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Invariant Connections and Vortices

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Abstract. We study the vortex equations on a line bundle over a compact Kähler manifold. These are a generalization of the classical vortex equations over $\mathbb{R}^2$. We first prove an invariant version of the theorem of Donaldson, Uhlenbeck and Yau relating the existence of a Hermitian–Yang–Mills metric on a holomorphic bundle to the stability of such a bundle. We then show that the vortex equations are a dimensional reduction of the Hermitian–Yang–Mills equation. Using this fact and the theorem above we give a new existence proof for the vortex equations and describe the moduli space of solutions.

Introduction

In this paper we shall study a direct generalization of the vortex equations on $\mathbb{R}^2$ in which the euclidean plane is replaced by a compact Kähler manifold.

The vortex equations on $\mathbb{R}^2$ were first introduced in 1950 by Ginsburg and Landau [9] in the study of superconductivity. Geometrically they correspond to the equations satisfied by the absolute minima of the Yang–Mills–Higgs functional, defined for a unitary connection A and a smooth section ϕ of a Hermitian line bundle over $\mathbb{R}^2$ as

$$\mathrm{YMH}(A, \phi) = \int_{\mathbb{R}^2} |F_A|^2 + |d_A\phi|^2 + \frac{1}{4}(1 - |\phi|^2)^2 .$$

Here F_A is the curvature of A and $d_A\phi$ is the covariant derivative of ϕ.

If we regard $\mathbb{R}^2$ as the complex plane we may decompose with respect to the complex structure to get $d_A = d'_A + d''_A$. Then by integration by parts we can show that the functional above is bounded below by $2\pi d$, where d is an integer called the *vortex number*, and this minimum is attained if and only if

$$\left.\begin{aligned} d''_A\phi &= 0 \\ F_A &= \frac{1}{2}*(1 - |\phi|^2) \end{aligned}\right\} .$$

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These equations are invariant under gauge transformations and the moduli space of solutions is described by the basic existence theorem of Jaffe and Taubes [13]. They proved that given d points $x_i \in \mathbb{R}^2$ (possibly with multiplicities) there exists a solution to the vortex equations, unique up to gauge equivalence, with $\phi(x_i) = 0$. This means that the moduli space of *vortices* is the space of unordered d-tuples, which coincides with the vector space $\mathbb{C}^d$.

The feature of the vortex equations we shall exploit is that they are a *dimensional reduction* of the (anti)-self-dual Yang–Mills equation. More precisely, consider an $SU(2)$ bundle E on a Riemannian 4-manifold M. Suppose that $SO(3)$ (or $SU(2)$) acts by isometries on M and that this action can be lifted to E. Then $SO(3)$ also acts on the space of connections on E, and there is a one-to-one correspondence between $SO(3)$-invariant connections $\mathbf{A}$ and pairs (A, ϕ), where A is a unitary connection on a Hermitian line bundle L over the orbit space $M/SO(3)$ and ϕ is a section of L. The pure Yang–Mills functional of an invariant connection reduces to the Yang–Mills–Higgs functional of (A, ϕ). Moreover, (A, ϕ) satisfies the vortex equations if and only if the corresponding invariant connection $\mathbf{A}$ satisfies the (anti)-self-dual Yang–Mills equation. In this way, taking $M = \mathbb{R}^2 \times S^2$ Taubes [20] gets the vortex equations over $\mathbb{R}^2$, and taking $M = \mathbb{R}^2_+ \times S^2$ Witten [22] gets the vortex equations over the hyperbolic plane $\mathbb{R}^2_+$.

Taking this invariant point of view we will be able to prove an existence theorem for the more general vortex equations studied in this paper.

In the first section of the paper we introduce these equations. Let L be a Hermitian line bundle over a compact Kähler manifold X. If A is a unitary connection on L which is *integrable* (that is, whose curvature has vanishing (0, 2)-part), and ϕ is a smooth section of L, one can define for the pair (A, ϕ) a generalized Yang–Mills–Higgs functional depending on a real parameter τ. As in the $\mathbb{R}^2$ case this functional is bounded below by $2\pi\tau d$, where d is the degree of L, and this bound is attained if and only if

$$\left.\begin{aligned} d''_A\phi &= 0 \\ \Lambda F_A - \frac{i}{2}|\phi|^2 + \frac{i}{2}\tau &= 0 \end{aligned}\right\},$$

where Λ is contraction by the Kähler form. These equations are called the *τ-vortex equations* (though the second equation alone is also sometimes called the *τ-vortex equation*).

It will be convenient to take the equivalent point of view of fixing a holomorphic structure $\bar\partial_L$ on L and fixing a holomorphic section ϕ of $\mathscr{L} = (L, \bar\partial_L)$. The τ-vortex equation becomes then the equation

$$\Lambda F_h - \frac{i}{2}|\phi|^2 + \frac{i}{2}\tau = 0$$

for a metric h on $\mathscr{L}$, where F_h is the curvature of the metric connection determined by $\mathscr{L}$ and h.

By integrating this equation, we see that a necessary condition for existence of solutions with $\phi \not\equiv 0$ is that

$$\deg L < \frac{\tau \operatorname{Vol} X}{4\pi}.$$

What is interesting is that this condition is also sufficient. Our strategy for proving this involves showing that the vortex equations appear as a dimensional reduction of the Hermitian–Yang–Mills equation, generalising the abovementioned results of Witten [22] and Taubes [20].

Recall that a hermitian metric on a holomorphic bundle is said to be *Hermitian–Einstein* or *Hermitian–Yang–Mills* if the curvature F of the metric connection (the unique connection compatible with both the metric and the holomorphic structure) satisfies

$$\Lambda F = \text{const.}\,\mathbf{I}\,.$$

In Sect. 2 we prove a G-invariant version of the theorem of Donaldson, Uhlenbeck and Yau [3, 4, 21] relating the existence of a Hermitian–Yang–Mills metric on a holomorphic bundle over a compact Kähler manifold to the *stability* of such a bundle. This theorem will be used in the proof of the existence theorem for the τ-vortex equation. Let $\mathscr{E}$ be a holomorphic bundle over a compact Kähler manifold, and suppose that a compact group G acts holomorphically on the manifold preserving the Kähler form. Suppose also that the action can be lifted holomorphically to $\mathscr{E}$. The sufficient condition for the existence of a G-invariant Hermitian–Yang–Mills metric is now that of G-invariant stability. This is like ordinary stability, but the numerical condition on the slopes applies only to G-invariant subsheaves of $\mathscr{E}$.

In Sect. 3 we show how the τ-vortex equation appears as a dimensional reduction under the action of $SU(2)$ of the Hermitian–Yang–Mills equation on a holomorphic rank two vector bundle $\mathscr{E}$ over $X \times \mathbb{P}^1$ associated to $(\mathscr{L}, \phi)$. Here $\mathbb{P}^1$ is the complex projective line, and $X \times \mathbb{P}^1$ is endowed with the product of the metric on X and the Fubini–Study metric on $\mathbb{P}^1$, with a coefficient which is essentially the inverse of τ.

Using the results of Sects. 2 and 3, in Sect. 4 we reduce the criterion for the existence of solutions to the vortex equation to the stability of $\mathscr{E}$; but this coincides with $\deg L < \tau \operatorname{Vol} X/4\pi$.

The τ-vortex equation has also been studied by Bradlow [1], who gives two different proofs of the existence of solutions and a description of the moduli space of solutions, as well as a number of interpretations of the parameter τ. Exploiting the fact that the vortex equations are moment map equations in the sense of symplectic geometry, we have given [7] another proof of the existence theorem in the case of a Riemann surface.

Our approach to the vortex equations also enables us to describe the moduli space of solutions. This moduli space can be described in terms of effective divisors on X like the description above of the moduli of vortices over $\mathbb{R}^2$. However the vortex moduli space coincides also with the fixed point set under the action of $SU(2)$ of the moduli space of stable holomorphic structures on the smooth bundle underlying $\mathscr{E}$. It is then embedded in the more familiar Donaldson moduli space. In particular it inherits the structure of a complex analytic space, with a Kähler metric outside of the set of singular points.

1. The Vortex Equations

In this section we introduce the *vortex equations* on line bundles. They appear as the equations satisfied by the absolute minima of the *Yang–Mills–Higgs* functional.

Bradlow [1, 2] has studied these equations in more generality, considering them on a vector bundle of arbitrary rank and we refer to him for details.

