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Braided Matrix Structure of the Sklyanin Algebra and of the Quantum Lorentz Group

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Abstract. Braided groups and braided matrices are novel algebraic structures living in braided or quasitensor categories. As such they are a generalization of super-groups and super-matrices to the case of braid statistics. Here we construct braided group versions of the standard quantum groups $U_q(g)$. They have the same FRT generators $l^{\pm}$ but a matrix braided-coproduct $\underline{\Delta} L = L \underline{\otimes} L$, where $L = l^+ S l^-$, and are self-dual. As an application, the degenerate Sklyanin algebra is shown to be isomorphic to the braided matrices $BM_q(2)$; it is a braided-commutative bialgebra in a braided category. As a second application, we show that the quantum double $D(U_q(sl_2))$ (also known as the "quantum Lorentz group") is the semidirect product as an algebra of two copies of $U_q(sl_2)$, and also a semidirect product as a coalgebra if we use braid statistics. We find various results of this type for the doubles of general quantum groups and their semi-classical limits as doubles of the Lie algebras of Poisson Lie groups.

1. Introduction

Historically, the existence of particles with bose and fermi statistics led physicists naturally to the study of super-algebras and super-groups. In a similar way, the existence in low-dimensional quantum field theory of particles with braid statistics [1, 2] surely motivates the study of novel braided algebraic structures. The formulation and study of precisely such new algebraic structures has been initiated in [3–10] and [11–14] under the heading "braided groups." They precisely generalize results about super-algebras and super-groups to a situation in which the super-transposition map $\Psi(b \otimes c) = (-1)^{|b||c|} c \otimes b$ on homogeneous elements, is replaced by a braided-transposition or braiding Ψ obeying the Yang-Baxter equations. This is formulated mathematically by means of the theory of braided or quasitensor categories and it is in such a category that a braided group lives (just as a super-algebra or super-Lie algebra lives in the category of super-spaces). Among the general results is that in

★ SERC Fellow and Drapers Fellow of Pembroke College, Cambridge every situation where there is a quasitensor category $\mathscr{C}$ (such as the super-selection structure in low-dimension quantum theory) there is to be found a braided group $\mathrm{Aut}(\mathscr{C})$ [11]. So such objects are surely relevant to physics [4, 8].

The motivation for this theory so far is however, of a very general nature. Here we develop two very specific applications of the general theory to the structure of two algebras of independent interest in physics, namely to the degenerate Sklyanin algebra and to the quantum Lorentz group, or more generally to the quantum dobule of a general quantum group. We will see that these algebraic structures are naturally endowed with braid statistics and that this braiding enables us to obtain new results about them. We also obtain in the semiclassical limit a new result about Poisson Lie algebras.

The precise definition of a braided group is recalled in the Preliminaries below. The first step is to formulate precisely what we mean by braid statistics using the theory of quasitensor categories. This is a collection of objects closed under a tensor product $\otimes$ and equipped with isomorphisms $\Psi_{V,W}: V \otimes W \to W \otimes V$ for any two objects V and W. These play the role of the usual transposition for vector spaces or the super-transposition for super-spaces mentioned above. The second idea is to remember that instead of working with groups or Lie algebras we can work with their corresponding cocommutative group or enveloping Hopf algebras. The same is known in the super-case where we can work with super-cocommutative Hopf algebras rather than with the super-group or super-Lie algebra itself. Likewise in the braided case we work directly with braided-cocommutative Hopf algebras living in a quasitensor category. We call such objects braided groups of enveloping algebra type. There also appears to be a general theory of braided Lie algebras themselves underlying these braided groups (we make some remarks about this in Sect. 2).

Because we are working with Hopf algebras (albeit living in a quasitensor category), the mathematical technology here is for the most part already familiar to physicists in the context of quantum groups. Thus, we have an algebra B and a braided-coproduct as a coassociative homomorphism $\underline{\Delta}: B \to B\underline{\otimes}B$. The crucial difference is that $B\underline{\otimes}B$ is not an ordinary tensor product but a braided one. It coincides with $B \otimes B$ as an object but its two subalgebras B do not commute, enjoying instead an exchange rule given by Ψ. Thus

$$(a \otimes b)(c \otimes d) = a\Psi(b \otimes c)d\,,$$

where $\Psi_{B,B}: B \otimes B \to B \otimes B$ is a Yang-Baxter operator. This is clearly a generalization of the super-tensor product of super-algebras and shows the use of Ψ as a kind of braid statistics. The braided tensor product construction here is very natural from the point of view of algebras living in a quasitensor category. Let us note that such quasitensor categories are closely related to the theory of link invariants [15, 16] and indeed many of the abstract braided group computations and proofs are best done by means of drawing braids and tangles, see [8, 12, 13]. For example, the proof of associativity of $B\underline{\otimes}B$ (or more generally $B\underline{\otimes}C$ for two algebras in the category) depends on the functoriality and hexagon identities for Ψ and is most easily done by such diagrammatic means. This is one of the novel features of these new algebraic structures.

Among the general results in [4, 12] is that every quantum group $(H, \mathscr{R})$ (by this we mean an ordinary Hopf algebra equipped with a quasitriangular structure or "universal R-matrix" [17], such as the $U_q(g)$ of Drinfeld and Jimbo [17, 18]) can be transmuted in a canonical way into a braided group. Only the coalgebra need be changed and Ψ is obtained in a standard way from $\mathscr{R}$ in the adjoint representation.

Likewise every dual quantum group $(A, \mathscr{R})$ [by which we mean a dual-quasitriangular Hopf algebra such as the quantum function algebra $SL_q(2)$] can be transmuted to a braided-group of function algebra type. This is braided-commutative in a certain sense (rather than braided-cocommutative as above). By means of these transmutation processes we can obviously formulate all questions about quantum groups in terms of their associated braided groups. In brief, braided groups generalize both super groups and (via transmutation) quantum groups.

We begin in Sect. 2 by computing the braided groups $BU_q(g)$ of enveloping algebra type associated to the familiar quantum groups $U_q(g)$. They turn out to have the same generators $l^{\pm}$ of $U_q(g)$ in FRT form [19] but a new braided-coproduct. Writing $L = l^+ sl^-$, the braided-coproduct comes out as the matrix one $\underline{\Delta} L = L \underline{\otimes} L$. Here s is the antipode for the quantum group. Exactly this combination of generators $L = l^+ sl^-$ is well-known in certain contexts [20–22] but until now the obvious matrix coalgebra $\underline{\Delta} L = L \underline{\otimes} L$ has had no role [the usual coproduct of $U_q(g)$ is not so simple in terms of L]. We see now that in the braided setting these generators are very natural. Our matrix braided-coproduct can be used in all the same ways as the usual coproduct provided only that one remembers always to work in the quasitensor category (by using the relevant Ψ in place of any usual transpositions).

Essential in this computation is the fact that these quantum groups $U_q(g)$ are factorizable in the sense [23] that they have a non-degenerate "quantum Killing form" $Q = \mathscr{R}_{21} \mathscr{R}_{12}$. We also note, perhaps surprisingly, that the corresponding braided groups are self-dual: the $BU_q(g)$ are isomorphic via Q to the corresponding braided groups of function algebra type already computed in [6] from an R-matrix. This is a purely quantum phenomenon in that it holds only for generic $q \neq 1$ and certain roots of unity. It has important consequences, some of which are already known in other contexts. For example, it means that the braided versions of the familiar quantum groups are both braided-cocommutative and braided-commutative and indeed self-dual, so more like $\mathbb{R}^n$ than anything else [24, 25]. Related to this, though we do not describe an abstract notion of braided Lie algebras, we note that the generators L of $BU_q(g)$ do enjoy a kind of Lie bracket (which we compute) based on the quantum adjoint action and obeying some Lie-algebra like identities.

Our first main application is in Sect. 3 where we study the degenerate Sklyanin algebra. The Sklyanin algebra was introduced in [26] as a way to generate representations of a certain bialgebra (by which we mean a Hopf algebra without antipode) related to the 8-vertex model. Apart from that, it has remarkable mathematical properties (for example, the same Poincaré series as the commutative ring of polynomials in 4 variables [27]) and is related to the theory of elliptic curves. It has three parameters governed by one constraint. The degenerate case in which one of the parameters vanishes has been of fundamental importance in the development of quantum groups because a quotient of it led to the quantum group $U_q(sl_2)$. Many authors have wondered accordingly if the Skylanin algebra is also a quantum group or a bialgebra. So far this has defied all attempts even in the degenerate case: the Skylanin algebra does not appear to be any kind of usual quantum group or bialgebra. We show that the degenerate Sklyanin algebra is, however, a bialgebra living in a quasitensor category. It is in fact isomorphic to the braided matrices $BM_q(2)$ introduced in [6, 7] with braided-coproduct again in matrix form. This means that it is indeed some kind of group-like object in the sense that we can tensor product its (braided) representations, act by it on algebras etc., just as we can for any group or quantum group. Moreover, the quotienting procedure to obtain the algebra of $U_q(sl_2)$ from the degenerate

Sklyanin algebra is understood now as setting equal to one the braided determinant of the braided-matrix generator.

Our second main application is in Sect. 4 where we study the quantum double of a quantum group, for example the quantum double of $U_q(sl_2)$. The quantum double $D(H)$ is a general construction for obtaining a new quantum group from any Hopf algebra H [17]. It is built on the linear space $H \otimes H^*$ with a doubly-twisted product (here H^* is dual to H). The case when H is quasitriangular was studied explicitly by the author in [28] and we shall build on the results there concerning the semidirect product structure of $D(H)$ in this case. The quantum double in this case, particularly $D(U_q(g))$ is of considerable interest in physics. One of the reasons for interest, which we will be able to illuminate, is the idea in [29] that such a double should be regarded as a kind of complexification of the quantum group $U_q(g)$. For example, $D(U_q(sl_2))$ should be regarded as (by definition) the quantum enveloping algebra of the Lorentz group $so(1,3)$. [29] obtained some arguments for this in the dual C^*-algebra context of matrix pseudogroups, but some puzzles remain regarding such an interpretation. In particular, the quantum double is *a priori* built on the tensor product of a quantum group of enveloping algebra type and one of function algebra type (dual to the first). So $D(U_q(sl_2))$ does not look at first like two copies of $U_q(sl_2)$. By means of results in Sect. 2 and the theory of braided groups we show that in fact $D(U_q(g))$ is indeed generated as an algebra by two copies of $U_q(g)$, with certain cross relations: it is a semidirect product of one copy acting on the other by the quantum adjoint action

$$D(U_q(g)) \cong U_q(g)_{\mathrm{Ad}} \rtimes U_q(g)$$

as an algebra. Explicitly, if $l^\pm$ are the FRT generators of one copy and, say $m^\pm$ of the other then the cross relations are

$$R_{12} l_2^+ M_1 = M_1 R_{12} l_2^+ , \qquad R_{21}^{-1} l_2^- M_1 = M_1 R_{21}^{-1} l_2^- ,$$

where $M = m^+ s m^-$. The notation here is the standard one for matrix generators and R is the appropriate R-matrix for the quantum group. This gives a very simple matrix description of the algebra of $D(U_q(g))$. Moreover, we show that the coalgebra of the quantum double in our description also has a semidirect coproduct form, namely

$$\Delta l^\pm = l^\pm \otimes l^\pm, \qquad \Delta M = \left(\sum M \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} M\right) \mathcal{R}_{21}^{-1} .$$

Here $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ lives in the tensor square of the copy of $U_q(g)$ generated by $l^\pm$. Note here the use of the braided-coproduct $\underline{\Delta} M = M \underline{\otimes} M$. These results about the structure of the quantum double are obtained in Corollary 4.3 as the assertion that

$$D(U_q(g)) \cong BU_q(g) \rtimes U_q(g)$$

as an algebra and coalgebra. This also implies that the dual of the quantum double is also isomorphic to a semidirect product, a fact which is far from evident in [29]. In general, such a semidirect product structure for the quantum double has far reaching consequences. For example its representation theory can be analysed by a Hopf algebra version of Mackey's construction for semidirect products. It is also relevant to recent approaches to the quantum differential calculus on quantum groups. Let us note that the quantum double has also been studied in [30, 23] among other places.

Here we concentrate on illuminating the semidirect product result by computing its semiclassical version. This is also to be found in Sect. 4. The semiclassical notion of a quantum group is a quasitriangular Lie bialgebra (g, r) as introduced by Drinfeld

[31]. here g is a Lie algebra and $r \in g \otimes g$ obeys the Classical Yang-Baxter equations. The corresponding Lie group has a compatible Poisson bracket obtained from r (it is a Poisson Lie group). In some sense, $U_q(g)$ is a "quantization" of such a (g, r). There is a classical double construction $D(g)$ also introduced by Drinfeld [31] built on $g \oplus g^*$ with a doubly-twisted Lie bracket. For the standard Drinfeld-Jimbo r-matrix it is known that g^* is a solvable Lie algebra (hence quite different from g) while $D(g)$ has a real form isomorphic to the complexification $g_{\mathbb{C}}$ and g, g^* are components in its Iwasawa decomposition. Factorizability of $U_q(g)$ corresponds at the semiclassical level to the linear isomorphism $g^* \to g$ provided by the Killing form when g is semisimple. Our semidirect product theorem for the quantum double now becomes

$$D(g) \cong g \rtimes g$$

whenever g is a quasitriangular Lie bialgebra with non-degenerate symmetric part of r. This result is obtained in Theorem 4.4 and is related to the Iwasawa decomposition of $g_{\mathbb{C}}$ in Corollary 4.5. Thus, at least at the semiclassical level our semidirect product result is compatible with the view of $D(g)$ as complexification of g.

This completes our outline of the main results of the paper. For completeness we also include some related results about the quantum double in relation to braided statistics. Firstly, one can replace the second $U_q(g)$ also by its braided version: we show that the semidirect product $BU_q(g) \rtimes BU_q(g)$ is an example of a quantum braided group (i.e. a quasitriangular Hopf algebra living a quasitensor category). As a coalgebra it is now a braided-tensor coproduct as explained in Corollary 4.7. Finally, because $D(H)$ is a quantum group, it too has its own associated braided group $BD(H)$. We compute this in the Appendix, with emphasis on the simplest case where $H = \mathbb{C}G$ the group algebra of a finite group. The importance of this structure in physics is less well-established but it can be expected to play a role in Chern-Simons theories with finite gauge group as in [32] or in theories exhibiting non-Abelian anyon statistics. We also give a braided interpretation of an old theorem of Radford [33] to the effect that if $H_1 \rightleftarrows H$ is any Hopf algebra projection then $H_1 \cong B \rtimes H$, where B is a Hopf algebra living [like $BD(H)$ above] in the quasitensor category of $D(H)$-representations. This kind of Hopf algebra projection arises for certain quantum homogneneous spaces.

A sequel to this paper is in [38] where we use the main result of Sect. 4 to give an interpretation of the quantum double $D(H)$ as deformed quantum mechanics. The same theorem also allows one to interpret the quantum double as a quantum group principal frame bundle as developed elsewhere with T. Brzeziński. A third sequel develops the notion of braided Lie algebra from Sect. 2.

