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# Total Cross Sections in $N$-body Problems: Finiteness and High Energy Asymptotics

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**Abstract.** We study the finiteness of total scattering cross sections from an arbitrary channel to a two-cluster channel and establish the high energy asymptotics for total scattering cross sections with initial two-cluster channel and those from an arbitrary channel to a two-cluster channel.

## 1. Introduction

The total scattering cross sections are usually defined, within a normalization in energy, as the square integral over all outgoing directions of the scattering amplitude ([1, 2, 10]). To study total scattering cross sections through this definition, one needs to know a priori some information on scattering amplitudes. In [5], Enss and Simon introduced another method to define total cross sections. Let $S$ be the scattering operator for the pair $(-\Delta, -\Delta + V(x))$ in $L^2(\mathbf{R}^d)$. For any $g \in C_0^\infty(\mathbf{R}_+)$, put

$$g_\omega(x) = (2\pi)^{-1/2} \int_{\mathbf{R}} e^{i\lambda x\cdot\omega} g(\lambda) d\lambda \, .$$

Then the total cross section, $\sigma(\lambda, \omega)$, with the incident direction $\omega$ is defined through the relation ([5]):

$$\int \sigma(\lambda, \omega) |g(\lambda)|^2 d\lambda = \|(S-1)g_\omega\|^2 \, , \tag{1.1}$$

so long as the right-hand side of (1.1) makes sense. It is clear that $g_\omega \notin L^2(\mathbf{R}^d)$, if $d > 1$. By considering $\|(S-1)g_\omega\|$ as the limit of a family of appropriate cut-off functions, Enss and Simon proved that if $V(x)$ decays like $O(\langle x\rangle^{-\rho})$ with $\rho > (d+1)/2$, the total cross section is finite when averaged over any energy interval. They also established similar results for total scattering cross sections with initial two-cluster scattering channel in many-body problems ([5]). In [14], using Enss and Simon's approach and studying the spectral representation for two-cluster scattering matrices, Robert and the author proved the pointwise finiteness

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of the total cross sections with two-cluster initial channel. The purpose of this work is to study the finiteness of total cross sections with an arbitrary initial channel and establish high energy asymptotics for total scattering cross sections. We find the approach of [5] particularly useful in handling total cross sections with arbitrary initial scattering channel. In fact, the definition (1.1) allows to study the total cross sections for each fixed incident direction $\omega$ and making use of microlocal resolvent estimates established in [19], we shall prove that the total scattering cross sections from an arbitrary initial channel to a two-cluster channel are finite for some distinguished directions $\omega$ and may be infinite for the other directions. See Theorem 3.1 and the remark following its statement.

Let us now introduce some notations. Let $\Delta$ be the Laplacian on the Euclidean space $X = \mathbf{R}^d$. Let $\mathscr{A}$ be the set of all cluster decompositions of an $N$-body system labelled by $\{1, 2, \ldots, N\}$. The $N$-body Schrödinger operator to be studied in this work is of the form:

$$P = -\Delta + \sum_{a \in \mathscr{A}} V_a(x^a) .$$

Here $x^a = \pi^a x$ with $\pi^a$ the orthogonal projection from $X$ onto some subspace $X^a$. Assume the following conventions on the collection $\{X^a, a \in \mathscr{A}\}$: (i). $\mathscr{A}$ is partially ordered by the relation: $a \subseteq b$ iff $X^a \subseteq X^b$; (ii). There are elements $a_{\max}$ and $a_{\min}$ in $\mathscr{A}$ such that $X^{a_{\max}} = X$ and $X^{a_{\min}} = \{0\}$; (iii). For any $a, b \in \mathscr{A}$, the union $a \cup b$ is defined in $\mathscr{A}$ with the property that $X^{a \cup b} = X^a + X^b$. Since $x^{a_{\min}} \equiv 0$, $V_{a_{\min}}$ is a constant. To fix the idea, we take $V_{a_{\min}} = 0$.

For each $a \in \mathscr{A}$, we denote by $X_a$ the orthogonal complement (with respect to the Euclidean structure on $X$) of $X^a$ in $X$: $X = X^a \oplus X_a$. Accordingly, a generic point $x \in X$ can be decomposed as: $x = x^a + x_a$. We denote $-\Delta^a$ ($-\Delta_a$, resp.) the Laplacian in $x^a$-variables ($x_a$-variables, resp.) and $D^a = -i\partial/\partial x^a$, $D_a = -i\partial/\partial x_a$. Put

$$P^a = -\Delta^a + \sum_{b \subseteq a} V_b(x^b), \quad P_a = P^a - \Delta_a ,$$

$$I_a(x) = \sum_{b \not\subseteq a} V_b(x^b) .$$

Then for any $a \in \mathscr{A}$, one has: $P = P_a + I_a(x)$. Let $\mathscr{T}$ denote the set of all thresholds (including the eigenvalues) of $P$:

$$\mathscr{T} = \bigcup_a \sigma_{\mathrm{pp}}(P^a) .$$

$S_a$ will denote the unit sphere in $X_a$. Put

$$\Sigma_a = S_a \setminus \bigcup_{b \not\subseteq a} X_b . \tag{1.2}$$

Due to the geometry of an $N$-body system, one can check that $\Sigma_a = S_a$ if $\# a = 2$ ($\# a$ being the number of clusters in $a$). The norm and the scalar product in $L^2(X_a)$ will be denoted by $\|\cdot\|_a$ and $\langle\cdot,\cdot\rangle_a$, while those in $L^2(X)$ will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

Let $a$ be a non-trivial cluster decomposition (i.e., $a \in \mathscr{A}$ with $\# a \geqq 2$). A scattering channel $\alpha$ stands for a collection of data: $\alpha = (a, E_\alpha, \phi_\alpha)$, where $E_\alpha \in \sigma_{\mathrm{pp}}(P^a)$ and $\phi_\alpha$ is an associated normalized eigenfunction:

$$P^a \phi_\alpha = E_\alpha \phi_\alpha, \quad \|\phi_\alpha\| = 1 .$$

When $a = a_{\min}$, one uses the convention that $P^a = 0$, $P_a = -\Delta$ and in this case, the only scattering channel is the free one: $\alpha = (a_{\min}, 0, 1)$. We shall say that $\alpha$ is a scattering channel with non-threshold energy, if

$$E_\alpha \in \sigma_{\mathrm{pp}}(P^a) \setminus \bigcup_{b \subset a} \sigma_{\mathrm{pp}}(P^b) .$$

Let $\mathscr{J}_\alpha$: $L^2(X_a) \to L^2(X)$ be the channel identification:

$$(\mathscr{J}_\alpha f)(x) = \phi_\alpha(x^a) f(x_a) .$$

To be simple, we assume that $\forall a \in \mathscr{A}$, $V_a$ is smooth on $X^a$ and

$$|\partial_y^\alpha V_a(y)| \leqq C_\alpha \langle y \rangle^{-\rho - |\alpha|} , \tag{1.3}$$

for any $\alpha \in \mathbf{N}^{n^a}$ ($n^a = \dim X^a$). Here $\rho > 0$ will be precised later. Let us indicate that in the main part of this work, local singularities of Coulomb type can be included. See Remark 4.1. Under the assumption (1.3) with $\rho > 1$, it is well known that the wave operators

$$W_\pm^\alpha = s - \lim_{t \to \pm\infty} U(t)^* U_a(t) \mathscr{J}_\alpha$$

exist for any scattering channel $\alpha$ and are complete ([15]). Here $U(t)$ and $U_a(t)$ are unitary groups generated by $P$ and $P_a$, respectively.

Now let $\alpha = (a, E_\alpha, \phi_\alpha)$ and $\beta = (b, E_\beta, \phi_\beta)$ be two given scattering channels. Let

$$S_{\beta\alpha} = W_+^{\beta *} W_-^\alpha$$

be the scattering operator from an initial channel $\alpha$ to a final channel $\beta$. As in [5, 14], we define the total scattering cross section $\sigma_{\beta\alpha}(\lambda, \omega)$ with incident direction $\omega \in S_a$, from an initial channel $\alpha$ to a final channel $\beta$ by

$$\int \sigma_{\beta\alpha}(\lambda, \omega) |g(\lambda)|^2 d\lambda = \|(S_{\beta\alpha} - \delta_{\beta\alpha}) g_\omega\|_b^2 . \tag{1.4}$$

Here $\|\cdot\|_b$ is the norm in $L^2(X_b)$, $g \in C_0^\infty(\mathbf{I}_\alpha)$ with $\mathbf{I}_\alpha = ]E_\alpha, \infty[$ and

$$g_\omega(x_a) = \frac{1}{2\sqrt{\pi}} \int_{\mathbf{R}} e^{i n_\alpha(\lambda) x_a \cdot \omega} \frac{g(\lambda)}{n_\alpha(\lambda)^{1/2}} d\lambda ,$$

where $n_\alpha(\lambda) = \sqrt{\lambda - E_\alpha}$ and $x_a \cdot \omega$ denotes the scalar product of $x_a$ and $\omega$. The right-hand side of (1.4) should be understood as follows: Let $\chi_R(\cdot) = \chi(\cdot/R)$ be a family of cut-off functions in $y = x_a - (x_a \cdot \omega)\omega$ variables with $\chi(0) = 1$. For example, we can take $\chi(y) = e^{-y^2}$ ([5]). If the limit

$$\lim_{R \to \infty} \|(S_{\beta\alpha} - \delta_{\beta\alpha}) \chi_R g_\omega\|_b$$

exists, we put

$$\|(S_{\beta\alpha} - \delta_{\beta\alpha}) g_\omega\|_b = \lim_{R \to \infty} \|(S_{\beta\alpha} - \delta_{\beta\alpha}) \chi_R g_\omega\|_b .$$

Equation (1.4) means that if the above limit exists for all $g \in C_0^\infty(\mathbf{I}_\alpha)$, the total cross section $\sigma_{\beta\alpha}(\lambda, \omega)$ is defined as a positive distribution for $\lambda \in \mathbf{I}_\alpha$.

*Remark 1.1.* Actually, our definition differs from that of [5] by the scale of energy: They took $\lambda^2$ as the energy, while here we take $\lambda$ as the energy. For further discussions about the equivalence between this definition and the usual one for total scattering cross sections, we refer to [5, 14].

Now let $\alpha$ be a two-cluster scattering channel with non-threshold energy and $\beta$ be an arbitrary scattering channel. Assume the condition (1.3) with $\rho > (n_a + 1)/2$, $n_a = \dim X_a$. In [14], it is proved that the total cross section for the initial channel $\alpha$

$$\sigma_\alpha(\lambda, \omega) = \sum_{\text{all } \beta} \sigma_{\beta\alpha}(\lambda, \omega)$$

is a well defined continuous function for $(\lambda, \omega) \in (\mathbf{I}_\alpha \backslash \mathcal{T}) \times S_a$. Assume (1.3) for $\rho > (n_{ab} + 1)/2$ with $n_{ab} = \max\{n_a, n_b\}$. We shall show in Sect. 3 that for an arbitrary $\beta$, the total cross section, $\sigma_{\alpha\beta}(\lambda, \omega)$, from $\beta$ to $\alpha$ is a continuous function in $(\lambda, \omega)$ for any $\omega \in \Sigma_b$ and $\lambda > |E_\beta| \operatorname{ctg}^2 \theta_\omega$ ($\theta_\omega$ being the opening angle between $\omega$ and $S_b \backslash \Sigma_b$). See Theorem 3.1. We believe that $\sigma_{\alpha\beta}(\lambda, \omega)$ should be finite for all $(\lambda, \omega) \in (\mathbf{I}_{\alpha\beta} \backslash \mathcal{T}) \times \Sigma_a$. Here

$$\mathbf{I}_{\alpha\beta} = ]\max\{E_\alpha, E_\beta\}, +\infty[ .$$

So far with the results on microlocal resolvent estimates of [19], we are only able to prove this conjecture for any $N$-body Schrödinger operator $P$ having the spectral structure of a three-body operator. See Remark 3.1.