Let X be a compact Kähler manifold of complex dimension n, with fixed Kähler metric and Kähler form ω. Let L be a complex line bundle over X, and fix a hermitian metric h on L. Let $\mathscr{A}$ be the space of unitary connections on (L, h), and let $\Omega^0(L)$ be the space of sections of L.

Definition. *We define the Yang–Mills–Higgs functional* $\mathrm{YMH}_\tau\colon \mathscr{A}\times\Omega^0(L)\to\mathbb{R}$ *by*

$$\mathrm{YMH}_\tau(A,\phi) = \|F_A\|^2 + \|d_A\phi\|^2 + \frac{1}{4}\|\,|\phi|_h^2 - \tau\|^2 . \tag{1}$$

Here $\|\ \|$ *denotes the* L^2 *norm,* $F_A\in\Omega_X^2$ *is the curvature of the connection* A, $d_A\phi\in\Omega^1(L)$ *is the covariant derivative of* ϕ, $|\phi|_h$ *is the norm of* ϕ *with respect to* h, *and* τ *is a real parameter.*

The functional YMH_τ is invariant under the standard action of the gauge group $\mathscr{G}$ of unitary transformations of (L, h), so it defines a functional on the space $(\mathscr{A}\times\Omega^0(L))/\mathscr{G}$.

Let $\mathscr{A}^{1,1}$ be the space of integrable unitary connections on (L, h), i.e. the space of $A\in\mathscr{A}$ such that $F_A^{0,2}=0$.

Proposition 1. *If* $(A,\phi)\in\mathscr{A}^{1,1}\times\Omega^0(L)$ *then*

$$YMH_\tau(A,\phi) = 2\|d_A''\phi\|^2 + \left\|i\Lambda F_A + \frac{1}{2}|\phi|_h^2 - \frac{\tau}{2}\right\|^2 + 2\pi\tau\deg L . \tag{2}$$

Here d_A'' *is the* (0, 1) *part of the connection,* $\Lambda F_A\in\Omega_X^0$ *is the contraction of* F_A *with the Kähler form, and* $\deg L$ *is the degree of* L *with respect to* ω.

Proof. We expand

$$\left\|i\Lambda F_A + \frac{1}{2}|\phi|_h^2 - \frac{\tau}{2}\right\|^2 = \|\Lambda F_A\|^2 + \frac{1}{4}\|\,|\phi|_h^2 - \tau\|^2 + \langle i\Lambda F_A, |\phi|_h^2\rangle - \langle i\Lambda F_A, \tau\rangle . \tag{3}$$

The result follows now from the identities

$$\langle i\Lambda F_A, |\phi|_h^2\rangle = -\|d_A''\phi\|^2 + \|d_A'\phi\|^2 ,$$

$$\|\Lambda F_A\|^2 = \|F_A\|^2 \quad\text{and}\quad \int_X i\Lambda F_A\frac{\omega^n}{n!} = 2\pi\deg L .$$

See [1] for details. □

We conclude then that the functional YMH_τ is bounded below by $2\pi\tau\deg L$. This lower bound is attained at $(A,\phi)\in\mathscr{A}^{1,1}\times\Omega^0(L)$ if and only if

$$\left.\begin{aligned} &d_A''\phi = 0\\ &\Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau = 0\end{aligned}\right\} . \tag{4}$$

These are the *τ-vortex equations*. The first equation says simply that ϕ is holomorphic with respect to the holomorphic structure on L induced by $A\in\mathscr{A}^{1,1}$, and we will refer to the second equation alone as the *τ-vortex equation.*

2. Invariant Stability and the Hermitian–Yang–Mills Equation

We shall prove now an invariant version of the theorem of Donaldson, Uhlenbeck and Yau [3, 4, 21] relating the existence of a *Hermitian–Yang–Mills* metric on a holomorphic vector bundle to the stability of the bundle. This theorem will be one of the main ingredients in our proof of an existence theorem for the τ-vortex equations. It is convenient to review first the notion of stability and the by now standard results.

Let M be a compact Kähler manifold of dimension m with a fixed Kähler metric having Kähler form ω, and let $\mathscr{E}$ be a holomorphic vector bundle over M. The degree of a coherent sheaf is defined as

$$\deg \mathscr{F} = \frac{1}{(m-1)!} \int_M c_1(\mathscr{F}) \wedge \omega^{m-1} ,$$

where $c_1(\mathscr{F}) = c_1(\det \mathscr{F})$, and $\det \mathscr{F}$ is a line bundle associated to any coherent sheaf, which coincides with the determinant line bundle when $\mathscr{F}$ is locally free (see [15, 17], for instance). The *slope* $\mu(\mathscr{F})$ is the number

$$\mu(\mathscr{F}) = \deg \mathscr{F} / \operatorname{rank} \mathscr{F} ,$$

where rank $\mathscr{F}$ is the rank of the vector bundle that $\mathscr{F}$, like any other coherent sheaf, determines outside of a subset of M. The smallest such subset is called the singularity set of $\mathscr{F}$ and has codimension at least one.

We say that $\mathscr{E}$ is *stable* with respect to ω if for every coherent subsheaf $\mathscr{F} \subset \mathscr{E}$ with $0 < \operatorname{rank} \mathscr{F} < \operatorname{rank} \mathscr{E}$,

$$\mu(\mathscr{F}) < \mu(\mathscr{E}) .$$

Likewise, $\mathscr{E}$ is *semistable* if for every coherent subsheaf $\mathscr{F} \subset \mathscr{E}$ with $0 < \operatorname{rank} \mathscr{F}$,

$$\mu(\mathscr{F}) \leqq \mu(\mathscr{E}) .$$

Remarks. 1. We identify $\mathscr{E}$ with its sheaf of germs of holomorphic sections.
2. One can prove that it suffices to check the (semi) stability condition for *saturated* subsheaves of $\mathscr{E}$, i.e. coherent subsheaves $\mathscr{F}$ whose quotient sheaf $\mathscr{E}/\mathscr{F}$ is torsion free.
3. The notion of (semi)stability can be extended to any torsion free coherent sheaf.

We say that a hermitian metric h on $\mathscr{E}$ is *Hermitian–Yang–Mills* or *Hermitian–Einstein* with respect to ω if

$$\Lambda F_h = \lambda \mathbf{I}_{\mathscr{E}} . \tag{5}$$

where $F_h \in \Omega^{1,1}(\operatorname{End} \mathscr{E})$ is the curvature of the metric connection, Λ is contraction with the Kähler form, $\mathbf{I}_{\mathscr{E}} \in \Omega^0(\operatorname{End} \mathscr{E})$ is the identity and λ is the constant

$$\lambda = \frac{-2\pi i}{\operatorname{Vol} M} \mu(\mathscr{E}) .$$

Equivalently, we could start with a smooth hermitian vector bundle E over M and say that an integrable unitary connection is *Hermitian–Yang–Mills* if

$$\Lambda F_A = \lambda \mathbf{I}_E . \tag{6}$$

For details see for example [15].

It is important to understand the precise correspondence between these two points of view – fixing the holomorphic structure and varying the metric, or fixing the metric and varying the holomorphic structure (or corresponding connection). The key point in this correspondence is that given two hermitian metrics h and $\tilde{h}$ on E there is an element g in $\mathscr{G}^{\mathbb{C}}$, the gauge group of general linear automorphisms of E, unique up to a unitary gauge transformation, such that $\tilde{h} = hg^*g$, i.e.

$$\tilde{h}(s, t) = h(gs, gt) \quad \text{for } s, t \in \Omega^0(E) .$$

Let $\bar{\partial}_E$ be a holomorphic structure on E and suppose that E has a hermitian metric $\tilde{h}$ such that the metric connection $\tilde{A}$ determined by $\bar{\partial}_E$ and $\tilde{h}$ satisfies [6]. Then we want to find an integrable connection, *unitary* with respect to h (up to unitary gauge equivalence), satisfying Eq. (6). Let A be the metric connection determined by $\bar{\partial}_E$ and h, and let $g \in \mathscr{G}^{\mathbb{C}}$ be such that $\tilde{h} = hg^*g$. The relation between A and $\tilde{A}$ is given by

$$d_{g(A)} = g \circ d_{\tilde{A}} \circ g^{-1} ,$$

where

$$d_{g(A)} = g \circ d''_A \circ g^{-1} + (g^*)^{-1} \circ d'_A \circ g^*$$

is the action of $\mathscr{G}^{\mathbb{C}}$ on $\mathscr{A}^{1,1}$ induced by the identification of $\mathscr{A}^{1,1}$ with the space of holomorphic structures on E (cf. [3]). This action extends that of the unitary gauge group

$$\mathscr{G} = \{g \in \mathscr{G}^{\mathbb{C}} | g^*g = 1\} .$$

It is easy to see that

$$F_{g(A)} = g \circ F_{\tilde{A}} \circ g^{-1} ;$$

$g(A)$ is then the desired solution to Eq. (6). For details see for example [3, 15].