Preliminaries. Braided monoidal or quasitensor categories have been formally introduced into category theory in [34]. A quasitensor category means for us $(\mathscr{C}, \otimes, \underline{1}, \Phi, \Psi)$, where $\mathscr{C}$ is a category equipped with a monoidal product $\otimes$ and identity object $\underline{1}$ (with some associated maps) and functorial associativity isomorphisms $\Phi_{V,W,Z} : V \otimes (W \otimes Z) \to (V \otimes W) \otimes Z$ for any three objects V, W, Z, obeying Maclane's pentagon identity [so $(\mathscr{C}, \otimes, \underline{1}, \Phi)$ is a monoidal category]. In addition for a quasitensor category we need functorial quasi-symmetry isomorphisms $\Psi_{V,W} : V \otimes W \to W \otimes V$ obeying two hexagon identities. If we omit Φ (it will be trivial in the examples of interest here) then the hexagons take the form

$$\Psi_{V,W\otimes Z} = \Psi_{V,Z} \circ \Psi_{V,W}\,, \qquad \Psi_{V\otimes W,Z} = \Psi_{V,Z} \circ \Psi_{W,Z}\,. \tag{1}$$

One can deduce that $\Psi_{V,\underline{1}} = \mathrm{id} = \Psi_{\underline{1},V}$ for any V. If $\Psi_{V,W} = \Psi^{-1}_{W,V}$ for all V, W ($\Psi^2 = \mathrm{id}$) then one of the hexagons is superfluous and we have an ordinary symmetric monoidal or tensor category as in [35]. In general $\Psi_{V,W}, \Psi^{-1}_{W,V}$ are distinct and commonly represented as distinct braid crossings connecting $V \otimes W$ to $W \otimes V$. The coherence theorem for quasitensor categories says that if a sequence of the Ψ, Ψ^{-1} written in this way correspond to the same braid then they compose to the same map. The quasisymmetry Ψ can be called a "braiding" or "braided-transposition" for this reason. Functoriality of Ψ asserts explicitly that Ψ commutes with morphisms in the sense

$$\Psi_{V,Z} \circ (\mathrm{id} \otimes \phi) = (\phi \otimes \mathrm{id}) \circ \Psi_{V,W}, \quad \forall \phi : W \to Z \tag{2}$$

on one input and similarly on the other input. In the diagrammatic notation, functoriality for Ψ, Ψ^{-1} asserts that any morphisms between objects commute in the sense that they can be pulled through braid crossings. Here a typical morphism with n inputs and m outputs is represented as an $n+m$-vertex. We draw all morphisms pointing downwards.

The idea of an algebra in a quasitensor category (indeed, in any monoidal one) is just the obvious one: it is an object B in the category and morphisms $\eta : \underline{1} \to B$, $\underline{\cdot} : B \otimes B \to B$ obeying the usual axiomns (as diagrams) for associativity and unity. We use the term "algebra" a little loosely here, however in our examples, all constructions will indeed be k-linear over a field or ring k. Of crucial importance for the theory of braided groups is that if B, C are two algebras in a quasitensor category then one can define a new algebra, which we call the *braided tensor product algebra* $B \underline{\otimes} C$ also living in the category. As an object it consists of $B \otimes C$ equipped now with the algebra structure

$$\underline{\cdot}_{B \underline{\otimes} C} = (\underline{\cdot}_B \otimes \underline{\cdot}_C)(\mathrm{id} \otimes \Psi_{C,B} \otimes \mathrm{id}). \tag{3}$$

The proof that this is associative is a good exercise in the diagrammatic notation mentioned above [8]. For completeness, we have recalled the relevant diagram in Fig. 1. The first equality is functoriality under the morphism $\underline{\cdot} : B \otimes B \to B$, the second is associativity of the products in B, C and the third is a functoriality under $\underline{\cdot} : C \otimes C \to C$. There is another opposite braided tensor product using $\Psi^{-1}_{B,C}$ instead of $\Psi_{C,B}$.

This braided tensor product is the crucial ingredient in the definition of a Hopf algebra living in a quasitensor category [3–8]. This means $(B, \underline{\Delta}, \underline{\varepsilon}, \underline{S})$, where B is an algebra living in the category and $\underline{\Delta} : B \to B \underline{\otimes} B$, $\underline{\varepsilon} : B \to \underline{1}$ are algebra homomorphisms. In addition $(B, \underline{\Delta}, \underline{\varepsilon})$ form a coalgebra in the usual way [so $(\underline{\Delta} \otimes \mathrm{id})\underline{\Delta} = (\mathrm{id} \otimes \underline{\Delta})\underline{\Delta}$, $(\underline{\varepsilon} \otimes \mathrm{id})\underline{\Delta} = \mathrm{id} = (\mathrm{id} \otimes \underline{\varepsilon})\underline{\Delta}$]. The antipode $\underline{S}$ if it exists also obeys the usual axioms [so $\underline{\cdot}(\underline{S} \otimes \mathrm{id})\underline{\Delta} = \eta\underline{\varepsilon} = \underline{\cdot}(\mathrm{id} \otimes \underline{S})\underline{\Delta}$]. These axioms are analogous to the usual axioms for a Hopf algebra as in [36] but now as morphisms in the category. If there is no antipode then we have merely a bialgebra in the category. The diagrammatic form for the bialgebra axiom for $\underline{\Delta}$ is shown in Fig. 2(a).

As explained in the introduction, a *braided group* means a Hopf algebra living in a quasitensor category, equipped with some further structure expressing a kind of braided-commutativity or braided-cocommutativity (so that it is like the function algebra or enveloping algebra respectively of a classical group). The notion of braided (co)-commutativity that we need is a little involved in the abstract setting (though it is clear enough in the concrete examples that we need below). The problem in the abstract setting is that the obvious notions of opposite product $\underline{\cdot} \circ \Psi_{B,B}, \underline{\cdot} \circ \Psi^{-1}_{B,B}$ or opposite coproduct $\Psi_{B,B} \circ \underline{\Delta}, \Psi^{-1}_{B,B} \circ \underline{\Delta}$ do not again define Hopf algebras in the

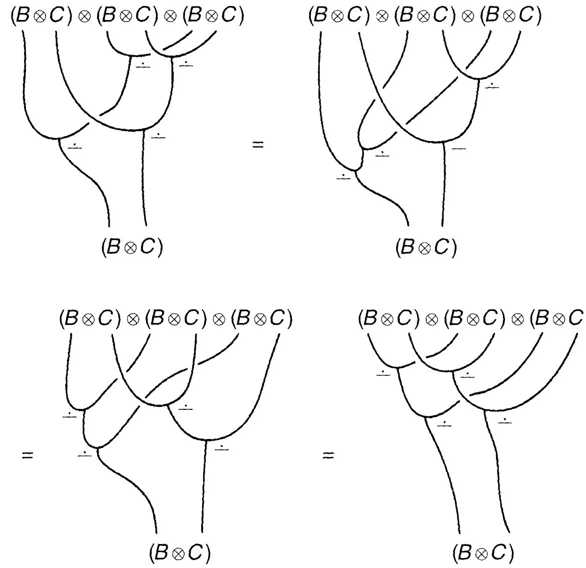

Fig. 1. Proof of associativity of the braided tensor product of algebras

quasitensor category when $\Psi^2 \neq \mathrm{id}$. For example, $\Psi^{-1}_{B,B} \circ \underline{\Delta}: B \to B \otimes B$ gives a homomorphism to the opposite braided tensor product algebra, and hence a Hopf algebra in the quasitensor category with opposite braiding. Again, this is easy to see using the diagrammatic notation. Since there is no intrinsic notion of opposite Hopf algebra in the braided case, there is no intrinsic way to assert that a Hopf algebra in the category coincides with its opposite.

In practice, we avoid this problem by working with a weaker (non-intrinsic) notion of braided-commutativity or braided-cocommutativity defined with respect to a class $\mathcal{C}$ of comodules or modules respectively. Here a B-module in the category is (V, α_V), where V is an object and $\alpha_V: B \otimes V \to V$ is a morphism such that $\alpha_V(\mathrm{id} \otimes \alpha_V) = \alpha_V(\underline{\cdot} \otimes \mathrm{id})$, $\alpha_V(\underline{\eta} \otimes \mathrm{id}) = \mathrm{id}$ as usual. Then

Definition 1.1 [4, 12]. A (weak) opposite coproduct for a bialgebra B in a quasitensor category, is a pair $(\underline{\Delta}^{\mathrm{op}}, \mathcal{C})$, where $\underline{\Delta}^{\mathrm{op}}: B \to B \underline{\otimes} B$ defines a second bialgebra structure for the same algebra B, and $\mathcal{C}$ is a class of B-modules such that the condition in Fig. 2(b) holds for all (V, α_V) in the class.

Definition 1.2 [4, 12]. A braided-cocommutative bialgebra is a pair $(B, \mathcal{C})$, where B is a bialgebra in the category and $(\underline{\Delta}, \mathcal{C})$ is an opposite coproduct. A braided

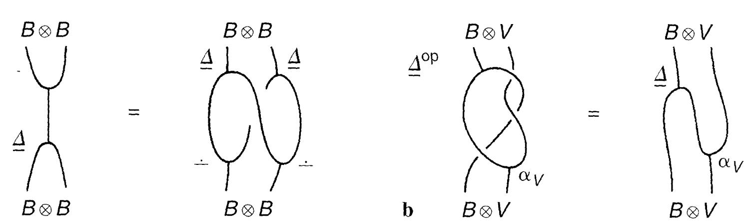

Fig. 2. (**a**) Bialgebra axiom, (**b**) Opposite coproduct axiom

group (of enveloping algebra type) is a pair $(B, \mathscr{O})$, where B is a Hopf algebra in a quasitensor category and $(B, \mathscr{O})$ is braided-cocommutative.

Thus a braided group is nothing other than a Hopf algebra B in a quasitensor category equipped with a class $\mathscr{O}$ with respect to which it behaves in a cocommutative way. In our examples, the class $\mathscr{O}$ is quite large and contains all useful modules (such as the braided-adjoint action of any Hopf algebra in the category on itself). For a given Hopf algebra B the class of all modules $\mathscr{O}(B)$ with respect to which it is cocommutative is closed under tensor product [12, Theorem 3.2] and under dualization if the quasitensor category has duals [12, Theorem 3.1]. The notion of opposite product is analogous, working now with a class of comodules in the category obeying an analogous condition (its diagram is just given by turning Fig. 2(b) upside down). It enables us to similarly define a braided-group of function algebra type as a braided-commutative Hopf algebra in a quasitensor category.

The above constructions in quasitensor categories are necessarily quite abstract. Fortunately, we will be concerned below only with the quasitensor categories that arise as representations of a quantum group, in which case explicit formulae are possible. Here a quantum group means for us the data $(H, \Delta, \varepsilon, s, \mathscr{R})$ where $\Delta: H \to H \otimes H$ is the coproduct, $\varepsilon: H \to k$ the counit and $s: H \to H$ the antipode forming an ordinary Hopf algebra over a field (or, with care, a commutative ring) k. For these we use the standard notation [36], notably $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ for the action of Δ on $h \in H$. The additional invertible element $\mathscr{R} \in H \otimes H$ is the quasitringular structure or "universal R-matrix" and obeys the axioms of Drinfeld [17]

$$(\Delta \otimes \mathrm{id})\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{23}\,, \qquad (\mathrm{id} \otimes \Delta)\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{12}\,, \qquad \Delta^{\mathrm{op}} = \mathscr{R}(\Delta\,)\mathscr{R}^{-1}\,, \tag{4}$$

where Δ^{op} denotes Δ followed by the usual transposition on $H \otimes H$, and $\mathscr{R}_{12} = \mathscr{R} \otimes 1$ in $H^{\otimes 3}$ as usual. A quantum group is said to be *triangular* if $\mathscr{R}_{21}\mathscr{R}_{12} = 1 \otimes 1$ [17] and strictly quasitriangular otherwise. We refer to [17, 18] for the standard (strictly) quasitriangular Hopf algebras $U_q(g)$. At generic $q = e^{\hbar/2}$ these are viewed over the ring of formal power series $\mathbb{C}[[\hbar]]$ [17].

These axioms are indeed such as to ensure that the category of representations of a quantum group form a quasitensor category. Some treatments of this topic appeared independently in [37, Sect. 7] and [16] as well as surely being known to experts at the time. Briefly, the objects in this category are the representations (modules) of H, while the tensor product of two such representations is given in the usual way by pull back along Δ. For any two such objects V, W the quasisymmetry or braiding is given by

$$\Psi_{V,W}(v \otimes w) = \sum \mathscr{R}^{(2)} \triangleright w \otimes \mathscr{R}^{(1)} \triangleright v\,, \tag{5}$$

where $\mathscr{R} = \sum \mathscr{R}^{(1)} \otimes \mathscr{R}^{(2)}$ and $\triangleright$ denotes the relevant actions. One can easily check that $\Psi_{V,W}$: $V \otimes W \to W \otimes V$ obeys (1) and is indeed an intertwiner for the action of H, precisely because $\mathscr{R}$ obeys (4). Moreover, the category is an ordinary symmetric one (with $\Psi^2 = \mathrm{id}$) precisely when the quantum group is triangular rather than strictly traingular. The role of the quantum group here is to generate the quasitensor category in which we work. For example, the finite-dimensional quantum group $(\mathbb{Z}_2', \mathscr{R})$ in [6, Sect. 6] generates the category of super-vector spaces as its representations. Here $\mathbb{Z}_2'$ denotes the group algebra of $\mathbb{Z}_2$ equipped with a certain non-trivial $\mathscr{R}$.

In [4, 7, 11] we gave a construction for braided groups in this kind of quasitensor category. In fact, there is a canonical construction for each type beginning from the quantum group itself that generates the category. Thus, if H is a quantum group, it has an associated braided group (of enveloping algebra type) $\underline{H}$ explicitly given as follows. Its linear space and algebra structure are unchanged, but are now viewed as living in the quasitensor category of H-modules by the quantum adjoint action. This, and the modified coalgebra and antipode are [4]

$$h \triangleright b = \sum h_{(1)} b s h_{(2)}, \qquad \underline{\Delta} b = \sum b_{(1)} s \mathscr{R}^{(2)} \otimes \mathscr{R}^{(1)} \triangleright b_{(2)}, \\ \underline{s} b = \sum \mathscr{R}^{(2)} s(\mathscr{R}^{(1)} \triangleright b) \tag{6}$$

for all $h \in H$ $b \in \underline{H}$. This gives the braided structures in terms of those of our initial H. For our class $\mathscr{O}$ we take the tautological $\underline{H}$-modules in the quasitensor category, where the action of $\underline{H}$ on an object V is given by the same linear map as the action of H that makes V an object. In this case the braided-cocommutativity of $\underline{H}$ then explicitly takes the concrete form [4],

$$\sum \Psi(b_{\underline{(1)}} \otimes Q^{(1)} \triangleright b_{\underline{(2)}}) Q^{(2)} = \sum b_{\underline{(1)}} \otimes b_{\underline{(2)}}, \tag{7}$$

where $\underline{\Delta} b = \sum b_{\underline{(1)}} \otimes b_{\underline{(2)}}$ and $Q = \mathscr{R}_{21} \mathscr{R}_{12}$. Here $Q^{(2)}$ multiplies the result of Ψ from the right as $(1 \otimes Q^{(2)})$. This implies the condition in Fig. 2(b) for all $\mathscr{O}$ and is essentially equivalent to it in the present context. Q corresponds to the double-twist $\Psi_{V,B} \circ \Psi_{B,V}$ and so is absent in the triangular case. For the example $H = U_q(g)$, we shall denote its associated braided group by $\underline{H} = BU_q(g)$.

Finally, we recall from [7, 11] that if $(A, \mathscr{R})$ is a dual quantum group [with dual quasitriangular structure $\mathscr{R}: A \otimes A \to k$ obeying some obvious axioms dual to (4)] then there is a corresponding braided group $\underline{A}$ (of function algebra type). As a coalgebra it coincides with A, while the modified product and antipode take the form [7, 11],

$$a \underline{\cdot} b = \sum a_{(2)} b_{(2)} \mathscr{R}((s a_{(1)}) a_{(3)}, s b_{(1)}), \qquad \underline{s} a = \sum s a_{(2)} \mathscr{R}((s^2 a_{(3)}) s a_{(1)}, a_{(4)}) \tag{8}$$

and there is a right adjoint coaction of A so that this $\underline{A}$ is a Hopf algebra in the category of A-comodules. If $A = H^*$ then the corresponding coadjoint action of H on $\underline{A}$ is $h \triangleright a = \sum a_{(2)} \langle h, (s a_{(1)}) a_{(3)} \rangle$. There is also a right action L^* of H on A defined by $L_h^*(a) = \sum \langle h, a_{(1)} \rangle a_{(2)}$. In terms of test-elements $g \in H$, these are

$$\langle h \triangleright a, g \rangle = \langle a, \sum (s h_{(1)}) g h_{(2)} \rangle, \qquad \langle L_h^*(a), g \rangle = \langle a, hg \rangle. \tag{9}$$

The braided-commutativity of $\underline{A}$ then takes the concrete form

$$a \underline{\cdot} b = \sum \underline{\cdot} \circ Q^{(2)} \triangleright \Psi(L^*_{Q^{(1)}}(a) \otimes b), \tag{10}$$

where $Q^{(2)}$ acts on the first factor of the result of Ψ. This implies the diagrammatic form of the braided-commutativity condition for all $\underline{A}$-comodules $\mathscr{O}$ for which the coaction of $\underline{A}$ is the tautological one. At least in the finite-dimensional case, the two braided groups $\underline{A}, \underline{H}$ are dually paired in a certain sense (cf. [24]). Here we consider both $\underline{A}, \underline{H}$ as living in the category of H-modules since every (right) A-comodule defines a (left) H-module by dualization. For the standard quantum function algebras G_q dual to the $U_q(g)$, we denote the associated braided groups of function algebra type by BG_q. The matrix bialgebras $A(R)$ [19] mapping onto G_q can also be converted in a similar way and give the braided-matrices $B(R)$ introduced in [6, 7]. They are braided-commutative bialgebras in the category of $U_q(g)$-modules.