Admitting the result on the finiteness of total cross sections, our results on the high energy asymptotics can be stated in the following two theorems.

**Theorem 1.1.** *Let $\alpha = (a, E_\alpha, \phi_\alpha)$ be a two-cluster channel with non-threshold energy. Assume the condition* (1.3) *with $\rho > (n_a + 1)/2$. Put*

$$\eta_a = \begin{cases} \frac{1}{2}, & \text{if } \rho > \frac{n_a + 3}{2}, \\ \frac{1}{2}\left(\rho - \frac{n_a + 1}{2} - \varepsilon\right), & \text{if } \frac{n_a + 1}{2} < \rho \leqq \frac{n_a + 3}{2}, \end{cases} \tag{1.5}$$

*for any $\varepsilon > 0$. Then one has:*

$$\sigma_\alpha(\lambda, \omega) = \frac{1}{4\lambda} \int_{X^a} |\phi_\alpha(x^a)|^2 \int_{\Pi_\omega} \left( \int_{\mathbf{R}} I_a(x^a, y + s\omega) ds \right)^2 dy\, dx^a + O(\lambda^{-1-\eta_a}), \tag{1.6}$$

*as $\lambda \to \infty$, uniformly in $\omega \in S_a$. Here $y = x_a - (x_a \cdot \omega)\omega$ and $\Pi_\omega = \{x_a; x_a \cdot \omega = 0\}$.*

*Remark 1.2.* (i) The high energy asymptotics of total scattering cross sections in two body scattering can be studied by Born approximation, which is essentially a perturbation theory around the free hamiltonian. See [10]. This, however, does not apply to many-body problems, because in the later case, the intercluster interactions do not decay on the whole configuration space $X$. To overcome this difficulty, we shall use microlocal resolvent estimates obtained in [19] (see Sect. 2). For rigorous results on total cross sections at high energies in two-body scattering, see [6, 9, 17, 21] and the references therein.

(ii) The semiclassical asymptotics of total cross sections, $\sigma_\alpha(\lambda, \omega, h)$ ($h \to 0$), with an initial two-cluster scattering channel is established in [14]. Seeing the relation $P - \lambda = \lambda(-h^2 \Delta + h^2 \sum V_a(x^a) - 1)$ with $h = \lambda^{-1/2}$, the reader may ask if one can directly apply the result (or the methods) of [14] in the semiclassical limit to obtain high energy asymptotics for $\sigma(\lambda, \omega)$. To answer this question, we just indicate that in the semiclassical limit, the contribution from small impact parameter $y$ (compared with $O(h^{-1})$) is negligible, while in high energy asymptotics, this contribution gives the leading term in (1.6). In addition, the final result in [14] is written down in terms of $I_a(0, x_a)$, while the substitution of $I_a(x^a, x_a)$ by $I_a(0, x_a)$ in (1.6) gives an error of the order $O(\lambda^{-1})$. Our viewpoint is that high energy asymptotics for total cross sections should be simpler than the semiclassical ones and with microlocal resolvent estimates established in [19], we can obtain better remainder estimates. Note that microlocal resolvent estimates are not needed in [14].

**Theorem 1.2.** *Let $\alpha = (a, E_\alpha, \phi_\alpha)$ be a two-cluster channel with non-threshold energy and $\beta = (b, E_\beta, \phi_\beta)$ be an arbitrary channel. Assume the condition* (1.3) *with $\rho > (n_{ab} + 1)/2$, $n_{ab} = \max\{n_a, n_b\}$. Then the following results hold.*

(i) *If $a = b$,*

$$\sigma_{\alpha\beta}(\lambda, \omega) = \frac{1}{4\lambda} \int_{\Pi_\omega} \left| \int_{\mathbf{R}} \int_{X^a} I_a(x^a, y + s\omega)\phi_\beta(x^a)\overline{\phi_\alpha(x^a)}\, dx^a ds \right|^2 dy + O(\lambda^{-1-\eta_a}), \quad (1.7)$$

*as $\lambda \to \infty$, for each $\omega \in \Sigma_b$.*

(ii) *If $a \neq b$,*

$$\sigma_{\alpha\beta}(\lambda, \omega) = O(\lambda^{-1-2\eta_b}), \quad \lambda \to \infty . \quad (1.8)$$

*Here $\eta_b$ is defined as $\eta_a$ with a replaced by b and the estimates* (1.7) *and* (1.8) *are locally uniform in $\omega \in \Sigma_b$.*

Note that in (i) of Theorem 1.2, $\beta$ may be a two-cluster channel with threshold energy. The result of (ii) of Theorem 1.2 shows that in high energy scattering, the probability for the particles to change clusters is small if the potentials are bounded.

*Remark 1.3.* (i) Equations (1.6) and (1.7) can be rewritten as follows. Define

$$I^\omega(x^a, y) = \int_{\mathbf{R}} I_a(x^a, y + s\omega) ds .$$

Then, (1.6) and (1.7) become

$$\sigma_\alpha(\lambda, \omega) = \frac{1}{4\lambda} \int_{\Pi_\omega} \| I^\omega(\cdot, y)\phi_\alpha \|^2_{L^2(X^a)} dy + O(\lambda^{-1-\eta_a}) , \quad (1.9)$$

$$\sigma_{\alpha\beta}(\lambda, \omega) = \frac{1}{4\lambda} \int_{\Pi_\omega} |\langle I^\omega(\cdot, y)\phi_\alpha, \phi_\beta \rangle_{L^2(X^a)}|^2 dy + O(\lambda^{-1-\eta_a}) . \quad (1.10)$$

Roughly speaking, this shows that when $a = b$, the leading term of $\sigma_{\beta\alpha}(\lambda, \omega)$ as $\lambda \to \infty$ is determined by the $\beta$-channel projection of the effective potential $I^\omega(x^a, y)\phi_\alpha(x^a)$.

(ii) If the potentials have local singularities, the methods of the proof for Theorems 1.1 and 1.2 still allow to give the leading term in high energy asymptotics for $\sigma_\alpha(\lambda, \omega)$ and $\sigma_{\alpha\beta}(\lambda, \omega)$. See the remark at the end of Sect. 4.

(iii) One can also study the asymptotics of total cross sections for $N$-body Schrödinger operators with a coupling constant $g \geqq 1$: $P(g) = -\Delta + gV$ in the regime $g/\lambda^{1/2} \to 0$. See[10] and [21] for two-body problems. To do this, we need to establish microlocal resolvent estimates as in [19] for $P(g)$ with $g = o(\sqrt{\lambda})$ as $\lambda \to \infty$. This is possible at least for bounded potentials, because for the conjugate operator constructed in [19], we have an explicit lower bound for the commutator. But we shall not go into details in this direction.

The main ideas in the proofs of Theorems 1.1 and 1.2 consist in using eikonal approximation (see also [10, 13, 17, 21] in the two-body case and [8, 14] in semi-classical asymptotics in the many-body case) to obtain the leading term and applying the microlocal resolvent estimates of [19] to establish precise remainder estimates. While we believe that the remainder estimates in (1.6) and (1.7) are optimal at least for $\rho > (n_a + 3)/2$ or $\rho > (n_{ab} + 3)/2$, it is still unknown in (ii) of Theorem 1.2 whether one can prove $\sigma_{\alpha\beta}(\lambda, \omega) = O(\lambda^{-\infty})$ or one can find a non-vanishing leading term at a finite order of $\lambda^{-1}$. The method of eikonal approximation used in this work does not allow us to answer this question.

The plan of this work is as follows: In Sect. 2, we recall from [19] some results on microlocal resolvent estimates. Section 3 is devoted to studying the finiteness of $\sigma_{\alpha\beta}(\lambda, \omega)$ when $\omega \in \Sigma_b$ and to establishing a representation formula for $\sigma_{\alpha\beta}(\lambda, \omega)$ when $\lambda$ is large. In Sect. 4, we prove Theorem 1.1 and in Sect. 5, we give a general upper bound for $\sigma_{\alpha\beta}(\lambda, \omega)$. The proof of Theorem 1.2 is completed in Sect. 6.

The results of this work are announced in [20].

## 2. Microlocal Resolvent Estimates

In this section, we recall some results on microlocal resolvent estimates on $N$-body Schrödinger operators which will play an important role in this work. We refer to [18, 19] for the proofs of these results.

Let $P$ be a generalized $N$-body Schrödinger operator: $P = -\Delta + \sum_{a \in \mathscr{A}} V_a(x^a)$. We write formally $V_a^0(x^a) = V_a(x^a)$ and $V_a^j(x^a) = (x^a \cdot \nabla^a) V_a^{j-1}(x^a)$, for $j = 1, 2, \ldots$ . To simplify the statement of our results, assume that the following conditions are satisfied for some $\rho > 0$, $R > 0$:

$$\begin{aligned} &\forall a \in \mathscr{A},\quad j \in \mathbf{N},\quad V_a^j(\cdot) \text{ is relatively compact in} \\ &L^2(X^a) \text{ with respect to } -\Delta^a \text{ and} \\ &\quad |\partial_y^\alpha V_a(y)| \leqq C_\alpha \langle y \rangle^{-\rho-|\alpha|}, \quad \forall \alpha \in \mathbf{N}^{n^a} \\ &\text{for } \quad |y| > R. \end{aligned} \tag{2.1}$$

Note that for physical $N$-body systems, Coulombic singularities are allowed in (2.1). Let $d(\lambda)$ denote the distance between $\lambda$ and $\mathscr{T} \cap ]-\infty, \lambda]$. Under the assumption (2.1), positive thresholds of $P$ are absent: $\mathscr{T} \subset ]-\infty, 0]$. See Theorem 4.19 in [4]. So one has for $\lambda > 0$, $d(\lambda) = \lambda$.

For $a \in \mathscr{A}$ with $a \neq a_{\max}$ and $\mu \in \mathbf{R}$, we denote by $S_\pm^a(\mu)$ the class of $\mu$-dependent bounded symbols, $a_\pm$, on $T^*X_a$ satisfying

$$\begin{aligned} &\operatorname{supp} a_\pm \subset \{(x_a, \xi_a);\ \pm x_a \cdot \xi_a \geqq \pm \mu \langle x_a \rangle\}, \\ &a_\pm \in C^\infty(T^*X_a),\quad |\partial_{x_a}^\alpha \partial_{\xi_a}^\beta a_\pm(x_a, \xi_a)| \leqq C_{\alpha\beta} \langle x_a \rangle^{-|\alpha|} \langle \xi_a \rangle^{-|\beta|}, \end{aligned} \tag{2.2}$$

uniformly in $\mu \in \mathbf{R}$.

In the following, we denote by $a(x, D)$ the pseudo-differential operator with symbol $a$ defined by

$$a(x, D)u(x) = \frac{1}{(2\pi)^d} \iint e^{i(x-y)\cdot\xi} a(x, \xi) u(y) dy d\xi$$

and by $R(z) = (P - z)^{-1}$ the resolvent of $P$.

Let $\mathscr{A}' = \{a \in \mathscr{A}; a \neq a_{\max}\}$. For $a \in \mathscr{A}'$ and $\eta > 0$, we define:

$$\Omega_a(\eta) = \{x; |x^a| < \eta|x| \text{ and } \forall b \not\subseteq a, |x^b| > \eta^2|x|\} \cup \{|x| < 1\} .$$

When $a = a_{\min}$, we shall write: $\Omega_a(\eta) = \Omega_0(\eta)$ to indicate the free cluster regions. Due to the geometrical structure of $N$-body systems, $\{\Omega_a(\eta); a \in \mathscr{A}'\}$ is a covering of the configuration space $X$, if $\eta > 0$ is small enough: $\bigcup_a \Omega_a(\eta) = X$.

**Theorem 2.1.** *Assume the condition* (2.1). *For* $\varepsilon > 0$, *denote by* $\mathscr{T}^\varepsilon$ *an* $\varepsilon$*-neighbourhood of* $\mathscr{T}$. *Then for any* $\varepsilon > 0$, *the following results hold uniformly in* $\kappa \in ]0, 1]$.