The main results relating the notions of stability and Hermitian–Yang–Mills metric are given by the following.

Theorem 2. *Let $\mathscr{E}$ be a holomorphic vector bundle over M as above. If $\mathscr{E}$ has a Hermitian–Yang–Mills metric h, then $\mathscr{E}$ is semistable, and $(\mathscr{E}, h)$ decomposes as a direct sum*

$$(\mathscr{E}, h) = \bigoplus_i (\mathscr{E}_i, h_i)$$

of stable vector bundles $\mathscr{E}_i$ with Hermitian–Yang–Mills metrics h_i, all with slope $\mu(\mathscr{E}_i) = \mu(\mathscr{E})$.

The proof is due independently to Kobayashi [14, 15] and Lübke [16].

Theorem 3. *Let $\mathscr{E}$ be a holomorphic vector bundle over M as above. If $\mathscr{E}$ is stable, then it admits a Hermitian–Yang–Mills metric which is unique up to scale.*

Donaldson proved it in the algebraic case [3, 4] and Uhlenbeck and Yau gave a proof for a general compact Kähler manifold [21].

Let M be a compact Kähler manifold as above. Suppose that a compact Lie group G acts holomorphically on M preserving the Kähler metric. Let $\mathscr{E}$ be a G-invariant holomorphic vector bundle: this means that the action of G can be lifted holomorphically to $\mathscr{E}$.

Definition. *The bundle $\mathscr{E}$ is G-invariantly stable with respect to ω if for every G-invariant coherent subsheaf $\mathscr{F}$ with* $0 < \operatorname{rank} \mathscr{F} < \operatorname{rank} \mathscr{E}$ *we have* $\mu(\mathscr{F}) < \mu(\mathscr{E})$.

The main goal of this section is to prove G-invariant versions of Theorems 2 and 3:

Theorem 4. *Let $\mathscr{E}$ be a G-invariant holomorphic vector bundle over a Kähler manifold M as above. If $\mathscr{E}$ has a G-invariant Hermitian–Yang–Mills metric h, then* $(\mathscr{E}, h) = \bigoplus_i (\mathscr{E}_i, h_i)$, *where $\mathscr{E}_i$ is G-invariantly stable having a G-invariant Hermitian–Yang–Mills metric h_i, and* $\mu(\mathscr{E}_i) = \mu(\mathscr{E})$.

Theorem 5. *Let $\mathscr{E}$ be a G-invariant holomorphic vector bundle as above. If $\mathscr{E}$ is G-invariantly stable, then it supports a G-invariant Hermitian–Yang-Mills metric.*

Remark. Theorem 5 can be obtained as a corollary of a more general theorem of Simpson [18] about the existence of Hermitian–Yang–Mills metrics on Higgs bundles with a group action. He combines the methods of Donaldson and Uhlenbeck and Yau to construct the Hermitian–Yang–Mills metric as the limit of a solution to a non-linear heat equation. The invariance of the metric under the action of the group follows from the invariance of the heat equation.

However, the situation we are considering is much simpler since there is no Higgs field, and a much easier proof can be given by reducing the G-invariant case to the ordinary one. We will do that by analysing the relation between G-invariant stability and ordinary stability. Using the uniqueness of the maximal destabilizing subsheaf for a non-semistable bundle we will show that if the bundle is G-invariantly stable then it must be semistable. Then, if it is not stable it has to contain a proper stable subsheaf. It turns out that this is a subbundle and the total bundle decomposes as a direct sum of this subbundle and other subbundles transformed of it by different elements of the group.

Theorem 6. *Let $\mathscr{E}$ be a G-invariant holomorphic vector bundle as above. Then $\mathscr{E}$ is G-invariantly stable if and only if it is G-indecomposable and is of the form* $\mathscr{E} = \bigoplus_{i=0}^n \mathscr{E}_i$, *where $\mathscr{E}_i$ is a stable bundle which is the transformed of $\mathscr{E}_0$ by an element of G.*

We first prove the following.

Proposition 7. *If $\mathscr{E}$ is G-invariantly stable, then it is semistable.*

Proof. Suppose that $\mathscr{E}$ is G-invariantly stable but not semistable. Then there exists a unique maximal destabilizing saturated semistable subsheaf $\mathscr{F}$ (see [15]) such that

$$\mu(\mathscr{S}) \leqq \mu(\mathscr{F})$$

for any subsheaf $\mathscr{S}$ of $\mathscr{E}$. In particular

$$\mu(\mathscr{E}) \leqq \mu(\mathscr{F})\,. \tag{7}$$

By uniqueness $\mathscr{F}$ is G-invariant, and (7) contradicts the G-invariant stability of $\mathscr{E}$. □

Lemma 8. *Let $\mathscr{E}$ be a holomorphic vector bundle over a compact Kähler manifold. Let $\mathscr{F}$ be a proper saturated subsheaf such that $\mu(\mathscr{F}) = \mu(\mathscr{E})$; then*

(a) $\mu(\mathscr{E}/\mathscr{F}) = \mu(\mathscr{F}) = \mu(\mathscr{E})$,
(b) *If $\mathscr{E}$ is semistable, then $\mathscr{F}$ and $\mathscr{E}/\mathscr{F}$ are semistable.*

Proof. (a) follows from the formula

$$\mu(\mathscr{E}) = \frac{\operatorname{rank} \mathscr{F}\, \mu(\mathscr{F}) + \operatorname{rank}(\mathscr{E}/\mathscr{F})\mu(\mathscr{E}/\mathscr{F})}{\operatorname{rank} \mathscr{F} + \operatorname{rank}(\mathscr{E}/\mathscr{F})}.$$

(b) is a direct consequence of (a) and the definition of semistability. □

Lemma 9. *Let $\mathscr{E}$ be a holomorphic vector bundle over a compact Kähler manifold. Suppose that $\mathscr{E}$ is semistable but not stable; then there exists a saturated subsheaf $\mathscr{F}$ with $0 < \operatorname{rank} \mathscr{F} < \operatorname{rank} \mathscr{E}$ such that*

(a) $\mu(\mathscr{F}) = \mu(\mathscr{E})$;
(b) *$\mathscr{F}$ is stable.*

Proof. If $\mathscr{E}$ is not stable there exists a saturated subsheaf $\mathscr{F}$ with $0 < \operatorname{rank} \mathscr{F} < \operatorname{rank} \mathscr{E}$ and $\mu(\mathscr{F}) = \mu(\mathscr{E})$. By Lemma 8, $\mathscr{F}$ is semistable. If it is not stable we can iterate, reducing finally to a rank one torsion free sheaf, which is always stable. □

Lemma 10. *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be torsion-free coherent sheaves over a compact Kähler manifold. Let $f\colon \mathscr{S}_1 \to \mathscr{S}_2$ be a non-zero homomorphism. Suppose that $\mathscr{S}_1$ is stable, $\mathscr{S}_2$ is semistable and $\mu(\mathscr{S}_1) = \mu(\mathscr{S}_2)$; then $\operatorname{rank} \mathscr{S}_1 = \operatorname{rank}(f(\mathscr{S}_1))$ and f is injective.*

Proof. See [15].

Proof of Theorem 6. Suppose that $\mathscr{E}$ is G-invariantly stable. Clearly $\mathscr{E}$ is G-indecomposable. On the other hand, by Proposition 7, $\mathscr{E}$ is semistable. Suppose that it is not stable. By Lemma 9 there exists a saturated subsheaf $\mathscr{F}$ such that $0 < \operatorname{rank} \mathscr{F} < \operatorname{rank} \mathscr{E}$, $\mu(\mathscr{F}) = \mu(\mathscr{E})$, and $\mathscr{F}$ is stable.

Obviously $\mathscr{F}$ cannot be G-invariant, since this would contradict the G-invariant stability of $\mathscr{E}$. So choose $g_1 \in G$ such that $\mathscr{F}^{g_1} \neq \mathscr{F}$, where $\mathscr{F}^{g_1}$ is the transformed of $\mathscr{F}$ by g_1.