2. Braided Group of $U_q(g)$ in "FRT" Form

This section is devoted to a computation of the braided groups associated to the quantum groups $U_q(g)$ in FRT form. The results will then be applied in later sections. Our first two technical results can in principle be motivated from category theory [39] along the lines of [24]. For our algebraic purposes (and in the form we need now) we give a direct algebraic treatment. The first establishes that $Q = \mathcal{R}_{21}\mathcal{R}_{12}$ defines a morphism $\underline{A} \to \underline{H}$ in the category of H-modules.

Proposition 2.1 cf. [23]. *Let H be a finite-dimensional quantum group and $A = H^*$ its corresponding dual quantum group. Then the map $Q: A \to H$ given by $Q(a) = \sum\langle Q^{(1)}, a\rangle Q^{(2)}$ is an intertwiner for the adjoint and coadjoint actions of H above.*

Proof. We compute the action of H using the definitions above and standard properties of quantum groups. $\mathcal{R}' = \sum \mathcal{R}'^{(1)} \otimes \mathcal{R}'^{(2)}$ denotes a second identical copy of $\mathcal{R}$. We have

$$\begin{aligned}
Q(h \triangleright a) &= \sum \mathcal{R}'^{(1)}\mathcal{R}^{(2)}\langle (sh_{(1)})\mathcal{R}'^{(2)}\mathcal{R}^{(1)}h_{(2)}, a\rangle \\
&= \sum \mathcal{R}'^{(1)}\mathcal{R}^{(2)}h_{(3)}sh_{(4)}\langle (sh_{(1)})\mathcal{R}'^{(2)}\mathcal{R}^{(1)}h_{(2)}, a\rangle \\
&= \sum \mathcal{R}'^{(1)}h_{(2)}\mathcal{R}^{(2)}sh_{(4)}\langle (sh_{(1)})\mathcal{R}'^{(2)}h_{(3)}\mathcal{R}^{(1)}, a\rangle \\
&= \sum h_{(3)}\mathcal{R}'^{(1)}\mathcal{R}^{(2)}sh_{(4)}\langle (sh_{(1)})h_{(2)}\mathcal{R}'^{(2)}\mathcal{R}^{(1)}, a\rangle \\
&= \sum h_{(1)}\mathcal{R}'^{(1)}\mathcal{R}^{(2)}sh_{(2)}\langle \mathcal{R}'^{(2)}\mathcal{R}^{(1)}, a\rangle = h \triangleright Q(a)\,.
\end{aligned}$$

Here the third and fourth equalities use the intertwining property of the quasitriangular structure $\mathcal{R}$ for Δ with its opposite. $\square$

Proposition 2.2. *The morphism $Q: \underline{A} \to \underline{H}$ established in Proposition 2.1 is a Hopf algebra homomorphism for the Hopf algebras $\underline{A}, \underline{H}$ in the category of H-modules. In particular, if H is factorizable in the sense of* [23], *we have $\underline{A} \cong \underline{H}$.*

Proof. We compute with the product defined in (8) to give

$$\begin{aligned}
Q(a\underline{\cdot}b) &= \sum \mathcal{R}^{(1)}\mathcal{R}'^{(2)}\langle a_{(2)}b_{(2)}, \mathcal{R}^{(2)}\mathcal{R}'^{(1)}\rangle \langle a_{(1)}, \mathcal{R}''^{(1)}\rangle \\
&\quad \times \langle a_{(3)}, \mathcal{R}'''^{(1)}\rangle \langle b_{(1)}, (s\mathcal{R}'''^{(2)})\mathcal{R}''^{(2)}\rangle \\
&= \sum \mathcal{R}^{(1)}\mathcal{R}'^{(2)}\langle a \otimes b, \mathcal{R}''^{(1)}\mathcal{R}^{(2)}{}_{(1)}\mathcal{R}'^{(1)}{}_{(1)}\mathcal{R}'''^{(1)} \\
&\quad \otimes (s\mathcal{R}'''^{(2)})\mathcal{R}''^{(2)}\mathcal{R}^{(2)}{}_{(2)}\mathcal{R}'^{(1)}{}_{(2)}\rangle\,.
\end{aligned}$$

Using the axioms for $\mathcal{R}$ this is the evaluation with $a \otimes b$ in the last two factors of the element $\sum X^{(1)} \otimes X^{(2)}\mathcal{R}'''^{(1)} \otimes (s\mathcal{R}'''^{(2)})X^{(3)}$, where $X = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\mathcal{R}_{21}\mathcal{R}_{31} = \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{21}\mathcal{R}_{31}$ by the quantum Yang-Baxter equations (QYBE) obeyed by $\mathcal{R}$. Writing $\mathcal{R}_{13} = (\mathrm{id} \otimes s)(\mathcal{R}_{13}^{-1})$, combining the elements on which s acts and using the QYBE again for them we obtain (after cancellations) the element $\mathcal{R}_{12}\mathcal{R}_{21}\mathcal{R}_{13}\mathcal{R}_{31}$. The pairing of this with $a \otimes b$ is just $Q(a)Q(b)$ as required. Next we compute with the coproduct defined in (6) to give

$$\underline{\Delta}Q(a) = \sum \mathcal{R}^{(1)}{}_{(1)}\mathcal{R}'^{(2)}{}_{(1)}\mathcal{R}''^{(2)}s\mathcal{R}'''^{(2)} \otimes \mathcal{R}'''^{(1)}\mathcal{R}^{(1)}{}_{(2)}\mathcal{R}'^{(2)}{}_{(2)}\mathcal{R}''^{(1)}\langle a, \mathcal{R}^{(2)}\mathcal{R}'^{(1)}\rangle\,.$$

This is evaluation of a on the third factor of the element $\sum X^{(1)}s\mathcal{R}'''^{(2)} \otimes \mathcal{R}'''^{(1)}X^{(2)} \otimes X^{(3)}$, where $X = \mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{32}\mathcal{R}_{31}\mathcal{R}_{21} = \mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{21}\mathcal{R}_{31}\mathcal{R}_{32}$ by the QYBE.

Writing $\mathscr{R}_{31} = (s \otimes \mathrm{id})(\mathscr{R}_{31}^{-1})$, combining the arguments of s, using the QYBE again and cancelling now gives the element $\mathscr{R}_{13}\mathscr{R}_{31}\mathscr{R}_{23}\mathscr{R}_{32}$. The pairing of this with a gives $(Q \otimes Q) \circ \Delta a$ as required. $\square$

Note that the notion of factorizability introduced in [23] is precisely that the linear map $Q: H^* \to H$ is a linear isomorphism. Our propositions say that in the braided setting it becomes a Hopf isomorphism. Quantum doubles as well as the $U_q(g)$ at least for generic q are known to be factorizable. For the latter we work as usual over formal power-series in a parameter $\hbar$, where $q = e^{\hbar/2}$ as in [17]. In this case there is a suitable dual to play the role of H^* in the propositions above. The algebraic proofs above clearly extend to this setting. We are therefore in a position to exploit the isomorphism Q for these Hopf algebras. To do this, it is convenient to work with the generators in "FRT" form as follows.

Firstly, [19] (cf. [17]) identified the duals of $U_q(g)$ as quotients of bialgebras $A(R)$ for certain R-matrices $R \in M_n \otimes M_n$ associated to the classical families of simple Lie algebras g. Here $A(R)$ is the bialgebra with generators $t^i{}_j$ and relations $Rt_1t_2 = t_2t_1R$ in standard notations. [19] also showed how to recover $U_q(g)$ in some form as an algebra with matrix generators $l^\pm$ and various relations. Among them are the matrix relations of the form $l_1^\pm l_2^\pm R = Rl_2^\pm l_1^\pm$ and $l_1^- l_2^+ R = Rl_2^+ l_1^-$, as well as many hidden relations among the $2n^2$ generators expressed in the form of an ansatz for the $l^\pm$ in terms of the familiar generators for the $U_q(g)$. We refer to this description of $U_q(g)$ by "matrix generators+ansatz" as *the* "FRT" *form* of $U_q(g)$. All our results below are intended for the $U_q(g)$ in this form, and we rely on [19] for details of their connection with other descriptions of $U_q(g)$ (this is known at least for the classical families of Lie algebras g). In fact, if the universal $\mathscr{R}$ for $U_q(g)$ is known in any given set of generators, it can be exploited to give the required ansatz easily according to

$$l^+ = \sum \mathscr{R}^{(1)}\langle t, \mathscr{R}^{(2)}\rangle\,, \qquad l^- = \sum \langle t, s\mathscr{R}^{(1)}\rangle \mathscr{R}^{(2)}\,, \tag{11}$$

see [40] where this method was used to generate the ansatz for $U_q(sl(3))$.

We also need the explicit description of the braided matrices $B(R)$ as introduced in [6]. They are given by matrix generators $u^i{}_j$ and certain matrix relations. The difference is that now, the $u^i{}_j$ span an object in the quasitensor category of $U_q(g)$-modules. The action is $l_2^+ \triangleright u_1 = R^{-1}u_1R$ and $l_1^- \triangleright u_2 = Ru_2R^{-1}$ [6]. The braided-coproduct, braiding Ψ and algebra relations take the form [6]

$$\underline{\Delta} u^i{}_j = u^i{}_k \otimes u^k{}_j\,, \qquad \Psi(u^I \otimes u^K) = u^L \otimes u^J \Psi^K{}_L{}^I{}_J\,,$$
$$u^I u^K = u^L u^J \Psi'^K{}_L{}^I{}_J\,, \tag{12}$$

where the u^I etc. are $u^i{}_j$ written with multi-indices, matrix Ψ comes from (5) and Ψ' is a variant corresponding (in the quotient Hopf algebra) to the right-hand side of (10). Explicitly, they are given by [6]

$$\Psi^I{}_J{}^K{}_L = R^{k_0}{}_a{}^d{}_{j_0} R^{-1a}{}_{l_0}{}^{j_1}{}_b R^{l_1}{}_c{}^b{}_{i_1} \tilde{R}^c{}_{k_1}{}^{i_0}{}_d\,, \tag{13}$$

$$\Psi'^I{}_J{}^K{}_L = R^{-1d}{}_{j_0}{}^{k_0}{}_a R^{j_1}{}_b{}^a{}_{l_0} R^{l_1}{}_c{}^b{}_{i_1} \tilde{R}^c{}_{k_1}{}^{i_0}{}_d\,, \tag{14}$$

where $\tilde{R} = ((R^{t_2})^{-1})^{t_2}$ with ^{t_2} denoting transposition in the second matrix factor. Another (more conventional) way to write the relations of $B(R)$ is to move two of the R's in Ψ' to the left-hand side, in which case the equations become equivalently

$$R_{21}u_1R_{12}u_2 = u_2R_{21}u_1R_{12}\,. \tag{15}$$

These are quite similar to some equations in [41] as well as being known in [20–22] to be discussed below. In our case they are nothing other than the braided-commutativity (10) in the case of $B(R)$. The coproduct $\underline{\Delta}$ extends to all of $B(R)$ as a bialgebra in this quasitensor category [6]. The construction is very general, but in the present case (for the standard R matrices) after further quotienting $B(R)$ by "braided-determinant" type relations, one obtains $\underline{A} = BG_q$, the braided group corresponding to the quantum group dual to $H = U_q(g)$ in the setting above. We are now ready to prove our main result of this section. It is a corollary of Proposition 2.2 in the form over $\mathbb{C}[[\hbar]]$.

Corollary 2.3. *Let $H = U_q(g)$ in* FRT *form* [19] *and $\underline{A} = BG_q$ the braided group of function algebra type as recalled above. Then the braided group $\underline{H} = BU_q(g)$ of enveloping algebra type corresponding to $U_q(g)$ has the same algebra structure as $U_q(g)$ but the new coproduct implied by*

$$\underline{\Delta} L^i{}_j = L^i{}_k \otimes L^k{}_j\,, \quad \textit{where}\ \ L = l^+ s l^- \,.$$

The space spanned by the $L^i{}_j$ is a $U_q(g)$-module via $l_2^+ \triangleright L_1 = R^{-1} L_1 R$ and $l_1^- \triangleright L_2 = R L_2 R^{-1}$ and the identification $L = u$ allows us to consider $BU_q(g) = BG_q$ as a self-dual braided group.

Proof. We compute the map Q in Proposition 2.2 as

$$\begin{aligned} Q(u^i{}_j) &= \sum \mathscr{R}^{(1)} \mathscr{R}'^{(2)} \langle u^i{}_j, \mathscr{R}^{(2)} \mathscr{R}'^{(1)} \rangle = \sum \mathscr{R}^{(1)} \mathscr{R}'^{(2)} \langle u^i{}_k, \mathscr{R}^{(2)} \rangle \, \langle u^k{}_j, \mathscr{R}'^{(1)} \rangle \\ &= \sum \mathscr{R}^{(1)} s \mathscr{R}'^{(2)} \langle u^i{}_k, \mathscr{R}^{(2)} \rangle \, \langle u^k{}_j, s\mathscr{R}'^{(1)} \rangle = l^{+i}{}_k s l^{-k}{}_j = L^i{}_j \,. \end{aligned}$$

The action is $l^{+i}{}_j \triangleright Q(u^k{}_l) = Q(l^{+i}{}_j \triangleright u^k{}_l) = R^{-1k}{}_m{}^i{}_a L^m{}_n R^n{}_l{}^a{}_j$ using Proposition 2.1 and the action on $u^k{}_l$ obtained from (9) above and the pairing $\langle u, l^+ \rangle = R$ from [19]. Note that the transmutation procedure used to define these braided groups in [6] is such that we can identify the generators u with the generators t of G_q (but not their products), and we have used this fact here. Similarly for the action of l^-. [The action on $B(R)$ above is similar, but in a more general setting.] Because of Proposition 2.1 this result on L must coincide with the quantum adjoint action as defined in (6). □

These new generators L for $U_q(g)$ are the ones in which the corresponding $BU_q(g)$ becomes explicitly identified with a quotient of $B(R)$ where the matrix coproduct is braided. Explicitly for $U_q(sl_2)$ they are easily computed as

$$L = \begin{pmatrix} q^H & q^{-1/2}(q-q^{-1}) q^{H/2} X_- \\ q^{-1/2}(q-q^{-1}) X_+ q^{H/2} & q^{-H} + q^{-1}(q-q^{-1})^2 X_+ X_- \end{pmatrix} \tag{16}$$

in the usual description for $U_q(sl_2)$ with the conventions $[X_+, X_-] = (q^H - q^{-H})/(q - q^{-1})$ of [18]. These combinations have been known in various contexts, notably [21, 22] and cf. [20, 41]. There it is known that the relations

$$R_{21} L_1 R_{12} L_2 = L_2 R_{21} L_1 R_{12} \tag{17}$$

also describe the relations of $U_q(sl_2)$ (for example) given the ansatze for $l^{\pm}$. We see from Corollary 2.3 that this is due to the factorizability of $U_q(sl_2)$ and hence holds quite generally as an expression of the braided-commutativity of BG_q as in (10) carried over to $BU_q(g)$ via the isomorphism $BU_q(g) \cong BG_q$. Likewise, we learn that BG_q is braided-cocommutative since $BU_q(g)$ is. Explicitly, this braided-cocommutativity takes the form

$$\Psi(L^i{}_a \otimes L^b{}_c)M^a{}_b{}^c{}_j = L^i{}_a \otimes L^a{}_j, \qquad M^a{}_b{}^c{}_j = l^{+c}{}_k(s^{-1}l^{+m}{}_b)l^{-a}{}_m sl^{-k}{}_j. \quad (18)$$

This is noted for completeness and is readily computed from (7) using the same technique as in the corollary above.