(i) *For any* $l \in \mathbf{N}$ *and* $s > l - 1/2$, *there exists* $C > 0$ *such that*

$$\|\langle x\rangle^{-s} R(\lambda \pm i\kappa)^l \langle x\rangle^{-s}\| \leqq C\langle\lambda\rangle^{-l/2}, \quad \forall \lambda \notin \mathscr{T}^\varepsilon . \tag{2.3}$$

(ii) *For any* $a \in \mathscr{A}'$, $l \in \mathbf{N}$, $s > l - 1/2$, $J_a$ *a bounded function with support contained in* $\Omega_a(\eta)$ *and* $b^a_\pm \in S^a_\pm(\mu_\pm)$, *there exists* $C > 0$ *such that*

$$\|\langle x\rangle^{s-l} J_a(x) b^a_\mp(x_a, D_a)(R(\lambda \pm i\kappa))^l \langle x\rangle^{-s}\| \leqq C\langle\lambda\rangle^{-l/2}, \quad \forall \lambda \notin \mathscr{T}^\varepsilon , \tag{2.4}$$

*uniformly in* $\mp \mu_\pm < (1 - \varepsilon)(d(\lambda))^{1/2}$.

(iii) *For any* $a, a' \in \mathscr{A}'$, $J_a$ *and* $J_{a'}$ *bounded functions supported in* $\Omega_a(\eta)$ *and* $\Omega_{a'}(\eta)$, *respectively, and for any* $l \in \mathbf{N}$, $s, r \in \mathbf{R}$ *and* $b^c_\pm \in S^c_\pm(\mu_\pm)$ *with* $\mu_+ > \mu_- + \varepsilon(d(\lambda))^{1/2}$ *for* $c = a$ *or* $a'$, *there exists* $C > 0$, *such that*

$$\|\langle x\rangle^s J_a(x) b^a_\mp(x_a, D_a)(R(\lambda \pm i\kappa))^l J_{a'}(x) b^{a'}_\pm(x_{a'}, D_{a'})\langle x\rangle^r\| \leqq C\langle\lambda\rangle^{-l/2}, \quad \lambda \notin \mathscr{T}^\varepsilon , \tag{2.5}$$

*uniformly in* $\pm \mu_\mp < (1 - \varepsilon)(d(\lambda))^{1/2}$.

In Theorem 2.1, we control the support of symbols in terms of the energy. In some cases, this allows to replace the microlocalizations with symbols in $S^a_\pm(\mu_\pm)$ by those with support in a largest possible outgoing or incoming region. For this purpose, let us introduce another class of symbols on $T^*X_a$. Let $S^a_\pm$ be the class of all smooth bounded symbols on $T^*X_a$ with the following support property: $b_\pm \in S^a_\pm$ iff there exist $\varepsilon > 0$ and $d > 0$ such that

$$\operatorname{supp} b_\pm \subset \{(x_a, \xi_a); \pm x_a \cdot \xi_a > -(1 - \varepsilon)|x_a||\xi_a| \text{ and } |\xi_a| \geqq d\} .$$

Introduce two subclasses of cluster decompositions in $\mathscr{A}$:

$$\mathscr{A}^+ = \{a \in \mathscr{A}; P^a \geqq 0\}$$

and

$$\mathscr{A}^+_2 = \{a \in \mathscr{A}; \#a = 2 \text{ and } \sigma_{\text{ess}}(P^a) = [0, +\infty[\} .$$

The following theorem shows that if $a \in \mathscr{A}^+ \cup \mathscr{A}^+_2$, we can replace the symbols $b^a_\pm \in S^a_\pm(\mu)$ by $b^a_\pm \in S^a_\pm$.

**Theorem 2.2.** *Assume the condition* (2.1).

(i) *For* $a \in \mathscr{A}$, *let* $J_a$ *be a bounded function supported in* $\Omega_a(\eta)$ *for some* $\eta > 0$. *Let* $b^a_\pm \in S^a_\pm$, *if* $a \in \mathscr{A}^+ \cup \mathscr{A}^+_2$, *and* $b^a_\pm \in S^a_\pm(\mu_\pm)$ *with* $\pm \mu_\pm > -(1 - \varepsilon)d(\lambda)^{1/2}$, *if* $a \notin \mathscr{A}^+ \cup \mathscr{A}^+_2$. *For any* $l \in \mathbf{N}$, $\varepsilon > 0$ *and* $s > l - 1/2$, *there exists* $C > 0$ *such that*

$$\|\langle x\rangle^{s-l} J_a(x) b^a_\mp(x_a, D_a)(R(\lambda \pm i\kappa))^l \langle x\rangle^{-s}\| \leqq C\langle\lambda\rangle^{-l/2}, \quad \forall \lambda \notin \mathscr{T}^\varepsilon , \tag{2.6}$$

*uniformly in* $0 < \kappa < 1$.

(ii) *For any* $a, a' \in \mathscr{A}$, *let* $J_c$ *and* $b^c_\pm$ *be taken as in* (i) *according to* $c \in \mathscr{A}^+ \cup \mathscr{A}_2^+$ *or* $c \notin \mathscr{A}^+ \cup \mathscr{A}_2^+$ *for* $c = a$ *or* $a'$. *Assume in addition that* $(b^a_-, b^{a'}_+)$ *and* $(b^a_+, b^{a'}_-)$ *are pairs of symbols with the property of disjoint support. Then, for any* $l \in \mathbf{N}$, $\varepsilon > 0$, $s, r \in \mathbf{R}$, *there exists* $C > 0$, *such that*

$$\| \langle x \rangle^s J_a(x) b^a_\mp(x_a, D_a)(R(\lambda \pm i\kappa))^l b^{a'}_\pm(x_{a'}, D_{a'}) J_{a'}(x) \langle x \rangle^r \| \leqq C \langle \lambda \rangle^{-l/2}, \quad \lambda \notin \mathscr{T}^\varepsilon , \tag{2.7}$$

*uniformly in* $0 < \kappa < 1$.

We refer to Sect. 2 in [19] for the definition of the notion of pairs of symbols with the property of disjoint support.

*Remark 2.1.* If an $N$-body Schrödinger operator has the spectral structure of three-body Schrödinger operators (i.e., there is no negative eigenvalue for any sub-Hamiltonian $P^a$ with $\# a \geqq 3$), then, $\mathscr{A}^+ \cup \mathscr{A}_2^+ = \mathscr{A}$. In this case Theorem 2.2 gives complete sharp microlocal resolvent estimates. In general case, our results are almost optimal in high energy estimates.

**Theorem 2.3.** *Assume the condition* (2.1).

(i) *For any* $\varepsilon > 0$, *there exists* $\lambda_0 > 0$ *such that for any* $a \in \mathscr{A}$, $b^a_\pm \in S^a_\pm$ *with*

$$\operatorname{supp} b^a_\pm \subset \{ \mp x_a \cdot \xi_a \leqq (1 - \varepsilon)|x_a||\xi_a| \} , \tag{2.8}$$

*estimates* (2.4) *hold for* $\lambda \geqq \lambda_0$.

(ii) *For any* $\varepsilon > 0$, *there exists* $\lambda_0 > 0$ *such that for any* $a, a' \in \mathscr{A}$ *and for any* $b^c_\pm \in S^c_\pm$, $c = a, a'$, *such that* $(b^a_\pm, b^{a'}_\mp)$ *are pairs of symbols with the property of disjoint support and that* (2.8) *is satisfied for supp* $b^c_\pm$, $c = a, a'$. *Then estimates* (2.5) *hold for* $\lambda \geqq \lambda_0$.

Theorems 2.1–2.3 are proved in [19] under less restrictive conditions on the regularity and the decay of potentials.

## 3. Finiteness of Total Cross Sections $\sigma_{\alpha\beta}(\lambda, \omega)$

Let $\alpha = (a, E_\alpha, \phi_\alpha)$ be a two-cluster channel with non-threshold energy. Let $\beta = (b, E_\beta, \phi_\beta)$ be an arbitrary scattering channel. The finiteness of total cross section with initial two-cluster scattering channel $\alpha$, $\sigma_\alpha(\lambda, \omega) = \sum_\beta \sigma_{\beta\alpha}(\lambda, \omega)$, is studied in [2, 3, 5, 14]. In [2, 3], an average was taken over all incident directions, while in [5] an average was taken over any energy interval. In [14], the pointwise finiteness of $\sigma_\alpha(\lambda, \omega)$ is proved for each $(\lambda, \omega) \in (\mathbf{I}_\alpha \setminus \mathscr{T}) \times S_a$. Here we want to give a pointwise meaning to the total scattering cross sections $\sigma_{\alpha\beta}(\lambda, \omega)$ from an arbitrary initial channel to a two-cluster channel with non-threshold energy defined through (1.4).

Fix an incident direction $\omega \in \Sigma_b$. Define $\theta_\omega$ by

$$\theta_\omega = \inf \left\{ \theta \in ]0, \pi/2]; \exists \omega' \in S_b \cap \left( \bigcup_{c \not\subset b} X_c \right) \text{ with } \cos\theta = |\omega \cdot \omega'| \right\} . \tag{3.1}$$

Since $\omega \in \Sigma_b$, $\theta_\omega > 0$ and for any $\varepsilon > 0$, the intersection between $\bigcup_{c \not\subset b} X_c$ and the cone $\{x; |x \cdot \omega| > \cos(\theta_\omega - \varepsilon)\}$ is void.

For a given initial channel $\beta = (b, E_\beta, \phi_\beta)$ and $\omega \in \Sigma_b$, we construct microlocalizations in the following way. Let $\tau_{\beta,\omega} = |E_\beta|\,|\mathrm{ctg}\,\theta_\omega|^2$. For $\lambda_0 > -E_\beta + \tau_{\beta,\omega}$, take $\delta > 0$ so small that

$$[(1-2\delta)\lambda_0, (1+2\delta)\lambda_0] \subset\ ]\tau_{\beta,\omega} - E_\beta, +\infty[\,.$$

Let $I = [(1-\delta)\lambda_0, (1+\delta)\lambda_0]$. For $\varepsilon > 0$ sufficiently small, take $\chi_b \in C_0^\infty(X_b^*)$ with

$$\chi_b(\xi_b) = \begin{cases} 1, & \text{for } |\xi_b|^2 \in I \text{ and } |\xi_b/|\xi_b| - \omega| < \varepsilon\,; \\ 0, & \text{for } |\xi_b|^2 \notin [(1-2\delta)\lambda_0, (1+2\delta)\lambda_0] \text{ or } |\xi_b/|\xi_b| - \omega| > 2\varepsilon\,. \end{cases}$$

Let $j \in C_0^\infty(\mathbf{R})$ with $j(t) = 0$ if $t < 1/2$ and $j(t) = 1$ if $t \geqq 1$. Put:

$$J_1(x) = \prod_{c \not\subseteq b} j\left(\frac{|x^c|}{\delta|x|}\right),$$

and

$$J_b(x) = J_1(x) j(|x|) + (1 - j(|x|))\,. \tag{3.2}$$

On the support of $\nabla J_b = \nabla J_1 j + (J_1 - 1)\nabla j$, there exists at least one $c \not\subseteq b$ with $|x^c| \leqq \delta|x|$. Recall that by the choice of $\theta_\omega$,

$$\min_{x;\, c \not\subseteq b} \{|\pi^c \hat{x}|;\, |\hat{x}\cdot\omega| \geqq \cos(\theta_\omega - \varepsilon)\} > 0\,.$$