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{F} & \to & \mathscr{E} & \to & \mathscr{Q} & \to 0 & \\ & & & & \uparrow & \overset{f_1}{\nearrow} & & & , \\ & & & & \mathscr{F}^{g_1} & & & & \end{array}$$

where $\mathscr{Q}$ is the quotient sheaf $\mathscr{E}/\mathscr{F}$ and f_1 is the projection of $\mathscr{F}^{g_1}$ to $\mathscr{Q}$. Since $\mathscr{F}$ is stable, so is $\mathscr{F}^{g_1}$, and $\mu(\mathscr{F}^{g_1}) = \mu(\mathscr{F})$. By Lemma 8 $\mu(\mathscr{Q}) = \mu(\mathscr{F})$, and $\mathscr{Q}$ is semistable. By Lemma 10 f_1 is injective. Hence $\mathscr{F} \cap \mathscr{F}^{g_1} = 0$, so $\mathscr{F} + \mathscr{F}^{g_1} \cong \mathscr{F} \oplus \mathscr{F}^{g_1}$. In particular, $\mu(\mathscr{F} + \mathscr{F}^{g_1}) = \mu(\mathscr{F})$.

We will consider separately the following two cases:

$$\mathscr{F}^{g_2} \subset \mathscr{F} + \mathscr{F}^{g_1} \quad \text{for all } g_2 \in G \text{ and } g_2 \neq g_1\,; \tag{8}$$

$$\mathscr{F}^{g_2} \not\subset \mathscr{F} + \mathscr{F}^{g_1} \quad \text{for some } g_2 \in G \text{ and } g_2 \neq g_1\,. \tag{9}$$

Suppose first that (8) holds. Then $\mathscr{F} + \mathscr{F}^{g_1}$ is a G-invariant subsheaf. Since $\mathscr{E}$ is G-invariantly stable and $\mu(\mathscr{F} + \mathscr{F}^{g_1}) = \mu(\mathscr{E})$, $\operatorname{rank}(\mathscr{F} + \mathscr{F}^{g_1}) = \operatorname{rank} \mathscr{E}$. Hence $\operatorname{rank} \mathscr{F}^{g_1} = \operatorname{rank} \mathscr{Q}$, so $\deg \mathscr{F}^{g_1} = \deg \mathscr{Q}$. Consequently, the torsion sheaf $\mathscr{T}$ in

$$0 \to \mathscr{F}^{g_1} \overset{f_1}{\to} \mathscr{Q} \to \mathscr{T} \to 0$$

has degree zero and hence the support of $\mathscr{T}$ must be of codimension $\geqq 2$. Since f_1 is an injection, we conclude that outside of a set S of codimension $\geqq 2$, f_1 is an isomorphism.

Let $M' = M - S$ and consider the exact sequence

$$0 \to \mathscr{F}|_{M'} \to \mathscr{E}|_{M'} \to \mathscr{Q}|_{M'} \to 0 . \tag{10}$$

Because $\mathscr{Q}|_{M'} \cong \mathscr{F}^{g_1}|_{M'}$, the injection $\mathscr{F}^{g_1} \overset{f_1}{\hookrightarrow} \mathscr{E}$ gives a holomorphic splitting of the sequence (10)

$$\mathscr{E}|_{M'} = \mathscr{F}|_{M'} \oplus \mathscr{Q}|_{M'} .$$

Hence, the sequence

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{Q} \to 0 \tag{11}$$

splits over M, as is shown by the following lemma.

Lemma 11. *Let S and M' be as before. If* (10) *splits holomorphically over M', then so does* (11), *and moreover $\mathscr{F}$ and $\mathscr{Q}$ are locally free,* i.e. *vector bundles.*

Proof. First recall that a coherent sheaf $\mathscr{S}$ is *reflexive* if $\mathscr{S} \cong \mathscr{S}^{**}$, or equivalently, if it is normal and torsion free. Here *normal* means that for every open set $U \subset M$, and every analytic set $A \subset U$ of codimension at least 2, the restriction map $\Gamma(U, \mathscr{S}) \to \Gamma(U - A, \mathscr{S})$ is an isomorphism.

Since $\mathscr{E}$ is reflexive and $\mathscr{Q}$ is torsion free, $\mathscr{F}$ is reflexive. Consequently $\operatorname{Hom}(\mathscr{E}, \mathscr{F})$ and $\operatorname{Hom}(\mathscr{F}, \mathscr{F})$ are also reflexive, and in particular, normal. Hence the splitting homomorphism $p' \in H^0(M', \operatorname{Hom}(\mathscr{E}, \mathscr{F}))$ with

$$p' \circ j = \mathrm{id}_{\mathscr{F}}|_{M'} \in H^0(M', \operatorname{Hom}(\mathscr{F}, \mathscr{F}))$$

extends uniquely to a splitting homomorphism $p \in H^0(M, \operatorname{Hom}(\mathscr{E}, \mathscr{F}))$ with

$$p \circ j = \mathrm{id}_{\mathscr{F}} \in H^0(M, \operatorname{Hom}(\mathscr{F}, \mathscr{F})) .$$

This proves that $\mathscr{E} = \mathscr{F} \oplus \mathscr{Q}$. Since $\mathscr{E}$ is locally free both $\mathscr{F}$ and $\mathscr{Q}$ are projective $\mathscr{O}_M$-modules, and hence locally free. □

Now suppose that the second case (9) holds, i.e.

$$\mathscr{F}^{g_2} \not\subset \mathscr{F} + \mathscr{F}^{g_1} \quad \text{for some} \quad g_2 \in G \text{ and } g_2 \neq g_1 .$$

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{F} + \mathscr{F}^{g_1} & \to & \mathscr{E} & \to & \mathscr{Q}' & \to & 0 . \\ & & & & \uparrow & \overset{f_2}{\nearrow} & & & \\ & & & & \mathscr{F}^{g_2} & & & & \end{array}$$

We first notice that $\mathcal{Q}'$ is torsion free. To see this consider the saturation of $\mathcal{F} + \mathcal{F}^{g_1}$, i.e. the smallest subsheaf $\mathcal{S}$ of $\mathcal{E}$ containing $\mathcal{F} + \mathcal{F}^{g_1}$ such that $\mathcal{E}/\mathcal{S}$ is torsion free.

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{F} & \to & \mathcal{S} & \to & \mathcal{R} & \to & 0\,. \\ & & & & \uparrow & \nearrow & & & \\ & & & & \mathcal{F}^{g_1} & & & & \end{array}$$

Applying similar arguments to the ones above and Lemma 11 one can see that $\mathcal{S} = \mathcal{F} + \mathcal{F}^{g_1}$.

As in the previous case, $\mathcal{Q}'$ is semistable. Since $\mathcal{F}^{g_2}$ is stable and $\mu(\mathcal{F}^{g_2}) = \mu(\mathcal{Q}')$, we can again apply Lemma 10 to conclude that f_2 is injective and hence that

$$(\mathcal{F} + \mathcal{F}^{g_1}) \cap \mathcal{F}^{g_2} = 0\,.$$

Iterating the previous argument, after a finite number of steps we prove that

$$\mathcal{E} = \bigoplus_{i=0}^{l} \mathcal{E}_i\,,$$

where $\mathcal{E}_i = \mathcal{F}^{g_i}$ for $g_i \in G$ all different and $\mathcal{E}_0 = \mathcal{F}$.

We now prove the other direction of the Theorem. If $\mathcal{E}$ is actually indecomposable we are finished. Suppose then that $\mathcal{E}$ is G-indecomposable and $\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{E}_i$, with $\mathcal{E}_i$ the transformed of $\mathcal{E}_0$ by an element of G. Since $\mathcal{E}$ is semistable it is G-invariantly semistable. Suppose that $\mathcal{E}$ is not G-invariantly stable; then there exists a G-invariant saturated subsheaf $\mathcal{F}$ with $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E}$ and

$$\mu(\mathcal{F}) = \mu(\mathcal{E})\,. \tag{12}$$

Since $\operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E}$ we can suppose without loss of generality that $\mathcal{F}_0 = \mathcal{F} \cap (\mathcal{E}_0 \oplus 0)$ satisfies $0 < \operatorname{rank} \mathcal{F}_0 < \operatorname{rank} \mathcal{E}_0$. We have the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{E}_0 & \to & \mathcal{E} & \to & \bigoplus_{i \neq 0} \mathcal{E}_i & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathcal{F}_0 & \to & \mathcal{F} & \to & \mathcal{F}' & \to & 0\,, \end{array}$$

where $\mathcal{F}'$ is the image of $\mathcal{F}$ under the projection of $\mathcal{E}$ to $\bigoplus_{i \neq 0} \mathcal{E}_i$.

