vec{m}]\,.$$

Proposition. *We have an exact sequence of $\mathbb{Z}$- graded associative algebras*:

$$0 \to J_s^{[\vec{m}]} \to \mathscr{D}^{a\mathscr{O}} \xrightarrow{\varphi_{\vec{s}}^{[\vec{m}]}} \widetilde{M}(\infty)[\vec{m}] \to 0\,,$$

where $J_{\vec{s}}^{[\vec{m}]} = \bigcap_{i=1}^{N} J_{s_i}^{[m_i]}$.

Proof. It is clear from (3.2.2) that $\operatorname{Ker}\varphi_{\vec{s}}^{[\vec{m}]} = J_{\vec{s}}^{[\vec{m}]}$. The surjectivity of $\varphi_{\vec{s}}^{[\vec{m}]}$ follows from the following well-known fact: for every discrete sequence of points in $\mathbb{C}$ and a non-negative integer m there exists $f(w) \in \mathscr{O}$ having prescribed values of its first m derivatives at these points.

3.3. We denote by $\widetilde{gl}(\infty)[m]$ the $\mathbb{Z}$-graded Lie algebra over $\mathbb{C}$ corresponding to the associative algebra $\widetilde{M}(\infty)[m]$ viewed as an algebra over $\mathbb{C}$. Consider the following 2-cocycle on $\widetilde{gl}(\infty)[m]$ with values in R_m:

$$C(A,B) = \operatorname{tr}[J,A]B\,, \tag{3.3.1}$$

where $J = \sum_{i\leq 0} E_{ii}$, and denote by $\widehat{gl}(\infty)[m] = \widetilde{gl}(\infty)[m] + R_m$ the corresponding central extension. The $\mathbb{Z}$-gradation of this Lie algebra extends from $\widetilde{gl}(\infty)[m]$ by letting $wt R_m = 0$.

The homomorphism $\varphi_s^{[m]}$ of the associative algebras defines a homomorphism of the corresponding Lie algebras, which we denote by the same letter:

$$\varphi_s^{[m]} : \mathscr{D} \to \widetilde{gl}(\infty)[m] \quad \text{and} \quad \varphi_s^{[m]} : \mathscr{D}^{\mathscr{O}} \to \widetilde{gl}(\infty)[m] .$$

Denote by $\Psi_s^{[m]}$ the restriction of the cocycle C given by (2.5.1) to $\varphi_s^{[m]}(\mathscr{D}^{\mathscr{O}})$. This gives us the following R_m-valued cocycle on $\mathscr{D}^{\mathscr{O}}$:

$$\Psi_s^{[m]} = \Psi + \Psi_{s,0} + \sum_{j=1}^{m} \Psi_s^{(j)} \frac{t^j}{j!} , \tag{3.3.2}$$

where the cocycle Ψ is given by (1.5.5) and the cocycles $\Psi_{s,0}$ and $\Psi_s^{(j)}$ are defined in Remark 2.2(c). Using (2.2.4–6), we thus obtain the following proposition.

Proposition. *The $\mathbb{C}$-linear map $\widehat{\varphi}_s^{[m]} : \widehat{\mathscr{D}} \to \widehat{gl}(\infty)[m]$ defined by*

$$\widehat{\varphi}_s^{[m]}|_{\widehat{\mathscr{D}}_j} = \varphi_s^{[m]}|_{\mathscr{D}_j} \quad \text{if } j \neq 0 , \tag{3.3.3}$$

$$\widehat{\varphi}_s^{[m]}(e^{xD}) = \varphi_s^{[m]}(e^{xD}) - \frac{e^{sx}-1}{e^x-1} - \sum_{j=1}^{m} \frac{x^j e^{sx}}{e^x-1} t^j/j! ,$$

$$\widehat{\varphi}_s^{[m]}(C) = 1 \in R_m \tag{3.3.4}$$

is a homomorphism of Lie algebras over $\mathbb{C}$.

3.4. Define an automorphism ν of the algebra $\widetilde{M}(\infty, \mathbb{C})$ by letting

$$\nu(E_{ij}) = E_{i+1,j+1} .$$

Let $\varphi_s = \varphi_{s,0} : \mathscr{D}^{a\mathscr{O}} \to \widetilde{M}(\infty, \mathbb{C})$ (see Sect. 3.1). Then we have

$$\varphi_{s+1}(z^k f(D)) = \nu \varphi_s(z^k f(D)) . \tag{3.4.1}$$

Definition. A *monodromic loop* is a map $f : \mathbb{C} \to \widetilde{M}(\infty, \mathbb{C})$ such that

(i) f is holomorphic on $\mathbb{C}$, i.e., $f(w) = \sum_{ij} f_{ij}(w) E_{ij}$, where $f_{ij} \in \mathscr{O}$,

(ii) $f(w+1) = \nu f(w)$, i.e., $f_{ij}(w+1) = f_{i+1,j+1}(w)$.

We let $\mathscr{L}_\nu \widetilde{M}(\infty)$ denote the associative algebra of all monodromic loops. It clearly inherits from $\widetilde{M}(\infty, \mathbb{C})$ the principal gradation.

Define a linear map $\varphi : \mathscr{D}^{a\mathscr{O}} \to \mathscr{L}_\nu \widetilde{M}(\infty)$ by letting

$$\varphi(E) = \operatorname{loop}\{s \to \varphi_s(E)\} . \tag{3.4.2}$$

This is a homomorphism of associative algebras. The inverse homomorphism φ^{-1} is constructed as follows. Given $f(w) = \sum_j f_j(w) E_{j,j+k} \in \mathscr{L}_\nu \widetilde{M}(\infty)$ (a monodromic loop concentrated on the k^{th} diagonal), we let

$$\varphi^{-1}(f(w)) = z^k f_0(D) . \tag{3.4.3}$$

Thus we obtain the following result.

Proposition. *The map φ is an isomorphism of $\mathbb{Z}$-graded associative algebras*

$$\varphi : \mathscr{D}^{a\mathscr{O}} \xrightarrow{\sim} \mathscr{L}_\nu \widetilde{M}(\infty)\,.$$

Remark. Monodromic loops are sections of the vector bundle on the cylinder $\mathbb{C}/\mathbb{Z}$ with fiber $\widetilde{M}(\infty, \mathbb{C})$ and transition function ν in a small neighbourhood of the line $\mathrm{Re}\, w = 1$.
Denote by $\mathscr{L}_\nu \widetilde{gl}(\infty)$ the Lie algebra obtained from $\mathscr{L}_\nu \widetilde{M}(\infty)$ by taking the usual bracket. For each $s \in \mathbb{C}$ define a 2-cocycle C_s on this Lie algebra by

$$C_s(f(w), g(w)) = C(f(s), g(s)), \quad \text{where } f(w), g(w) \in \mathscr{L}_\nu \widetilde{gl}(\infty)\,. \tag{3.4.4}$$

It is easy to see that under the isomorphism

$$\varphi : \mathscr{D}^{\mathscr{O}} \xrightarrow{\sim} \widetilde{gl}(\infty)$$

given by Proposition 2.4, the cocycle C_s induces the following cocycle on $\mathscr{D}^{\mathscr{O}}$:

$$\Psi_s(z^k f(D), z^m g(D)) = \begin{cases} \sum\limits_{-k \leq j \leq -1} f(j+s)g(j+k+s) & \text{if } k = -m \geq 0\,, \\ 0 & \text{if } k + m \neq 0\,. \end{cases}$$

Denote by $\mathscr{L}_\nu \widetilde{gl}(\infty)^\wedge$ the central extension of $\mathscr{L}_\nu \widetilde{gl}(\infty)$ corresponding to the cocycle C_0. Then the isomorphism $\varphi : \mathscr{D}^{\mathscr{O}} \xrightarrow{\sim} \mathscr{L}_\nu \widetilde{gl}(\infty)$ lifts to the isomorphism $\widehat{\varphi} : \widehat{\mathscr{D}^{\mathscr{O}}} \xrightarrow{\sim} \mathscr{L}_\nu \widetilde{gl}(\infty)^\wedge$.

