Commun. Math. Phys. 155, 561–568 (1993)

Communications in
Mathematical
Physics

Invariants of 2+1 Gravity

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Received September 2, 1992; in revised form February 5, 1993

Abstract. In [1, 2] we established and dicussed the algebra of observables for $2+1$ gravity at both the classical and quantum level. Here our treatment broadens and extends previous results to any genus g with a systematic discussion of the centre of the algebra. The reduction of the number of independent observables to $6g-6 (g>1)$ is treated in detail with a precise classification for $g=1$ and $g=2$.

1. Introduction

In previous articles [1, 2] we analysed the algebra of quantum observables for $2+1$ gravity on an initial data Riemann surface of genus g. The homotopy group $\pi_1(\Sigma)$ of the surface is defined by generators t_i, $i=1\dots 2g+2$ and presentation:

$$\begin{aligned} t_1 t_2 \dots t_{2g+2} &= 1\,, \\ t_1 t_3 \dots t_{2g+1} &= 1\,, \\ t_2 t_4 \dots t_{2g+2} &= 1\,. \end{aligned} \tag{1.1}$$

The integrated anti-De Sitter connection in the surface defines a representation $S:\pi_1(\Sigma)\to SL(2,R)$. The $n(n-1)/2$ gauge invariant trace elements

$$\alpha_{ij}=\alpha_{ji}=\frac{1}{2}\,\mathrm{Tr}(S(t_i t_{i+1}\dots t_{j-1}))$$

generate the abstract algebra $K(n)$, where $n=2g+2$, $\alpha_{ii}=1$ and $i,j\in Z_n$, that is, endowed with an explicit cyclical symmetry of order n. The sequence $1\dots n$ is by convention anticlockwise, see Fig. 1. Consider 4 anticlockwise points m, j, l, k. The corresponding quantum operators a_{ij} and algebra $A(n)$ are defined by the commutation relations:

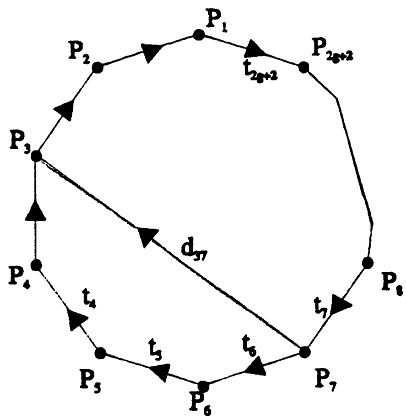

Fig. 1.

$$(a_{mk}, a_{jl}) = (a_{mj}, a_{kl}) = 0\,, \tag{1.2}$$

$$(a_{jk}, a_{km}) = \left(\frac{1}{K} - 1\right)(a_{jm} - a_{jk}a_{km})\,, \tag{1.3}$$

$$(a_{jk}, a_{kl}) = \left(1 - \frac{1}{K}\right)(a_{jl} - a_{kl}a_{jk})\,, \tag{1.4}$$

$$(a_{jk}, a_{lm}) = \left(K - \frac{1}{K}\right)(a_{jl}a_{km} - a_{kl}a_{jm})\,, \tag{1.5}$$

where $K = \dfrac{4\alpha - ih}{4\alpha + ih} = e^{i\theta}$, $\Lambda = -\dfrac{1}{3\alpha^2}$ is the cosmological constant and h is Planck's constant.

The classical limit $K \to 1$, $a_{ih} \to \alpha_{ih}$ of [1.2–1.5] are the Poisson brackets $[\,,]$ defined by

$$\frac{(A, B)}{K - 1} \to [A, B]\,.$$

The operators in (1.2–1.5) are ordered with the convention that $s(a_{ij})$ is increasing from left to right where

$$s(a_{ij}) = \frac{(i-1)(2n-2-i)}{2} + j - 1\,.$$

The algebra of observables of $2+1$ classical gravity for an initial data surface Σ of genus g can be identified with a particular factor algebra of $K(2g+2)$ as explained in [2]. There exists a group $\mathbf{D}(n)$ of automorphisms on $K(n)$ and $A(n)$ implemented through exponentiated canonical transformations and induced by the mapping class group [3] as follows:

Let $a_{jk} = \dfrac{\cos(\psi_{jk})}{\cos\left(\dfrac{\theta}{2}\right)}$. The algebra $A(n)$ can be enlarged to $B(n)$ by including non-periodic functions of ψ_{jk}, $\mathbf{D}(n)$ is then an inner group of automorphisms in $B(n)$. Define:

$$F(\psi_{jk}) = \exp\left(- \frac{i\psi_{jk}^2}{2\theta} \right); \tag{1.6}$$

then the induced transformations are:

$$B \xrightarrow{D_{jk}} F(\psi_{jk}) B F(\psi_{jk})^{-1}, \quad \text{where} \quad B \in B(n).$$

$\mathbf{D}(n)$ is generated by the maps $D_{jk} = D_{kj}$. Here follows a table of images under the map D_{jk}.

Element in $A(n)$	Image
a_{kl}	$(1+K)a_{jk}a_{kl} - Ka_{jl}$
a_{km}	a_{mj}
a_{mj}	$(1+K)a_{jk}a_{mj} - Ka_{km}$
a_{lj}	a_{kl}
a_{lm}	$a_{lm} - (1+K)a_{kl}a_{km} - (K+K^2)a_{jl}a_{jm} + (1+K^2)a_{jk}a_{kl}a_{mj}$.

(1.7)

The elements a_{pq} not listed in the table and a_{jk} are invariant under D_{jk}. Given $\chi \in \mathbf{D}(n)$ we denote by $D(W,\chi)$ the image of W under the map χ.

The action of $\mathbf{D}(n)$ on $K(n)$ follows from (1.7) in the classical limit $K \to 1$, $a_{ih} \to \alpha_{ih}$, $i,h = 1\ldots n$.

In Sect. 2 we determine for each n a set of p linearly independent central [i.e. invariant under (1.7)] elements $\mathbf{A}_{nm}$, $m = 1\ldots p$ in $K(n)$, where $n = 2p$ or $n = 2p+1$. In Sect. 3 we analyse the trace identities which follow from the presentation (1.1) of the homotopy group $\pi_1(\Sigma)$ and a set of rank identities with focus on $g = 1,2$. These identities together generate an ideal $I(n) \subset K(n)$. Finally in Sect. 4 we discuss the quantum counterpart of $I(n)$ and of the central elements (Casimirs) $\mathbf{A}_{nm}$ in $B(n)$.

2. The Centre of $K(n)$

Consider the $n \times n$ classical matrix $C(\beta)$ with elements:

$$\begin{aligned} C_{ij} &= e^{i\beta}\alpha_{ij}, \qquad i > j, \\ C_{ij} &= e^{-i\beta}\alpha_{ij}, \qquad i < j, \\ C_{ii} &= \cos(\beta), \end{aligned} \tag{2.1}$$

where β is real and arbitrary. Note that

$$C(\beta) = -C(\beta+\pi) = C(\beta)^+, \qquad C(-\beta) = C(\beta)^T. \tag{2.2}$$

and that $C\left(\frac{\pi}{2}\right) = -C\left(\frac{\pi}{2}\right)^T$ so that $\mathrm{Det}\, C(\beta)$ is real, even in β and $\mathrm{Det}\, C(\beta+\pi) = (-1)^n\, \mathrm{Det}\, C(\beta)$. $C(0)$ has at most rank 4 (see Sect. 3). The Fourier expansion of $\mathrm{Det}\, C(\beta)$ is

$$\mathrm{Det}\, C(\beta) = 2^{1-n} \cos(n\beta) + \sum_{m=1}^{p} \cos((n-2m)\beta)\, \mathbf{A}_{nm}\,, \tag{2.3}$$

where $p = \frac{n}{2}$ or $p = \frac{n-1}{2}$.

Let $\Omega(\gamma)$ be the $n \times n$ matrix defined recursively by

$$\Omega([\eta,\gamma]) = [\Omega(\eta),\gamma] - [\Omega(\gamma),\eta] + \Omega(\eta)\,\Omega(\gamma) - \Omega(\gamma)\,\Omega(\eta)\,, \tag{2.4}$$

where $\eta, \gamma \in K(n)$ and by the initial conditions

$$\begin{gathered} \Omega(\alpha_{i,i+1})_{km} = 0\,, \qquad k \neq i, i+1 \quad \text{or} \quad m \neq i, i+1\,, \\ \Omega(\alpha_{i,i+1})_{ii} = \alpha_{i,i+1}\,, \qquad \Omega(\alpha_{i,i+1})_{i+1,i+1} = -\alpha_{i,i+1}\,, \\ \Omega(\alpha_{i,i+1})_{i,i+1} = -1\,, \qquad \Omega(\alpha_{i,i+1})_{i+1,1} = 1\,. \end{gathered} \tag{2.5}$$