Consider first the case $\mathcal{F}' = 0$. Then $\mathcal{F} \subseteq \mathcal{E}_0$; we claim that in fact $\mathcal{F} = \mathcal{E}_0$. First, $\operatorname{rank} \mathcal{F} = \operatorname{rank} \mathcal{E}_0$, for otherwise the stability of $\mathcal{E}_0$ would imply that $\mu(\mathcal{F}) < \mu(\mathcal{E}_0) = \mu$, contradicting (12). But since $\mathcal{F}$ is saturated, it follows that $\mathcal{F} = \mathcal{E}_0$. Since $\mathcal{F}$ is G-invariant, so is $\mathcal{E}_0$, contradicting the hypothesis that $\mathcal{E}$ is G-indecomposable.

Next suppose that $\mathcal{F}' \neq 0$. The semistability of $\mathcal{E}_i$ implies that

$$\deg \mathcal{F}_0 \leqq \mu \operatorname{rank} \mathcal{F}_0 \quad \text{and} \quad \deg \mathcal{F}' \leqq \mu \operatorname{rank} \mathcal{F}'\,.$$

On the other hand, since $\operatorname{rank} \mathcal{F}_0 < \operatorname{rank} \mathcal{E}_0$, by the stability of $\mathcal{E}_0$, $\deg \mathcal{F}_0 < \mu \operatorname{rank} \mathcal{F}_0$, so

$$\mu(\mathcal{F}) = \frac{\deg \mathcal{F}_0 + \deg \mathcal{F}'}{\operatorname{rank} \mathcal{F}_0 + \operatorname{rank} \mathcal{F}'} < \mu\,, \tag{13}$$

again contradicting (12). This completes the proof of Theorem 6. □

We are ready now for our main theorems.

Proof of Theorem 4. By Theorem 2, $\mathscr{E} = \bigoplus_l \mathscr{F}_l$ with $\mathscr{F}_l$ stable and $\mu(\mathscr{F}_l) = \mu(\mathscr{E})$. Suppose that $\mathscr{F}_1$ is not G-invariant; then there exists $g_1 \in G$ such that $\mathscr{F}_1^{g_1} \neq \mathscr{F}_1$, so there is a non-trivial diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{F}_1 & \to & \mathscr{E} & \xrightarrow{} & \bigoplus_{l \neq 1} \mathscr{F}_l & \to & 0\,. \\ & & & & \uparrow & \nearrow_{f_1} & & & \\ & & & & \mathscr{F}_1^{g_1} & & & & \end{array}$$

By Lemma 10, f_1 is an injection, and $\mathscr{F}_1 \cap \mathscr{F}_1^{g_1} = 0$. We repeat this argument, considering as many $g_k \in G$ as necessary, to get $\mathscr{E}_1 = \mathscr{F}_1 \oplus \mathscr{F}_1^{g_1} \oplus \ldots \oplus \mathscr{F}_1^{g_k}$, G-indecomposable. We repeat it again for another $\mathscr{F}_i$ not G-invariant and not contained in $\mathscr{E}_1$, till we get $\mathscr{E} = \bigoplus \mathscr{E}_i$. Now Theorem 6 applies to each $\mathscr{E}_i$. □

Proof of Theorem 5. By Theorem 6, $\mathscr{E}$ is of the form $\mathscr{E} = \bigoplus_{i=0}^{n} \mathscr{E}_i$, where $\mathscr{E}_i$ is a stable bundle which is the transformed of $\mathscr{E}_0$ by an element of G. By Theorem 3 there exists a Hermitian–Yang–Mills metric h_i on $\mathscr{E}_i$. Since $\mu(\mathscr{E}_i) = \mu(\mathscr{E})$ for all i, the direct sum $h = \bigoplus_i h_i$ is a Hermitian–Yang–Mills metric on $\mathscr{E}$. The transformed of h by an element of G is also a Hermitian–Yang–Mills metric and doing the average over G we get the desired G-invariant Hermitian–Yang–Mills metric on $\mathscr{E}$. □

3. The Vortex Equation as a Dimensional Reduction of the Hermitian–Yang–Mills Equation

In order to study the existence of solutions to the system of Eqs. (4) it is convenient to look at it as an equation for a hermitian metric on L. For this equivalent point of view we fix a holomorphic structure $\bar{\partial}_L$ on L. We will denote L together with this holomorphic structure by $\mathscr{L}$. We also fix ϕ, a holomorphic section of $\mathscr{L}$. Then we are looking for a hermitian metric h on $\mathscr{L}$ satisfying

$$\Lambda F_h - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau = 0\,, \tag{14}$$

where F_h is the curvature of the metric connection.

In Sect. 2 we explained the equivalence between the two different ways of dealing with the Hermitian–Yang–Mills equation. The situation here is very similar. Suppose that $\tilde{h}$ is a metric on $\mathscr{L}$ satisfying the τ-vortex equation (14). Let $\tilde{A}$ be the metric connection determined by $\bar{\partial}_L$ and $\tilde{h}$. Then

$$\Lambda F_{\tilde{A}} - \frac{i}{2}|\phi|_{\tilde{h}}^2 + \frac{i}{2}\tau = 0\,.$$

But we want of course a pair $(A, \phi) \in \mathscr{A}^{1,1} \times \Omega^0(L)$ satisfying (4). As in Sect. 2 let $g \in \mathscr{G}^{\mathbb{C}}$, so that $\tilde{h} = hg^*g$, and let A be the metric connection determined by $\bar{\partial}_L$ and h. We saw that

$$F_{g(A)} = g \circ F_{\tilde{A}} \circ g^{-1}\,.$$

On the other hand,

$$|\phi|_{\tilde{h}}^2 = |g\phi|_h^2 ,$$

where the action of $\mathscr{G}^{\mathbb{C}}$ on $\Omega^0(L)$ is given by multiplication. This is because $g:(L, \tilde{h}) \to (L, h)$ is an isometry; indeed

$$\langle g\psi, g\eta\rangle_h = \langle g^* g\psi, \eta\rangle_h = \langle \psi, \eta\rangle_{\tilde{h}} .$$

We conclude that $(g(A), g\phi)$ is the desired solution to Eqs. (4).

We will now show that the vortex equation (14) for a metric on a holomorphic line bundle $\mathscr{L}$ over X with a prescribed holomorphic section ϕ can be obtained as a dimensional reduction of the Hermitian–Yang–Mills equation on a rank two vector bundle over $X \times \mathbb{P}^1$, where $\mathbb{P}^1$ is the complex projective line. This generalises the results of Witten [22] and Taubes [20] for the classical vortex equation over the hyperbolic and euclidean planes respectively.

Let $\mathscr{L}$ be a holomorphic line bundle over X and let ϕ be a holomorphic section of $\mathscr{L}$. There is canonically associated to $(\mathscr{L}, \phi)$ a rank two holomorphic vector bundle $\mathscr{E}$ over $X \times \mathbb{P}^1$ given as an extension

$$0 \to p^*\mathscr{L} \to \mathscr{E} \to q^*\mathscr{O}(2) \to 0 . \tag{15}$$

Here p and q are the projections from $X \times \mathbb{P}^1$ to X and $\mathbb{P}^1$ respectively. We denote by $\mathscr{O}$ the structure sheaf of $\mathbb{P}^1$ and by $\mathscr{O}_X$ the structure sheaf of X. By $\mathscr{O}(2)$ we denote as usual the holomorphic line bundle with Chern class 2 on $\mathbb{P}^1$, isomorphic to the holomorphic tangent bundle of $\mathbb{P}^1$.

Extensions as above are parametrized by

$$\begin{aligned} H^1(X \times \mathbb{P}^1, p^*\mathscr{L} \otimes q^*\mathscr{O}(-2)) &\cong H^0(X, \mathscr{L}) \otimes H^1(\mathbb{P}^1, \mathscr{O}(-2)) \\ &\cong H^0(X, \mathscr{L}) \end{aligned} \tag{16}$$

since $H^0(\mathbb{P}^1, \mathscr{O}(-2)) = 0$ and $H^1(\mathbb{P}^1, \mathscr{O}(-2)) \cong H^0(\mathbb{P}^1, \mathscr{O})^* \cong \mathbb{C}$; we choose $\mathscr{E}$ to be the extension determined by ϕ.