4. Quasifinite Highest Weight Modules over $\widehat{\mathscr{D}}$

4.1. Let b be a monic polynomial and let $\lambda \in \widehat{\mathscr{D}}_0^*$ be such that $\lambda|_{\widehat{\mathscr{D}}_0^b} = 0$ (see Sect. 2.5). Consider the parabolic subalgebra $\mathfrak{p}$ whose first characteristic polynomial is b and denote by $M(\lambda; b)$ the generalized Verma module $M(\widehat{\mathscr{D}}, \mathfrak{p}, \lambda)$.

Definition. A Verma module $M(\lambda)$ over $\widehat{\mathscr{D}}$ is called *highly degenerate* if there exists a singular vector in $M(\lambda)_{-1}$.

The following proposition follows from Propositions 2.4 and 2.5 and formula (2.4.1).

Proposition. *The following conditions on $\lambda \in \widehat{\mathscr{D}}_0^*$ are equivalent:*

(i) $M(\lambda)$ *is highly degenerate;*
(ii) $L(\lambda)$ *is quasifinite;*
(iii) $L(\lambda)$ *is a quotient of a generalized Verma module $M(\lambda; b)$ for some monic polynomial b.* $\square$

Let $L(\lambda)$ be a quasifinite irreducible highest weight module over $\widehat{\mathscr{D}}$. According to Proposition 4.1, $(z^{-1}b(D))v_\lambda = 0$ for some monic polynomial $b(w)$. Such monic polynomial of minimal degree is called the *characteristic polynomial* of $L(\lambda)$. Note that $L(\lambda)$ is a quotient of $M(\lambda; b)$, where b is the characteristic polynomial of $L(\lambda)$.

4.2. We shall characterize $\lambda \in \widehat{\mathscr{D}}_0^*$ by its labels $\Delta_n = -\lambda(D^n)$ and the central charge $c = \lambda(C)$. Introduce the generating series

$$\Delta_\lambda(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Delta_n .$$

Recall that a *quasipolynomial* is a linear combination of functions of the form $p(x)e^{\alpha x}$, where $p(x)$ is a polynomial and $\alpha \in \mathbb{C}$. Recall the following well-known fact.

Lemma. *A formal power series is a quasipolynomial if and only if it satisfies a non-trivial linear differential equation with constant coefficients.*

Theorem. *A $\widehat{\mathscr{D}}$-module $L(\lambda)$ is quasifinite if and only if*

$$\Delta_\lambda(x) = \frac{\phi(x)}{e^x - 1} ,$$

where $\phi(x)$ is a quasipolynomial such that $\phi(0) = 0$.

Proof. It follows from Propositions 4.1 and 2.5 that $L(\lambda)$ is quasifinite if and only if there exists a monic polynomial

$$b(w) = w^N + f_{N-1}w^{N-1} + \cdots + f_0$$

such that for all $s = 0, 1, \ldots$ we have:

$$\lambda(D^s b(D) - (D+1)^s b(D+1) + b(0)\delta_{s,0}c) = 0 .$$

This condition can be rewritten as follows:

$$\sum_{n=0}^{N} f_n F_{n+s} = 0 \quad \text{for all } s = 0, 1, \ldots , \tag{4.2.1}$$

where

$$F_n = \delta_{n,0}c + \sum_{j=0}^{n-1} \binom{n}{j} \Delta_j . \tag{4.2.2}$$

Introducing the generating series

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} F_n ,$$

we may rewrite (4.2.1) in the form

$$\left(\sum_{n=0}^{N} f_n \left(\frac{d}{dx} \right)^n \right) F(x) = 0 . \tag{4.2.3}$$

Thus, by Lemma 4.2, $L(\lambda)$ is quasifinite if and only if $F(x)$ is a quasipolynomial.

But (4.2.2) can be rewritten in terms of generating series as follows:

$$c - F(x) = (e^x - 1)\Delta_\lambda(x). \tag{4.2.4}$$

The theorem follows. $\square$

From the proof of the theorem we obtain the following.

Corollary. *Let $L(\lambda)$ be a quasifinite irreducible highest weight module over $\widehat{\mathscr{D}}$, and let $b(w)$ be its characteristic polynomial. By Theorem 4.2, $F(x) = (1 - e^x)\Delta_\lambda(x) + c$ is a quasipolynomial. Let $F^{(N)} + f_{N-1}F^{(N-1)} + \cdots + f_0 = 0$ be the minimal order linear differential equation with constant coefficients satisfied by $F(x)$. Then $b(w) = w^N + f_{N-1}w^{N-1} + \cdots + f_0$.* $\square$

4.3. In this section we show that any quasifinite $\widehat{\mathscr{D}}$-module V may be extended "by continuity" at lest to all the $\widehat{\mathscr{D}_k^{\mathscr{O}}}$ for $k \neq 0$.
We shall need the following fact.

Lemma. *The map $\varphi : \mathscr{O} \to \mathscr{O}$ given by*

$$\varphi\left(\sum_{n=0}^{\infty} f_n z^n\right) = \sum_{n=0}^{\infty} |f_n| z^n$$

is continuous.

Proof. Given $f = \sum_{n=0}^{\infty} f_n z^n \in \mathscr{O}$, where $f_n = |f_n|e^{i\theta_n}$ and $\theta_n \in \mathbb{R} \in \mathbb{C}$, we let

$$f^0(z) = \sum_{n=0}^{\infty} e^{-i\theta_n} z^n, \quad f^*(z) = \sum_{n=0}^{\infty} |f_n| z^n.$$

Let $B_R = \{z \in \mathbb{C} |\ |z| \leq R\}$ denote the disk of radius R and let C_R be its boundary. Note that $f^0(z)$ is holomorphic in each $B_{1-\varepsilon}$ for $0 < \varepsilon < 1$ and that $\max_{B_{1-\varepsilon}} |f^0(z)| \leq \frac{1}{\varepsilon}$.
We need to estimate $|f^*(z)|$ on each disk B_R. Take $R_1 > R$ and note that for $|w| < R_1$ we have:

$$f^*(w) = \frac{1}{2\pi i} \int_{C_{R_1}} f(z) f^0\left(\frac{w}{z}\right) \frac{dz}{z}. \tag{4.3.1}$$