Also, in various other contexts it has been noted that such combinations as in (16) are indeed fixed under the quantum adjoint action for $U_q(sl_2)$. However, Corollary 2.3 ensures these desirable features hold quite generally, giving us a fundamental ad-invariant subspace of $U_q(g)$ for all the standard Lie algebras g. This suggests that this subspace should have properties resembling some kind of "quantum Lie algebra" (or "braided Lie algebra") for $U_q(g)$. Recall that for an ordinary Lie algebra the vector space of g is a g-module by the adjoint action, and this action as a map coincides with the Lie bracket. We can therefore likewise take the quantum adjoint action on the space spanned by the $L^i{}_j$ as a "quantum Lie bracket" or "braided Lie bracket."

Proposition 2.4. *Let $\mathscr{L} \subset U_q(g)$ be the subspace spanned by the $\{L^i{}_j\}$ in Corollary 2.3. We let $[\,,\,]: U_q(g) \otimes U_q(g) \to U_q(g)$ be defined by $[h, g] = h \triangleright g$, where $\triangleright$ is the quantum adjoint action in (6). This "quantum Lie bracket" enjoys the properties of closure and "Jacobi" identities*

(L0) $[\xi, \eta] \in \mathscr{L}$ for $\xi, \eta \in \mathscr{L}$,

(L1) $[\xi, [\eta, \zeta]] = \sum [[\xi_{(1)}, \eta], [\xi_{(2)}, \zeta]]$,

(L2) $[[\xi, \eta], \zeta] = \sum [\xi_{(1)}, [\eta, [s\xi_{(2)}, \zeta]]]$,

where $\Delta\xi = \sum \xi_{(1)} \otimes \xi_{(2)} \in U_q(g) \otimes U_q(g)$ is the usual coproduct, and s is the usual antipode. On the vectors L^I where $I = (i_0, i_1)$ we have

$$[L^I, L^J] = c^{IJ}{}_K L^K, \qquad c^{IJ}{}_K = \tilde{R}^a{}_{i_1}{}^{j_0}{}_b R^{-1b}{}_{k_0}{}^{i_0}{}_c Q^c{}_a{}^{k_1}{}_{j_1},$$

where $Q = R_{21}R_{12}$.

Proof. The identity (L1) is an expression of the fact that the quantum adjoint action is an intertwiner for itself, i.e. in the general setting above it is a morphism $\underline{H} \otimes \underline{H} \to \underline{H}$ in the category of H-modules. This a general feature of the braided groups $\underline{H}$ associated to H [12]. This, and the identity (L2) follow easily from the definition of the quantum adjoint action in (6). For the explicit form of the $c^{IJ}{}_K$ we use the same method as in Corollary 2.3 to compute the action of sl^- on L. It comes out as $(sl^{-i}{}_j) \triangleright L^k{}_l = \tilde{R}^a{}_j{}^k{}_m L^m{}_n R^i{}_a{}^n{}_l$. Using this and the action of l^+ already given, we compute $[L^I, L^J] = L^I \triangleright L^J$. $\square$

Note that an easy computation gives $\Delta L^i{}_j = l^{+i}{}_a sl^{-b}{}_j \otimes L^a{}_b$ (as already noted in [21]) so that the ordinary coproduct on the right-hand side of (L1), (L2) does not have its image in $\mathscr{L} \otimes \mathscr{L}$. Hence these identities do not make sense for $\mathscr{L}$ in isolation from its quantum group $U_q(g)$. For this reason we do not formalize these as axioms for an abstract Lie algebra. Nevertheless, if we imagine that ξ is primitive as for a classical lie algebra, i.e. $\Delta\xi = \xi \otimes 1 + 1 \otimes \xi$ (and $s\xi = -\xi$) and note that (unusually) $[1, \xi] = 1 \triangleright \xi = \xi$, then the above do reduce to two ordinary Jacobi identities for the two left-hand sides. These two ordinary Jacobi identities imply on adding that $[\eta, [\xi, \zeta]] + [[\xi, \zeta], \eta] = 0$, i.e. in the semisimple case they imply antisymmetry. Thus (L1), (L2) together play the role of one usual Jacobi identity and antisymmetry. On the other hand, not all elements of $\mathscr{L}$ are primitive, even as $q \to 1$, and indeed the properties of the generators $\mathscr{L}$ also express group-like as well as Lie-algebra like features.

In addition, there are numerous other identities inherited from the structure of $U_q(g)$ and its braided group, expressing the joint role of these as "enveloping algebra" for $\mathscr{L}$. These include

$$[\xi\eta,\zeta]=[\xi,[\eta,\zeta]]\,,\qquad [\xi,\eta\zeta]=\sum[\xi_{(1)},\eta]\,[\xi_{(2)},\zeta]\,,$$

$$\sum(s\xi_{(1)})\,[\xi_{(2)},\eta]=\sum[\eta_{(1)},s\xi]\eta_{(2)}\,,\qquad [\xi,\eta]=\sum\xi_{(1)}\eta s\xi_{(2)}\,,$$

$$\Big(\sum[\xi_{(1)},\ ]\otimes[\xi_{(2)},\ ]\Big)\underline{\Delta}=\underline{\Delta}[\xi,\ ]\,,\qquad (\mathrm{id}\otimes\underline{\Delta})\underline{\Delta}=(\underline{\Delta}\otimes\mathrm{id})\underline{\Delta}\,.$$

If we imagine ξ, etc. primitive as before, the second identity becomes $[\xi,\eta\zeta]=[\xi,\eta]\zeta+\eta[\xi,\zeta]$, the third becomes $[\xi,\eta]=[\eta,-\xi]$ and the fourth becomes $[\xi,\eta]=\xi\eta-\eta\xi$. The last line refers to the braided coproduct of $BU_q(g)$, which restricts to $\underline{\Delta}:\mathscr{L}\to\mathscr{L}\otimes\mathscr{L}$ as a generalization of the coproduct in the universal enveloping algebra of a Lie algebra. The identity corresponds to the fact there that if ξ,η are primitive, then $[\xi,\eta]$ is also primitive. The last identity is the coassociativity inherited from that of $BU_q(g)$. Thus, it is this $\underline{\Delta}$ that preserves $\mathscr{L}$ and extends to products of the generators as a (braided) Hopf algebra.

Finally, in the remaining sections of the paper we will focus for concreteness on the example of the above for $H=U_q(sl_2)$. There is a standard matrix R for this in the FRT approach. Then $B(R)=BM_q(2)$ (the braided matrices of sl_2 type) has the action on $u=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ given by [6]

$$q^{H/2}\triangleright\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&q^{-1}b\\qc&d\end{pmatrix},\tag{19}$$

$$\begin{aligned}X_+\triangleright\begin{pmatrix}a&b\\c&d\end{pmatrix}&=\begin{pmatrix}-q^{3/2}c&-q^{1/2}(d-a)\\0&q^{-1/2}c\end{pmatrix},\\ X_-\triangleright\begin{pmatrix}a&b\\c&d\end{pmatrix}&=\begin{pmatrix}q^{1/2}b&0\\q^{-1/2}(d-a)&-q^{-3/2}b\end{pmatrix}.\end{aligned}\tag{20}$$

The algebra relations are comparable to those of the quantum matrices $M_q(2)$ and come out as

$$ba=q^2ab\,,\qquad ca=q^{-2}ac\,,\qquad da=ad\,,\qquad bc=cb+(1-q^{-2})a(d-a)\,,\tag{21}$$

$$db=bd+(1-q^{-2})ab\,,\qquad cd=dc+(1-q^{-2})ca\,.\tag{22}$$

The additional "braided-determinant" relation

$$ad-q^2cb=1\tag{23}$$

gives the braided group $BSL_q(2)$ [the braided group version of $SL_q(2)$]. This is a braided group of "function algebra" type and is $\underline{A}$ in the setting above when $H=U_q(sl_2)$. Our results above imply that this can be identified as $u=L$ with the braided group $BU_q(sl_2)$ of enveloping algebra type. Note that $BSL_q(2)$ has a bosonic central element $q^{-1}a+qd$ as explained in [6]. It is the spin 0 generator in the identification $\mathscr{L}=1\oplus 3$, where the remaining generators form a 3-dimensional spin 1 representation of $U_q(sl_2)$. The element is bosonic in the sense that $\Psi((q^{-1}a+qd)\otimes f)=f\otimes(q^{-1}a+qd)$ for all f since the action of $\mathscr{R}$ in (5) is trivial. We see from the identification $u = L$ that this element is just the quadratic Casimir of $BU_q(sl_2)$.

In summary, we have shown that the generators $L = l^+ sl^-$ of $U_q(g)$ are convenient for the description of the corresponding braided group. This braided group is at the same time the braided-cocommutative braided group of enveloping algebra type consisting of $U_q(g)$ with a modified coproduct, and the braided-commutative braided group of function algebra type dual to this. Moreover, these generators L exhibit a number of Lie-algebra type properties inherited from these structures. In particular, $BU_q(sl_2) = BSL_q(2)$ can be identified.

3. Braided Matrix Structure of the Degenerate Sklyanin Algebra

The Sklyanin algebra was intrdouced in [26, 42] in connection with an ansatz for the 8-vertex model. As an algebra it has been extensively studied by ring-theorists for its remarkable properties, see [27] and elsewhere. The algebra has four generators S_0, S_α, $\alpha = 1, 2, 3$ and three structure constants J_{12}, J_{23}, J_{31} subject to the constraint $J_{12} + J_{23} + J_{31} + J_{12}J_{23}J_{31} = 0$ and the relations

$$[S_0, S_\alpha] = \iota J_{\beta\gamma}\{S_\beta, \sigma_\gamma\}, \qquad [S_\alpha, S_\beta] = \iota\{S_0, S_\gamma\}, \tag{24}$$

where α, β, γ are from the set $1, 2, 3$ in cyclic order and $\{\ ,\ \}$ denotes anticommutator. There are two Casimir elements

$$C_1 = S_0^2 + \sum_\alpha S_\alpha^2, \qquad C_2 = \sum_\alpha S_\alpha^2 J_\alpha, \tag{25}$$

where $J_{\alpha\beta} = -\dfrac{J_\alpha - J_\beta}{J_\gamma}$. The degenerate case where (say) $J_{12} = 0$ is well-known to be closely connected with the quantum group $U_q(sl_2)$. We write $S_\pm = S_1 \pm \iota S_2$ and $K_\pm = S_0 \pm tS_3$, where $t = \sqrt{J_{23}}$ (a fixed square root). Then the relations become

$$\begin{gathered}[K_+, S_\pm] = \pm t\{K_+, S_\pm\}, \qquad [K_-, S_\pm] = \mp t\{K_-, S_\pm\}, \\ [K_+, K_-] = 0, \qquad [S_+, S_-] = \frac{1}{t}(K_+^2 - K_-^2).\end{gathered} \tag{26}$$

Writing $q = \dfrac{1+t}{1-t}$ and $Y_\pm = \frac{1}{2}\sqrt{1-t^2}S_\pm$ we have

$$\begin{gathered}[K_+, K_-] = 0, \qquad K_+Y_\pm = q^{\pm 1}Y_\pm K_+, \\ K_-Y_\pm = q^{\mp 1}Y_\pm K_-, \qquad [Y_+, Y_-] = \frac{K_+^2 - K_-^2}{q - q^{-1}}\end{gathered} \tag{27}$$

while two independent linear combinations of the Casimir elements are (with $J_1 = J_2 = 1$, $J_3 = 1 + J_{23}$)

$$\begin{gathered}C_1 - C_2 = K_+K_-, \\ C \equiv 2\left(\frac{(1+t^2)}{(1-t^2)}C_1 - C_2\right) = q^{-1}K_+^2 + qK_-^2 + (q - q^{-1})^2 Y_+Y_-.\end{gathered} \tag{28}$$

Thus, with these changes of variables we see that the further quotient $K_+K_- = 1$ gives us the familiar algebra of $U_q(sl_2)$ (in Jimbo's conventions) and the combination

C shown becomes its familiar quadratic Casimir element. $U_q(sl_2)$ is of course a Hopf algebra but, as far as is known, the full degenerate Sklyanin algebra itself it not. Instead we have

Theorem 3.1. *The degenerate Sklyanin algebra as described is isomorphic to the braided matrices* $BM_q(2)$ *of* sl_2 *type. Explicitly the generators* $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *of the latter take the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} K_+^2 & q^{-1/2}(q-q^{-1})K_+Y_- \\ q^{-1/2}(q-q^{-1})Y_+K_+ & K_-^2 + q^{-1}(q-q^{-1})^2 Y_+Y_- \end{pmatrix}$$

and we allow $a, d - ca^{-1}b$ *of* $BM_q(2)$ *to be invertible and have square roots. Hence the degenerate Sklyanin algebra has the structure of a bialgebra in the quasitensor category of* $U_q(sl_2)$*-modules. The bosonic central elements* $q^{-1}a + qd$, $ad - q^2cb$ *of* $BM_q(2)$ *are explicitly*

$$q^{-1}a + qd = C\,, \qquad ad - q^2cb = K_+^2K_-^2\,.$$

Proof. Since the braided group $BU_q(sl_2)$ has the same structure as an algebra as the quantum group $U_q(sl_2)$, we know that the quotient of the degenerate Sklyanin algebra by $K_+K_- = 1$ is isomorphic to this also, and hence by Corollary 2.3 isomorphic to the braided group $BSL_q(2)$. But this is a quotient of $BM_q(2)$ by the braided-determinant $ad - q^2cb = 1$, so we are motivated to make an ansatz for $BM_q(2)$ in the form stated. The ansatz is then verified by explicit computations which we leave to the reader. Clearly, the generators $K_\pm^2, K_+Y_\pm$ can be recovered from the a, b, c, d. Thus, it is more precisely this form of the Sklyanin algebra (rather than generators $K_\pm, Y_\pm$) that is isomorphic to $BM_q(2)$. This is not, however, an important distinction when we work over $\mathbb{C}[[\hbar]]$ with $K_\pm = q^{H\pm/2}$ say with $q = e^{\hbar/2}$ [as for $U_q(sl_2)$]. □

We compute the braided structure in the Sklyanin algebra implied by this theorem, as follows.

Proposition 3.2. *The action of* $U_q(sl_2)$ *on the degenerate Sklyanin algebra as a bialgebra in the category of* $U_q(sl_2)$*-modules is explicitly*

$$\begin{gathered} q^{H/2} \triangleright K_\pm = K_\pm\,, \qquad q^{H/2} \triangleright Y_\pm = q^{\pm 1}Y_\pm\,, \\ X_\pm \triangleright K_+ = (1-q^{\pm 1})Y_\pm\,, \qquad X_\pm \triangleright K_- = (q^{\pm 1}-1)K_+^{-1}Y_\pm K_-\,, \\ X_+ \triangleright Y_+ = (1-q^{-1})Y_+^2K_+^{-1}, \qquad X_+ \triangleright Y_- = K_+^{-1}(Y_+Y_- - qY_-Y_+)\,, \\ X_- \triangleright Y_- = (1-q)Y_-^2K_+^{-1}, \qquad X_- \triangleright Y_+ = K_+^{-1}(Y_-Y_+ - q^{-1}Y_+Y_-)\,. \end{gathered}$$

The degenerate Sklyanin algebra is invariant under this action in the sense $h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$, *where* $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ *is the usual coproduct of* $U_q(sl_2)$.