Here $\hat{x} = x/|x|$. Taking

$$0 < \delta < \min_{x;\, c \not\subseteq b} \{|\pi^c \hat{x}|;\, |\hat{x}\cdot\omega| \geqq \cos(\theta_\omega - \varepsilon)\}$$

sufficiently small, the support of $\nabla J_b$ is disjoint from the cone $\{x;\, |\hat{x}\cdot\omega| \geqq \cos(\theta_\omega - \varepsilon)\}$. Consequently, for $x \in \operatorname{supp} \nabla J_b$ and $\xi_b \in \operatorname{supp} \chi_b$, one has

$$|\xi_b - |\xi_b|\omega| < 2\varepsilon|\xi_b|\,,$$

$$|x_b \cdot \xi_b| < (2\varepsilon|x_b| + |x_b\cdot\omega|)|\xi_b| \leqq (2\varepsilon + \cos(\theta_\omega - \varepsilon))|x_b|\sqrt{(1+2\delta)\lambda_0}\,.$$

Now let $\lambda_1 = \lambda_0 + E_\beta$. It can be checked that

$$(\varepsilon + \cos(\theta_\omega - \varepsilon))(1+2\delta)^{1/2}\left(1 - \frac{E_\beta}{\lambda_1}\right)^{1/2} \leqq (1-\varepsilon)\,,$$

for $\lambda_1 > (1-2\delta)^{-1}\tau_{\beta,\omega}$, if $\varepsilon, \delta > 0$ are chosen sufficiently small. This proves that on the support of $\nabla J_b(x)\chi_b(\xi_b)$,

$$|x_b \cdot \xi_b| \leqq (1-\varepsilon)\lambda_1^{1/2}|x_b| \leqq (1-\varepsilon')d(\lambda)^{1/2}|x_b|, \quad \varepsilon' > 0\,, \tag{3.3}$$

for $\lambda \in \mathbf{I}_1 \equiv [(1-\delta)\lambda_1, (1+\delta)\lambda_1]$ if $\delta > 0$ is small. Choosing appropriately $\chi_b$, we can verify that

$$\nabla J_b(\cdot)\chi_b(\cdot) \in S_+^b(\mu_+) \cap S_-^b(\mu_-), \quad \lambda \in \mathbf{I}_1\,.$$

Here $\mu_\pm = \mp(1-\varepsilon')d(\lambda)^{1/2}$. Therefore, we can apply the results in Theorem 2.1 to this microlocalization and obtain, for instance, for any $s > 1/2$,

$$\|\langle x\rangle^s \nabla J_b(x)\chi_b(D_b)R(\lambda \pm i0)\langle x\rangle^{-s}\| \leqq C\langle\lambda\rangle^{-1}, \quad \lambda \in \mathbf{I}_1, \tag{3.4}$$

for all $\lambda_1 > \tau_{\beta,\omega}$. Remark that if $b \in \mathscr{A}^+ \cup \mathscr{A}_2^+$, (3.4) is true for all $\lambda \notin \mathscr{T}^\varepsilon$, because

$$\nabla J_b(\cdot)\chi_b(\cdot) \in S_+^b \cap S_-^b, \quad \forall \lambda .$$

Our result on the finiteness of $\sigma_{\alpha\beta}(\lambda, \omega)$ is following

**Theorem 3.1.** *Assume the condition* (2.1) *with* $\rho > (n_{ab} + 1)/2$. *Let* $\sigma_{\alpha\beta}(\lambda, \omega)$ *denote the total cross section from an arbitrary channel* $\beta$ *to a two-cluster channel* $\alpha$ *with non-threshold energy. For each* $\omega \in \Sigma_b$, *let* $\lambda_1 > \tau_{\beta,\omega}$. *Construct* $J_b, \chi_b, \mathbf{I}_1$ *as above. Then the total scattering cross section from* $\beta$ *to* $\alpha$ *is finite on* $\mathbf{I}_1$ *and one has, for any* $g \in C_0^\infty(\mathbf{I}_1)$,

$$\int \sigma_{\alpha\beta}(\lambda, \omega)|g(\lambda)|^2 d\lambda$$

$$= \int \pi \| F_\alpha(\lambda) \mathscr{J}_\alpha^* \{1 - I_a R(\lambda + i0)\} Q_b e_\beta(\lambda, \omega) \|_{L^2(S_a)}^2 \frac{|g(\lambda)|^2}{(\lambda - E_\beta)^{1/2}} d\lambda . \tag{3.5}$$

*Here* $F_\alpha(\lambda)$ *is the spectral representation for* $-\Delta_a + E_\alpha$ *(see* (3.9)),

$$e_\beta(\lambda, \omega) = \phi_\beta(x^b) e^{i\sqrt{\lambda - E_\beta}\,\omega \cdot x_b}$$

*and* $Q_b$ *is defined by*

$$Q_b = I_b(x) J_b(x) \chi_b(D_b) + [-\Delta, J_b(x)] \chi_b(D_b) . \tag{3.6}$$

*In particular, the total cross section* $\sigma_{\alpha\beta}(\lambda, \omega)$ *defined by* (1.4) *extends to a continuous function in* $(\lambda, \omega)$ *for* $(\lambda, \omega) \in \{(\mu, \theta) \in \mathbf{R} \times S_b; \theta \in \Sigma_{b,\mu} > \tau_{\beta,\theta}\}$:

$$\sigma_{\alpha\beta}(\lambda, \omega) = \frac{\pi}{(\lambda - E_\beta)^{1/2}} \| F_\alpha(\lambda) \mathscr{T}_\alpha^* \{1 - I_a R(\lambda + i0)\} Q_b e_\beta(\lambda, \omega) \|_{L^2(S_a)}^2 . \tag{3.7}$$

*Remark 3.1.* (a) Under the condition of Theorem 3.1, we expect that for any initial channel $\beta$, $\sigma_{\alpha\beta}(\lambda, \omega)$ is finite for any $(\lambda, \omega) \in (\mathbf{I}_{\alpha\beta} \backslash \mathscr{T}) \times \Sigma_b$. The result of Theorem 3.1 implies that this is true if $\beta$ is a scattering channel with energy 0: $E_\beta = 0$. More generally, if $b \in \mathscr{A}^+ \cup \mathscr{A}_2^+$, (3.4) is true for any $\lambda \notin \mathscr{T}^\varepsilon$. We can then apply the method of the proof for Theorem 3.1 to prove that for any $b \in \mathscr{A}^+ \cup \mathscr{A}_2^+$, $\sigma_{\alpha\beta}(\lambda, \omega)$ is finite for $(\lambda, \omega) \in (\mathbf{I}_{\alpha\beta} \backslash \mathscr{T}) \times \Sigma_b$ and is given by (3.7). In particular, this shows that if $P$ is an $N$-body Schrödinger operator with a three-body spectral structure, then $\forall \beta$, $\sigma_{\alpha\beta}(\lambda, \omega)$ is finite for any $(\lambda, \omega) \in (\mathbf{I}_{\alpha\beta} \backslash \mathscr{T}) \times \Sigma_b$. Our conjecture may be compared with that on the smoothness of scattering amplitudes raised in [16]. The methods of the present work show that to study total cross sections, one can avoid studying the properties of scattering amplitudes.

(b) For $\omega \in S_b \backslash \Sigma_b = S_b \cap (\bigcup_{c \not\subseteq b} X_c)$, the finiteness of $\sigma_{\alpha\beta}(\lambda, \omega)$ is a subtle question. To see this, let us recall that the structure of the scattering amplitude, $S_{0\alpha}(\lambda; \omega, \omega')$, from a two-cluster channel with non-threshold energy to the free channel in three-body scattering was studied in [7]. Since $S_{\alpha 0}$ is formally equal to $S_{0\alpha}^*$ with a reverse of the time, one can apply the same method of [7] to verify that $S_{\alpha 0}(\lambda; \omega', \omega)$ has the same structure as $\overline{S_{0\alpha}(\omega, \omega')}$. As a consequence of Theorem 1.1 in [7], we see that $S_{\alpha 0}(\lambda; \omega, \omega')$ is continuous for $(\omega, \omega') \in S_a \times \Sigma_b$ with $b = a_{\min}$ and when $\omega' \in \Sigma_b$, $\omega' \to S_b \cap X_c$ for some $c \in \mathscr{A}$ with $\# c = 2$, $S_{\alpha 0}(\lambda; \omega, \omega')$ has a singularity of the form:

$$S_{\alpha 0}(\lambda; \omega, \omega') = |\omega'^c|^{-1} A_{-1} + A_0, \quad \text{as } \omega'^c = \pi^c \omega' \to 0 , \tag{3.8}$$

where $A_{-1}$ and $A_0$ are continuous functions and $A_{-1} = 0$ if 0 is neither eigenvalue nor resonance of $P^c$. Therefore, we can conclude that if 0 is neither eigenvalue nor resonance of $P^c$, the total scattering cross section $\sigma_{\alpha\beta}(\lambda, \omega)$ ($\beta$ being the free channel) extends to a continuous function near $S_b \cap X_c$ and if 0 is an eigenvalue or a resonance of $P^c$, $\sigma_{\alpha\beta}(\lambda, \omega)$ tends to infinity when $\omega^c \to 0$.

The proof of Theorem 3.1 is divided into three steps.

(a) As the first step, we study the spectral representation of the scattering matrices for $T_{\alpha\beta}$.

Let $F_\beta \colon L^2(X_b) \to \mathbf{H}_\beta \equiv L^2(\mathbf{I}_\beta; L^2(S_b))$ be defined by:

$$(F_\beta f)(\lambda, \theta) = c_b(\lambda) \int e^{-i\sqrt{(\lambda - E_\beta)}\theta \cdot x_b} f(x_b) dx_b \,, \tag{3.9}$$

where

$$c_b(\lambda) = (2\pi)^{-n_b/2}(\lambda - E_\beta)^{(n_b - 2)/4}$$

with $n_b = \dim X_b$. Then we can verify that $\|F_\beta f\|_{\mathbf{H}_\beta} = \|f\|_b$. Put $\mathscr{F}_\beta = F_\beta \mathscr{J}_\beta^*$. Then $\mathscr{F}_\beta P_b \mathscr{F}_\beta^*$ acts as the multiplication by $\lambda$ in $\mathbf{H}_\beta$. By Sobolev's lemma, $F_\beta$ defines a family of maps, $F_\beta(\lambda)$, $\lambda \in \mathbf{I}_\beta$, from $L^{2,s}(X_b)$, $s > 1/2$, to $L^2(S_b)$:

$$(F_\beta(\lambda) f)(\theta) = (F_\beta f)(\lambda, \theta) \,.$$

Here $L^{2,s}$ is the weighted $L^2$ space $L^{2,s}(X_b) = L^2(X_b, \langle x_b \rangle^{2s} dx_b)$. Similarly, we can construct a spectral representation $\mathscr{F}_\alpha$ for $P_a$. Then $F_\alpha T_{\alpha\beta} F_\beta^*$ can be represented by a family of operators $\{T_{\alpha\beta}(\lambda) = F_\alpha(\lambda) T_{\alpha\beta} F_\beta(\lambda)^*; \lambda \in \mathbf{I}_{\alpha\beta}\}$ mapping $L^2(S_b)$ to $L^2(S_a)$. Note that $T_{\alpha\beta}(\lambda)$ is a priori only defined a.e. in $\lambda$.

**Proposition 3.2.** *Assume the condition* (2.1) *for some* $\rho > 1$. *For* $\omega \in \Sigma_b$, *let* $\chi$, $J_b$ and $\mathbf{I}_1$ *be constructed as before. For any* $f_c \in \mathscr{S}(X_c)$ *with* $c = a$ *or* $b$, *we denote*: $f_b(\lambda, \theta) = (F_\beta f_b)(\lambda, \theta)$ *and* $f_a(\lambda, \theta') = (F_\alpha f_a)(\lambda, \theta')$. *Assume that* $f_c(\lambda, \cdot) = 0$ *for* $\lambda$ *outside* $\mathbf{I}_1$. *Then one has*:

$$\langle T_{\alpha\beta} \chi(D_b) f_b, f_a \rangle_a = \int_{\mathbf{I}_1} \langle T^\chi_{\alpha\beta}(\lambda) f_b(\lambda, \cdot), f_a(\lambda, \cdot) \rangle_{L^2(S_a)} d\lambda \,, \tag{3.10}$$

*where*

$$T^\chi_{\alpha\beta}(\lambda) = -2\pi i F_\alpha(\lambda) \mathscr{J}_\alpha^* \{Q_b - I_a R(\lambda + i0) Q_b\} \mathscr{J}_\beta F_\beta(\lambda)^* \,, \tag{3.11}$$

*for* $\lambda \notin \mathscr{T}$. *Here* $Q_b$ *is defined by* (3.6). *In particular, the localized scattering matrix* $T^\chi_{\alpha\beta}(\lambda) = F_\alpha(\lambda) T_{\alpha\beta} \chi(D_b) F_\beta(\lambda)^*$ *is continuous for* $\lambda \in \mathbf{I}_1$.