From (4.3.1) we see that

$$\max_{B_R} |f^*(w)| \leq \max_{B_{R_1}} |f(z)| \cdot \max_{\substack{w \in B_R \\ |z| = R_1}} \left| f^0\left(\frac{w}{z}\right)\right| \leq \frac{1}{1 - R/R_1} \max_{B_{R_1}} |f(z)|. \qquad \square$$

Proposition. *Let V be a quasifinite $\widehat{\mathscr{D}}$-module. Then the action of $\widehat{\mathscr{D}}$ on V naturally extends to the action of $\widehat{\mathscr{D}_k^{\mathscr{O}}}$ on V for any $k \neq 0$.*

Proof. Let $V = \bigoplus_p V_p$ be the $\mathbb{Z}$-gradation of V, $\dim V_p < \infty$ for all p. Consider the space

$$\mathrm{Hom}(V, V) = \bigoplus_{p,q} \mathrm{Hom}(V_p, V_q)$$

with the topology of direct sum of finite dimensional spaces $\mathrm{Hom}(V_p, V_q)$. We can assume that the V_p are normed spaces, and spaces $\mathrm{Hom}(V_p, V_q)$ have induced norms $||, ||_{p,q}$.

We will show that map $\widehat{\mathscr{D}_k} \to \mathrm{Hom}(V, V)$ for $k \neq 0$ is continuous. To do this we have to estimate the norm of the operator $z^k D^n$ in the space $\mathrm{Hom}(V_p, V_{p+k})$ for fixed k and p and for arbitrary n. We have:

$$z^k D^n = \frac{1}{(2k)^n}(\mathrm{ad}\, D^2 - k^2)^n z^k . \tag{4.3.2}$$

The operator $B = \mathrm{ad}\, D^2 - k^2 : \mathrm{Hom}(V_p, V_{p+k}) \to \mathrm{Hom}(V_p, V_{p+k})$ acts between finite-dimensional normed spaces, hence we obtain from (4.3.2):

$$\left|\left|z^k D^n\right|\right|_{p,p+k} \leq A \cdot \alpha^n, \quad \text{where} \quad A = ||z^k||, \quad \alpha = ||B/2k|| . \tag{4.3.3}$$

It follows that

$$\begin{aligned} ||z^k f(D)||_{p,p+k} &= \left|\left| \sum_{n\geq 0} f_n z^k D^n \right|\right|_{p,p+k} \leq \sum_{n\geq 0} |f_n| \; ||z^k D^n||_{p,p+k} \\ &\leq A \cdot \sum_{n\geq 0} |f_n| \alpha^n = A\varphi(f)(\alpha) . \end{aligned}$$

Thus, by Lemma 4.3, the map $\widehat{\mathscr{D}_k} \to \mathrm{Hom}(V, V)$ is continuous for $k \neq 0$. Hence this map may be extended to the completion: $\widehat{\mathscr{D}_k^{\mathscr{O}}} \to \mathrm{Hom}(V, V)$ (the completion of $\mathbb{C}[w]$ in topology of uniform convergence on compact sets is $\mathscr{O}$). $\square$

4.4. We return now to the $\mathbb{Z}$-graded complex Lie algebra $\mathfrak{g}^{[m]} := \widehat{gl}(\infty)[m] = \widetilde{gl}(\infty, R_m) + \underset{\sim}{R_m}$ introduced in Sect. 3.3. Recall that it is a central extension of the Lie algebra $\widetilde{gl}(\infty, R_m)$ over $\mathbb{C}$ by the $m+1$-dimensional space R_m.

An element $\lambda \in (\mathfrak{g}_0^{[m]})^*$ is usually given by its *labels*

$$\lambda_k^{(j)} = \lambda(t^j E_{kk}), \quad k \in \mathbb{Z}, \; j = 0, \ldots, m ,$$

and *central charges*

$$c_j = \lambda(t^j), \quad j = 0, 1, \ldots, m .$$

Let

$$h_k^{(j)} = \lambda_k^{(j)} - \lambda_{k+1}^{(j)} + \delta_{k,0} c_j, \quad k \in \mathbb{Z}, \; j = 0, \ldots, m . \tag{4.4.1}$$

As usual, we have the irreducible highest weight $\mathfrak{g}^{[m]}$-module $L(\mathfrak{g}^{[m]}, \lambda)$ associated to λ. The proof of the following proposition is standard:

Proposition. *The $\mathfrak{g}_{[m]}$-module $L(\mathfrak{g}^{[m]}, \lambda)$ is quasifinite if and only if for each $j = 0, 1, \ldots, m$ all but finitely many of the $h_k^{(j)}$ are zero.*

4.5. Given $\overrightarrow{m} = (m_1, \ldots, m_N) \in \mathbb{Z}_+^N$, we let $\mathfrak{g}^{[\overrightarrow{m}]} = \widehat{gl}(\infty)[\overrightarrow{m}] = \bigoplus_{i=1}^N \mathfrak{g}^{[m_i]}$. By Proposition 3.3, we have a surjective homomorphism of Lie algebras over $\mathbb{C}$:

$$\widehat{\varphi}_{\vec{s}}^{[\overrightarrow{m}]} = \bigoplus_{i=1}^N \widehat{\varphi}_{s_i}^{[m_i]} : \widehat{\mathscr{D}} \to \mathfrak{g}^{[\overrightarrow{m}]} , \tag{4.5.1}$$

Choose a quasifinite $\lambda(i) \in \left(\mathfrak{g}_0^{[m_i]}\right)^*$ and let $L(\mathfrak{g}^{[m_i]}, \lambda(i))$ be the corresponding irreducible $\mathfrak{g}^{[m_i]}$-module. Then

$$L\big(\mathfrak{g}^{[\vec{m}]}, \vec{\lambda}\big) = \bigotimes_{i=1}^{N} L\big(\mathfrak{g}^{[m_i]}, \lambda(i)\big)$$

is an irreducible $\mathfrak{g}^{[\vec{m}]}$-module. Using the homomorphism (4.5.1), we make $L(\mathfrak{g}^{[\vec{m}]}, \vec{\lambda})$ a $\widehat{\mathscr{D}}$-module, which we shall denote by $L_{\vec{s}}^{[\vec{m}]}(\vec{\lambda})$.
We can prove now the following important Theorem.