Proof. This is determined from the form of the isomorphism in the preceding theorem and the known action of $U_q(sl_2)$ on $BM_q(2)$ recalled in (19)–(20) from [6]. The action of $q^{H/2}$ is easily determined first since this element is group-like so that $q^{H/2} \triangleright (Y_+K_+) = (q^{H/2} \triangleright Y_+)(q^{H/2} \triangleright K_+)$ etc. Similarly $X_+ \triangleright (Y_+K_+) = 0$ according to (20), but $X_+ \triangleright (Y_+K_+) = (X_+ \triangleright Y_+)(q^{H/2} \triangleright K_+) + (q^{-H/2} \triangleright Y_+)(X_+ \triangleright K_+) = (X_+ \triangleright Y_+)K_+ + q^{-1}Y_+(X_+ \triangleright K_+)$ using the standard coproduct of X_+. This determines $X_+ \triangleright Y_+$ once $X_+ \triangleright K_+$ is known. This is extracted from knowledge of $X_+ \triangleright K_+^2$ from (20) and a similar computation. In the same way the action $X_+ \triangleright Y_-$ is extracted from $X_+ \triangleright b$ in (20). Finally, $X_+ \triangleright K_-$ can be extracted more easily from $K_+^2 K_-^2$ bosonic. The action of X_- is then obtained by a symmetry principle. These computations have been made and the resulting action verified using the algebra package REDUCE. □

In principle, we can similarly extract the form of the braiding Ψ and the braided coproduct $\underline{\Delta}$ on the generators $K_\pm, Y_\pm$ from the braiding and matrix coproduct $\underline{\Delta}$ for $BM_q(2)$. From the theorem and [6], some examples of $\Psi, \underline{\Delta}$ are

$$\Psi(K_+^2 \otimes K_+Y_-) = K_+Y_- \otimes K_+^2, \quad \Psi(Y_+K_+ \otimes K_+Y_-) = q^{-2}K_+Y_- \otimes Y_+K_+, \tag{29}$$

$$\underline{\Delta}K_+^2 = K_+^2 \otimes K_+^2 + q^{-1}(q-q^{-1})^2 K_+Y_- \otimes Y_+K_+, \tag{30}$$

$$\underline{\Delta}K_+Y_- = K_+^2 \otimes K_+Y_- - q^{-2}K_+Y_- \otimes K_+^2 + q^{-1}K_+Y_- \otimes C\,, \tag{31}$$

$$\underline{\Delta}Y_+K_+ = Y_+K_+ \otimes K_+^2 - q^{-2}K_+^2 \otimes Y_+K_+ + q^{-1}C \otimes Y_+K_+. \tag{32}$$

However, in practice it is rather hard to proceed further to compute Ψ and $\underline{\Delta}$ explicitly on $K_\pm, Y_\pm$ alone. This is because they do not transform in a simple way among themselves under the action in Proposition 3.2 so that the braidings $\Psi(K_+ \otimes Y_+)$ etc. as determined by $\mathcal{R}$ in (5) are given by infinite power-series rather than finite combinations of the generators. This in turn means that the braider tensor product algebra structure does not compute in closed form. Rather, we see that the generators $K_+^2, q^{-1/2}(q-q^{-1})Y_+K_+, q^{-1/2}(q-q^{-1})K_+Y_-, C$ do transform among themselves and are more convenient for the description of the braiding and braided coproduct.

4. Braided Structure in the Quantum Lorentz Group

In this section we give a second application of Proposition 2.2 and Corollary 2.3, this time to the algebraic structure of the quantum double of the quantum groups $U_q(g)$. A physically interesting example of such a double, namely the quantum double of $U_q(su(2))$ has been called the "quantum Lorentz group" in [29] on the basis of a Hopf-C^*-algebraic "quantum Iwasawa decomposition." Recall that the ordinary Lorentz group can be identified (at the Lie algebra level) with $sl_2(\mathbb{C})$ regarded as a real Lie algebra, i.e. the complexification of $su(2)$: the quantum double $D(U_q(su(2)))$ can likewise be regarded as a kind of "complexification" of $U_q(su(2))$ as a Hopf $*$-algebra (or rather, in the dual form as a C^*-algebra). In fact, there are some quite general algebraic arguments to arrive at this same conclusion, based on the author's Yang-Baxter-theoretic proof of the Iwasawa decomposition for ordinary Lie algebras appearing in [43]. We recall this first.

Let u be a compact real form of a complex semisimple Lie algebra g. The latter is the complexification of u and forms a real Lie algebra of twice the dimension: $g = \imath u \oplus u$ with the Lie bracket of u extended linearly to g. The Iwasawa decomposition states that there is a splitting $g = k \oplus u$ as vector spaces into two sub-Lie algebras, with k solvable. We observed in [43, sect. 2] that this Lie algebra k could be identified with the Lie algebra structure on u^* associated to the Drinfeld-Jimbo solution r of the Classical Yang-Baxter Equations (CYBE) as follows. Choosing a Cartan-Weyl basis for g, the solution $r \in g \otimes g$ takes the form

$$r = \sum_\lambda \mathrm{sgn}(\lambda) \frac{E_\lambda \otimes E_{-\lambda}}{K(E_\lambda, E_{-\lambda})} + K^{-1}\,, \tag{33}$$

where the sum is over root vectors E_λ and K denotes the Killing form with inverse K^{-1}. In Drinfeld's theory in [31] this defines a quasitriangular Lie bialgebra (g, δ, r), where $\delta: g \to g \otimes g$ is defined by $\delta\xi = \sum [\xi, r^{(1)}] \otimes r^{(2)} + r^{(1)} \otimes [\xi, r^{(2)}]$. Now, just as every finite-dimensional Hopf algebra has a dual one built on the dual linear space, every finite dimensional Lie bialgebra has a dual g^*. Its Lie bracket is defined from δ by $\langle [\eta, \eta'], \xi\rangle = \langle \eta \otimes \eta', \delta\xi\rangle$ for $\eta, \eta' \in g^*$. The key observation in [43, Sect. 2] is that in our case, r in (33) has its first term (its antisymmetric part) lying entirely in $\iota u \otimes u$, while the second term (its symmetric part) lies in $u \otimes u$. Using this we showed that the subspace $u^* \subset g^*$ defined by $u^* = \iota K(u, \ )$ is fixed under this Lie bracket on g^*. The coadjoint actions of g on g^* and g^* on g restrict to mutual actions of u, u^* on each other, and finally Drinfeld's Lie bialgebra double $D(g)$ built on the linear space of $g^* \oplus g$ [31] restricts to a Lie bialgebra $u^{*\mathrm{op}} \bowtie u$ built on $u^* \oplus u \subset g^* \oplus g$. For later use, the Lie algebra structure of $D(g)$ is explicitly given by

$$\begin{aligned}[][\eta \oplus \xi, \eta' \oplus \xi'] = ([\eta', \eta] + \sum \eta'_{[1]}\langle \eta'_{[2]}, \xi\rangle - \eta_{[1]}\langle \eta_{[2]}, \xi'\rangle) \\ \oplus ([\xi, \xi'] + \sum \xi_{[1]}\langle \eta', \xi_{[2]}\rangle - \xi'_{[1]}\langle \eta, \xi'_{[2]}\rangle), \end{aligned} \tag{34}$$

where $\delta\xi = \sum \xi_{[1]} \otimes \xi_{[2]}$, etc. is our explicit notation for the cobrackets. The cobracket on $D(g)$ is the tensor product of those on g and g^*. We note that $u^{*\mathrm{op}}$, etc. denotes (in the present conventions) u^* with its opposite (reversed) Lie bracket, while the notation $u^{*\mathrm{op}} \bowtie u$ derives from a general "double semidirect sum" construction for a Lie algebra from a pair mutually acting on each other in a compatible way (a "matched pair" of Lie algebras). We introduced this notion in [44, Sect. 4] where we showed that $D(g) = g^{*\mathrm{op}} \bowtie g$ by the mutual coadjoint actions. Other authors have also arrived at similar notions of Lie algebra matched pairs, notably [45, 46]. Finally, there is an isomorphism [43, Sect. 2]

$$\phi: u^{*\mathrm{op}} \bowtie u \cong g, \qquad \phi(\eta \oplus \xi) = \sum r^{(1)}\langle \eta, r^{(2)}\rangle + \xi. \tag{35}$$

The isomorphism is our Yang-Baxter theoretic description of the Iwasawa decomposition of g. Both the solvable Lie algebra $k = u^{*\mathrm{op}}$ and the decomposition itself are derived from the Drinfeld-Jimbo solution (33).

In [43, Sect. 3] we proceeded to construct a "matched pair" of Lie groups $U, U^{*\mathrm{op}}$ (say) by building from the mutual actions between $u, u^{*\mathrm{op}}$ a matching pair of gauge fields over $U^{*\mathrm{op}}, U$ and using their parallel transport to exponentiate to global actions of the groups. The resulting group double-semidirect product $U^{*\mathrm{op}} \bowtie U \cong G$ (where G is the simply-connected Lie group of g) provided a new constructive proof of the group Iwasawa decomposition. In fact, the constructions were quite general, allowing for the exponentiation of any Lie algebra splitting or "Manin triple" to a Lie group one provided some technical criteria were satisfied (this was the main result of [43]).

We can however, go in another direction, namely to deform to the quantum group setting. Here the relevant notion, the "double cross product" of mutually acting Hopf algebras (matched pairs of Hopf algebras) was introduced in [44, sect. 3]. Every factorization of a Hopf algebra into sub-Hopf algebras as defined in [44] can be reconstructed from its factors by this double cross product construction. Once again, we showed that Drinfeld's quantum double $D(H)$, where H is any (say, finite-dimensional) Hopf algebra, is simply a Hopf algebra double cross product $D(H) = H^{*\mathrm{op}} \bowtie H$, this time by mutual quantum coadjoint actions. In the conventions that we need below, $H^{*\mathrm{op}}$ denotes H^* with the opposite product (more usually, one takes here the opposite coproduct [17], but this is isomorphic via the antipode s). We have not discussed in [44] the question of real forms ($*$-structures) but it is clear that just as $u^{*\mathrm{op}} \bowtie u \cong \iota u \oplus u$ in (35) is a real form of $g^{*\mathrm{op}} \bowtie g = D(g)$, so $D(U_q(g))$ should be regarded as a Hopf algebra whose real form is the complexification of the real form $U_q(u)$ of $U_q(g)$. Our algebraic results below are thus a further step towards an Iwasawa decomposition theorem for quantum groups. We will obtain an analogue of the formula (35) with the role of r played by the universal R-matrix of the quantum group.

Finally, working over $\mathbb{C}$ as we do brings out some further structure not visible over $\mathbb{R}$. Namely, when regarding $g = \iota u \oplus u$, there is a sense in which the elements of ιu (the pure boosts in the case of the Lorentz group) are acted upon by the elements of u (the rotations). I.e., the Lorentz group Lie algebra (in addition to its numerous other descriptions) has the flavour of a semidirect sum where the rotations act on the boosts by commutation. On the other hand, this cannot be literally so since the boosts do not close under commutation. In fact, we will see that $g = \iota u \oplus u$ can be embedded in a natural way in a semidirect sum $g \rtimes g$, with the "boosts" acted upon by "rotations." This is related to our result $D(g) \cong g \rtimes g$ below. We will also see the latter result in the quantum case for $D(H)$.

We begin (as in Sect. 2) with a general result for finite-dimensional quantum groups H. Its origins are in a result in [28] that the quantum double $D(H)$ in this case (when H is quasitriangular) has the structure of a semidirect product. The result was obtained before the notion of braided groups had been introduced. We need the following more explicit variant. In the conventions that we need, we build the quantum double $D(H)$ on $H^* \otimes H$ as follows. The coproduct, counit and unit for $D(H)$ are the tensor product ones while the product of $D(H)$ comes out as

$$(a \otimes h)(b \otimes g) = \sum b_{(2)} a \otimes h_{(2)} g \langle s h_{(1)}, b_{(1)} \rangle \langle h_{(3)}, b_{(3)} \rangle \tag{36}$$

for h, g in H and a, b in H^*. Its antipode is $s(a \otimes h) = (1 \otimes sh)(s^{-1} a \otimes 1)$. We have

Proposition 4.1. *Let H be a finite-dimensional quasitriangular Hopf algebra with quantum double $D(H)$, $A = H^*$ and $\underline{A}$ the associated braided group of function algebra type. Then $D(H) \cong \underline{A} \rtimes H$ as a semidirect product by the coadjoint action of H on $\underline{A}$ and as a semidirect coproduct with the H coaction induced by $\mathcal{R}: A \to H$. Explicitly, the semidirect product and coproduct on $\underline{A} \rtimes H$ are*

$$(a \otimes h)(b \otimes g) = \sum a \underline{\cdot} (h_{(1)} \triangleright b) \otimes h_{(2)} g\,,$$

$$\Delta(a \otimes h) = \sum a_{(1)} \otimes \mathcal{R}^{(2)} h_{(1)} \otimes \mathcal{R}^{(1)} \triangleright a_{(2)} \otimes h_{(2)}$$

and the required isomorphism $\theta: \underline{A} \rtimes H \to D(H)$ is $\theta(a \otimes h) = \sum a_{(1)} \langle \mathcal{R}^{(1)}, a_{(2)} \rangle \otimes \mathcal{R}^{(2)} h$.

Proof. An abstract category-theoretic explanation of this result has recently been given in [14]. However, for our present purposes we need a completely explicit algebraic version as stated. Firstly, let us note that if $\beta: \underline{A} \to H \otimes \underline{A}$ is any left comodule structure respecting $\underline{A}$ as a coalgebra, the semidirect coproduct coalgebra is $\Delta(a \otimes h) = \sum a_{(1)} \otimes a_{(2)}{}^{(\bar{1})} h_{(1)} \otimes a_{(2)}{}^{(\bar{2})} \otimes h_{(2)}$, where $\beta(a) = \sum a^{(\bar{1})} \otimes a^{(\bar{2})}$ denotes β explicitly. This is a standard construction dual to the equally standard semidirect product algebra construction stated. In the present case the action is the one on $\underline{A}$ in (9) and the coaction is $\beta(a) = \sum \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \triangleright a$ (this is the way that any action of a quasitriangular Hopf algebra is converted by $\mathcal{R}$ to a coaction [28]). We now verify that θ is an isomorphism of coalgebras by computing

$$
\begin{aligned}
(\theta\otimes\theta)\Delta(a\otimes h) &= \sum a_{(1)}\langle \mathcal{R}^{(1)}, a_{(2)}\rangle \otimes \mathcal{R}^{(2)}\mathcal{R}''^{(2)}h_{(1)} \otimes a_{(4)}\langle \mathcal{R}'^{(1)}, a_{(5)}\rangle \\
&\quad \otimes \mathcal{R}'^{(2)}h_{(2)}\langle \mathcal{R}''^{(1)}, (sa_{(3)})a_{(6)}\rangle \\
&= \sum a_{(1)}\langle \mathcal{R}^{(1)}{}_{(1)}, a_{(2)}\rangle \otimes \mathcal{R}^{(2)}h_{(1)} \otimes a_{(4)}\langle \mathcal{R}'^{(1)}, a_{(5)}\rangle \\
&\quad \otimes \mathcal{R}'^{(2)}h_{(2)}\langle \mathcal{R}^{(1)}{}_{(2)}, (sa_{(3)})a_{(6)}\rangle \\
&= \sum a_{(1)}\langle \mathcal{R}^{(1)}, a_{(4)}\rangle \otimes \mathcal{R}^{(2)}h_{(1)} \otimes a_{(2)}\langle \mathcal{R}'^{(1)}, a_{(3)}\rangle \otimes \mathcal{R}'^{(2)}h_{(2)} \\
&= \sum a_{(1)} \otimes \mathcal{R}^{(2)}h_{(1)} \otimes a_{(2)} \otimes \mathcal{R}'^{(2)}h_{(2)}\langle \mathcal{R}'^{(1)}\mathcal{R}^{(1)}, a_{(3)}\rangle \\
&= \sum a_{(1)} \otimes \mathcal{R}^{(2)}{}_{(1)}h_{(1)} \otimes a_{(2)} \otimes \mathcal{R}^{(2)}{}_{(2)}h_{(2)}\langle \mathcal{R}^{(1)}, a_{(3)}\rangle \\
&= \Delta_{A\otimes H}\theta(a\otimes h)\,.
\end{aligned}
$$