*Proof.* Notice that $J_b$ is supported in $\Omega_b = \{x; \forall c \not\subseteq b, |x^c| > \delta|x|\}$ for some suitable $\eta, \delta > 0$, such that $J_b(x) = 1$ for $|x_b \cdot \xi_b| > (1 - \varepsilon_0)|x_b||\xi_b|$, $\forall \xi_b \in \operatorname{supp} \chi$. Making use of microlocal propagation estimates for $U_b(t)\mathscr{J}_\beta = e^{-it(-\Delta_b + E_\beta)}\mathscr{J}_\beta$, we can verify:

$$\text{s-}\lim_{t \to \pm\infty} (1 - J_b)\chi(D_b) U_b(t) \mathscr{J}_\beta = 0 \,.$$

Therefore,

$$W_\beta^\pm \chi(D_b) = \text{s-}\lim_{t \to \pm\infty} U(t)^* J_b \chi(D_b) U_b(t) \mathscr{J}_\beta \,,$$

where $U(t) = e^{-itP}$. We denote $B = J_b \chi(D_b)$. Then

$$\begin{aligned} T_{\alpha\beta} \chi(D_b) f_b &= W_\alpha^{+*}(W_\beta^- - W_\beta^+)\chi(D_b) \\ &= -\int_{\mathbf{R}} W_\alpha^{+*} U(t)^* i Q_b U_b(t) \phi_\beta f_b dt \,, \end{aligned}$$

where

$$Q_b = PB - BP_b = I_b B + [-\Delta, J_b]\chi(D_b).$$

Since $\operatorname{supp} J_b \subset \Omega_b$, it is clear $I_b B = O(\langle x\rangle^{-\rho})$. Put

$$B_1 = [-\Delta, J_b]\chi(D_b).$$

The symbol of $B_1$ is supported in

$$\Omega_b \cap \{|x_b \cdot \xi_b| \leqq (1-\varepsilon_0)|x_b||\xi_b|\}, \tag{3.12}$$

and is of the order $O(\langle x\rangle^{-1})$. Introducing a convergent factor and making use of the intertwining property of the wave operator $W_\alpha^+$, we can verify that

$$\begin{aligned}
&\langle T_{\alpha\beta}\chi(D_b)f_b, f_a\rangle_a \\
&= -\int_{\mathbf{R}} \langle W_\alpha^{+*} U(t)^* iQ_b U_b(t)\phi_\beta f_b, f_a\rangle_a dt \\
&= -\lim_{\varepsilon\to 0}\int_{\mathbf{R}} \left\langle \mathscr{J}_\alpha^* U_a(t)^* \left\{1 - i\int_{\mathbf{R}_+} e^{-\varepsilon s} U_a(s)^* I_a U(s)ds\right\} iQ_b U_b(t)\phi_\beta f_b, f_a\right\rangle_a dt \\
&= -2\pi \lim_{\varepsilon\to 0}\int \langle F_\alpha(\lambda)\mathscr{J}_\alpha^*\{1 - I_a R(\lambda + i\varepsilon)\} iQ_b \mathscr{J}_\beta F_\beta(\lambda)^* f_b(\lambda,\cdot), f_a(\lambda,\cdot)\rangle_{L^2(S_a)} d\lambda.
\end{aligned} \tag{3.13}$$

Applying (3.4), one sees that the last limit in the above equations exists. This proves (3.10). Let $s = \rho/2 > 1/2$. Then

$$\lambda \to \mathscr{J}_\beta F_\beta(\lambda)^* \in \mathscr{L}(L^2(S_b); L^{2,-s}(X))$$

is continuous. Since $\phi_\alpha$ is rapidly decreasing in $x^a$, we can see that $\mathscr{J}_\alpha Q_b \langle x\rangle^s = O(\langle x_a\rangle^{-s})$. Therefore,

$$\lambda \to F_\alpha(\lambda)\mathscr{J}_\alpha^* Q_b \mathscr{J}_\beta F_\beta(\lambda)$$

is continuous for $\lambda \in \mathbf{I}_{\alpha\beta}$. By (a) of Theorem 2.1 and (3.4) with $s = \rho/2$, it is easy to see that

$$\lambda \to F_\alpha(\lambda)\mathscr{J}_\alpha^* I_a R(\lambda + i\varepsilon) Q_b \mathscr{J}_\beta F_\beta(\lambda)^*$$

is continuous for $\lambda \in \mathbf{I}_1$. From this it follows that $\lambda \to T_{\alpha\beta}^\chi(\lambda) \in \mathscr{L}(L^2(S_b); L^2(S_a))$ is continuous for $\lambda \in \mathbf{I}_1$. □

(b) The second step of the proof of Theorem 3.1 is to insert a localization by $\chi(D_b)$ in the definition (1.4), if $\operatorname{supp} g \subset \mathbf{I}_1$.

We shall work locally in $\lambda \in \mathbf{I}_1$. Let

$$\varepsilon_\beta = e^{in_\beta(\lambda)\omega\cdot x_b}, \quad \text{with } n_\beta(\lambda) = \sqrt{\lambda - E_\beta},$$

$\varepsilon_\beta \in L^{2,-s}(X_b)$ for any $s > n_b/2$. Since $\chi(D_b)$ is continuous on $L^{2,r}(X_b)$, for any $r \in \mathbf{R}$, and since $\chi(n_\beta(\lambda)\omega) = 1$ for $\lambda \in \mathbf{I}_1$ (by the choice of $\chi$), one can verify that for $\lambda \in \mathbf{I}_1$,

$$\chi(D_b)\varepsilon_\beta = \varepsilon_\beta, \quad \text{in } L^{2,-s}(X_b), \tag{3.14}$$

for any $s > n_b/2$. In the rest of this work, we shall freely use this relation.

**Lemma 3.3.** *Let $h_R$ be the family of cut-off functions defined by:*

$$h_R = e^{-(x_b - \langle x_b, \omega\rangle\omega)^2/R}.$$

*Then one has*:

$$\lim_{R\to\infty} (\chi(D_b)h_R g_\omega - h_R g_\omega) = 0, \quad in\ L^2(X_b)\,, \tag{3.15}$$

*for any* $g \in C_0^\infty(\mathbf{I}_1)$.

*Proof.* We assume that $\omega$ is pointed at the $x_1$ direction and write $x_b = (x_1, x')$. Let $\hat{f}$ denote the Fourier transform of $f$. Then the Fourier transform of $\chi(D_b)h_R g_\omega$ is equal to $\chi(\xi_b)\hat{h}_R(\xi')\hat{g}_\omega(\xi_1)$. We can compute:

$$\hat{g}_\omega(\xi_1) = c(\xi_1)g(\xi_1^2 + E_\beta)\,,$$

where $c(\xi_1)$ is some bounded function which we do not need to compute explicitly. By the choice of $\chi$, $\chi(\xi_b) = 1$ for $\xi_1$ in the support of $g(\xi_1^2 + E_\beta)$ and $|\xi'| < \varepsilon$ with $\varepsilon > 0$ small enough. If $|\xi'| > \varepsilon$, $|\hat{h}_R(\xi')| \leqq CR^{N_0}e^{-\varepsilon R}$, for some $N_0 > 0$. Therefore, $\lim_{R\to\infty} \|(1 - \chi(D_b))h_R g_\omega\|_b = 0$ for any $g \in C_0^\infty(\mathbf{I}_1)$. □

By Lemma 3.3, (1.4) is equivalent with the definition:

$$\int \sigma_{\alpha\beta}(\lambda, \omega)|g(\lambda)|^2 d\lambda = \|T_{\alpha\beta}\chi(D_b)g_\omega\|^2\,, \tag{3.16}$$

for any $g \in C_0^\infty(\mathbf{I}_1)$. The right-hand side of (3.16) is again taken as the limit:

$$\lim_{R\to\infty} \|T_{\alpha\beta}\chi(D_b)h_R g_\omega\|^2\,,$$

if it does exist.

(c) Finally, we can finish the proof of Theorem 3.1 by studying the right-hand side of (3.16) by means of the time-dependent method.

*Proof of Theorem 3.1.* To prove the finiteness of $\|T_{\alpha\beta}\chi(D_b)g_\omega\|$, we use the equality

$$\|T_{\alpha\beta}\chi(D_b)g_\omega\|^2 = \int_{\mathbf{I}_{\alpha\beta}} \|T_{\alpha\beta}^\chi(\lambda)(F_\beta g_\omega)(\lambda,\cdot)\|_{L^2(S_a)}^2 d\lambda$$

and the properties of localized scattering matrices.

By the microlocal resolvent estimate (3.4) and the decay assumption (2.1) on the potentials, one sees that $T_{\alpha\beta}^\chi(\lambda)$: $L^2(S_b) \to L^2(S_a)$ given by (3.11) extends to a bounded operator from $H^{-\rho+s}(S_b)$ to $L^2(S_a)$, for some $s > 1/2$. Here $H^r(S_b)$ is the Sobolev space on $S_b$ of order $r \in \mathbf{R}$. By a direct computation, one can verify that $F_\beta(\lambda)g_\omega$ is a distribution in $H^{-\rho'}(S_b)$, for any $\rho' > (n_b - 1)/2$ and $\lim_{R\to\infty} F_\beta(\lambda)(h_R g_\omega) = F_\beta(\lambda)g_\omega$ in $H^{-\rho'}(S_b)$ for any $\rho' > (n_b - 1)/2$. This shows that the right-hand side of (3.16) is finite.

To prove (3.5), we proceed to calculate

$$\int_{I_{\alpha\beta}} \|T_{\alpha\beta}^\chi(\lambda)F_\beta(\lambda)g_\omega\|_{L^2(S_a)}^2 d\lambda\,,$$

by the time-dependent method which is easier to justify. Let $\chi_1$ be a cut-off function on $\mathbf{R}$ so that $\chi_1(\xi_b^2 + E_\beta)\chi(\xi_b) = \chi(\xi_b)$. Using the intertwining properties of wave operators, we have:

$$T_{\alpha\beta}\chi(D_b)h_R g_\omega$$
$$= \int_{\mathbf{R}} \mathscr{J}_\alpha^* U_a(s)^* \left\{ -i\chi_1(P) - \int_{\mathbf{R}_+} U_a(t)^* I_a \chi_1(P) U(t)\,dt \right\} Q_b U_b(s)(\phi_\beta h_R g_\omega)\,ds\,.$$

Note that $F_\alpha(\lambda)\mathscr{J}_\alpha^* U_a(t) = e^{-it\lambda}F_\alpha(\lambda)\mathscr{J}_\alpha^*$ and that for any $\rho' > n_b/2$,

$$\lim_{R\to\infty} \langle x_b\rangle^{-\rho'} U_b(t)(\phi_\beta h_R g_\omega) = \int_{\mathbf{R}} e^{-i\lambda t}\langle x_b\rangle^{-\rho'} e_\beta(\lambda,\omega)\frac{g(\lambda)}{2(\pi)^{1/2}(\lambda-E_\beta)^{1/4}}\,d\lambda\,.$$

