Commun. Math. Phys. 155, 325–383 (1993)

Communications in
Mathematical
Physics

Construction of YM_4 with an Infrared Cutoff

Jacques Magnen, Vincent Rivasseau and Roland Sénéor

Centre de Physique Théorique, CNRS, UPR14, Ecole Polytechnique,
F-91128 Palaiseau Cedex, France

Received February 17, 1992; in revised form September 8, 1992

Abstract. We provide the basis for a rigorous construction of the Schwinger functions of the pure SU(2) Yang-Mills field theory in four dimensions (in the trivial topological sector) with a fixed infrared cutoff but no ultraviolet cutoff, in a regularized axial gauge. The construction exploits the positivity of the axial gauge at large field. For small fields, a different gauge, more suited to perturbative computations is used; this gauge and the corresponding propagator depends on large background fields of lower momenta. The crucial point is to control (in a non-perturbative way) the combined effect of the functional integrals over small field regions associated to a large background field and of the counterterms which restore the gauge invariance broken by the cutoff. We prove that this combined effect is stabilizing if we use cutoffs of a certain type in momentum space. We check the validity of the construction by showing that Slavnov identities (which express infinitesimal gauge invariance) do hold non-perturbatively.

I. Introduction and Outline

Non-abelian gauge theories form the core of modern high energy physics, and in the recent years they have been very important in pure mathematics too. Perhaps the main reason for this success lies in the discovery that these theories (at the perturbative level) are renormalizable and asymptotically free. Therefore most physicists are convinced that the ultraviolet problem in non-abelian gauge theories is well understood and void of any surprises. However it remains to substantiate this belief rigorously beyond perturbation theory.

The first rigorous program of study of this problem is the one of Balaban [B]. This program defines a sequence of block-spin transformations for the pure Yang-Mills theory in a finite volume on the lattice and shows that, as the lattice spacing tends to 0 and these transformations are iterated many times, the resulting effective action on the unit lattice remains bounded. From this result the existence of an ultraviolet limit for *gauge invariant* observables such as "smoothed Wilson loops" should follow, at least through a compactness argument using a subsequence of approximations; but the limit is not necessarily unique. Clearly this is a point which requires further work. Although very impressive, Balaban's work is not easily accessible, partly because the use of the lattice regularization is the source of many technical complications and partly because the results are scattered over many publications; hence to check the consistency of all the arguments is very difficult. Also it does not address the problem of constructing the expectation values of products of the field operators in a particular gauge (the Schwinger functions), because these are not gauge invariant observables. It is true that physical quantities should be gauge invariant. Nevertheless the gauge fixed framework is obviously the most convenient for perturbative computations, and one can consider in fact that the ultraviolet problem for the Yang-Mills *field* theory is not yet understood until this point is clarified.

A related program was also undertaken by Federbush [F].

In this paper we provide another approach to the same problem, by constructing the Schwinger functions of the pure SU(2) Yang-Mills field in the axial gauge, with an infrared cutoff such as a finite volume box. We outline the construction in a single self-contained paper, but it remains admittedly still very complicated and technical. It certainly requires some knowledge of constructive theory; we assume familiarity of the reader with a reference on the construction of just renormalizable models such as [R]. We do not repeat most of the arguments which are already contained there. We do not claim to provide here the proofs of convergence of our expansion in all detail. However we think that this paper, which summarizes many years of efforts and trials on this problem, both provides a detailed outline of these proofs and remains relatively short and (hopefully!) readable.

Beside these remarks our program has in fact a lot in common with the one of Balaban. We are indebted to him because his pioneering efforts encouraged us to attack this problem; we did not take our technical tools directly in his works, but meeting similar difficulties we think we found often very similar solutions. We do not use the lattice cutoff but a momentum cutoff of a certain type which breaks gauge invariance and requires gauge restoring counterterms which stabilize the field at sizes of order λ^{-1}. We think that the role of this stabilizing cutoff is quite similar to the role played by compactness of integration over the gauge group in Balaban: it provides us with the initial information that the field variables in the Lie algebra are of order at most λ^{-1}. This information by itself is not enough to start perturbation theory, but we have found that if we combine it with the positivity of the axial gauge, then the field variables in the Lie algebra become of order roughly $\lambda^{-1/2}$. The fact that the field is of order roughly $\lambda^{-1/2}$ is however true only in probability. To exploit this fact we have to make a division of the phase space for the field into small field regions and large field regions, and expect that the large field regions are so rare that they can be resummed and controlled. This makes an explicit change of the gauge possible in the small field regions, and in turn this change of gauge allows the use of perturbation theory there (remember that in the axial gauge alone, perturbation theory is sick). However because not all couplings between high and low momenta are of a form which can be dominated in the technical constructive sense, we have to use a background dependent gauge and a background dependent propagator for the analysis of the small field perturbative region. The background field at a given scale and position is roughly speaking made of all the large fields of lower frequencies located at this position.