For the first euality we used the definitions of θ and the stated coproduct on $\underline{A}\rtimes H$. For the second and fifth we used axioms of the quasitriangular structure $\mathcal{R}$, for the third we used the duality between H and A and the antipode axioms. We verify that θ is an isomorphism of algebras by computing

$$
\begin{aligned}
\theta((a\otimes h)(b\otimes g)) &= \theta\Big(\sum (h_{(1)}\triangleright b)^{(\bar 1)}a_{(1)} \otimes h_{(2)}g\langle \mathcal{R}, a_{(2)}\otimes (h_{(1)}\triangleright b)^{(\bar 2)}\rangle\Big) \\
&= \sum \theta((\mathcal{R}^{(2)}h_{(1)}\triangleright b)a_{(1)} \otimes h_{(2)}g)\langle \mathcal{R}^{(1)}, a_{(2)}\rangle \\
&= \sum (\mathcal{R}^{(2)}h_{(1)}\triangleright b)_{(1)}a_{(1)} \otimes \mathcal{R}'^{(2)}h_{(2)}g \\
&\quad \times \langle \mathcal{R}'^{(1)}, (\mathcal{R}^{(2)}h_{(1)}\triangleright b)_{(2)}a_{(2)}\rangle\, \langle \mathcal{R}^{(1)}, a_{(3)}\rangle \\
&= \sum (\mathcal{R}^{(2)}h_{(1)}\triangleright b)_{(1)}a_{(1)} \otimes \mathcal{R}'^{(2)}h_{(2)}g \\
&\quad \times \langle \mathcal{R}'^{(1)}{}_{(2)}\mathcal{R}^{(1)}, a_{(2)}\rangle\, \langle \mathcal{R}'^{(1)}{}_{(1)}, (\mathcal{R}^{(2)}h_{(1)}\triangleright b)_{(2)}\rangle \\
&= \sum b_{(2)}a_{(1)} \otimes \mathcal{R}'^{(2)}h_{(2)}g\langle \mathcal{R}'^{(1)}{}_{(2)}\mathcal{R}^{(1)}, a_{(2)}\rangle \\
&\quad \times \langle \mathcal{R}'^{(1)}{}_{(1)}, b_{(3)}\rangle\, \langle \mathcal{R}^{(2)}h_{(1)}, (sb_{(1)})b_{(4)}\rangle \\
&= \sum b_{(2)}a_{(1)} \otimes \mathcal{R}'^{(2)}\mathcal{R}''^{(2)}h_{(2)}g\langle \mathcal{R}''^{(1)}\mathcal{R}^{(1)}, a_{(2)}\rangle \\
&\quad \times \langle \mathcal{R}'^{(1)}, b_{(3)}\rangle\, \langle \mathcal{R}^{(2)}h_{(1)}, (sb_{(1)})b_{(4)}\rangle \\
&= \sum b_{(2)}a_{(1)} \otimes \mathcal{R}'^{(2)}\mathcal{R}''^{(2)}h_{(3)}g\langle \mathcal{R}''^{(1)}\mathcal{R}'''^{(1)}\mathcal{R}^{(1)}, a_{(2)}\rangle \\
&\quad \times \langle \mathcal{R}^{(2)}h_{(1)}, sb_{(1)}\rangle\, \langle \mathcal{R}'^{(1)}, b_{(3)}\rangle\, \langle \mathcal{R}'''^{(2)}h_{(2)}, b_{(4)}\rangle \\
&= \sum b_{(2)}a_{(1)} \otimes \mathcal{R}'^{(2)}\mathcal{R}''^{(2)}h_{(3)}g\langle \mathcal{R}''^{(1)}\mathcal{R}'''^{(1)}\mathcal{R}^{(1)}, a_{(2)}\rangle \\
&\quad \times \langle \mathcal{R}^{(2)}h_{(1)}, sb_{(1)}\rangle\, \langle \mathcal{R}'^{(1)}\mathcal{R}'''^{(2)}h_{(2)}, b_{(3)}\rangle.
\end{aligned}
$$

The first equality uses (8) in the definition of the semidirect product algebra structure on $\underline{A}\rtimes H$, writing (8) explicitly in terms of the right adjoint coaction corresponding to the coadjoint action $\triangleright$ in (9). That $\triangleright$ is an action then gives the second equality. The fifth equality uses the coadjoint coaction again, in explicit form $b\mapsto \sum b_{(2)}\otimes (sb_{(1)})b_{(3)}$.

The sixth and seventh equalities use the axioms for $\mathcal{R}$. On the other side we compute with the product $\cdot$ in $D(H)$ from (33), the expression

$$\begin{aligned}\theta(a\otimes h)\cdot\theta(b\otimes g) &= \sum(a_{(1)}\otimes\mathcal{R}^{(2)}h)\cdot(b_{(1)}\otimes\mathcal{R}'^{(2)}g)\langle\mathcal{R}^{(1)},a_{(2)}\rangle\,\langle\mathcal{R}'^{(1)},b_{(2)}\rangle\\ &= \sum b_{(2)}a_{(1)}\otimes\mathcal{R}^{(2)}{}_{(2)}h_{(2)}\mathcal{R}'^{(2)}g\langle\mathcal{R}^{(1)},a_{(2)}\rangle\,\langle\mathcal{R}'^{(1)},b_{(4)}\rangle\\ &\quad\times\langle\mathcal{R}^{(2)}{}_{(1)}h_{(1)},sb_{(1)}\rangle\,\langle\mathcal{R}^{(2)}{}_{(3)}h_{(3)},b_{(3)}\rangle\\ &= \sum b_{(2)}a_{(1)}\otimes\mathcal{R}''^{(2)}h_{(2)}\mathcal{R}'^{(2)}g\langle\mathcal{R}'''^{(1)}\mathcal{R}''^{(1)}\mathcal{R}^{(1)},a_{(2)}\rangle\\ &\quad\times\langle\mathcal{R}^{(2)}h_{(1)},sb_{(1)}\rangle\,\langle\mathcal{R}'''^{(2)}h_{(3)}\mathcal{R}'^{(1)},b_{(3)}\rangle\\ &= \sum b_{(2)}a_{(1)}\otimes\mathcal{R}''^{(2)}\mathcal{R}'^{(2)}h_{(3)}g\langle\mathcal{R}'''^{(1)}\mathcal{R}''^{(1)}\mathcal{R}^{(1)},a_{(2)}\rangle\\ &\quad\times\langle\mathcal{R}^{(2)}h_{(1)},sb_{(1)}\rangle\,\langle\mathcal{R}'''^{(2)}\mathcal{R}'^{(1)}h_{(2)},b_{(3)}\rangle.\end{aligned}$$

For the last two equalities we used the axioms for $\mathcal{R}$. Comparing these two results we see that $\theta(a\otimes h)\cdot\theta(b\otimes g)=\theta((a\otimes h)(b\otimes g))$ in virtue of the QYBE in the form $\mathcal{R}'''_{23}\mathcal{R}''_{21}\mathcal{R}'_{31}=\mathcal{R}'_{31}\mathcal{R}''_{21}\mathcal{R}'''_{23}$. The other facts such as the unit and counit are easy. □

Corollary 4.2. *If H is a finite-dimensional factorizable quantum group then there is an isomorphism $\phi=Q\circ\theta^{-1}:D(H)\cong \underline{H}\rtimes H$, where the semidirect product is by the quantum adjoint action $\triangleright$ of H on $\underline{H}$. Explicitly, $\phi(a\otimes h)=\sum Q(a_{(1)})\langle\mathcal{R}^{-(1)},a_{(2)}\rangle\otimes\mathcal{R}^{-(2)}h$.*

This is immediate from Proposition 4.1 and Proposition 2.2 in Sect. 2. Once again, we can apply this more generally if we have a suitable notion of dual Hopf algebra. In the setting where $H=U_q(g)$ we have

Corollary 4.3. *Let $U_q(g)$ be in* "FRT" *forms as in Corollary 2.3. Let t be the matrix generator of G_q, $L=l^+sl^-$ that of $U_q(g)$ and $M=m^+sm^-$ that of $BU_q(g)$ (it is the same algebra). Under the isomorphism $\phi:D(H)\cong BU_q(g)\rtimes U_q(g)$ the element $t^i{}_j\otimes L^k{}_l$ corresponds to $M^i{}_a\otimes l^{-a}{}_jL^k{}_l$.*

Explicitly, the structure of $BU_q(g)\rtimes U_q(g)$ consists of the two copies of $U_q(g)$ generated by L,M as subalgebras with cross relations and coproduct

$$R_{12}l_2^+M_1=M_1R_{12}l_2^+\,,\qquad R_{21}^{-1}l_2^-M_1=M_1R_{21}^{-1}l_2^-\,,$$
$$\Delta l^\pm=l^\pm\otimes l^\pm\,,\qquad \Delta M=\Big(\sum M\mathcal{R}^{(2)}\otimes\mathcal{R}^{(1)}M\Big)\mathcal{R}_{21}^{-1}\,,$$

where $\mathcal{R}\in U_q(g)\otimes U_q(g)$ as generated by L.

Proof. This follows at once from the explicit form of ϕ in Corollary 4.2 and (11). The explicit form of the cross relations is nothing other than the quantum adjoint action $l^\pm\triangleright M$ computed as explained in the proof of Corollary 2.3. The coproduct structure is the one in Proposition 4.1 computed in the present case with the aid of the semidirect product algebra structure and (4). □

These results show in particular that the quantum Lorentz group can be put into semidirect product form, $D(U_q(sl_2))=BU_q(sl_2)\rtimes U_q(sl_2)$. The price we pay for keeping this more familiar semidirect product form is that the algebra containing the "boosts" must be treated with braid statistics as the braided group $BU_q(sl_2)$ [as an algebra, it coincides with $U_q(sl_2)$]. It is not any kind of ordinary Hopf algebra, but a braided one in the category of $U_q(sl_2)$-modules. The quantum "rotations" $U_q(sl_2)$

can remain unchanged as an ordinary Hopf algebra and the result (as a quantum double) is again a factorizable ordinary Hopf algebra. In order to better understand this interpretation of Corollary 4.3, we pause now to compute its classical meaning at the level of Lie bialgebras.

Theorem 4.4. *Let (g, δ, r) be a quasitriangular Lie bialgebra with non-degenerate ad-invariant symmetric part r_+ of r, and $D(g)$ its double. Then there is an isomorphism $\phi : D(g) \cong g \rtimes g$, where $g \rtimes g$ is the semidirect sum by the adjoint action of g on itself. Explicitly, it is given by*

$$\phi(\eta \oplus \xi) = 2r_+(\eta) \oplus \left(\xi - \sum \langle \eta, r^{(1)} \rangle r^{(2)}\right).$$

Here we view r_+ as a linear map $r_+ : g^* \to g$.

Proof. The proof is by direct computation from (34) using the maps shown. An introduction to the necessary methods of Lie bialgebras is to be found in [44, Sect. 1]. Firstly, let us recall that the structure of a semidirect sum by (in our case) the adjoint action means

$$[\zeta \oplus \xi, \zeta' \oplus \xi'] = ([\zeta, \zeta'] + \alpha_\xi(\zeta') - \alpha_{\xi'}(\zeta)) \oplus [\xi, \xi'] ; \qquad \alpha_\xi(\zeta) = [\xi, \zeta] \tag{37}$$

for $\zeta \oplus \xi, \zeta' \oplus \xi' \in g \rtimes g$. Using this, we have

$$\begin{aligned}[\phi(\eta \oplus \xi), \phi(\eta' \oplus \xi')] = ([2r_+(\eta), 2r_+(\eta')] + [\xi - r(\eta), 2r_+(\eta')] \\ - [\xi' - r(\eta'), 2r_+(\eta)]) \oplus [\xi - r(\eta), \xi' - r(\eta')] ,\end{aligned}$$

where we have written $\sum \langle \eta, r^{(1)} \rangle r^{(2)} = r(\eta)$. On the other side we compute using (34),

$$\begin{aligned}\phi([\eta \oplus \xi, \eta' \oplus \xi']) = 2r_+([\eta', \eta] + \eta'_{[1]} \langle \eta'_{[2]}, \xi \rangle - \eta_{[1]} \langle \eta_{[2]}, \xi' \rangle) \\ \oplus ([\xi, \xi'] + \xi_{[1]} \langle \eta', \xi_{[2]} \rangle - \xi'_{[1]} \langle \eta, \xi'_{[2]} \rangle - r([\eta', \eta]) \\ - r(\eta'_{[1]}) \langle \xi, \eta'_{[2]} \rangle + r(\eta_{[1]}) \langle \xi', \eta_{[2]} \rangle) .\end{aligned}$$

For brevity, we omit summation signs. These two displayed expressions are equal as follows. Firstly, $r([\eta', \eta]) = [r(\eta'), r(\eta)]$ is simply the CYBE when r is viewed as a map $g^* \to g$ as we do here and the bracket $[\eta', \eta]$ is the one on g^* also defined by r via δ on g (this is equivalent to the more usual form $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ of the CYBE). Secondly, we have $\xi_{[1]} \langle \eta', \xi_{[2]} \rangle - r(\eta'_{[1]}) \langle \xi, \eta'_{[2]} \rangle = -\xi_{[2]} \langle \eta', \xi_{[1]} \rangle + \langle \eta', [\xi, r^{(1)}] \rangle r^{(2)} = -[\xi, r^{(2)}] \langle r^{(1)}, \eta' \rangle = -[\xi, r(\eta')]$. Here we used antisymmetry of $\delta\xi$ and then its explicit form $[\xi, r^{(1)}] \otimes r^{(2)} + r^{(1)} \otimes [\xi, r^{(2)}]$. Thirdly, we have ${r_+}^{(1)} \langle \eta', [{r_+}^{(2)}, \xi] \rangle = -[{r_+}^{(2)}, \xi] \langle \eta', {r_+}^{(1)} \rangle = [\xi, r_+(\eta')]$ by ad-invariance of r_+. Fourthly we compute

$$\begin{aligned}r_+([\eta', \eta]) &= \langle \delta {r_+}^{(1)}, \eta' \otimes \eta \rangle {r_+}^{(2)} \\ &= \langle [{r_+}^{(1)}, r^{(1)}] \otimes r^{(2)} + r^{(1)} \otimes [{r_+}^{(1)}, r^{(2)}], \eta' \otimes \eta \rangle {r_+}^{(2)} \\ &= \langle -[{r_+}^{(1)}, r^{(2)}] \otimes r^{(1)} + 2[{r_+}^{(1)}, {r'_+}^{(1)}] \otimes {r'_+}^{(2)}, \eta' \otimes \eta \rangle {r_+}^{(2)} \\ &\quad + [r(\eta'), r_+(\eta)] \\ &= [r(\eta'), r_+(\eta)] - [r(\eta), r_+(\eta')] + 2[r_+(\eta), r_+(\eta')] ,\end{aligned}$$

where we used the definitions of the bracket in g^* in terms of δ on g. For the last term in the third equality we used the previous (third) observation applied to

$\xi = r(\eta')$. Similarly for the final result. After these four observations we see that the expressions for $[\phi(\eta \oplus \xi), \phi(\eta' \oplus \xi')]$ and $\phi([\eta \oplus \xi, \eta' \oplus \xi'])$ coincide, i.e. ϕ is a Lie algebra homomorphism. □

Recall that the notion of a Lie bialgebra was introduced by Drinfeld as the infinitesimal notion of a Hopf algebra. Thus, if we write $\Delta\xi = \xi\otimes 1 + 1\otimes\xi + \frac{1}{2}\delta\xi + \ldots$, where we consider δ a deformation of order $\hbar$, then to lowest order in $\hbar$ the formulae (36) reduce to the structure of the Lie bialgebra double $D(g)$ in (34). The formulae for the preceding theorem were obtained in the same way from Corollary 4.2 with $\mathcal{R} = 1 + r + \ldots$, where r is also considered of order $\hbar$. For example $Q = \mathcal{R}_{21}\mathcal{R}_{12} = 1 + r_{21} + r_{12} + \ldots$. Thus, the notion of "factorizability" of quantum groups is, at the level of Lie bialgebras just the notion that the ad-invariant symmetric part of r be non-degenerate. Thus the role of $Q : \underline{A} \to \underline{H}$ in Proposition 2.1 is precisely played by $2r_+ : g^* \to g$. For the solution (33) this is given by twice the inverse Killing form $K^{-1} : g^* \to g$. Note that the isomorphism ϕ in Theorem 4.4 works also at the level of cobrackets in the form $D(g) \cong \underline{g} \rtimes g$, where $\underline{g}$ denotes the Lie algebra g equipped with a certain modified ("braided") cobracket $\underline{\delta}$. Finally, the isomorphism ϕ clearly resembles the Iwasawa decomposition (35), with ϕ in Corollaries 4.2 and 4.3 as quantum analogues. Indeed,

Corollary 4.5. *Let $g = \iota u \oplus u$ denote the complexification of a real Lie algebra u. There is a canonical embedding $g \subset \underline{g} \rtimes g$ such that the restriction of ϕ in Theorem 4.4 to $u^{*\mathrm{op}} \bowtie u \subset D(g)$ recovers the Iwasawa decomposition* (35). *It is*

$$g \subset \underline{g} \rtimes g\,, \qquad \xi_1 + \iota\xi_2 \mapsto 2\iota\xi_2 \oplus (\xi_1 - \iota\xi_2)\,.$$