Theorem. *Consider the embedding $\widehat{\varphi}_{\vec{s}}^{[\vec{m}]} : \widehat{\mathscr{D}} \to \widehat{gl}(\infty)[\vec{m}]$ (recall that $s_i - s_j \notin \mathbb{Z}$ if $i \neq j$) and let V be a quasifinite $\widehat{gl}(\infty)[\vec{m}]$-module. Then any $\widehat{\mathscr{D}}$-submodule of V is a $\widehat{gl}(\infty)[\vec{m}]$-submodule as well. In particular, the $\widehat{\mathscr{D}}$-modules $L_{\vec{s}}^{[\vec{m}]}(\vec{\lambda})$ are irreducible.*

Proof. Let U be a ($\mathbb{Z}$-graded) $\widehat{\mathscr{D}}$- submodule of V. U is a quasifinite $\widehat{\mathscr{D}}$-module as well, hence, by Proposition 4.3, it can be extended to $\widehat{\mathscr{D}_k^{\mathscr{O}}}$ for any $k \neq 0$. But by Proposition 3.2, the map $\varphi_s^{[\vec{m}]} : \mathscr{D}_k^{\mathscr{O}} \to \widetilde{gl}(\infty)[\vec{m}]_k$ is surjective for any $k \neq 0$. Thus U is invariant with respect to all members of the principal gradation $\widetilde{gl}(\infty)[\vec{m}]_k$ with $k \neq 0$. Since $\widehat{gl}(\infty)[\vec{m}]$ coincides with its derived algebra, this proves the theorem. $\square$

4.8. By Proposition 4.4 and Theorem 4.5, the $\widehat{\mathscr{D}}$-modules $L_{\vec{s}}^{[\vec{m}]}(\vec{\lambda})$ are irreducible quasifinite highest weight modules. Using formulas (3.2.1) and (3.3.4), it is easy to calculate the generating series $\Delta_{\vec{m},\vec{s},\vec{\lambda}}(x) = \sum_{n\geq 0} \Delta_n x^n/n!$ of the highest weight and the central charge c of the $\widehat{\mathscr{D}}$-module $L_{\vec{s}}^{[\vec{m}]}(\vec{\lambda})$. We have

$$\Delta_{m,s,\lambda}(x) = -\sum_{j=0}^{m}\sum_{i\in\mathbb{Z}} \big(\lambda_i^{(j)}/j!\big)x^j e^{(s-i)x} + \frac{c_0(e^{sx}-1) + \sum_{j=1}^{m}(c_j/j!)x^j e^{sx}}{e^x - 1} \tag{4.6.1}$$

$$c = c_0 \tag{4.6.2}$$

and

$$\Delta_{\vec{m},\vec{s},\vec{\lambda}}(x) = \sum_i \Delta_{m_i,s_i,\lambda(i)}(x), \quad c = \sum_i c_0(i)\,. \tag{4.6.3}$$

Introduce the polynomials (see (4.4.1)):

$$g_k(x) = \sum_{j=0}^{m} h_k^{(j)} x^j/j! \quad (k \in \mathbb{Z})\,. \tag{4.6.4}$$

Then (4.6.1) can be rewritten as follows:

$$\Delta_{m,s,\lambda}(x) = \frac{\sum_{k\in\mathbb{Z}} e^{(s-k)x} g_k(x) - c_0}{e^x - 1}\,. \tag{4.6.5}$$

Using these formulas, it is not difficult to see that any irreducible quasifinite highest weight module $L(\widehat{\mathscr{D}}, \lambda)$ can be obtained in as above an essentially unique way. More precisely, we have the following.

Theorem. *Let $L = L(\widehat{\mathscr{D}}, \lambda)$ be an irreducible quasifinite highest weight module with central charge c and $\Delta_\lambda(x) = \phi(x)/(e^x - 1)$, where $\phi(x)$ is a quasipolynomial such that $\phi(0) = 0$ (see Theorem 4.2). We write $\phi(x) + c = \sum\limits_{s\in\mathbb{C}} p_s(x)e^{sx}$, where $p_s(x)$ are polynomials. We decompose the set $\{s \in \mathbb{C} | p_s(x) \neq 0\}$ in a disjoint union of congruence classes* $\operatorname{mod}\mathbb{Z}$. *Let $S = \{s, s + k_1, s + k_2, \ldots\}$ be such a congruence class, let $m = \max\limits_{s\in S} \deg p_s$ and let $h_{k_r}^{(j)} = \left(\frac{d}{dx}\right)^j p_{s+k_r}(0)$. We associate to S the $\widetilde{gl}(\infty)[m]$-module $L^{[m]}(\lambda_S)$ with the central charges*

$$c_j = \sum_{k_r} h_{k_r}^{(j)}, \tag{4.6.6}$$

and labels

$$\lambda_i^{(j)} = \sum_{k_r \geq i} \widetilde{h}_{k_r}^{(j)}, \tag{4.6.7}$$

where $\widetilde{h}_k^{(j)} = h_k^{(j)} - c_j\delta_{k,0}$. Then the $\widehat{\mathscr{D}}$-module L is isomorphic to the tensor product of all the modules $L_s^{[m]}(\lambda_S)$.

Proof. The tensor product L' of all the modules $L_s^{[m]}(\lambda_S)$ is irreducible due to Theorem 4.5. It remains to show that L' has the same highest weight as L does. This is done by exploiting the observation (used already before) that $-\Delta(x)$ is the value of the highest weight of L' on e^{xD}, and using the formulas (4.6.2–5). □

Remark. Changing the representative s in S amounts to the shift ν^j of $gl(\infty)[m]$. Up to these shifts the above construction of L via the embedding $\widehat{\mathscr{D}} \to gl(\infty)[\vec{m}]$ is unique.

5. *Unitary Quasifinite Highest Weight Modules over $\widehat{\mathscr{D}}$*

5.1. It is easy to see that any anti-involution ω of the associative algebra $\mathscr{D}^a$, such that $\omega(\mathscr{D}_j^a) = \mathscr{D}_{-j}^a$ and $\omega(D) = D$, by a rescaling of z can be brought to the following form:

$$\omega(z^k f(D)) = \overline{f}(D)z^{-k} = z^{-k}\overline{f}(D - k), \tag{5.1.1}$$

where for $f(D) = \sum\limits_i f_i D^i$ we let $\overline{f}(D) = \sum\limits_i \overline{f}_i D^i (f_i \in \mathbb{C})$. The involution ω given by (5.1.1) extends to the whole algebra $\mathscr{D}^{a\mathscr{O}}$.

Note that

$$\Psi(\omega(A), \omega(B)) = \Psi(B, A), \quad A, B \in \mathscr{D}^{\mathscr{O}}. \tag{5.1.2}$$

Hence the anti-involution ω of the Lie algebras $\mathscr{D}$ and $\mathscr{D}^{\mathscr{O}}$ lifts to an anti-involution of their central extensions $\widehat{\mathscr{D}}$ and $\widehat{\mathscr{D}^a}$, such that $\omega(C) = C$, which we again denote by ω.

Remark. (a) The Virasoro subalgebra $\mathrm{Vir}(\beta)$ [defined by (2.1.7)] is ω-stable if and only if $\beta = \frac{1}{2}$

(b) Under the homomorphism $\varphi_s = \varphi_{s,0} : \mathscr{D}^{a\mathscr{O}} \to \widetilde{M}(\infty, \mathbb{C})$ we have

$$\left(\varphi_s(z^k f(D))\right)^* = \varphi_{\bar{s}}\left(\omega(z^i f(D))\right).$$

Here A^* stands for the complex conjugate transpose of the matrix $A \in \widetilde{M}(\infty, \mathbb{C})$.
(c) (see e.g. [K]) For the involution ω of $\widehat{gl}(\infty, \mathbb{C})$ defined by $\omega(A) = {}^t\overline{A}$, $\omega(1) = 1$, a highest weight $\widehat{gl}(\infty, \mathbb{C})$-module with highest weight λ and central charge c is unitary if and only if the numbers $h_i^{(0)}$ (see (4.4.1)) are non-negative integers and $c = \sum_i h_i^{(0)}$.