Let $SU(2)$ act on $X \times \mathbb{P}^1$, trivially on X, and in the standard way on $\mathbb{P}^1 \cong SU(2)/U(1)$. This action can be lifted to an action on $\mathscr{E}$, trivial on $p^*\mathscr{L}$ and standard on $q^*\mathscr{O}(2)$. Since the induced actions on $H^0(X, \mathscr{L})$ and $H^0(\mathbb{P}^1, \mathscr{O})$ are trivial, $\mathscr{E}$ is an $SU(2)$-invariant holomorphic vector bundle.

For $\sigma \in \mathbb{R}^+$, consider the $SU(2)$-invariant Kähler metric on $X \times \mathbb{P}^1$ whose Kähler form is $\Omega_\sigma = p^*\omega + q^*\omega_\sigma$, where ω is the Kähler form on X and ω_σ is the *Fubini–Study* metric with coefficient σ: in co-ordinates

$$\omega_\sigma = \frac{i\sigma}{2\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}$$

so that

$$\int_{\mathbb{P}^1} \omega_\sigma = \sigma .$$

We can now state the main result of this section.

Proposition 12. *Let $\mathscr{L}$ be a holomorphic line bundle over X and ϕ be a holomorphic section. Let $\mathscr{E}$ be the holomorphic vector bundle over $X \times \mathbb{P}^1$ defined by $(\mathscr{L}, \phi)$, and* *let $\sigma = 8\pi/\tau > 0$. Then $\mathscr{L}$ admits a hermitian metric satisfying the τ-vortex equation if and only if $\mathscr{E}$ admits an $SU(2)$-invariant Hermitian–Yang–Mills metric with respect to Ω_σ.*

Proof. Suppose that $\mathscr{E}$ admits an $SU(2)$-invariant Hermitian–Yang–Mills metric $\mathbf{h}$ with respect to Ω_σ. This means that

$$\Lambda_\sigma F_{\mathbf{h}} = \lambda \mathbf{I}_{\mathscr{E}} , \tag{17}$$

where $\Lambda_\sigma = p^*\Lambda + q^*\Lambda_\sigma$ is contraction by the Kähler form Ω_σ and λ is the constant

$$\begin{aligned}\lambda &= -\pi i \frac{\deg_\sigma \mathscr{E}}{\mathrm{Vol}(X \times \mathbb{P}^1)} \\ &= -\pi i \frac{\sigma \deg \mathscr{L} + 2\mathrm{Vol}\, X}{\sigma \,\mathrm{Vol}\, X} ,\end{aligned} \tag{18}$$

since

$$\begin{aligned}\deg_\sigma \mathscr{E} &= \frac{1}{n!} \int_{X \times \mathbb{P}^1} c_1(\mathscr{E}) \wedge \Omega_\sigma^n \\ &= \frac{1}{n!} \int_{X \times \mathbb{P}^1} (c_1(\mathscr{L}) + c_1(\mathscr{O}(2))) \wedge (\omega^n + n\omega^{n-1} \wedge \omega_\sigma) \\ &= \sigma \deg \mathscr{L} + 2\mathrm{Vol}\, X .\end{aligned} \tag{19}$$

Since $\mathbf{h}$ is $SU(2)$-invariant and the actions of $SU(2)$ on $p^*\mathscr{L}$ and $q^*\mathscr{O}(2)$ correspond to different weights, $\mathbf{h}$ is of the form

$$\mathbf{h} = \mathbf{h}_1 \oplus \mathbf{h}_2 ,$$

for $SU(2)$-invariant metrics $\mathbf{h}_1$ and $\mathbf{h}_2$ on $p^*\mathscr{L}$ and $q^*\mathscr{O}(2)$ respectively. Moreover

$$\mathbf{h}_1 = p^*h_1 \quad \text{and} \quad \mathbf{h}_2 = p^*h_2 \otimes q^*h_2' ,$$

where h_1 and h_2 are metrics on $\mathscr{L}$ and $\mathscr{O}_X$ and h_2' is an $SU(2)$-invariant metric on $\mathscr{O}(2)$.

The metric connection of $(\mathscr{E}, \mathbf{h})$ can be written as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \beta \\ -\beta^* & \mathbf{A}_2 \end{pmatrix} , \tag{20}$$

where $\mathbf{A}_1, \mathbf{A}_2$ are the metric connections of $(p^*\mathscr{L}, \mathbf{h}_1)$ and $(q^*\mathscr{O}(2), \mathbf{h}_2)$ and $\beta \in \Omega^{0,1}(X \times \mathbb{P}^1, p^*\mathscr{L} \otimes q^*\mathscr{O}(-2))$ is a representative of the extension class in $H^1(X \times \mathbb{P}^1, p^*\mathscr{L} \otimes q^*\mathscr{O}(-2))$. Then $\beta^* \in \Omega^{1,0}(X \times \mathbb{P}^1, p^*\mathscr{L}^* \otimes q^*\mathscr{O}(2))$ is the *second fundamental form* of $p^*\mathscr{L}$ in $(\mathscr{E}, \mathbf{h})$.

The corresponding curvature matrix is

$$F_{\mathbf{h}} = F_{\mathbf{A}} = \begin{pmatrix} F_{\mathbf{A}_1} - \beta \wedge \beta^* & D'\beta \\ -D''\beta^* & F_{\mathbf{A}_2} - \beta^* \wedge \beta \end{pmatrix} , \tag{21}$$

where $D : \Omega^1(p^*\mathscr{L} \otimes q^*\mathscr{O}(-2)) \to \Omega^2(p^*\mathscr{L} \otimes q^*\mathscr{O}(-2))$ is built from $\mathbf{A}_1$ and $\mathbf{A}_2$ (see [15], for example).

By $SU(2)$-invariance the connections $\mathbf{A}_1$ and $\mathbf{A}_2$ are of the form

$$\mathbf{A}_1 = p^*A_1 \quad \text{and} \quad \mathbf{A}_2 = p^*A_2 * q^*A'_2 ,$$

where A_1, A_2 and A'_2 are the metric connections of $(\mathscr{L}, h_1)$, $(\mathcal{O}_X, h_2)$ and $(\mathcal{O}(2), h'_2)$ respectively. Then

$$F_{\mathbf{A}_1} = p^*F_{h_1} \quad \text{and} \quad F_{\mathbf{A}_2} = p^*F_{h_2} + q^*F_{h'_2} .$$

Notice that because of the isomorphism (16) ϕ determines an extension class $[\mathscr{E}]$ over $X \times \mathbb{P}^1$. We are taking an $SU(2)$-invariant representative in this extension class; as one can easily see, it is given by

$$\beta = p^*\phi \otimes q^*\alpha ,$$

where $\alpha \in \Omega^{0,1}(\mathbb{P}^1, \mathcal{O}(-2))$ is $SU(2)$-invariant. In other words, up to a constant to be fixed later, α is given in co-ordinates by

$$\alpha = \frac{dz}{(1+|z|^2)^2}\, d\bar{z} .$$

Denote by β^* the adjoint of $\beta \in \Omega^{0,1}(\mathrm{Hom}(q^*\mathcal{O}(2), p^*\mathscr{L}))$ with respect, of course, to the metrics $\mathbf{h}_1$ and $\mathbf{h}_2$. If we write $h = h_1 \otimes h_2^*$, where h_2^* is the metric dual to h_2, then

$$\beta^* = p^*\phi^{*h} \otimes q^*\alpha^* ,$$

where ϕ^{*h} denotes the adjoint of $\phi \in \Omega^0(X, \mathrm{Hom}(\mathcal{O}_X, \mathscr{L}))$ with respect to the metrics h_1 and h_2, and α^* is the adjoint of $\alpha \in \Omega^{0,1}(\mathbb{P}^1, \mathrm{Hom}(\mathcal{O}(2), \mathcal{O}))$ with respect to a constant metric on $\mathcal{O}$ and the metric h'_2 on $\mathcal{O}(2)$. Let

$$\alpha = \gamma \frac{dz}{(1+|z|^2)^2}\, d\bar{z} \quad \text{for } \gamma \in \mathbb{C} .$$