Proof. This is obtained by computing

$$\phi^{-1}(\xi_1 + \iota\xi_2) = \iota K(\xi_2, \;) \oplus \left(\xi_1 + \sum \iota K(r_-{}^{(1)}, \xi_2) r_-{}^{(2)}\right)$$

as the inverse of the Iwasawa decomposition (35). We then apply to this the map ϕ in Theorem 4.4 which, for the Drinfeld-Jimbo solutioin (33) takes the form

$$\phi(\eta \oplus \xi) = 2K^{-1}(\eta) \oplus \left(\xi - K^{-1}(\eta) - \sum \langle \eta, r_-{}^{(1)} \rangle r_-{}^{(2)}\right).$$

Applying this gives $2\iota\xi_2 \oplus (\xi_1 - \iota\xi_2)$ as stated. Note that once found, one can easily verify this embedding $g \subset \underline{g} \rtimes g$ by elementary means (it holds for any real Lie algebra u), and hence regard the Iwasawa decomposition as merely the restriction of ϕ in Theorem 4.4 to a "real part." This is the reason we have denoted both maps by ϕ. We leave the direct proof that the stated embedding $g \subset \underline{g} \rtimes g$ is a Lie algebra homomorphism to the reader. g has the Lie algebra structure of u extended linearly, while $\underline{g} \rtimes g$ has the semidirect product one in (37). □

We are now in a position to make precise our remarks about the Lorentz group. We take $u = su(2)$ and $g = o(1,3) = sl_2(\mathbb{C}) = \iota u \oplus u$. Physically, the real u has compact generators J_i (say) of angular momentum (rotation) while ιu has non-compact generators K_i (say), the Lorentz boosts. Their commutation relations induced from those of u by complexification are of course

$$[J_i, J_j] = \varepsilon_{ijk} J_k\,, \qquad [J_i, K_j] = \varepsilon_{ijk} K_k\,, \qquad [K_i, K_j] = -\varepsilon_{ijk} J_k\,. \tag{38}$$

The semidirect sum $o(1,3) \rtimes o(1,3)$ is built on $o(1,3) \oplus o(1,3)$ in the usual way by (37) and the embedding in Corollary 4.5 by

$$J_i \mapsto 0 \oplus J_i\,, \qquad K_i \mapsto 2K_i \oplus (-K_i)\,. \tag{39}$$

Thus it embeds rotations as rotations in the second $o(1,3)$ and boosts as boosts in the first $o(1,3)$ along with a "compensating" negative boost in the second. Apart from this compensation, the embedded boosts are acted upon by the rotations as part of the semidirect sum. This unusual embedding corresponds to the "real" part $su(2)^{*\mathrm{op}} \bowtie su(2) \subset D(o((1,3))$, where $o(1,3) \cong su(2)^{*\mathrm{op}} \bowtie su(2)$ is the Iwasawa decomposition and $D(o(1,3)) \cong o(1,3) \rtimes o(1,3)$ is from Theorem 4.4. This embedding is of course quite distinct from the more usual identification $o(1,3)_{\mathbb{C}} \cong sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C})$ made in physics. The latter is special while our embedding, although less familiar, is canonical in the sense that a corresponding one holds for all complexifications g.

This completes our study of the algebraic structure of Drinfeld's quantum double for the case of $D(U_q(g))$. We gave the general theory, the quantum group setting and the classical limit. We return now to the general setting and note that we can identify the semidirect structure in Proposition 4.1 and Corollary 4.2 as examples of algebraic "bosonization" [13, Sect. 4]. There we show that if B is any Hopf algebra in the quasitensor category of modules of a quantum group H then the semidirect product and coproduct $B \rtimes H$ along the lines of Proposition 4.1 is an ordinary Hopf algebra. Thus our result is that for a quantum group H, the Drinfeld quantum double $D(H)$ is the bosonization of $\underline{A}$ or (in the factorizable case) of $\underline{H}$.

We can also use some more of the general theory in [12] to go further and transmute the Hopf algebra $D(H)$ itself into a braided one. The general transmutation principle in [12] asserts that if $H \to H_1$ is a Hopf algebra map between ordinary Hopf algebras (with H a quantum group, i.e. with universal R-matrix) then H_1 with the same algebra acquires the additional structure of a Hopf algebra in the quasitensor category of H-modules, denoted $B(H, H_1)$.

Proposition 4.6. *Let H be a finite-dimensional quantum group and $D(H)$ its double. The transmutation $B(H, D(H))$ of $D(H)$ into a Hopf algebra in the quasitensor category of H-modules is $B(H, D(H)) \cong \underline{A} \rtimes \underline{H}$, a semidirect product algebra (with tensor product coalgebra) in the category. If H is factorizable then $B(H, D(H)) \cong \underline{H} \rtimes \underline{H}$.*

Proof. This follows from the identification in Proposition 4.1 of $D(H)$ as the result of bosonization of $\underline{A}$. In general, the bosonization theorem of [13] proceeds by forming the tautological semidirect product $B \rtimes \underline{H}$ of the Hopf algebra B in the category by the braided group $\underline{H}$ acting by the same action of H by which B is an object in the category. The construction works because $\underline{H}$ really behaves as a "group" in the sense that it is braided-cocommutative. The resulting cross product is then identified in [13] as the result of the transmutation of an ordinary Hopf algebra H_1 by a map $H \to H_1$, H_1 being the bosonization. We work the argument in reverse. $\square$

The semidirect product in the preceding proposition is not any complicated quantum group semidirect product as in Proposition 4.1 and Corollary 4.2: it is precisely the semidirect product by a group in the usual way (with no twisting of the coproduct) except that all objects are treated with braid statistics. It is precisely like the semidirect product by a super-group for example, with the role of ± 1 played by Ψ. We have,

Corollary 4.7. *Let $H = U_q(g)$ in* "FRT" *form. Then $\underline{H} \rtimes \underline{H}$ explicitly has the structure on $\underline{H} \otimes \underline{H}$ as follows. We denote $L = 1 \otimes L$ and $M = M \otimes 1$ for the generators of the two copies of $\underline{H}$. These are embedded as sub-Hopf algebras in the quasitensor category of H-modules, with the cross relations and coproduct*

$$L^k{}_l M^i{}_j = \cdot L^k{}_m \triangleright \Psi(L^m{}_l \otimes M^i{}_j), \quad \underline{\Delta}(M^i{}_j \otimes L^k{}_l) = M^i{}_m \otimes \Psi(M^m{}_j \otimes L^k{}_n) \otimes L^n{}_l .$$

Proof. This immediate from the definition of cross products in quasitensor categories studied in [13, Sect. 2]. The element $L^k{}_m$ acts on the left factor of the result of Ψ by the braided group adjoint action $\triangleright$ which, in the present case, coincides as a linear map with the quantum adjoint action (6). □

Thus, if we are prepared to work with the "quantum Lorentz group" entirely in the quasitensor category of $U_q(sl_2)$-modules, then it takes a very natural form as simply the semidirect product of two identical copies of the braided group $BU_q(sl_2)$ with one of them (containing the "braided boosts") acted upon (via the adjoint action) by the other (the "braided rotations").

This algebraic analysis of the structure of the quantum Lorentz group (for example) raises an interesting problem: what is the right notion of $*$-structure for Hopf algebras in quasitensor categories? This is not a simple problem since for a worthwhile notion of $*$-structure one has to consider also what should be a "braided Hilbert space" and the corresponding adjoint operation, before the right notion of unitarity etc. in this braided setting is found. (The situation is complicated by the fact that $\Psi^2 \neq \mathrm{id}$.) We can hope that there can be found such a notion such that we can compute quantum and braided real forms of the above results along the lines of (33), and perhaps making contact with the approach of [29]. See also [47]. This is a direction for further work.

A. Braided Groups of Quantum Doubles

In this section we study one of the simplest examples of a factorizable Hopf algebra, namely the braided group of the quantum double $D(H)$ of a general Hopf algebra H. It is useful to see how some of the general theory of braided groups, such as the self-duality in Proposition 2.2 looks in this case. This is even more transparent for the simplest case of all, namely $D(G)$, where G is a finite group. Here the self-duality appears like the self-duality of $\mathbb{R}$ as expressed by $C_0(\mathbb{R}) \cong C^*(\mathbb{R})$ (the Fourier convolution theorem). Moreover, quantum doubles $D(G)$ (and quasi-Hopf algebra extensions of them) have been identified in certain non-Abelian anyonic systems and in the context of orbifold-based rational conformal field theories [48]. In both cases one can work equally well with the corresponding braided group. The category of $D(G)$-modules in which the braided-groups live is also an interesting one and includes the category of crossed G-sets as introduced by Whitehead [49]. By developing our results for this simple discrete quantum group $D(G)$ we hope to provide an antidote to the more abstract results in the text.

We begin however, with the braided version of general $D(H)$, before passing to our example. Thus H denotes an arbitrary finite-dimensional Hopf algebra. The structure of $D(H) = H^{*\mathrm{op}} \bowtie H$ was recalled in (36) above. It contains both H and $H^{*\mathrm{op}}$ as factors, where the latter is H^* with (in our conventions) the opposite product. This means that a left $D(H)$-module is precisely a vector space V on which H and $H^{*\mathrm{op}}$ are represented in a compatible way on the left, or equivalently on which H, H^*

act from the left and right respectively. Denoting the actions $\triangleright, \triangleleft$, the compatibility condition is [28]

$$\sum \langle a_{(1)}, h_{(1)}\rangle h_{(2)} \triangleright (v \triangleleft a_{(2)}) = \sum (h_{(1)} \triangleright v) \triangleleft a_{(1)} \langle a_{(2)}, h_{(2)}\rangle\,, \qquad v \in V,\ a \in H^*,\ h \in H\,. \tag{40}$$

The action of $D(H)$ is then $(a \otimes h) \triangleright v = (h \triangleright v) \triangleleft a$. Note that a right V^*-module corresponds in the finite-dimensional case to a left H-comodule. So we can equally well think of V as a left H-module and H-comodule in a compatible way. This category is then the category of H-crossed " bimodules" studied in [50] as well as subsequently by other authors. It clearly makes sense in this form for any Hopf algebra or bialgebra (not necessarily finite dimensional).

This $D(H)$ is a quantum group as explained by Drinfeld [17] with (in our conventions) $\mathscr{R} = \sum (f^a \otimes 1) \otimes (1 \otimes e_a)$, where $\{e_a\}$ is a basis of H and $\{f^a\}$ a dual one. Moreover, it is also known that it is factorizable [23], so we can apply Proposition 2.2 etc. For example, the map $Q : D(H)^* \to D(H)$ provided by $\mathscr{R}_{21}\mathscr{R}_{12}$ easily comes out from (36) as

$$Q(h \otimes a) = \sum \langle a, e_{a(2)}\rangle f^a \otimes (se_{a(1)}) h e_{a(3)}\,, \tag{41}$$

$$Q^{-1}(a \otimes h) = \sum e_{a(1)} h s e_{a(3)} \otimes f^a \langle a, e_{a(2)}\rangle\,. \tag{42}$$

Equally easily, the quantum adjoint action on $D(H)$ comes out as

$$h \triangleright (a \otimes g) = \sum a_{(2)} \otimes h_{(2)} g s h_{(4)} \langle a_{(1)}, s h_{(1)}\rangle \langle a_{(3)}, h_{(3)}\rangle\,, \tag{43}$$

$$(b \otimes h) \triangleleft a = \sum (s^{-1} a_{(3)}) b a_{(1)} \otimes h_{(2)} \langle a_{(4)}, h_{(1)}\rangle \langle s^{-1} a_{(2)}, h_{(3)}\rangle\,. \tag{44}$$

By these actions the braided group $\underline{D(H)}$ canonically associated to $D(H)$ by the construction in (6) lives in the quasitensor category of $D(H)$-modules. We denote it $BD(H)$.

Proposition A.1. *Let H be a finite-dimensional Hopf algebra. The braided group $BD(H)$ associated to $D(H)$ has the same product* (36) *but modified coproduct, inverse antipode and braiding given by*

$$\underline{\Delta}(a \otimes h) = \sum a_{(1)} \otimes e_a \otimes f^a{}_{(1)} a_{(2)} s f^a{}_{(3)} \otimes h_{(1)} \langle f^a{}_{(2)}, h_{(2)}\rangle\,,$$
$$\underline{s}^{-1}(a \otimes h) = \sum (s f^a{}_{(1)})(s a_{(2)}) f^a{}_{(3)} \otimes e_a \langle a_{(1)} (s f^a{}_{(2)}) s a_{(3)}, h\rangle\,,$$
$$\Psi((a \otimes h) \otimes (b \otimes g)) = \Sigma e_a \triangleright (b \otimes g) \otimes (a \otimes h) \triangleleft f^a\,.$$

Proof. The braiding is simply from (5) and the known form of $\mathscr{R}$ for $D(H)$ (in some examples we can fruitfully evaluate it further). The braided-coproduct from (6) comes out as

$$\begin{aligned}
\underline{\Delta}(a \otimes h) &= \sum (a_{(1)} \otimes h_{(1)})(1 \otimes s e_a) \otimes (a_{(2)} \otimes h_{(2)}) \triangleleft f^a \\
&= \sum a_{(1)} \otimes h_{(1)} s e_a \otimes (s^{-1} f^a{}_{(3)}) a_{(2)} f^a{}_{(1)} \\
&\quad \otimes h_{(3)} \langle f^a{}_{(4)}, h_{(2)}\rangle \langle s^{-1} f^a{}_{(2)}, h_{(4)}\rangle \\
&= \sum a_{(1)} \otimes h_{(1)} e_a \otimes f^a{}_{(2)} a_{(2)} s f^a{}_{(4)} \otimes h_{(3)} \langle s f^a{}_{(1)}, h_{(2)}\rangle \langle f^a{}_{(3)}, h_{(4)}\rangle\,,
\end{aligned}$$

where the second equality is from (36) and the third by a change of basis and dual basis. This gives the expression stated in the proposition because $\sum h_{(1)}e_a \otimes \langle sf^a{}_{(1)}, h_{(2)}\rangle f^a{}_{(2)} = \sum e_a \otimes \varepsilon(h)f^a$ for all $h \in H$ (this is easily seen by evaluating on a test function in H^*). Likewise, the inverse braided-antipode is computed from a formula similar to (6) for $\underline{s}$ as

$$\begin{aligned}\underline{s}^{-1}(a\otimes h) &= \sum [s^{-1}((a\otimes h) \lhd f^a)]\,(1\otimes e_a)\\ &= \sum [s^{-1}((s^{-1}f^a{}_{(3)})af^a{}_{(1)} \otimes h_{(2)})]\,(1\otimes e_a)\,\langle f^a{}_{(4)}, h_{(1)}\rangle\,\langle s^{-1}f^a{}_{(2)}, h_{(3)}\rangle\\ &= \sum (1\otimes s^{-1}h_{(2)})\,((sf^a{}_{(1)})(sa)f^a{}_{(3)}\otimes e_a)\,\langle f^a{}_{(4)}, h_{(1)}\rangle\,\langle s^{-1}f^a{}_{(2)}, h_{(3)}\rangle\\ &= \sum (sf^a{}_{(2)})\,(sa_{(2)})f^a{}_{(4)}\\ &\quad \otimes (s^{-1}h_{(2)})e_a\langle a_{(1)}f^a{}_{(1)}, h_{(1)}\rangle\,\langle (sf^a{}_{(3)})\,(sa_{(3)}), h_{(3)}\rangle\\ &= \sum (sf^a{}_{(1)})\,(sa_{(2)})f^a{}_{(3)}\otimes e_a\langle a_{(1)}, h_{(1)}\rangle\,\langle (sf^a{}_{(2)})\,(sa_{(3)}), h_{(2)}\rangle\end{aligned}$$

as required. For the second equality we used (44), for the third the (inverse) antipode in $D(H)$ and for fourth the product in $D(H)$, see (36). For the last equality we used $\sum (se_a)h_{(2)} \otimes \langle f^a{}_{(2)}, h_{(1)}\rangle f^a{}_{(2)} = \sum se_a \otimes \varepsilon(h)f^a$ for any h. □