5.2. In this section we shall classify all unitary (irreducible) quasifinite highest weight modules over the Lie algebra $\widehat{\mathscr{D}}$ with respect to the anti-involution ω.

Lemma. *Let V be a unitary quasifinite highest weight module over $\widehat{\mathscr{D}}$ and let $b(w)$ be its characteristic polynomial. Then $b(w)$ has only simple real roots.*

Proof. Let v_λ be a highest weight vector of V and let $\Delta_j = -\lambda(D^j) \in \mathbb{R}$ be the labels of λ. Consider the element $S = -\frac{1}{2}(D^2 - \Delta_2 - 1) \in \widehat{\mathscr{D}}$. It is easy to check that for any $j \in \mathbb{Z}_+$ we have:

$$S^j(z^{-1}v_\lambda) = (z^{-1}D^j)v_\lambda\,. \tag{5.2.1}$$

By definition of the characteristic polynomial we have:

$$(z^{-1}b(D))v_\lambda = 0\,, \tag{5.2.2}$$

$$\left\{(z^{-1}D^j)v_\lambda | 0 \leq j < n\right\} \quad \text{is a basis of } V_{-1}\,, \tag{5.2.3}$$

where $n = \deg b(w)$. It follows from (5.2.1) and (5.2.2) that

$$b(S)(z^{-1}v_\lambda) = 0\,, \tag{5.2.4}$$

and it follows from (5.2.3) that

$$\left\{S^j(z^{-1}v_\lambda) | 0 \leq j < n\right\} \quad \text{is a basis of } V_{-1}\,. \tag{5.2.5}$$

We conclude from (5.2.4) and (5.2.5) that $b(w)$ is the characteristic polynomial of the operator S on V_{-1}. Operator S is selfadjoint, hence roots of $b(w)$ are real.
Let μ be a root of $b(w)$ of multiplicity m, so that $b(w) = c(w)(w-\mu)^m$, $c(w) \in \mathbb{C}[w]$. Then

$$v := (S - \mu)^{m-1}c(S)(z^{-1}v)$$

is a non-zero vector in V_{-1}, but

$$h(v, v) = h(c(S)(z^{-1}v),\, (S-\mu)^{2m-2}c(S)(z^{-1}v)) = 0 \quad \text{if } m \geq 2\,,$$

by (5.2.4). Hence the unitarity forces $m = 1$. $\square$

Theorem. (a) *A quasifinite irreducible highest weight module $L(\widehat{\mathscr{D}}, \lambda)$ is unitary if and only if*

$$\Delta_\lambda(x) = \sum_i n_i \frac{e^{s_i x} - 1}{e^x - 1} \tag{5.2.6}$$

for some positive integers n_i and real numbers s_i, such that

$$c = \sum_i n_i \,. \tag{5.2.7}$$

(b) *Any unitary quasifinite $\widehat{\mathscr{D}}$-module is obtained by taking tensor product of N unitary irreducible quasifinite highest weight modules over $\widehat{gl}(\infty, \mathbb{C})$ and restricting to $\widehat{\mathscr{D}}$ via an embedding $\widehat{\varphi}_{\vec{s}}^{[0]}$, where $\vec{s} = (s_1, \ldots, s_N)$ is a real vector and $s_i - s_j \notin \mathbb{Z}$ if $i \neq j$.*

Proof. By Proposition 4.6, being a quasifinite irreducible highest weight $\widehat{\mathscr{D}}$-module, V is isomorphic to one of the modules $L_{\vec{s}}^{[\vec{m}]}(\vec{\lambda})$. It follows from Lemma 5.2 and Corollary 4.6 that $\vec{m} = 0$. Now the claim (b) follows from Remarks 5.1(b) and (c). The claim (a) follows from (b) and (4.6.1 and 2). □

Corollary. *Suppose that only finitely many labels Δ_n of λ are non-zero. Then the $\widehat{\mathscr{D}}$-module $L(\lambda)$ is unitary if and only if*

$$c = \Delta_0 \in \mathbb{Z}_+ \quad \textit{and} \quad \Delta_j = 0 \text{ for } j > 0\,. \tag{5.2.8}$$

Proof. By the hypothesis, $\Delta_\lambda(x)$ is a polynomial of degree N, where $N = \max\{n | \Delta_n \neq 0\}$. By Corollary 4.2, it follows that the characteristic polynomial of $L(\lambda)$ is w^{N+1}. Hence, by Lemma 5.2, $N = 0$, i.e., $\Delta_j = 0$ for $j > 0$. The rest of (5.2.8) follows from Theorem 5.2. □

Remark. The $\widehat{\mathscr{D}}$-modules of Corollary 5.2 are obtained by taking the embedding $\widehat{\varphi}_0^{[0]} : \widehat{\mathscr{D}} \to \widehat{gl}(\infty, \mathbb{C})$ and composing it with the irreducible highest weight $\widehat{gl}(\infty, \mathbb{C})$-module with a non-negative integral central charge and zero labels.

Example. Consider the following parabolic subalgebra of the Lie algebra $\mathfrak{g} = \widehat{gl}(\infty, \mathbb{C})$:

$$\mathfrak{p} = \{(a_{ij})_{i,j \in \mathbb{C}} + \mathbb{C}C \mid a_{ij} = 0 \text{ if } i > 0 \geq j\}$$

and let $F_0 = E_{10}$. Given $c \in \mathbb{C}$, denote by M_c the generalized Verma module $M(\mathfrak{g}, \mathfrak{p}, \lambda_0)$, where λ_0 is the highest weight such that $\lambda_0(E_{jj}) = 0$ for all j and $\lambda_0(C) = c$. Then we have

$$L(\mathfrak{g}, \lambda_0) = M_c \quad \text{if } c \notin \mathbb{Z}_+\,,$$
$$L(\mathfrak{g}, \lambda_0) = M_c / \mathscr{U}(\mathfrak{g})(F_0^{c+1} v_{\lambda_0}) \quad \text{if } c \in \mathbb{Z}_+\,.$$

Consider the homomorphism $\widehat{\varphi}_0 : \widehat{\mathscr{D}} \to \mathfrak{g}$ given by:

$$\widehat{\varphi}_0(z^k f(D)) = \sum_{j \in \mathbb{Z}} f(-j) E_{j-k,j}, \widehat{\varphi}_0(C) = 1\,.$$