Making use of microlocal resolvent estimates established in [19] (see Sect. 2), we can compute,

$$\begin{aligned}&\lim_{R\to\infty} F_\alpha(\lambda)T_{\alpha\beta}\chi(D_b)h_R g_\omega\\
&= \int_{\mathbf{R}} e^{i\lambda s}F_\alpha(\lambda)\mathscr{J}_\alpha^*\left\{-i\chi_1(P) - \int_{\mathbf{R}_+} e^{i\lambda t}I_a\chi_1(P)U(t)dt\right\}\\
&\quad\times Q_b\int e^{-i\mu s}g(\mu)e_\beta(\mu,\omega)\frac{1}{2(\pi)^{1/2}(\mu-E_\beta)^{1/4}}\,d\mu\,ds\\
&= (\pi)^{1/2}F_\alpha(\lambda)\mathscr{J}_\alpha^*\left\{-i\chi_1(P) - \int_{\mathbf{R}_+} e^{i\lambda t}I_a\chi_1 U(t)dt\right\}Q_b g(\lambda)e_\beta(\lambda,\omega)\frac{1}{(\lambda-E_\beta)^{1/4}}\\
&= (\pi)^{1/2}F_\alpha(\lambda)\mathscr{J}_\alpha^*\{-i + iI_aR(\lambda+i0)\}Q_b e_\beta(\lambda,\omega)g(\lambda)\frac{1}{(\lambda-E_\beta)^{1/4}}\,.\end{aligned}$$

In the last equality, we used the fact that $\chi_1(\lambda) = 1$ on $\operatorname{supp} g$. We indicate in particular that the last expression is well defined. In fact, by the definition of $Q_b$ (see (3.6)), we can verify that

$$\mathscr{J}_\alpha^* Q_b e_\beta = O(\langle x_a\rangle^{-\rho}) + \langle B_1 e_\beta, \phi_\alpha\rangle_{L^2(X^a)}\,,$$

where $B_1 = [-\Delta, J_b]\chi(D_b)$. Recall that $\alpha$ is a two-cluster channel with non-threshold energy. If $a \not\subseteq b$, one has $|x^a| \geqq c|x|$ on supp $J_b$. If $a \subseteq b$, then, $b = a$, since $\# a = 2$. On the support of $\nabla J_b$, there exists a $c \not\subseteq b = a$ with

$$\delta|x|/2 \leqq |x^c| \leqq \delta|x|\,.$$

By the geometry of $N$-body systems, there exists $c_0 > 0$ such that $|x^a| + |x^c| \geqq c_0|x|$. This shows that on the range of $B_1$, one always has: $|x^a| \geqq c_1|x|$, $c_1 > 0$, if $\delta > 0$ is chosen small enough. Consequently, $F_\alpha(\lambda)\mathscr{J}_\alpha^* Q_b e_\beta \in L^2(S_a)$ and is continuous in $\lambda$. Applying the microlocal resolvent estimates (3.4) with $n_b/2 - 1 < s < \rho - 1 - 1/2$, we can also derive that $\mathscr{J}_\alpha^* I_a R(\lambda + i0)Q_b e_\beta(\lambda,\omega) \in L^{2,r}(X_a)$ for some $r > 1/2$ and is continuous in $\lambda$. This proves Theorem 3.1. □

## 4. High Energy Asymptotics of $\sigma_\alpha(\lambda, \omega)$

In this section, we assume that (1.3) is satisfied for some $\rho > (n_a + 1)/2$. Let $\alpha = (a, E_\alpha, \phi_\alpha)$ be a two-cluster scattering channel with non-threshold energy. Let us first recall a result from [14] on the pointwise finiteness of $\sigma_\alpha(\lambda,\omega) = \sum_\beta \sigma_{\alpha\beta}(\lambda,\omega)$ formally defined through (1.4). The finiteness of $\sigma_\alpha(\lambda,\omega)$ when integrated over the energy was proved in [5]. Let $g \in C_0^\infty(\mathbf{I}_\alpha)$ with $\operatorname{supp} g \cap \mathscr{T} = \emptyset$. Then one has:

$$\int \sigma_\alpha(\lambda,\omega)|g(\lambda)|^2 d\lambda = \int \frac{1}{n_\alpha(\lambda)}\Im\langle R(\lambda+i0)I_a e_\alpha, I_a e_\alpha\rangle|g(\lambda)|^2 d\lambda\,. \tag{4.1}$$

Here $R(\lambda \pm i0)$ are the boundary values of the resolvent $(P - z)^{-1}$ and $e_\alpha$ is defined by:

$$e_\alpha(x, \lambda, \omega) = \phi_\alpha(x^a)e^{in_\alpha(\lambda)x_a \cdot \omega} .$$

with $n_\alpha(\lambda) = (\lambda - E_\alpha)^{1/2}$. In particular, $\sigma_\alpha(\lambda, \omega)$ is a continuous function in $(\lambda, \omega)$:

$$\sigma_\alpha(\lambda, \omega) = \frac{1}{n_\alpha(\lambda)} \Im\langle R(\lambda + i0, h)I_a e_\alpha, I_a e_\alpha\rangle , \tag{4.2}$$

for $(\lambda, \omega) \in (\mathbf{I}_\alpha \backslash \mathscr{T}) \times S_a$. The study of high energy asymptotics for $\sigma_\alpha(\lambda, \omega)$ is based on the formula (4.2). In the two-body case, this formula is reduced to the following:

$$\sigma(\lambda, \omega) = \frac{1}{\sqrt{\lambda}} \Im\langle R(\lambda + i0) Ve(\lambda, \omega), Ve(\lambda, \omega)\rangle . \tag{4.3}$$

where now $V$ is a two-body potential and $e(\lambda, \omega) = e^{i\sqrt{\lambda}\omega \cdot x}$. The high energy asymptotics (or Born approximation) for $\sigma(\lambda, \omega)$ is usually carried out by a perturbation around the free Hamiltonian $-\Delta$. In fact, one can write, for short range potential $V$,

$$VR(\lambda + i0)V = VR_0(\lambda + i0)V + (VR_0(\lambda + i0))^2 V + \cdots .$$

Here $R_0(z) = (-\Delta - z)^{-1}$. The leading term of $\sigma(\lambda, \omega)$ can be derived by inserting this expression into (4.2). See [9, 10, 21]. However, this argument does not apply to many-body problems, since $I_a(x) = \sum_{b \not\subset a} V_b(x^b)$ does not decay on the whole configuration space $X$. We shall construct an eikonal approximation for the outgoing wave function $R(\lambda + i0)I_a e_\alpha(\lambda, \omega)$ and use microlocal resolvent estimates in Sect. 2 to estimate remainders.

Notice first that $\phi_\alpha$ is rapidly decreasing in $x^a$. Under the assumption (1.3) for $\rho > (n_a + 1)/2$, we have for any $1/2 < s < \rho - n_a/2$

$$\| \langle x\rangle^s I_a e_\alpha(\lambda, \omega) \| \leqq C_s ,$$

uniformly in $\lambda, \omega$. Therefore as a consequence of Theorem 2.1,

$$\sigma_\alpha(\lambda, \omega) = O(\lambda^{-1}), \quad \lambda \to \infty.$$

**Lemma 4.1.** *Let* $\chi_1(x_a, \lambda) = \Theta(x_a/\lambda^{1/2})$, *where* $\Theta$ *is a smooth function with* $\operatorname{supp} \Theta \subset \{|x_a| \leqq 1\}$ *and* $\Theta = 1$ *for* $|x_a| \leqq 1/2$. *Then*

$$\sigma_\alpha(\lambda, \omega) = \frac{1}{n_\alpha(\lambda)} \Im\langle R(\lambda + i0)\chi_1 I_a e_\alpha, I_a e_\alpha\rangle + O(\lambda^{-1-(\rho-(n_a+1)/2-\varepsilon)/2}) , \tag{4.4}$$

*for any* $\varepsilon > 0$.

*Proof.* It suffices to apply (a) of Theorem 2.1 and the estimate $\| \langle x\rangle^s(1 - \chi_1(x_a))I_a e_\alpha \| = O(\lambda^{-(\rho-n_a/2-s-\varepsilon/2)/2})$ for $s = (1 + \varepsilon)/2$. □

Let $f = \chi_1 I_a$. Define:

$$g(x) = \frac{1}{n'_\alpha(\lambda)} \int_0^\infty f(x^a, x_a - t\omega)e^{-\frac{i}{n'_\alpha(\lambda)}\int_0^t I_a(x^a, x_a - (t-s)\omega)ds} dt \tag{4.5}$$

with $n'_\alpha(\lambda) = 2n_\alpha(\lambda)$. $g(\cdot)$ is a well defined smooth function, since the integration is just taken over a finite interval in $t$. We can verify that

$$n'_\alpha(\lambda)(\omega \cdot \nabla_a)g + iI_a g = f . \tag{4.6}$$

Put $\chi_2(x_a, \lambda) = \Theta(x_a/(M\lambda^{1/2}))$ with $M \gg 1$ to be chosen later. Since $\chi_2 = 1$ on supp $f$, one has:

$$(P - \lambda)(i\chi_2 g e_\alpha) = fe_\alpha + i[-\Delta_a, \chi_2]ge_\alpha - i\chi_2\{e_\alpha \Delta g + 2\nabla^a g \cdot \nabla^a e_\alpha\}$$
$$\equiv fe_\alpha - r_1 - r_2 . \tag{4.7}$$

This shows $R(\lambda + i0)(fe_\alpha) = i\chi_2 g e_\alpha + R(\lambda + i0)(r_1 + r_2)$.

**Lemma 4.2.** *With the above notations, one has*

$$|\langle R(\lambda + i0)r_j, I_a e_\alpha\rangle| = O(\lambda^{-1/2-\eta_a}) ,$$

*for $j = 1, 2$. Here $\eta_a$ is defined in Theorem* 1.1.

*Proof.* Since $x^a \cdot \omega = 0$, $\phi_\alpha g$ is rapidly decreasing in $x^a$. Let $y = x_a - (\xi_a \cdot \omega)\omega$. We can also check that

$$|\phi_\alpha(x^a)g(x)| \leqq C_N \lambda^{-1/2}\langle x^a\rangle^{-N}\langle y\rangle^{-\rho+1}, \quad \text{for any } N > 0. \tag{4.8}$$

Let $\{\theta_1, \theta_2\}$ be a partition of unity in $\hat{x}_a \cdot \omega \in \mathbf{R}$ with supp $\theta_1 \subset \{\hat{x}_a \cdot \omega \in ]-\infty, 1 - \varepsilon/2[\}$ and supp $\theta_2 \subset \{\hat{x}_a \cdot \omega \in ]1 - \varepsilon, \infty[\}$. On the support of $\theta_1$, $|x_a - t\omega| \geqq c_\varepsilon(|x_a| + t)$ for $t > 0$. Consequently we can verify that on the support of $\theta_1$,

$$|\phi_\alpha(x^a)\partial^\beta_{x_a} g(x)| \leqq C_{N\beta}\lambda^{-1/2}\langle x_a\rangle^{-\rho-|\beta|+1}\langle x^a\rangle^{-N}, \quad \forall \beta \in \mathbf{N}^{n_a} . \tag{4.9}$$

Let us first estimate the remainder related to $r_2$.

$$|\langle R(\lambda + i0)r_2, I_a e_\alpha\rangle| \leqq C\|\langle x\rangle^{-\rho+n_a/2+\varepsilon}R(\lambda + i0)(\theta_1 + \theta_2 r_2)\|$$
$$\leqq C\{\lambda^{-1}\|\langle x_a\rangle^{1-\rho+1/2+\varepsilon}\chi_2 \varepsilon_\alpha\|$$
$$+ \|\langle x\rangle^{-\rho+n_a/2+\varepsilon}R(\lambda + i0)\theta_2 r_2\|\}$$
$$\leqq C\{\lambda^{-1/2-\eta_a} + \|\langle x\rangle^{-\rho+n_a/2+\varepsilon}R(\lambda + i0)\theta_2\chi(D_a)r_2\|\} .$$

Here $\chi$ is supported near $\xi_a = n_\alpha(\lambda)\omega$ and we used the fact $\chi(D_a)e_\alpha = e_\alpha$ in $L^2(X^a) \times L^{2,-n_a/2-\varepsilon}(X_a)$ and that the terms related to the commutators between $\chi(D_a)$ and the derivatives of $g$ can be bounded by $O(\lambda^{-1/2-\eta_a})$. On the support of $\theta_2(x_a)\chi(\xi_a)$, we have: $x_a \cdot \xi_a \geqq 0$. Applying (b) of Theorem 2.1 with $s = \rho - n_a/2 - \varepsilon$, we obtain

$$\|\langle x\rangle^{-\rho+n_a/2+\varepsilon}R(\lambda + i0)\theta_2\chi(D_a)r_2\| \leqq C\lambda^{-1/2}\|\langle x\rangle^{-\rho+n_a/2+1+\varepsilon}r_2\| = O(\lambda^{-1/2-\eta_a}) .$$

The desired estimate for $r_2$ follows.