Let $\gamma \in K(n)$ and $[C(\beta),\gamma]$ the $n \times n$ matrix of elements $[C(\beta)_{km}, \gamma]$, then it can be verified that

$$[C(\beta), \alpha_{i,i+1}] = \Omega(\alpha_{i,i+1})\, C(\beta) + C(\beta)\, \Omega(\alpha_{i,i+1})^T\,, \tag{2.6}$$

and by recursion that

$$[C(\beta),\gamma] = \Omega(\gamma)\, C(\beta) + C(\beta)\, \Omega(\gamma)^T\,. \tag{2.7}$$

From the identity

$$[\mathrm{Det}\, M, \gamma] = \mathrm{Det}\, M\, \mathrm{Tr}(M^{-1}[M,\gamma]) \tag{2.8}$$

and (2.6) it follows that

$$[\mathrm{Det}\, C(\beta), \alpha_{i,i+1}] = \mathrm{Det}\, C(\beta)\, \mathrm{Tr}(\Omega(\alpha_{i,i+1}) + \Omega(\alpha_{i,i+1})^T) = 0\,. \tag{2.9}$$

But since the $\alpha_{i,i+1}$ generate $K(n)$ through their Poisson brackets we obtain the general result $[\mathrm{Det}\, C(\beta), \gamma] = 0$ and therefore from (2.3) $[\mathbf{A}_{nm}, \gamma] = 0$.

Similarly let $\mathbf{Y}(\lambda)$ be the $n \times n$ matrix defined recursively by

$$\mathbf{Y}(\eta\lambda) = \mathbf{Y}(\eta)\, \mathbf{Y}(D(\lambda,\eta))\,, \qquad \lambda, \eta \in D(n)$$

and $\mathbf{Y}(D_{i,i+1})_{km} = 0$ if $k \neq m$ and ($k \neq i, i+1$ or $m \neq i, i+1$),

$$\begin{gathered} \mathbf{Y}(D_{i,i+1})_{ii} = 2\alpha_{ii+1}\,, \qquad \mathbf{Y}(D_{i,i+1})_{i+1,i+1} = 0\,, \\ \mathbf{Y}(D_{i,i+1})_{i,i+1} = 1\,, \qquad \mathbf{Y}(D_{i,i+1})_{i+1,i} = -1\,, \\ \mathbf{Y}(D_{i,i+1})_{kk} = 1 \quad \text{if} \quad k \neq i, i+1\,, \end{gathered}$$

then

$$D(C(\beta), D_{i,i+1}) = \mathbf{Y}(D_{i,i+1})^T\, C(\beta)\, \mathbf{Y}(D_{i,i+1})\,, \tag{2.10}$$

$$D(C(\beta), \lambda) = \mathbf{Y}(\lambda)^T\, C(\beta)\, \mathbf{Y}(\lambda)\,. \tag{2.11}$$

Since $\mathrm{Det}\, \mathbf{Y}(\lambda) = 1$, from (2.10–2.11) it follows that $\mathrm{Det}\, C(\beta)$ is invariant under the action of $\mathbf{D}(n)$.

For $n = 3$ there is only one central element:

$$\mathbf{A}_{31} = \frac{3}{4} - \alpha_{12}^2 - \alpha_{13}^2 - \alpha_{23}^2 + 2\alpha_{12}\alpha_{23}\alpha_{31}\,, \tag{2.12}$$

whereas for $g = 1$ and $n = 4$ there are 2 independent central elements given by:

$$\begin{aligned}\mathbf{A}_{41} &= \frac{1}{2}(1 - \alpha_{12}^2 - \alpha_{13}^2 - \alpha_{14}^2 - \alpha_{23}^2 - \alpha_{24}^2 - \alpha_{34}^2) \\ &\quad + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{12}\alpha_{24}\alpha_{41} + \alpha_{13}\alpha_{34}\alpha_{41} + \alpha_{23}\alpha_{34}\alpha_{42} \\ &\quad - 2\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{41}\,, \\ \mathbf{A}_{42} &= \mathbf{A}_{41} - \frac{1}{8} + \Pi_1^2\,,\end{aligned} \tag{2.13}$$

where $\Pi_1 = \alpha_{12}\alpha_{34} + \alpha_{14}\alpha_{23} - \alpha_{13}\alpha_{24}$ is also a central element.