We can assume that the metric h'_2 is given by

$$h'_2(z) = \frac{\sigma}{(1+|z|^2)^2}\, dz \otimes d\bar{z} .$$

Then

$$\alpha^* = h'^*_2 \bar{\alpha} = \frac{\pi\bar{\gamma}}{\sigma} \frac{\partial}{\partial z}\, dz \in \Omega^{1,0}(\mathbb{P}^1, \mathcal{O}(2)) ,$$

where h'^*_2 is the metric on $\mathcal{O}(-2)$ dual to h'_2. Thus if $\gamma = \sigma/\sqrt{8\pi}$,

$$\alpha \wedge \alpha^* = \frac{i}{4}\omega_\sigma , \tag{22}$$

and so

$$\beta \wedge \beta^* = \frac{i}{4} p^*|\phi|_h^2 \otimes q^*\omega_\sigma , \tag{23}$$

$$\beta^* \wedge \beta = -\frac{i}{4} p^*|\phi|_h^2 \otimes q^*\omega_\sigma . \tag{24}$$

In terms of (21), Eq. (17) implies that

$$\left.\begin{aligned} \Lambda_\sigma(F_{\mathbf{A}_1} - \beta \wedge \beta^*) = \lambda \\ \Lambda_\sigma(F_{\mathbf{A}_2} - \beta^* \wedge \beta) = \lambda \end{aligned}\right\} . \tag{25}$$

Substituting (22), (23) and (24) in (25), we get

$$\left.\begin{aligned} p^*\Lambda F_{h_1} - p^*|\phi|_h^2 \otimes \Lambda_\sigma(\alpha \wedge \alpha^*) = p^*\Lambda F_{h_1} - \frac{i}{4} p^*|\phi|_h^2 = \lambda \\ p^*\Lambda F_{h_2} + q^*\Lambda_\sigma F_{h_2'} - p^*|\phi|_h^2 \otimes \Lambda_\sigma(\alpha^* \wedge \alpha) = p^*\Lambda F_{h_2} + q^*\Lambda_\sigma F_{h_2'} + \frac{i}{4} p^*|\phi|_h^2 = \lambda \end{aligned}\right\} \tag{26}$$

since $\Lambda_\sigma(\omega_\sigma) = 1$. However, $\Lambda_\sigma F_{h_2'} = -4\pi i/\sigma$, so (26) becomes

$$\left.\begin{aligned} \Lambda F_{h_1} - \frac{i}{4}|\phi|_h^2 = \lambda \\ \Lambda F_{h_2} + \frac{i}{4}|\phi|_h^2 - \frac{4\pi i}{\sigma} = \lambda \end{aligned}\right\} . \tag{27}$$

Subtracting these two equations, and noticing that $F_h = F_{h_1} + F_{h_2^*} = F_{h_1} - F_{h_2}$, we obtain

$$\Lambda F_h - \frac{i}{2}|\phi|_h^2 + \frac{4\pi i}{\sigma} = 0 .$$

Since $\sigma = 8\pi/\tau$, we conclude that h is a solution to the τ-vortex equation.

To prove the other direction of the proposition, suppose that h is a solution to the τ-vortex equation and consider the metric

$$\mathbf{h} = p^*h_1 \otimes p^*h_2 \otimes q^*h_2' ,$$

where $h_1 = h_2 \otimes h$, for h_2 a metric on $\mathcal{O}_X$ to be determined later on and h_2' an $SU(2)$-invariant metric on $\mathcal{O}(2)$.

We then need to solve Eq. (17) or, equivalently, the system of equations

$$\left.\begin{aligned} \Lambda F_{h_1} - \frac{i}{4}|\phi|_h^2 = \lambda \\ \Lambda F_{h_2} + \frac{i}{4}|\phi|_h^2 - \frac{4\pi i}{\sigma} = \lambda \\ \Lambda_\sigma(D'\beta) = 0 \\ \Lambda_\sigma(D''B^*) = 0 \end{aligned}\right\} . \tag{28}$$

To solve the first two equations of (28) is equivalent to solving the system of equations

$$\left.\begin{aligned} \Lambda F_h - \frac{i}{2}|\phi|_h^2 + \frac{4\pi i}{\sigma} = 0 \\ 2\Lambda F_{h_2} + \Lambda F_h - \frac{4\pi i}{\sigma} = 2\lambda \end{aligned}\right\} . \tag{29}$$

But since $\sigma = 8\pi/\tau$, the first equation of (29) is the τ-vortex equation. To solve the second, note that since h_2 is a metric on $\mathcal{O}_X$, $h_2 = e^f$, for f a function on X. Then

$$\Lambda F_{h_2} = i\Delta_{\bar\partial} f$$

and the second equation of (29) becomes

$$i\Delta_{\bar\partial} f = \frac{1}{2}\left(2\lambda - \Lambda F_h + \frac{4\pi i}{\sigma}\right). \tag{30}$$

By Hodge theory, the necessary and sufficient condition for the existence of a solution of (30) is

$$\int_X \left(2\lambda - \Lambda F_h + \frac{4\pi i}{\sigma}\right) = 0\,,$$

but this is satisfied since it is precisely equivalent to the expression (18) that determines λ.

Finally we shall solve the last two equations of (28),

$$D'\beta = p^* D'\phi \otimes q^*\alpha + p^*\phi \otimes q^* D'\alpha\,,$$

$$D''\beta^* = p^* D''\phi^* \otimes q^*\alpha^* + p^*\phi^* \otimes q^* D''\alpha^*\,.$$

One can easily see that

$$D'\alpha = 0 \quad \text{and} \quad D''\alpha^* = 0\,.$$

On the other hand,

$$\Lambda_\sigma(p^* D'\phi \otimes q^*\alpha) = 0 \quad \text{and} \quad \Lambda_\sigma(p^* D''\phi^* \otimes q^*\alpha^*) = 0\,,$$

since the (1, 1)-forms inside have mixed contributions from X and $\mathbb{P}^1$. □

4. An Existence Theorem for the Vortex Equation

In this section we prove an existence theorem for solutions to the vortex equation based on the results of Sects. 2 and 3. This proof complements the two others given by Bradlow [1, 2] and the one given by the author in the case of a Riemann surface [7].

Theorem 13. *Let $\mathscr{L}$ be a holomorphic line bundle over a compact Kähler manifold X and $\phi \not\equiv 0$ be a holomorphic section, and let $\tau > 0$. Then $\mathscr{L}$ admits a smooth hermitian metric h satisfying the τ-vortex equation*

$$\Lambda F_h - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau = 0 \tag{31}$$

if and only if

$$\deg \mathscr{L} < \frac{\tau \operatorname{Vol} X}{4\pi}\,. \tag{32}$$

Proof. By integrating Eq. (31) and using the Chern–Weil formula for $\deg\mathscr{L}$ one can easily see that (32) is a necessary condition for existence of solutions. To see that it is also sufficient we first prove the following

Proposition 14. *Let $\mathscr{E}$ be the $SU(2)$-invariant holomorphic vector bundle over $X \times \mathbb{P}^1$ determined by $(\mathscr{L}, \phi)$ as the extension*

$$0 \to p^*\mathscr{L} \to \mathscr{E} \to q^*\mathscr{O}(2) \to 0\,. \tag{33}$$

Let $\sigma = 8\pi/\tau > 0$; *then* $\mathscr{E}$ *is stable with respect to the Kähler form* Ω_σ *on* $X \times \mathbb{P}^1$ *defined in Sect.* 3 *if and only if*

$$\deg \mathscr{L} < \frac{\tau \operatorname{Vol} X}{4\pi} .$$

Proof. If $\mathscr{E}$ is stable, then

$$\mu_\sigma(p^* \mathscr{L}) < \mu_\sigma(\mathscr{E}) , \tag{34}$$

where μ_σ is the slope with respect to Ω_σ; but

$$\mu_\sigma(p^* \mathscr{L}) = \sigma \deg \mathscr{L} \quad \text{and} \quad \mu_\sigma(\mathscr{E}) = \frac{\sigma}{2} \deg \mathscr{L} + \operatorname{Vol} X ,$$

which are easily seen to imply that (34) is equivalent to

$$\deg \mathscr{L} < \frac{2 \operatorname{Vol} X}{\sigma} = \frac{\tau \operatorname{Vol} X}{4\pi} .$$

To prove the other direction of the proposition we will show first that $\mathscr{E}$ is $SU(2)$-invariantly stable. We will then apply Theorem 6 to show that it is in fact stable. Suppose that $\mathscr{E}$ is not $SU(2)$-invariantly stable and let $\mathscr{F}$ a destablizing subsheaf, i.e. a rank one $SU(2)$-invariant subsheaf of $\mathscr{E}$ with torsion free quotient such that

$$\mu_\sigma(\mathscr{F}) \geqq \mu_\sigma(\mathscr{E}) . \tag{35}$$

Consider for such an $\mathscr{F}$ the diagram

$$\begin{array}{ccccccccc} 0 & \to & p^*\mathscr{L} & \to & \mathscr{E} & \to & q^*\mathcal{O}(2) & \to & 0 , \\ & & & & \uparrow & \overset{f}{\nearrow} & & & \\ & & & & \mathscr{F} & & & & \end{array}$$

where the map f is the composition of the inclusion $\mathscr{F} \to \mathscr{E}$ and the projection $\mathscr{E} \to q^*\mathcal{O}(2)$.