For $r_1$, we write it as

$$r_1 = -2n_\alpha(\lambda)\omega \cdot \nabla_a\chi_2 g e_\alpha + 2i\nabla_a g \cdot \nabla_a\chi_2 e_\alpha + i\Delta_a\chi_2 g e_\alpha .$$

The pieces corresponding to the last two terms can be treated as above, while for the piece corresponding to the first term, we need a new argument since there is an additional factor $n_\alpha(\lambda)$. Note that supp $\nabla\chi_2 \subset \{M\lambda^{1/2}/2 \leqq |x_a| \leqq M\lambda^{1/2}\}$ and $\text{supp}_{x_a} f(x^a, \cdot - t\omega)$ is contained in $\{|x_a - t\omega| \leqq \lambda^{1/2}\}$. Consequently, for $x_a$ in the support of $\omega \cdot \nabla_a\chi_2 g$, one has for some $t \geqq 0$,

$$M\lambda^{1/2}/2 \leqq |x_a| \leqq M\lambda^{1/2} , \tag{4.10}$$

$$|x_a - t\omega| \leqq \lambda^{1/2} . \tag{4.11}$$

Writing $x_a = (x_1, y)$ with $x_1 = x_a \cdot \omega$, (4.11) implies $|y| \leqq \lambda^{1/2}$ and $|x_1 - t| \leqq \lambda^{1/2}$. Equation (4.10) gives

$$|x_1| \geqq M\lambda^{1/2}/2 - |y| \geqq (M/2 - 1)\lambda^{1/2} .$$

Taking $M > 4$, it follows from the estimates $|x_1| \geqq (M/2 - 1)\lambda^{1/2} > \lambda^{1/2}$ and $|x_1 - t| \leqq \lambda^{1/2}$ $(t \geqq 0)$ that $x_1 > 0$ and therefore, $x_1 \geqq \lambda^{1/2} \geqq |x_a|/M$. Let $\chi(\xi_a)$ be chosen as in the first part of the proof. The support of $\omega \cdot \nabla_a \chi_2(x_a) g(x) \chi(\xi_a)$ is contained in an outgoing region. Since $\nabla_a \chi_2 = O(\lambda^{-1/2})$, we can apply (b) of Theorem 2.1 and the arguments already used to treat $r_2$ to finish the proof of Lemma 4.2. □

From Lemmas 4.1 and 4.2, we obtain

$$\sigma_\alpha(\lambda, \omega) = \frac{1}{n_\alpha(\lambda)} \Im \langle i\chi_2 g e_\alpha(\lambda, \omega), I_a e_\alpha(\lambda, \omega)\rangle + O(\lambda^{-1-\eta_a}) . \tag{4.12}$$

*Proof of Theorem 1.1.* It remains to calculate the asymptotics of

$$\Im \langle i\chi_2 g e_\alpha, I_a e_\alpha\rangle = \Re \int_X \chi_2(x_a) g(x) I_a(x) |\phi_\alpha(x^a)|^2 dx^a dx_a$$

as $\lambda \to \infty$. Decompose $X_a$ as $X_a = \mathbf{R} \times \Pi_\omega$ with $\Pi_\omega = \{x_a; x_a \cdot \omega = 0\}$. Write $x_a = (s, y) \in \mathbf{R} \times \Pi_\omega$ and let $x^a$ be fixed (the indication of $x^a$ variables is omitted in the following formulas). Making use of (4.5), we can compute the integral

$$\begin{aligned}
&\Re \int_{X_a} \chi_2 g I_a dx_a \\
&= \frac{1}{n'_\alpha(\lambda)} \Re \int_{\Pi_\omega} \int_{\mathbf{R}} (\chi_2 I_a)(y + s\omega) \left\{ \int_{-\infty}^{s} (\chi_1 I_a)(y + t\omega) e^{-\frac{i}{n'_\alpha(\lambda)} \int_t^s I_a(y + t'\omega) dt'} dt \right\} ds dy \\
&= \frac{1}{2\lambda^{1/2}} \int_{\Pi_\omega} \int_{\mathbf{R}} (\chi_2 I_a)(y + s\omega) \left\{ \int_{-\infty}^{s} (\chi_1 I_a)(y + t\omega) dt \right\} ds dy + O(\lambda^{-1}) \\
&= \frac{1}{2\lambda^{1/2}} \int_{\Pi_\omega} \int_{\mathbf{R}} I_a(y + s\omega) \left\{ \int_{-\infty}^{s} I_a(y + t\omega) dt \right\} ds dy + O(\lambda^{-1/2-\eta_a}) ,
\end{aligned}$$

uniformly in $x^a$. Here we used the fact that $\chi_j(x_a) = 1, j = 1, 2$, for $|x_a| \leqq \lambda^{1/2}/2$ and that for any $b \in \mathscr{A}$ with $b \nsubseteq a$, one has $|x^b| \geqq \delta |x_a|$ for some $\delta > 0$. From (4.12), it follows that

$$\sigma_\alpha(\lambda, \omega) = \frac{1}{4\lambda} \int_{X^a} |\phi_\alpha(x^a)|^2 \int_{\Pi_\omega} \left( \int_{\mathbf{R}} I_a(x^a, y + s\omega) ds \right)^2 dy dx^a + O(\lambda^{-1-\eta_a}) ,$$

as $\lambda \to \infty$. □

*Remark 4.1.* The smoothness assumption on potentials is more than necessary. In fact, if each potential has only a local singularity at the origin so that the assumption (2.1) (with $\rho > (n_a + 1)/2$) is satisfied outside 0 and for each $b \nsubseteq a$,

$$V_b \in L^2_{\mathrm{loc}}(X^{n^b}) , \tag{4.13}$$

we can still establish the high energy asymptotics for $\sigma_\alpha(\lambda, \omega)$:

$$\sigma_\alpha(\lambda, \omega) = \frac{1}{4\lambda} \int_{X^a} |\phi_\alpha(x^a)|^2 \int_{\Pi_\omega} \left( \int_{\mathbf{R}} I_a(x^a, y + s\omega) ds \right)^2 dy dx^a + o(\lambda^{-1}), \quad \text{as } \lambda \to \infty. \tag{4.14}$$

To see this, let $\chi_\varepsilon$ be a cut-off function for the set $\{|y| \leqq \varepsilon\}$ and let $I_a^\varepsilon(x) = \sum_{b \nsubseteq a} (1 - \chi_\varepsilon(x^b)) V_b(x^b)$. By Theorem 2.1, one has:

$$\sigma_\alpha(\lambda, \omega) = \frac{1}{n_\alpha(\lambda)} \Im \langle R(\lambda + i0) \chi_1 I_a^\varepsilon e_\alpha, I_a^\varepsilon e_\alpha \rangle + o(1) \lambda^{-1}.$$

Here $o(1) \to 0$ uniformly in $\lambda$, as $\varepsilon \to 0$. Now $I_a^\varepsilon$ is smooth, Repeating the proof of Theorem 1.1, we obtain:

$$\frac{1}{n_\alpha(\lambda)} \Im \langle R(\lambda + i0) \chi_1 I_a^\varepsilon e_\alpha, I_a^\varepsilon e_\alpha \rangle$$
$$= \frac{1}{4\lambda} \int_{X^a} |\phi_\alpha(x^a)|^2 \int_{\Pi_\omega} \left( \int_{\mathbf{R}} I_a^\varepsilon(x^a, y + s\omega) ds \right)^2 dy dx^a + O_\varepsilon(\lambda^{-1-\eta_a}) + o(1)\lambda^{-1}.$$

Equation (4.14) follows from the fact

$$\lim_{\varepsilon \to 0} \int_{X^a} |\phi_\alpha(x^a)|^2 \int_{\Pi_\omega} \left( \int_{\mathbf{R}} I_a^\varepsilon(x^a, y + s\omega) ds \right)^2 dy dx^a$$
$$= \int_{X^a} |\phi_\alpha(x^a)|^2 \int_{\Pi_\omega} \left( \int_{\mathbf{R}} I_a(x^a, y + s\omega) ds \right)^2 dy dx^a.$$

Note that according to (4.13), the last integral is finite. □

## 5. Upper Bounds on $\sigma_{\alpha\beta}(\lambda, \omega)$

From now on, we assume that the condition (1.3) is satisfied for $\rho > (n_{ab} + 1)/2$. Let $\alpha$ be a two-cluster channel with non-threshold energy and $\beta$ be an arbitrary channel. With the notations of Theorem 3.1, one has

$$\sigma_{\alpha\beta}(\lambda, \omega) = \frac{\pi}{(\lambda - E_\beta)^{1/2}} \| F_a(\lambda) \mathscr{J}_\alpha^* \{1 - I_a R(\lambda + i0)\} Q_b e_\beta(\lambda, \omega) \|^2_{L^2(S_a)},$$

for $\lambda \in \mathbf{I}_1 = ](1 - \delta)\lambda_1, (1 + \delta)\lambda_1[$, $\lambda_1 > \tau_{\beta,\omega}/(1 - \delta)$. To study $\sigma_{\alpha\beta}(\lambda, \omega)$ in the limit $\lambda \to \infty$, we replace $J_b$ by $J_b^{\lambda_1}(x) = J_b(x/\lambda_1^{1/2})$ and denote by $Q_b^{\lambda_1}$ the operator defined by (3.6) with $J_b$ replaced by $J_b^{\lambda_1}$. It is clear from the proof of Theorem 3.1 that (3.9) still holds for $\lambda \in \mathbf{I}_1$. Setting $\lambda = \lambda_1$, we obtain

$$\sigma_{\alpha\beta}(\lambda, \omega) = \frac{\pi}{(\lambda - E_\beta)^{1/2}} \| F_a(\lambda) \mathscr{J}_\alpha^* \{1 - I_a R(\lambda + i0)\} Q_b^\lambda e_\beta(\lambda, \omega) \|^2_{L^2(S_a)}, \tag{5.1}$$

for all $\lambda > \tau_{\beta,\omega}/(1 - \delta)$. Define

$$f_{\alpha\beta} = \mathscr{J}_\alpha^* \{1 - I_a R(\lambda + i0)\} Q_b e_\beta(\lambda, \omega). \tag{5.2}$$

By microlocal resolvent estimates (Theorem 2.1), $f_{\alpha\beta} \in L^{2,s}(X_a)$ for some $s > 1/2$. Applying Stone's formula (see for example [12]), we see that

$$\| F_\alpha(\lambda) f_{\alpha\beta} \|^2_{L^2(S_a)} = \langle F_\alpha(\lambda)^* F_\alpha(\lambda) f_{\alpha\beta}, f_{\alpha\beta} \rangle_{L^2(S_a)}$$
$$= \left\langle \frac{1}{2i\pi} (R_a(\lambda + i0) - R_a(\lambda - i0)) f_{\alpha\beta}, f_{\alpha\beta} \right\rangle_a$$
$$= \frac{1}{\pi} \Im \langle R_a(\lambda + i0) f_{\alpha\beta}, f_{\alpha\beta} \rangle_a .$$

Here $R_a(z)$ denotes the resolvent for $-\Delta + E_\alpha$. This proves

$$\sigma_{\alpha\beta}(\lambda, \omega) = \frac{1}{n_\beta(\lambda)} \Im \langle R_a(\lambda + i0) f_{\alpha\beta}, f_{\alpha\beta} \rangle_a . \tag{5.3}$$

The formula (5.3) looks similar with (4.2), but the dependence of $f_{\alpha\beta}$ on $\lambda$ is rather complicated. We decompose

$$f_{\alpha\beta} = g_{\alpha\beta} + r_{\alpha\beta}, \quad \text{with } g_{\alpha\beta} = \mathscr{J}^*_\alpha Q^\lambda_b e_\beta(\lambda, \omega) . \tag{5.4}$$

**Lemma 5.1.** *Let* $1/2 < s < \rho - n_a/2$.

(i) *If* $a = b$, $\| \langle x_a \rangle^s g_{\alpha\beta} \|_a \leqq C$ *uniformly in* $\lambda > \tau_{\beta,\omega}/(1 - \delta)$.
(ii) *If* $a \neq b$, *then for any* $M > 0$, $\| \langle x_a \rangle^s g_{\alpha\beta} \|_a \leqq C_M \lambda^{-M}$, *for* $\lambda > \tau_{\beta,\omega}/(1 - \delta)$.