Similarly we have a corresponding central element Π_g of degree $g + 1$ in the α_{jk} for any genus g whose square is $\operatorname{Det} C\left(\frac{\pi}{2}\right)$. For $g = 2$. $n = 6$ this is

$$\begin{aligned}\Pi_2 &= \alpha_{16}\alpha_{25}\alpha_{34} - \alpha_{15}\alpha_{26}\alpha_{34} - \alpha_{16}\alpha_{24}\alpha_{35} + \alpha_{14}\alpha_{26}\alpha_{35} + \alpha_{15}\alpha_{24}\alpha_{36} \\ &\quad - \alpha_{14}\alpha_{25}\alpha_{36} + \alpha_{16}\alpha_{23}\alpha_{45} - \alpha_{13}\alpha_{26}\alpha_{45} + \alpha_{12}\alpha_{36}\alpha_{45} - \alpha_{15}\alpha_{23}\alpha_{46} \\ &\quad + \alpha_{13}\alpha_{25}\alpha_{46} - \alpha_{12}\alpha_{35}\alpha_{46} + \alpha_{14}\alpha_{23}\alpha_{56} - \alpha_{13}\alpha_{24}\alpha_{56} + \alpha_{12}\alpha_{34}\alpha_{56}\,,\end{aligned} \tag{2.14}$$

whereas the remaining 2 Casimirs follow from (2.3).

3. The Ideal $I(n)$

3.1. Arbitrary genus

Let $d_{ik} = t_i t_{i+1} \dots t_{k-1}$ represent the diagonal (see the figure) from P_k to P_i with $d_{ii} = 1$, $d_{j,j+1} = t_j$, $d_{ik}d_{km} = d_{im}$, $d_{ki} = d_{ik}^{-1}$.

Let q be a fixed but otherwise arbitrary point and write

$$\alpha_{ik} = \frac{1}{2}\operatorname{Tr}(S(d_{iq})S(d_{kq}^{-1}))\,, \qquad i, k, q \in Z_n\,. \tag{3.1}$$

This definition is consistent only if we assume the first relator $d_{1,n+1} = 1$ in (1.1), which fixes $n = 2g + 2$.

The set of generic real 2×2 matrices forms a linear space R^4 with scalar product

$$(u, v) = \frac{1}{2}(\operatorname{Tr}(u)\operatorname{Tr}(v) - \operatorname{Tr}(uv)) \tag{3.2}$$

which reduces to $\frac{1}{2}\operatorname{Tr}(uv^{-1})$ with $(u, u) = 1$ for $u, v \in SL(2, R)$.

For $u \in R^4$ the vector $\tilde{u}$ defined by

$$\tilde{u} = -u + \operatorname{Tr}(u)\,1$$

reduces to u^{-1} for $u \in SL(2, R)$. Given $u, v, x, y \in R^4$ the alternating trace

$$T(u, v, x, y) = \operatorname{Tr}[u\tilde{v}x\tilde{y}] - \operatorname{Tr}[\tilde{u}v\tilde{x}y] \tag{3.3}$$

is multilinear and completely antisymmetric in its arguments and therefore proportional to the determinant of the 4 four-vectors u, v, x, y. $T(u, v, x, y) = 0$ is a sufficient and necessary condition for the existence of a linear homogeneous relation among u, v, x, y. Another useful identity is

$$\begin{aligned} &\mathrm{Tr}(u\tilde{v}x) + \mathrm{Tr}(\tilde{u}v\tilde{x}) \\ &\quad = 2(\mathrm{Tr}(\tilde{u}v)\,\mathrm{Tr}(x) + \mathrm{Tr}(\tilde{v}x)\,\mathrm{Tr}(u) - \mathrm{Tr}(\tilde{u}x)\,\mathrm{Tr}(v)). \end{aligned} \tag{3.4}$$