From Proposition 2.2 we know that this braided group of enveloping algebra type is also isomorphic to the braided group $\underline{D(H)^*}$ of function algebra type (via Q), i.e. $BD(H)$ is self-dual. In our case this is manifest for the product on $BD(H)$ is the same as that of $D(H)$ in (36): an easy computation from (36) gives its dual [the coproduct on $D(H)^*$ and $\underline{D(H)^*}$] as

$$\Delta_{D(H)^*}(h\otimes a) = \sum h_{(2)}\otimes (sf^a{}_{(1)})a_{(1)}f^a{}_{(3)}\otimes e_a\otimes a_{(2)}\langle f^a{}_{(2)}, h_{(1)}\rangle \tag{45}$$

which can be compared with the preceding proposition. Thus the product and coproduct on $BD(H)$ are manifestly isomorphic when compared by dualizing one of them, i.e. $BD(H)$ is in a certain sense "linearized." This is a general feature of the braided groups associated to quantum groups, and allows for them properties usually reserved for $\mathbb{R}^n$. For example, there is an operation $\mathscr{S}$ of "quantum Fourier transform" from the braided group to itself given by [24] $\mathscr{S} = \sum sQ^{(1)}\mu(Q^{(2)}(\ ))$, where μ is a suitably normalized left invariant integral and $Q = \mathscr{R}_{21}\mathscr{R}_{12}$. Just as the square of the Fourier transform on $\mathbb{R}^n$ is the parity operator (inversion on the group), we have $\mathscr{S}^2 = \underline{s}^{-1}$ [24]. Moreover, if the original quantum group is a ribbon Hopf algebra [16] then there is also an operator $\mathscr{T}$ given by left product by the inverse ribbon element, and $(\mathscr{S}\mathscr{T})^3 = \lambda\mathscr{S}^2$ for some constant λ. For $BD(H)$ the "quantum Fourier transform" is easily computed from the above as

$$\mathscr{S}(a\otimes h) = \sum f^a \otimes s\mu_{1(2)}\mu_2(s^{-1}e_{a(2)}h)\,\langle a, e_{a(3)}\mu_{1(1)}s^{-1}e_{a(1)}\rangle\,, \tag{46}$$

where μ_1 and μ_2 are left integrals on H^* and H respectively, suitably normalized. For example, the left integral on H is characterized by $\sum h_{(1)}\otimes \mu_2(h_{(2)}) = 1\mu_2(h)$ for all $h \in H$.

Let us note also that the structure of $D(H)$ and $BD(H)$ can also be expressed fruitfully in terms of $\mathrm{Hom}_k(H, H)$ rather than as we have computed on $H^*\otimes H$. This new form is slightly more general and its structure is a twisted version of the standard convolution bialgebra on $\mathrm{Hom}_k(H, H)$. To be explicit, the structure of $D(H)$ and $BD(H)$ in these terms is

$$(\phi\psi)(g) = \sum \phi(g_{(2)})_{(2)}\psi((s\phi(g_{(2)})_{(1)})g_{(1)}\phi(g_{(2)})_{(3)}),$$
$$(\Delta\phi)(g \otimes h) = \sum \phi(gh)_{(1)} \otimes \phi(gh)_{(2)}, \tag{47}$$
$$(s\phi)(g) = \sum se_{a(2)}\langle f^a, \phi(e_{a(1)}(s^{-1}g)se_{a(3)})\rangle, \tag{48}$$
$$(h \triangleright \phi)(g) = \sum h_{(2)}\phi((sh_{(1)})gh_{(3)})sh_{(4)}, \tag{49}$$
$$(\phi \triangleleft a)(g) = \sum a(g_{(3)}(s^{-1}\phi(g_{(2)})_{(3)})(s^{-1}g_{(1)})\phi(g_{(2)})_{(1)})\phi(g_{(2)})_{(2)}, \tag{50}$$
$$(\underline{\Delta}\phi)(g \otimes h) = \sum h_{(1)}\phi(gh_{(2)})_{(2)}sh_{(3)} \otimes \phi(gh_{(2)})_{(1)}, \tag{51}$$
$$(\underline{s}^{-1}\phi)(g) = \sum (sg_{(1)})(se_{a(2)})g_{(3)}\langle f^a, \phi(e_{a(1)}(sg_{(2)})se_{a(3)})\rangle. \tag{52}$$

Other structures such as Ψ and $\mathcal{S}$ are easily computed from those already computed above in the $H^* \otimes H$ form so we leave these to the reader. The strategy is to replace $h\langle a, \ \rangle$ [in (46) for example] by $\phi(\ )$ where ϕ is the linear map corresponding to $(a \otimes h)$. Both the original form and this "twisted convolution" form are useful, see below.

To conclude the general theory we mention that there are plenty of other algebraic structures naturally living in the present category of $D(H)$-modules. The following is a version of a theorem of Radford [33], translated into the present context.

Proposition A.2. *Let $H_1 \underset{i}{\overset{p}{\rightleftarrows}} H$ be a Hopf algebra projection (i.e., p, i are Hopf algebra homomorphisms between two Hopf algebras and $p \circ i = \mathrm{id}$). For simplicity we suppose H finite dimensional. Then there is a Hopf algebra B living in the quasitensor category of $D(H)$-modules such that $B \rtimes H \cong H_1$. Explicitly, B is a subalgebra of H_1 and a $D(H)$-module by*

$$B = \left\{ b \in H_1 \middle| \sum b_{(1)} \otimes p(b_{(2)}) = b \otimes 1 \right\},$$
$$h \triangleright b = \sum i(h_{(1)})bs \circ i(h_{(2)}), \qquad b \triangleleft a = \sum \langle a, p(b_{(1)})\rangle b_{(2)},$$

where $h \in H$ and $a \in H^*$. The braided-coproduct, braided-antipode and braiding of B are

$$\underline{\Delta}b = \sum b_{(1)}s \circ i \circ p(b_{(2)}) \otimes b_{(3)}, \qquad \underline{s}b = \sum i \circ p(b_{(1)})sb_{(2)}$$
$$\Psi(b \otimes c) = \sum p(b_{(1)}) \triangleright c \otimes b_{(2)}.$$

The structure of $B \rtimes H$ is the standard semidirect product by the action $\triangleright$ of H and the coaction of H corresponding to $\triangleleft$ as stated. The isomorphism $\theta : B \rtimes H \to H_1$ is $\theta(b \otimes h) = bi(h)$, with inverse $\theta^{-1}(a) = \sum a_{(1)}s \circ i \circ p(a_{(2)}) \otimes p(a_{(3)})$ for $a \in H_1$.

Proof. The only new part beyond [33] is the identification of the "twisted Hopf algebra" B now as a Hopf algebra living in a quasitensor category, and some slightly more explicit formulae for its structure. The set B coincides with the image of the projection $\Pi : H_1 \to H_1$ defined by $\Pi(a) = \sum a_{(1)}s \circ i \circ p(a_{(2)})$ in [33], while the pushed-out left adjoint coaction of H on B then reduces to the left coaction $b \mapsto \sum b^{(\bar{1})} \otimes b^{(\bar{2})} = \sum p(b_{(1)}) \otimes b_{(2)}$ as used to define $\triangleleft$ in the proposition. Note that the restriction to H finite-dimensional is avoided if we work throughout with this coaction rather than $\triangleleft$ as explained above (this is a reason why comodules are preferred in [33]). In the present terms, we obtain an action of $D(H)$. The braiding from (5) then comes out as $\Psi(b\otimes c)=\sum e_a \triangleright c\otimes b\triangleleft f^a=\sum b^{(\bar{1})}\triangleright c\otimes b^{(\bar{2})}$ giving the form shown. The axioms of a Hopf algebra in a quasitensor category require that $\underline{\Delta}: B\to B\otimes B$ is an algebra homomorphism with respect to the braided tensor product algebra structure on $B\otimes B$. Writing $\underline{\Delta} b=\sum b_{\underline{(1)}}\otimes b_{\underline{(2)}}$, this reads

$$\underline{\Delta}(bc)=\sum b_{\underline{(1)}}\Psi(b_{\underline{(2)}}\otimes c_{\underline{(1)}})c_{\underline{(2)}}=\sum b_{\underline{(1)}}(b_{\underline{(2)}}{}^{(\bar{1})}\triangleright c_{\underline{(1)}})\otimes b_{\underline{(2)}}{}^{(\bar{2})}c_{\underline{(2)}}$$

which is the condition in [33]. This, along with the other axioms of a Hopf algebra in our quasitensor category can also be easily verified directly from the formulae stated. Finally, the structure on $B\rtimes H$ is the standard semidirect product one, $(b\otimes h)(c\otimes g)=\sum b(h_{(1)}\triangleright c)\otimes h_{(2)}g$ and the standard semidirect coproduct by $\triangleleft$ as a coaction. Explicitly, the coproduct on $B\rtimes H$ comes out as $\Delta(b\otimes h)=\Sigma b_{\underline{(1)}}\otimes p(b_{\underline{(2)}(1)})h_{(1)}\otimes b_{\underline{(2)}(2)}\otimes h_{(2)}$. Applying θ to these structures and evluating further gives θ as a Hopf algebra isomorphism. □

We now further compute some of these constructions for an important class of examples, namely for $D(G)=D(kG)$, where G is a finite group and kG is its group algebra over a field k. Firstly, a $D(G)$-module means a vector space V on which G and $k(G)$ (functions on G with pointwise product) act. As is well-known, an action of $k(G)$ simply means a G-grading, see for example [51]. Indeed, writing $v\triangleleft a=\sum_g a(g)\beta_g(v)$ for some operators $\beta_g: V\to V$, the requirement of an action means that $\beta_g\beta_{g'}=\beta_g\delta_{g,g'}$. Hence $V=\bigoplus_g V_g$ for homogeneous subspaces V_g, where a acts by $v\triangleleft a=va(g)$. We say that $v\in V_g$ has degree $|v|=g$. Note that G may be non-Abelian. A $D(G)$-module then means a G-graded space on which G also acts, in a compatible way according to (40). This clearly reduces in our example to the condition

$$|g\triangleright v|=g|v|g^{-1}\,,\quad \forall v\in V,\, g\in G\,. \tag{53}$$

Thus a $D(G)$-module is a G-graded G-module obeying (53). Note that $D(G)$ is an ordinary semidirect product (even without appealing to Proposition 4.1) because kG is cocommutative, so that (36) simplifies. The braiding from (5) comes out as

$$\Psi(v\otimes w)=|v|\triangleright w\otimes v \tag{54}$$

for v homogeneous of degree $|v|$.

A large class of $D(G)$-modules is provided by the crossed G-sets of Whitehead [49]. A crossed G-set is a set M on which G acts, together with a map $\partial: M\to G$ such that $\partial(g\triangleright m)=g(\partial m)g^{-1}$ for all $g\in G$, $m\in M$. In this case the vector space kM with basis $m\in M$ clearly becomes a $D(G)$-module with $\triangleright$ extended linearly and degree $|m|=\partial m$. If M is a group one usually demands that ∂ is a group homomorphism. In this case it is easy to see that the group algebra kM is an algebra in the category of $D(G)$-modules [i.e. the product is $D(G)$-equivariant]. The braiding (54) is also well known to algebraic topologists. Note that a further natural demand to make in this context is $(\partial m)\triangleright n=mnm^{-1}$ for all $m,n\in M$ [49].

Bialgebras or Hopf algebras in this category of $D(G)$-modules obey the usual axioms after allowing for Ψ in (54) when defining the braided tensor product algebra structure. We now describe the braided group $BD(G)$ associated to $D(G)$. It lives in this category of $D(G)$-modules. Firstly we adopt a convenient description of $D(G)$ itself. Namely, we identify its underlying linear space as $k(G)\otimes kG = k(G,kG)$, i.e. kG-valued functions. The group-valued functions $\phi_{g,h} = f^g\otimes h$ for $g,h\in G$ provide a basis. Here $\phi_{g,h}(g') = \delta_{g,g'}h$. We will say that a function $\phi\in k(G,kG)$ is *group-like* if the value at any g is grouplike, $\Delta(\phi(g)) = \phi(g)\otimes\phi(g)$. They are certain allowed sums of the basis functions, and can be thought of as maps from $G\cup 0\to G\cup 0$ where 0 is adjoined. In this notation (there are plenty of others) the quantum group structure of $D(G)$ looks like

$$(\phi\psi)(g) = \phi(g)\psi(\phi(g)^{-1}g\phi(g)),\qquad (\Delta\phi)(g,h) = \phi(gh)\otimes\phi(gh),$$
$$s\phi_{g,h} = \phi_{h^{-1}g^{-1}h,h^{-1}}\,. \tag{55}$$

These are twisted variants of the usual algebra with point-wise product and values in kG. The antipode looks as stated on one of the basis functions. On a general group-like function it takes the form $(s\phi)(g) = \sum_{h\in G} h^{-1}\delta_{h,\phi(hg^{-1}h^{-1})}$, which is basically an inversion of ϕ as a map (the resulting function is not in general group-like, however).

Proposition A.3. *The braided group $BD(G)$ associated to $D(G)$ for G a finite group has the following structure for group-like functions $\phi,\psi: G\cup 0\to G\cup 0$,*

$$(\underline{\Delta}\phi)(g,h) = h\phi(gh)h^{-1}\otimes\phi(gh),\qquad (\underline{s}^{-1}\phi)(g) = g^{-1}(s\phi)(g)g,$$
$$(h\triangleright\phi)(g) = h\phi(h^{-1}gh)h^{-1},\qquad (\phi\triangleleft a)(g) = a([g,\phi(g)])\phi(g),$$
$$(\Psi(\phi\otimes\psi))(g,h) = ([h,\phi(h)]\triangleright\psi)(g)\otimes\phi(h),$$

where $[g,\phi(g)]\equiv g\phi(g)^{-1}g^{-1}\phi(g)$ is the group commutator. Thus the basis functions $\phi_{g,h}$ are homogeneous of degree $[g,h]$.

Proof. This, as well as (55), follows more easily from the "twisted convolution" form in (47)–(52) by simply dropping the $^{(1)},{}^{(2)}$ etc. suffices for the coproduct in H (this is the meaning of the group-like assumption). The original form on $H^*\otimes H$ is more useful for the antipode and braided inverse antipode on the basis functions $\phi_{g,h}$. Another form for the latter is $\underline{s}^{-1}\phi_{g,h} = \phi_{h^{-1}g^{-1}h,h^{-1}[g,h]}$. $\square$

Finally, as an application we note the quantum Fourier transform operator $\mathscr{S}$ from [24]. In fact, $D(G)$ is a ribbon Hopf algebra so that there is also an operator $\mathscr{T}$ as explained. The inverse ribbon element is simply the identity map $G\cup 0\to G\cup 0$ in $D(G)$. The integral on $D(G)$ is the tensor product one, $\mu(\phi) = \sum_{g\in G}\delta_{e,\phi(g)}$ (the number of points in the inverse image of the identity e under ϕ). From these observations along with (46) and a direct computation for the normalizations, we have

$$\mathscr{S}\phi_{g,h} = \phi_{h,h^{-1}g^{-1}h},\qquad (\mathscr{T}\phi)(g) = g\phi(g),\qquad (\mathscr{S}\mathscr{T})^3 = \mathscr{S}^2. \tag{56}$$

The action of $\mathscr{S}$ on group-like functions is $(\mathscr{S}\phi)(g) = \sum_h h^{-1}\delta_{g,\phi(ghg^{-1})}$. These expressions seem at first far removed from ordinary Fourier transforms yet they have similar abstract properties and moreover, when the quantum Fourier transform $\mathscr{S}$ is computed for the quantum deformations of ordinary Lie groups (such as $\mathbb{R}^n$), it does recover a deformation of ordinary Fourier transform [24]. The $D(G)$ example has a different character from these quantum deformations, being a twisted tensor product of $k(G) = k\hat{G}$ (in the Abelian case) and kG. In this case we see in (56) that $\mathscr{S}$ interchanges the roles of $k(G)$ and kG in $\phi_{g,h}$. In the Abelian case it is $\mathscr{S}\phi_{g,h} = \phi_{h,g^{-1}}$. Thus there are some similarities with ordinary Fourier transformation in its role of interchanging "position" and "momentum."

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