*Proof.* Recall that $\phi_\alpha(\cdot)$ is rapidly decreasing in $x^a$ and

$$g_{\alpha\beta}(x_a) = \int_{X^a} (Q^\lambda_b e_\beta)(x^a, x_a) \overline{\phi_\alpha(x^a)}\, dx^a ,$$

where $Q^\lambda_b = [-\Delta, J^\lambda_b] \chi(D_b) + I_b J^\lambda_b \chi(D_b)$. As in the proof of Theorem 3.1, we can show that on $\operatorname{supp} \nabla J^\lambda_b$, $|x^a| \geqq c|x|$, $c > 0$, and $|x| > \lambda^{1/2}/2$, due to the dilation in $\lambda$. Consequently,

$$\int_{X^a} [-\Delta, J^\lambda_b] \chi(D_b) e_\beta \overline{\phi_\alpha(x^a)} dx^a = O(\langle x_a \rangle^{-\infty} \lambda^{-\infty}) .$$

In the case $a = b$, $|I_b(x) \phi_\alpha(x^a)| \leqq C_M \langle x_a \rangle^{-\rho} \langle x^a \rangle^{-M}$, $\forall M \gg 1$. It follows that

$$\int_{Xa} I_b J^\lambda_b \chi(D) e_\beta \overline{\phi_\alpha}\, dx^a = O(\langle x_a \rangle^{-\rho})$$

uniformly in $\lambda$. This proves (i).

In the case $a \neq b$, we have $a \not\subset b$, because $\# a = 2$. This means that

$$\omega^a = \pi^a \omega \neq 0, \quad \text{for } \omega \in \Sigma_b . \tag{5.5}$$

Writing $\omega \cdot x_b = \omega \cdot x = \omega^a \cdot x^a + \omega_a \cdot x_a$, we have in this case

$$\int_{X^a} I_b J^\lambda_b \chi(D_b) e_\beta \overline{\phi_\alpha}\, dx^a = e^{in_\beta(\lambda)\omega_a \cdot x_a} \int_{X^a} e^{in_\beta(\lambda)\omega^a \cdot x^a} I_b J^\lambda_b \phi_\beta(x^b) \overline{\phi_\alpha(x^a)}\, dx^a . \tag{4.6}$$

Since $\omega^a \neq 0$, we have an oscillatory integral with non-stationary phase. Making use of the relation

$$\left( \frac{\omega^a \cdot D^a}{n_\beta(\lambda) |\omega^a|^2} \right)^M e^{in_\beta(\lambda)\omega^a \cdot x^a} = e^{in_\beta(\lambda)\omega^a \cdot x^a}, \quad \forall M \in \mathbf{N} ,$$

we obtain by integration by parts that

$$\int_{X^a} e^{in_\beta(\lambda)\omega^a\cdot x^a} I_b J_b^\lambda \phi_\beta(x^b)\overline{\phi_\alpha(x^a)}dx^a = O(\langle x_a\rangle^{-\rho}\lambda^{-M}),$$

for any $M > 1$. Here we used the smoothness of the potentials and the eigenfunctions. (ii) is proved. □

**Lemma 5.2.** *Let* $\eta_b$ *be defined as in Theorem* 1.2 *and* $1/2 < s < \rho - n_a/2$. *One has*: $\|\langle x_a\rangle^s r_{\alpha\beta}\|_a \leqq C\lambda^{-\eta_b}$.

*Proof.* By the decay assumption on the potentials, one has:

$$\|\langle x_a\rangle^s r_{\alpha\beta}\|_a \leqq C\|\langle x\rangle^{-\rho+s}R(\lambda+i0)Q_b^\lambda e_\beta(\lambda,\omega)\| \tag{5.7}$$

and

$$Q_b^\lambda = O(\langle x\rangle^{-\rho}) + I_b(1-J_1(x))(1-j(|x|/\lambda^{1/2})\chi(D_b) + B_1$$

with $B_1 = [-\Delta, J_b^\lambda]\chi(D_a)$.

The contribution from the term $O(\langle x\rangle^{-\rho})$ can be estimated by $O(\lambda^{-1/2})$, using (a) of Theorem 2.1. $(1-J_1)(1-j(\cdot/\lambda^{1/2}))\chi(D_b)$ is a pseudo-differential operator with symbol in $S_+^b(-(1-\varepsilon)\lambda^{1/2})$. We can apply (b) of Theorem 2.1 to show that the contribution from this term is bounded by

$$C\lambda^{-1/2}\|\langle x\rangle^{-\rho+s+1}(1-j(|x|/\lambda^{1/2}))I_b e_\beta\| \leqq C\lambda^{-\eta_b}.$$

Here we have used the fact that supp $(1-j(\cdot/\lambda^{1/2}))$ is contained in $\{|x| \leqq \lambda^{1/2}\}$ to estimate

$$\|\langle x\rangle^{-\rho+s+1}(1-j(|x|/\lambda^{1/2}))V_c e_\beta\| \leqq \begin{cases} C, & \text{if } \rho-s-1 > n_b/2\,; \\ C\lambda^{((n_b+1)/2+1+\varepsilon-\rho)/2}, & \text{if } \rho-s-1 \leqq n_b/2\,. \end{cases}$$

The term $\|\langle x\rangle^{s-\rho}R(\lambda+i0)B_1 e_\beta\|$ can be estimated by applying (b) of Theorem 2.1 and the estimate $\nabla J_b = O(\lambda^{-1/2})$. The details are omitted. □

As a consequence of Lemmas 5.1 and 5.2, we obtain an upper bound on $\sigma_{\alpha\beta}(\lambda,\omega)$ which implies (ii) of Theorem 1.2.

**Corollary 5.3.** *Assume the condition* (1.3) *for* $\rho > (n_{ab}+1)/2$. *For any* $\omega \in \Sigma_b$, *one has*:

$$\sigma_{\alpha\beta}(\lambda,\omega) \leqq \begin{cases} C\lambda^{-1}, & \text{if } a = b\,, \\ C\lambda^{-1-2\eta_b}, & \text{if } a \neq b\,, \end{cases} \tag{5.8}$$

*for* $\lambda > (1+\delta)\tau_{\beta,\omega}$.

## 6. High Energy Asymptotics of $\sigma_{\alpha\beta}(\lambda,\omega)$ with $a = b$

From the results established in the preceding section, one sees that

$$\sigma_{\alpha\beta}(\lambda,\omega) = \begin{cases} \dfrac{1}{n_\beta(\lambda)}\Im\langle R_a(\lambda+i0)g_{\alpha\beta}, g_{\alpha\beta}\rangle_a + O(\lambda^{-1-\eta_b}), & \text{if } a = b\,, \\[2ex] \dfrac{1}{n_\beta(\lambda)}\Im\langle R_a(\lambda+i0)r_{\alpha\beta}, r_{\alpha\beta}\rangle_a + O(\lambda^{-\infty}), & \text{if } a \neq b\,. \end{cases} \tag{6.1}$$

Note that by the method of non-stationary phase used in the proof of Lemma 5.1, the leading term in the eikonal approximation to $\sigma_{\alpha\beta}(\lambda, \omega)$ with $a \neq b$ gives a contribution of the order $O(\lambda^{-\infty})$, while the remainder terms are only of finite order in $\lambda^{-1}$. This suggests that one should use other methods to obtain the high energy asymptotics of $\sigma_{\alpha\beta}(\lambda, \omega)$ with $a \neq b$.

To study the high energy asymptotics of $\sigma_{\alpha\beta}(\lambda, \omega)$ when $a = b$, we only need to look at $\Im \langle R_a(\lambda + i0) g_{\alpha\beta}, g_{\alpha\beta} \rangle_a$. By the argument used in the proof of Lemma 5.1, we see that

$$\begin{aligned} g_{\alpha\beta}(x_a) &= \int_{X^a} Q_b^\lambda e_\beta \overline{\phi_\alpha(x^a)} dx^a \\ &= \int_{X^a} I_a J_a^\lambda e_\beta \overline{\phi_\alpha(x^a)} dx^a + O(\langle x_a \rangle^{-\infty} \lambda^{-\infty}) \\ &= e^{in_\beta(\lambda)\omega \cdot x_a} l(x_a) + O(\langle x_a \rangle^{-\infty} \lambda^{-\infty}), \end{aligned} \tag{6.2}$$

where

$$l(x_a) = \int_{X^a} I_a(x) \phi_\beta(x^a) \overline{\phi_\alpha(x^a)} dx^a .$$

In the last equality in (6.2), we have used the fact that on the support of $1 - J_a^\lambda = (1 - J_1) j(\cdot/\lambda^{1/2})$, $|x^a| \geqq c|x|$ for some $c > 0$ and $|x| > \lambda^{1/2}/2$. This gives

$$\sigma_{\alpha\beta}(\lambda, \omega) = \frac{1}{n_\beta(\lambda)} \Im \langle R_a(\lambda + i0) \varepsilon_\beta(\lambda, \omega) l, \varepsilon_\beta(\lambda, \omega) l \rangle_a + O(\lambda^{-1-\eta_a}) . \tag{6.3}$$

*Proof of* (i) *of Theorem 1.2*. Since $l(x_a) = O(\langle x_a \rangle^{-\rho})$, we can apply the method of the proof of Theorem 1.1 to construct an eikonal approximation for

$$\frac{1}{n_\beta(\lambda)} \Im \langle R_a(\lambda + i0) e^{in_\beta(\lambda)\omega \cdot x_a} l, e^{in_\beta(\lambda)\omega \cdot x_a} l \rangle_a .$$

In fact, it is now easier since $R_a(\lambda + i0)$ is a free resolvent. Define

$$g(x_a) = \int_0^\infty l(x_a - 2n_\beta(\lambda) t\omega) e^{-it(E_\alpha - E_\beta)} dt .$$

Following the line of the proof given in Sect. 4, we can estimate:

$$\begin{aligned} \sigma_{\alpha\beta}(\lambda, \omega) &= \frac{1}{\lambda^{1/2}} \Re \langle g, l \rangle_a + O(\lambda^{-1-\eta_a}) \\ &= \frac{1}{2\lambda} \Re \int_{\mathbf{R} \times \Pi_\omega} \overline{l(y + t\omega)} \left\{ \int_{-\infty}^t l(y + s\omega) ds \right\} dt dy + O(\lambda^{-1-\eta_a}) \\ &= \frac{1}{4\lambda} \int_{\Pi_\omega} \left| \int_{\mathbf{R}} l(y + s\omega) ds \right|^2 dy + O(\lambda^{-1-\eta_a}) . \end{aligned}$$

The details are omitted. □

**Note added in proof.** After the submission of this paper, the author received a preprint from H. Ito: "High energy behavior of total scattering cross sections for 3-body quantum systems", in which he proved Theorem 1.1 in the three-body case under the assumption that 0 is not eigenvalue of any two-body subhamiltonians.

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