Now consider n generic vectors $v_i \in R^4$, where $i = 1 \ldots n$. The Gram matrix with elements (v_i, v_k) is then of maximum rank 4. Then matrix $C(0)$ (2.1) with $v_i = S(d_{iq})$ is of this type. Given any $n \times n$ matrix $M(\beta)$ of rank $< n - q$ for $\beta = 0$ then

$$\left.\frac{\partial^k \,\mathrm{Det}\, M(\beta)}{\partial \beta^k}\right|_{\beta=0} = 0\,, \qquad 0 \le k \le q\,.$$

It follows that

$$\left.\frac{\partial^k \,\mathrm{Det}\, C(\beta)}{\partial \beta^k}\right|_{\beta=0} = 0\,, \qquad 0 \le k \le q\,, \tag{3.5}$$

and in particular

$$\mathrm{Det}(C(0)) = 2^{-2g-1} + \sum_{m=1}^{g+1} \mathbf{A}_{2g+2,m} = 0 \tag{3.6}$$

for $g > 1$. Equation (3.5) reduces the number of linearly independent Casimirs $\mathbf{A}_{2g+2,m}$ from $g + 1$ to 2.

Lowering the rank of an $n \times n$ symmetric matrix to $n - k$ implies $\frac{k(k+1)}{2}$ independent algebraic conditions on the matrix elements. Here $k = n - 4 = 2g - 2$ and the number of independent traces α_{ij} is $\frac{n(n-1)}{2} - \frac{(n-4)(n-3)}{2} = 6g$. We call these conditions rank identities.

A second class of identities follows from tracing the elements generated by the remaining relators in (1.1). If $R = 1$ is a relator then $\mathrm{Tr}(S(tR) - S(t)) = 0$ with $t \in \pi_1(\Sigma)$ yields a trace relation. Since $S(R) = 1$ poses only 3 conditions we obtain only 3 independent identities for each relator for a total of 6. This leads to $6g - 6$ independent traces α_{ik}, the number of independent moduli on a Riemann surface of genus g.

The ideal $I(n)$ generated by the rank and trace identities is closed under the Poisson brackets. If $I(n)$ is maximal (see Sect. 5) then $K(n)/I(n)$ is identified with the algebra of classical observables.

3.2. The Torus

Here there is no rank identity but the relators imply that $t_3 = t_1^{-1}$, $t_4 = t_2^{-1}$ and therefore $t_1 t_2 = t_2 t_1$. Thus from (3.3) we have $T(1, t_1, t_2, t_1 t_2) = 0$. This reduces the rank of $C(0)$ to 3 so that anyway $\mathrm{Det}(C(0)) = 0$.

The trace identities are then $\alpha_{12} - \alpha_{34} = 0$, $\alpha_{23} - \alpha_{14} = 0$ and $\alpha_{13} + \alpha_{24} - 2\alpha_{12}\alpha_{23} = 0$. The number of independent traces is then reduced from 6 to 2. All these conditions imply that $\mathbf{A}_{41} = -\frac{1}{2}$, $\mathbf{A}_{42} = \frac{3}{8}$, $\Pi_1 = 1$.

$I(n) = I(4)$ is isomorphic to $K(3)/I$, where I is the ideal generated by the single central element $\mathbf{A}_{31} + \frac{1}{4}$ with $\mathbf{A}_{31}$ given by (2.12).

3.3. $g = 2$

The relators $t_1 t_3 t_5 = 1, t_2 t_4 t_6 = 1$ can be rewritten as

$$d_{12}^{-1} d_{13} d_{14}^{-1} d_{15} d_{16}^{-1} = 1\,, \qquad d_{12} d_{13}^{-1} d_{14} d_{15}^{-1} d_{16} = 1\,,$$

or as

$$d_{12}^{-1} d_{13} d_{14}^{-1} = d_{16} d_{15}^{-1}\,, \qquad d_{12} d_{13}^{-1} d_{14} = d_{16}^{-1} d_{15}\,. \tag{3.7}$$

Tracing both of (3.7) and taking the difference and using now (3.3) with $u = 1$, $v = d_{12}$, $x = d_{13}$, $y = d_{14}$ we see that there is a linear homogeneous relation in u, v, x, y and that the Gram determinant of these 4 vectors vanishes. This is precisely the minor of $C(0)$ restricted to α_{km}, $k, m = 1 \dots 4$. By applying $\mathbf{D}(n)$ to this minor we find that any diagonal minor of $C(0)$ of dimension 4 must vanish. Therefore also all off-diagonal minors of $C(0)$ of dimension 4 vanish and the rank of $C(0)$ reduces to 3. The lowering of the rank increases the number of rank identities to 6 and reduces the number of independent traces to 9.