We first notice that $\ker f = \{0\}$, since otherwise $\mathscr{F}$ is injected in $p^*\mathscr{L}$ and, since $\mathscr{E}/\mathscr{F}$ is torsion free, $p^*\mathscr{L}/\mathscr{F}$ is torsion free, implying that $\mathscr{F} \cong p^*\mathscr{L}$. If $\deg \mathscr{L} < \tau \operatorname{Vol} X/4\pi$, then

$$\mu_\sigma(\mathscr{F}) = \mu_\sigma(p^*\mathscr{L}) < \mu_\sigma(\mathscr{E}) ,$$

contradicting (35).

We conclude that $\operatorname{im} f$ is a rank one, $SU(2)$-invariant subsheaf of $q^*\mathcal{O}(2)$, which is of course torsion free. Hence outside of a set S of codimension $\geqq 2$, $\operatorname{im} f$ is a line bundle. Nevertheless,

$$\operatorname{im} f|_{(X \times \mathbb{P}^1)\setminus S} \to q^*\mathcal{O}(2)|_{(X \times \mathbb{P}^1)\setminus S}$$

is not necessarily an injection of line bundles (i.e. an isomorphism); we also need to remove a set S' of codimension at least 1, the support of the torsion sheaf $q^*\mathcal{O}(2)/\operatorname{im} f$. By $SU(2)$-invariance the singularity set is of the form

$$S \cup S' = \tilde{S} \times \mathbb{P}^1 ,$$

where $\tilde{S} \subset X$ is a set of codimension $\geqq 1$.

Then, outside of the set $\tilde{S}\times\mathbb{P}^1$, $\operatorname{im} f$ is isomorphic to $q^*\mathcal{O}(2)$, and we have a splitting of the sequence (33) when restricted to $X\backslash\tilde{S}\times\mathbb{P}^1$. This implies that for a generic $x\in X (x\in X\backslash\tilde{S})$ the restriction of the sequence (33) to $\{x\}\times\mathbb{P}^1$ splits and hence is the trivial extension

$$0\to\mathcal{O}\to\mathcal{O}\oplus\mathcal{O}(2)\to\mathcal{O}(2)\to 0\,. \tag{36}$$

But this is impossible since, by construction, (33) only splits when restricted to $D\times\mathbb{P}^1$, where $D=(\phi)$ is the divisor determined by the holomorphic section ϕ. Indeed, since D has condimension 1 in X, for a generic $x\in X$ $(x\in X\backslash D)$ the restriction of (33) to $\{x\}\times\mathbb{P}^1$ is the non-trivial extension

$$0\to\mathcal{O}\to\mathcal{O}(1)\oplus\mathcal{O}(1)\to\mathcal{O}(2)\to 0\,. \tag{37}$$

We then get a contradiction, so there cannot exist $\mathscr{F}$ satisfying (35), proving the $SU(2)$-invariant stability of $\mathscr{E}$.

Suppose now that $\mathscr{E}$ is not stable. By Theorem 6, $\mathscr{E}\cong\mathscr{L}_1\oplus\mathscr{L}_2$, for $\mathscr{L}_1$ and $\mathscr{L}_2$ line bundles over $X\times\mathbb{P}^1$ and $\mathscr{L}_2$ is the transformed of $\mathscr{L}_1$ by an element of $SU(2)$. Then $\mathscr{L}_1\otimes\mathscr{L}_2\cong\det\mathscr{E}\cong p^*\mathscr{L}\otimes q^*\mathcal{O}(2)$. Hence

$$\mathscr{L}_1|_{\{x\}\times\mathbb{P}^1}\cong\mathscr{L}_2|_{\{x\}\times\mathbb{P}^1}\cong\mathcal{O}(1)\quad\text{for every } x\in X\,,$$

and then

$$\mathscr{E}|_{\{x\}\times\mathbb{P}^1}\cong\mathcal{O}(1)\oplus\mathcal{O}(1)\quad\text{for every } x\in X\,,$$

which fails to be true for those x such that $\phi(x)=0$. □

To finish the proof of Theorem 13, suppose that (32) holds. By the previous Proposition $\mathscr{E}$ is stable and in particular $SU(2)$-invariantly stable with respect to Ω_σ. Then by Theorem 5 there exists an $SU(2)$-invariant Hermitian–Yang–Mills metric with respect to Ω_σ on $\mathscr{E}$, and finally by Proposition 12 we get the desired solution to the τ-vortex equation. □

Consider the set-up of Sect. 1. We define the moduli space of τ-vortices $\mathfrak{V}_\tau$ as the quotient space of solutions to Eqs. (4) modulo the unitary gauge group $\mathscr{G}$. A description of this moduli space for an arbitrary Kähler manifold has been first given by Bradlow [1]. Exploiting our point of view we can equip the moduli space of vortices with the structure of a complex analytic space, with a Kähler metric outside of the singular points.

Consider the set

$$\mathscr{N}=\{(A,\phi)\in\mathscr{A}^{1,1}\times\Omega^0(L)\,|\,\phi\not\equiv 0 \text{ and } d''_A\phi=0\}\,.$$

The complex gauge group acts on $\mathscr{N}$ and the quotient space $\mathfrak{D}=\mathscr{N}/\mathscr{G}^{\mathbb{C}}$ can be identified with the space of effective divisors D such that the underlying smooth bundle to $[D]$, the holomorphic line bundle determined by D, is L. This amounts to the very standard fact that a holomorphic line bundle is the line bundle of an effective divisor if and only if it has a non-trivial holomorphic section and that moreover the divisor is given by the zeros of this section (see [10], for example).

Assume that (32) is satisfied. It is clear that a vortex $[(A,\phi)]\in\mathfrak{V}_\tau$ determines an element of $\mathfrak{D}$, namely the zero set of the holomorphic section ϕ. The converse is a reformulation of Theorem 13. let $D\in\mathfrak{D}\cong\mathscr{N}/\mathscr{G}^{\mathbb{C}}$, and choose a representative $(A,\phi)\in\mathscr{N}$ of D. The connection A determines a holomorphic structure d''_A on L, and ϕ is a holomorphic section. We can solve for a metric $\tilde{h}$ satisfying the τ-vortex equation. As shown at the end of Sect. 1, if $\tilde{h}$ is related to h by $\tilde{h} = hg^*g$ for $g \in \mathscr{G}^{\mathbb{C}}$, unique up to a unitary gauge transformation, then $[(g(A), g\phi)] \in \mathfrak{B}_\tau$.

Consider the C^∞ rank two vector bundle $\mathbf{E} = p^*L \oplus q^*H^{\otimes 2}$ over $X \times \mathbb{P}^1$, where H is the line bundle of Chern class 1 over $\mathbb{P}^1$ and let $\mathscr{M}$ be the moduli space of stable holomorphic structures with respect to Ω_σ on $\mathbf{E}$. Let $D \in \mathfrak{D}$ and ϕ be a holomorphic section of $[D]$, whose associated divisor is D. Let $\mathscr{E}_D$ be the bundle over $X \times \mathbb{P}^1$ determined by $([D], \phi)$ as the extension

$$0 \to p^*[D] \to \mathscr{E}_D \to q^*\mathscr{O}(2) \to 0 .$$

From Proposition 14, the correspondence $D \mapsto \mathscr{E}_D$ defines an injective map $\mathfrak{D} \subsetneq \mathscr{M}$ whose image can be essentially identified with the fixed-point set of $\mathscr{M}$ under the action of $SU(2)$. From here one can prove the following (see [8] for details).

Theorem 15. *The space $\mathfrak{D}$ is a complex analytic space, non-singular at all points for which $H^1(X, [D] \otimes \mathscr{O}_D) = 0$ and with a Kähler structure outside of the singular points.*

Remark. Notice that if X is a Riemann surface this condition is satisfied, recovering the fact that the symmetric products of X are smooth complex manifolds.

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