When restricted to $\widehat{\mathscr{D}}$, the module $L(\mathfrak{g}, \lambda_0)$ remains an irreducible quasifinite highest weight module with zero labels and central charge c. The singular vector $F_0^{c+1} v_{\lambda_0}$

of the $\mathfrak{g}$-module M_c remains singular for the $\widehat{\mathscr{D}}$-module M_c. It is a multiple of the following vector:

$$\left(z^{-1}\prod_{s=1}^{c}(D^2-s^2)\right)^{c+1} v_{\lambda_0}\,.$$

6. Quasifinite Highest Weight Modules over Quantum Pseudo-Differential Operators

6.1. The q-analogue of the algebra $\mathscr{D}^a$ is the algebra $\mathscr{D}^a_q$ of all regular difference operators on $\mathbb{C}^\times$ (see Sect. 1.4). However, a more important algebra is the algebra of quantum pseudo-differential operators $\mathscr{S}^a_q$ (which contains $\mathscr{D}^a_q$ as a subalgebra). This associative algebra is obtained by the construction explained in Sect. 1.1 by taking the algebra $A=\mathbb{C}[w,w^{-1}]$ and its automorphism σ defined by $\sigma(w)=qw$, where $q\in\mathbb{C}^\times$:

$$\mathscr{S}^a_q = A_\sigma[z,z^{-1}]\,.$$

Explicitly, let T_q denote the following operator on $\mathbb{C}[z,z^{-1}]$, where $q\in\mathbb{C}^\times$:

$$T_q f(z)=f(qz)\,.$$

Then $\mathscr{S}^a_q$ is the associative algebra of all operators on $\mathbb{C}[z,z^{-1}]$ of the form

$$E=\sum_{k\in\mathbb{Z}} e_k(z)T_q^k, \quad \text{where } e_k(z)\in\mathbb{C}[z,z^{-1}] \text{ and sum is finite.}$$

As before, we write such an operator as a linear combination of operators of the form $z^k f(T_q)$, where f is a Laurent polynomial in T_q. Then the product is given by

$$(z^m f(T_q))(z^k g(T_q)) = z^{m+k} f(q^k T_q) g(T_q)\,. \tag{6.1.1}$$

Let $\mathscr{S}_q$ denote the Lie algebra obtained from $\mathscr{S}^a_q$ by taking the usual bracket. Let $\mathscr{S}'_q=[\mathscr{S}_q,\mathscr{S}_q]$. We have:

$$\mathscr{S}_q=\mathscr{S}'_q\oplus\mathbb{C}T^0_q \quad \text{(direct sum of ideals).}$$

Thus, representation theory of $\mathscr{S}_q$ reduces to that of $\mathscr{S}'_q$.

Taking the trace form $\mathrm{tr}_0\left(\sum_j c_j w^j\right)=c_0$, we obtain by the general construction of Sect. 1.3 the following 2-cocycle on $\mathscr{S}_q$:

$$\Psi(z^m f(T_q), z^k g(T_q)) = m\delta_{m,-k}\,\mathrm{tr}_0\, f(q^{-m}w)q(w)\,. \tag{6.1.2}$$

The associated central extension of $\mathscr{S}'_q$ is denoted by $\widehat{\mathscr{S}_q}=\mathscr{S}'_q+\mathbb{C}C$. As we have mentioned in Remark 1.6(b), this is a well-known Lie algebra studied by many authors. We will show that the representation theory of the Lie algebra $\widehat{\mathscr{S}_q}$ with $|q|\neq 1$ is quite similar to that of $\widehat{\mathscr{D}}$. Details of most of the proofs will be omitted, being similar as well.

6.2. Let $\mathcal{O}$ denote (in this section) the algebra of all holomorphic functions in $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. We define a completion $\mathcal{S}_q^{a\mathcal{O}}$ of the algebra $\mathcal{S}_q^a$ by considering operators of the form $z^k f(T_q)$, where $f \in \mathcal{O}$. We extend the product (6.1.1) to $\mathcal{S}_q^{a\mathcal{O}}$ and denote by $\mathcal{S}_q^{a\mathcal{O}}$ the corresponding Lie algebra. The cocycle (6.1.2) extends to $\mathcal{S}_q^{\mathcal{O}}$ and we let $\widehat{\mathcal{S}_q^{\mathcal{O}}} = \mathcal{S}_q^{\mathcal{O}\prime} + \mathbb{C}C$ be the corresponding central extension.

Consider the associative algebra $R_m = \mathbb{C}[t]/(t^{m+1})$ and let $s \in \mathbb{C}$. Then we have the following embedding $\varphi_{s,t} : \mathcal{S}_q^a \to \widetilde{M}(\infty, R_m)$ of $\mathbb{Z}$-graded associative algebras over $\mathbb{C}$ (cf. [GL]):

$$\varphi_s^{[m]}(z^k f(T_q)) = \sum_{j\in\mathbb{Z}} f(sq^{-j+t})E_{j-k,j} \,. \tag{6.2.1}$$

It extends to a homomorphism $\varphi_s^{[m]} : \mathcal{S}_q^{a\mathcal{O}} \to \widetilde{M}(\infty, R)$.

Lemma. *The homomorphism $\varphi_s^{[m]}$ is surjective provided that $|q| \neq 1$.*

Let $\varphi_s = \varphi_s^{[0]} : \mathcal{S}_q^{a\mathcal{O}} \to \widetilde{M}; (\infty, \mathbb{C})$. We have (cf. Sect. 3.4):

$$\varphi_{qs}(z^k f(T_q)) = \nu\varphi_s(z^k f(T_q)) \,. \tag{6.2.2}$$

We define a *quantum monodromic loop* to be a holomorphic map $f : \mathbb{C}^\times \to \widetilde{M}(\infty, \mathbb{C})$ such that $f(qw) = \nu f(w)$. Denote by $\mathcal{L}_{q,\nu}\widetilde{M}(\infty)$ the associative algebra of all quantum monodromic loops. Then we have an isomorphism

$$\varphi : \mathcal{S}_q^{a\mathcal{O}} \xrightarrow{\sim} \mathcal{L}_{q,\nu}\widetilde{M}(\infty) \tag{6.2.3}$$

defined by the same formula as (3.4.2). Note that the quantum monodromic loops are sections of the vector bundle on the tours $\mathbb{C}^\times/\{q^n | n \in \mathbb{Z}\}$ (with modular parameter $(\log q)/(2\pi i)$) with fiber $\widetilde{M}(\infty, \mathbb{C})$ and transition function ν in a small neighbourhood of the circle $|w| = |q|$.

Denote by $\mathcal{L}_{q,\nu}\widetilde{gl}(\infty)$ the Lie algebra obtained from $\mathcal{L}_{q,\nu}\widetilde{M}(\infty)$ by taking the usual bracket. Considering the Laurent expansion at 0:

$$C(f(w), g(w)) = \sum_{n\in\mathbb{Z}} C_n(f, g)w^n \,,$$

we obtain $\mathbb{C}$-valued 2-cocycles C_n on this Lie algebra. Denote by $\mathcal{L}_{q,\nu}\widetilde{gl}(\infty)^\wedge$ the corresponding to C_0 central extension. Then the isomorphism $\varphi : \mathcal{S}_q^{\mathcal{O}} \xrightarrow{\sim} \mathcal{L}_{q,\nu}\widetilde{gl}(\infty)$ lifts to the isomorphism $\widehat{\varphi} : \widehat{\mathcal{S}_q^{\mathcal{O}}} \xrightarrow{\sim} \mathcal{L}_{q,\nu}\widetilde{gl}(\infty)^\wedge$.