The sum of the traces in (3.7) together with (3.4) lead to the remaining 3 conditions which all follow from

$$\Pi_1 - \alpha_{56} = \alpha_{12}\alpha_{34} + \alpha_{14}\alpha_{23} - \alpha_{13}\alpha_{24} - \alpha_{56} = 0 \tag{3.8}$$

and its images under $\mathbf{D}(n)$.

The ideal $I(6)$ is then generated by 3 trace identities (3.8) and the 6 rank identities reducing the number of independent traces from 15 to 6. Besides (3.8) the ideal includes all its images by the action of $\mathbf{D}(6)$. By using $I(6)$ we can express α_{il}, for fixed $l, i = 1 \dots 6$ as polynomials in the α_{km}, $k \neq l$, $m \neq l$ and show that $\Pi_2 = 1$. We conjecture that $\Pi_g = 1 \bmod I(2g+2)$.

Further, *all* the traces α_{km} can be expressed as polynomials or algebraic functions involving square roots, in terms of the single trace α_{j5} for some fixed j and the restricted set α_{km}, $k, m = 1 \dots 4$. However this restricted set satisfies the condition that the minor of $C(0)$ restricted to $k, m = 1 \dots 4$ vanishes. Therefore the number of independent traces is precisely 6.

4. The Quantum Algebra

There are quantum Casimirs (ordered as in Sect. 1) Q_{nm}, T_g which have $\mathbf{A}_{nm}$ and Π_g as the classical limit but we have not been able to derive a generating function for Q_{nm} similar to $\operatorname{Det} C(\beta)$. By direct check we found the following quantum Casimirs

$$\begin{aligned}
Q_{31} &= \frac{3}{4} - a_{12}^2 - a_{23}^2 - a_{13}^2 K^{-2} + \frac{1+K}{K}\, a_{12}a_{13}a_{23}\,, \\
Q_{41} &= \frac{1}{2}\Bigg(1 - a_{12}^2 - a_{23}^2 - a_{34}^2 - a_{13}^2 K^{-2} - a_{24}^2 K^{-2} - a_{14}^2 K^{-4} \\
&\quad + \frac{K+1}{K}\,(a_{12}a_{13}a_{23} + a_{23}a_{24}a_{34} + K^{-2} a_{12}a_{14}a_{24} \\
&\quad + K^{-2} a_{13}a_{14}a_{34}) - \frac{(1+K)^2}{K^3}\, a_{12}a_{23}a_{14}a_{34}\Bigg)\,, \\
T_1 &= a_{12}a_{34} + K^{-2} a_{23}a_{14} - K^{-1} a_{13}a_{24}\,.
\end{aligned} \tag{4.1}$$

The above Casimirs are only given up to additive constants with zero classical limits.

The quantum ideal for $g = 1$ can be generated by the elements

$$a_{12} - a_{34}\,, \qquad a_{14} - a_{23}\,, \qquad K a_{24} + a_{13} - (K+1) a_{12} a_{23} \tag{4.2}$$

which reduces the algebra A(4) to A(3).

The corresponding ideal for $g = 2$ is generated by

$$T_1 - (1 + K^{-2} - K^{-1}) a_{56} = a_{12} a_{34} + K^{-2} a_{23} a_{14} - K^{-1} a_{13} a_{24} - a_{56} \tag{4.3}$$

and its images under the cyclical group on the indices $1 \ldots 6$.

5. Outlook

The quantum analogue of the matrix $C(\beta)$ and its transformations under $\mathbf{D}(n)$ (1.7) will be discussed elsewhere [4]. We expect this quantum matrix to be related to the quantum Casimirs found so far. The connection between the algebra of observables and the space of moduli must be elucidated. Quantum representations for this algebra will be constructed and the relation with quantum groups, already understood for $g = 1$ [5] will be extended. We are grateful to the referee for raising the important question of whether or not $I(n)$ is maximal. We have heuristic arguments in the affirmative but will return to this more rigorously elsewhere.

Acknowledgement. We thank S. Carlip for stimulating discussions.

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Communicated by N. Yu. Reshetikhin