6.3. Let $\mathfrak{p} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{p}_j$ be a parabolic subalgebra of the Lie algebra $\widehat{\mathcal{S}_q}$ (i.e., $\mathfrak{p}_j = (\widehat{\mathcal{S}_q})_j$ for $j \geq 0$ and $(\widehat{\mathcal{S}_q})_j \neq 0$ for some $j < 0$). Then for each positive integer k we have $\mathfrak{p}_{-k} = z^{-k} I_{-k}$, where I_{-k} is a non-zero ideal of $\mathbb{C}[w, w^{-1}]$. Let $b_k(w)$ be the monic polynomial with $b_k(0) \neq 0$ which is a generator of the ideal I_{-k}. The polynomials b_k, $k = 1, 2, \ldots$ are called the characteristic polynomials of $\mathfrak{p}$.

Given a monic polynomial $b(w)$ with $b(0) \neq 0$, we let

$$b_k^{\min}(w) = b(w)b(q^{-1}w) \cdots b(q^{-k+1}w),$$
$$b_k^{\max} = \mathrm{lcm}\{b(w), b(q^{-1}w), \dots, b(q^{-k+1})w)\}.$$

There exist unique parabolic subalgebras of $\widehat{\mathscr{S}_q}$, which we denote by $\mathfrak{p}_{\min}(b)$ and $\mathfrak{p}_{\max}(b)$, for which the characteristic polynomials are $\{b_k^{\min}(w)\}$ and $\{b_k^{\max}(w)\}$ respectively. We have

$$\dim(\widehat{S}_q)_{-k}/\mathfrak{p}_{\min}(b)_{-k} = k \deg b(w),$$

and an analogue of Proposition 2.4 holds verbatim.
Also, we have for a parabolic subalgebra $\mathfrak{p}$ with the first characteristic polynomial $b(w)$:

$$[\mathfrak{p}, \mathfrak{p}] = \left(\bigoplus_{k\neq 0} \mathfrak{p}_k\right) \bigoplus (\widehat{S}_q)_0^b,$$

where

$$(\widehat{\mathscr{S}_q})_0^b = \{b(T_q)g(T_q) - b(qT_q)g(gT_q) + (\mathrm{tr}_0\, b(w)g(w))C \mid g(w) \in \mathbb{C}[w, w^{-1}]\}.$$

6.4. All the results of Sect. 4.1 hold for the Lie algebra $\widehat{\mathscr{S}_q}$ verbatim. However, the generating series $\Delta_\lambda(x)$ is defined differently.
We shall characterize $\lambda \in (\widehat{\mathscr{S}_q})_0^*$ by labels $\Delta_n = \lambda(T_q^n)$ $(n \neq 0)$ and central charge $c = \lambda(C)$. Introduce the generating series

$$\Delta_\lambda(x) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \Delta_n x^{-n}.$$

Theorem. (a) *An irreducible highest weight module $L(\widehat{S}_q, \lambda)$ is quasifinite if and only if one of the following equivalent conditions holds*:

(i) *There exists a non-zero polynomial $b(x)$ such that*

$$b(x)(\Delta_\lambda(x) - \Delta_\lambda(q^{-1}x) + c) = 0. \tag{6.4.1}$$

(ii) *There exists a quasipolynomial $P(x)$ such that*

$$(1 - q^n)\Delta_n = P(n) \quad \text{for } n \neq 0 \text{ and } c = P(0).$$

(b) *The monic polynomial of minimal degree satisfying (6.4.1) is the characteristic polynomial of a quasifinite module $L(\widehat{\mathscr{S}_q}, \lambda)$.*

Proof. According to Sect. 6.3, $L(\widehat{\mathscr{S}_q}, \lambda)$ is quasifinite if and only if there exists a non-zero polynomial $b(w)$ such that

$$\lambda(g(T_q)b(T_q) - g(qT_q)b(qT_q)) + c\,\mathrm{tr}_0(g(w)b(w)) = 0$$

for each $g(w) \in \mathbb{C}[w, w^{-1}]$. Taking $g(w) = w^n$, and letting $b(w) = \sum_j f_j w^j$, this can be rewritten as follows:

$$\sum_j f_j \Delta_{n+j}(1 - q^{n+j}) + f_{-n} c = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Multiplying both sides of this equality by x^{-n} and summing over $n \in \mathbb{Z}$, we obtain (6.4.1).
The equivalence of (i) and (ii), as well as (b) are clear. □

6.5. Choose a branch of $\log q$. Let $\tau = (\log q)/(2\pi i)$. Then any $s \in \mathbb{C}$ is uniquely written as $s = q^a$, $a \in \mathbb{C}/\tau^{-1}\mathbb{Z}$. The homomorphism $\varphi_s^{[m]} : \mathcal{S}_q' \to \widetilde{gl}(\infty)[m]$ defined by (6.2.1) lifts to a homomorphism $\widehat{\mathcal{S}_q} \to \widehat{gl}(\infty)[m]$ of central extensions, denoted by $\widehat{\varphi}_a^{[m]}$, by

$$\begin{aligned}
\widehat{\varphi}_a^{[m]}\big|_{(\widehat{\mathcal{S}_q})_j} &= \varphi_a^{[m]}\big|_{(\widehat{\mathcal{S}_q})_j} \quad \text{if } j \neq 0, \\
\widehat{\varphi}_a^{[m]}(T_q^n) &= \sum_{r \in \mathbb{Z}} q^{(a-r)} E_{rr} + \frac{q^{an}}{1 - q^n} \sum_{j=0}^{m} (n \log q)^j t^j / j! \quad (n \neq 0), \\
\widehat{\varphi}_a^{[m]}(C) &= 1 \in R_m .
\end{aligned}$$

We have results similar to Theorems 4.5, 4.6, and 5.2:

Theorem. *Assume that $|q| \neq 1$. Consider the embedding $\widehat{\varphi}_{\vec{a}}^{[\vec{m}]} : \widehat{\mathcal{S}_q} \to \widehat{gl}(\infty)[\vec{m}]$, where $a_i - a_j \notin \mathbb{Z} + \tau^{-1}\mathbb{Z}$ if $i \neq j$. Denote the quasifinite $\widehat{gl}(\infty)[\vec{m}]$-module $L^{[\vec{m}]}(\vec{\lambda})$, viewed as a $\widehat{\mathcal{S}_q}$-module via this embedding, by $L_{\vec{a}}^{[\vec{m}]}(\vec{\lambda})$.*
(a) *If V is a quasifinite $\widehat{gl}(\infty)[\vec{m}]$-module, then any submodule of the module V, viewed as a $\widehat{\mathcal{S}_q}$-module via the embedding $\widehat{\varphi}_{\vec{a}}^{[\vec{m}]}$, is a $\widehat{gl}(\infty)[\vec{m}]$ submodule as well. In particular, the $\widehat{\mathcal{S}_q}$-modules $L_{\vec{a}}^{[\vec{m}]}(\vec{\lambda})$ are irreducible.*
(b) *Any irreducible quasifinite highest weight module over $\widehat{\mathcal{S}_q}$ is isomorphic to one of the modules $L_{\vec{a}}^{[\vec{m}]}(\vec{\lambda})$.*
(c) *Let $q \in \mathbb{R}$, and let ω be the anti-involution of $\widehat{\mathcal{S}_q}$ defined by*

$$\omega(z^k f(T_q)) = z^{-k} \overline{f}(q^{-k} T_q).$$

Then a quasifinite highest weight module over $\widehat{\mathcal{S}_q}$ is unitary with respect to ω if and only if

$$\Delta_n = \sum_j \frac{n_j q^{a_j n}}{1 - q^n} \quad \textit{(finite sum) for all } n \neq 0,$$

where the n_j are positive integers and the a_j are real numbers. (In particular, unitarity implies that $c = \sum_j n_j \in \mathbb{Z}_+$.) Any unitary quasifinite $\widehat{\mathcal{S}_q}$-module is obtained by taking the tensor product of N unitary irreducible quasifinite highest weight modules over $\widehat{gl}(\infty, \mathbb{C})$ and restricting to $\widehat{\mathcal{S}_q}$ via an embedding $\widehat{\varphi}_{\vec{a}}^{[0]}$, $\vec{a} \in \mathbb{R}^N$.

Remarks. (a) The labels and the central charge of the $\widehat{\mathscr{S}_q}$-module $L_{\vec{a}}^{[m]}(\lambda)$ are given by the following formulas (see (4.6.4)):

$$\Delta_n = \sum_{k\in\mathbb{Z}} q^{(a-k)n} g_k(n\log q)/(1-q^n), \quad n \neq 0; \;\; c = c_0 .$$

(b) A vertex operator construction of unitary quasifinite highest weight modules over $\widehat{\mathscr{S}_q}$ with $c=1$ is given in [GL].

7. Matrix Case

Let us consider the Lie algebras $M_n\mathscr{D} = \mathrm{Mat}_n(\mathscr{D})$ and $M_n\mathscr{S}_q$. They are twisted Laurent polynomial algebras with $A = \mathrm{Mat}_n[w]$, $\sigma(w) = w+1$ and $A = \mathrm{Mat}_n[w, w^{-1}]$, $\sigma(w) = qw$ respectively.
The canonical central extension $M_n\widehat{\mathscr{S}_q}$ is defined via (1.3.1) with respect to the trace functional $\mathrm{tr}_0 : \mathrm{Mat}_n[w, w^{-1}] \to \mathbb{C}$ defined by

$$\mathrm{tr}_0\left(\sum_i m_i w^i\right) = \mathrm{tr}\, m_0 .$$

Restriction of tr_0 to $\mathrm{Mat}_n[w]$ gives rise to the canonical central extension $M_n\mathscr{D}^\wedge$.
The isomorphism $\mathbb{C}^n[z, z^{-1}] \xrightarrow{\sim} \mathbb{C}[z, z^{-1}]$ defined by

$$e_i z^j \longrightarrow z^{jn+i}$$

defines the isomorphism $\mathrm{Mat}_n(\widetilde{M}(\infty)) \xrightarrow{\sim} \widetilde{M}(\infty)$. Combined with isomorphism $\widehat{\mathscr{D}^{\mathscr{O}}} \xrightarrow{\sim} \mathscr{L}_\nu \widetilde{gl}(\infty)^\wedge$ and $\mathscr{S}_q^{\mathscr{O}} \xrightarrow{\sim} \mathscr{L}_{q,\nu}\widetilde{gl}(\infty)^\wedge$ it gives Lie algebra isomorphisms:

$$\begin{aligned} M_n\mathscr{D}^{\mathscr{O}\wedge} &\xrightarrow{\sim} \mathscr{L}_\nu \,\mathrm{Mat}_n\, \widetilde{gl}(\infty)^\wedge &&\xrightarrow{\sim} \mathscr{L}_{\nu^n}\widetilde{gl}(\infty)^\wedge ,\\ M_n\mathscr{S}_q^\wedge &\xrightarrow{\sim} \mathscr{L}_{q,\nu}\,\mathrm{Mat}_n\, \widetilde{gl}(\infty)^\wedge &&\xrightarrow{\sim} \mathscr{L}_{q,\nu^n}\widetilde{gl}(\infty)^\wedge . \end{aligned}$$

The representation theory of the Lie algebras $M_n\mathscr{D}^\wedge$ and $M_n\mathscr{S}_q^\wedge$ is similar to that of $\widehat{\mathscr{D}}$ and $\widehat{\mathscr{S}_q}$. All irreducible quasifinite highest weight modules over $M_n\mathscr{D}^\wedge$ and $M_n\mathscr{S}_q^\wedge$ are constructed by embedding in $\widehat{gl}(\infty)[\vec{m}]$ and restricting an irreducible quasifinite highest weight module over the latter.
An anti-involution of an algebra B combined with the matrix transposition defines an anti-involution of $\mathrm{Mat}_n(B)$. All quasifinite unitary highest weight modules over $M_n\mathscr{D}^\wedge$ and $M_n\mathscr{S}_q^\wedge$ are modules over $gl(\infty)[\vec{0}]^\wedge$. In particular, unitary modules over $M_n\mathscr{D}^\wedge$ and $M_n\mathscr{S}_q^\wedge$ have positive integer central charge.

References

[F] Feigin, B.L.: The Lie algebras $gl(\lambda)$ and the cohomology of the Lie algebra of differential operators. Usp. Math. Nauk **35**, No. 2, 157–158 (1988)

[FL] Luk'yanov, S.L., Fateev, V.A.: Conformally invariant models of two-dimensional quantum field theory with Z_n symmetry. Zh. Esp. Theor. Fiz. **94**, No. 3, 23–37 (1988)

[FFZ] Fairlie, D., Fletcher, P., Zachos, C.: Phys. Lett. **218B**, 203 (1989)

[GL] Golenishcheva-Kutuzova, M., Lebedev, D.: Vertex operator representation of some quantum tori Lie algebras. Commun. Math. Phys. **148**, 403–416 (1992)

[K] Kac, V.G.: Infinite-dimensional Lie algebras. 3-d edition, Cambridge: Cambridge University Press, 1990

[KP] Kac, V.G., Peterson, D.H.: Spin and wedge representations of infinite-dimensional Lie algebras and groups, Proc. Natl. Acad. Sci. USA **78**, 3308–3312 (1981)

[KhZ] Khesin, B., Zakharevich, I.: Poisson-Lie group of pseudodifferential symbols and fractional KP-KdV hiearchies. C.R. Acad. Sci. Paris, **t.316**, Serie 1, 621–626 (1993)

[Li] Li, W.L.: 2-cocycles on the algebra of differential operators. J. Algebra **122**, 64–80 (1989)

[PSR] Pope, C.N., Shen, X., Romans, L.J.: W_∞ and the Racah-Wigner Algebra. Nucl. Phys. **339B**, 191–122 (1990)

[R] Radul, A.O.: Lie algebras of differential operators, their central extensions and W-algebras. Funct. Anal. Appl. **25**, No. 1, 86–91 (1991)

[RV] Radul, A.O., Vaysburd, I.: Differential operators and W-algebras. Phys. Lett. **274B**, 317–322 (1992)

[Z] Zamolodchikov, A.B.: Infinite additional symmetries in 2-dimensional conformal quantum field theory. Teor. Mat. Phys. **65**, 1205–1213 (1985)

Communicated by A. Jaffe