The use of these background dependent gauge and propagator is a source of technical complications for the cluster expansions of constructive theory and it is also the source of a new difficulty with the evaluation of the large field regions; their functional integral is "renormalized" or "dressed" by their coupling to higher momenta small field regions. As could be understood quite intuitively, this coupling results in a determinant which reflects the difference in normalization between the Gaussian measure with a given background field, and the ordinary Gaussian measure when this background field is 0. One of the main points of this paper is to prove that this determinant can be controlled; it turns out that this is true in the case of the stabilizing cutoffs that we use.

As a justification that our construction is correct we show that the Schwinger functions that we construct satisfy the Slavnov identities which are the remnant of gauge invariance under small gauge transformations at the level of Schwinger functions. This is our main result, formulated at the end of Sect. VIII.

The drawbacks of our approach are that for the moment it is limited to the axial gauge (Feynman gauge or similar ones which are Euclidean invariant and convenient for perturbative computations, and which we use in the small field regions, cannot be used directly at the beginning because of their lack of positivity). Also the stability property that we require for our ultraviolet cutoff allows many different cutoffs but certainly for the moment rules out many others. It would be nice to understand in a deeper way why some cutoffs stabilize the theory and others do not.

We do not investigate invariance under large gauge transformations and non-trivial topological effects such as instantons; also we do not try to remove the infrared cutoff, since this would lead to large values of the coupling constant, and presumably to so-called non-perturbative effects corresponding to confinement. These problems are for the moment still out of the realm of our constructive methods, especially because there is no easy solvable model of these phenomenons around which to expand. Concerning the axioms of quantum field theory, we cannot study the complete set of Osterwalder-Schrader axioms, also because we do not remove our fixed infrared cutoff. However we think that the main axiom, the OS positivity, could be shown to hold with some additional work in the following way. It was proved in [L] and [OS] that OS positivity holds in the lattice gauge theory relative to the hyperplanes of the lattice. We could then use as a first cutoff a lattice cutoff, then use the momentum cutoffs of this paper for slicing and analyzing the theory. If we start the slicing at a scale quite below the inverse of the lattice spacing we think that the gauge-restoring counterterms are close to what they are in the ansatz of this paper. We think that in this way OS positivity can be proved.

In conclusion our statement can be formulated as follows:

The ultraviolet limit as $\varrho \to \infty$ of the Schwinger functions which are the moments of the bare measure defined below in (II.77) *exists; futhermore these functions in the ultraviolet limit satisfy the Slavnov identities* (VIII.6) *adapted to the particular infrared cutoff chosen.*

In this paper we do not provide a detailed proof of this statement but we give all the elements necessary to write such a proof, emphasizing those aspects which we consider technically the most difficult and original. We think that the writing up of such a detailed self-contained proof, which would presumably be several hundred pages long, remains an extremely valuable task.

II. The Starting Ansatz

A) The Model, Notations

We consider the pure Yang-Mills theory with an infrared cutoff, which we never try to lift. This cutoff may be imposed on the propagator, or we could consider the theory on a finite volume with some boundary conditions, or on the sphere S^4, the torus $\Lambda = \mathbb{R}^4/\mathbb{Z}^4$ or another compact Riemannian fourdimensional manifold. Naive infrared regularization breaks gauge invariance, but compactification of space and the choice of a particular bundle with fiber G defines an unbroken group of gauge transformations. For instance in the case of the torus with the trivial SU(2) bundle, the gauge transformation are simply the functions $x \to g(x)$ from $\mathbb{R}^4$ to G which are periodic with period lattice $\mathbb{Z}^4$. The momentum space corresponds to discrete Fourier analysis on the dual lattice $\Lambda^* = \mathbb{Z}^4$. Moreover the constant fields or the zero mode in Fourier space is deleted in all our functional integrals, hence there is no infrared problem.

For the pure SU(2) Yang-Mills theory the vector potential is a field A with components A^a_μ, $\mu = 0,1,2,3$, $a = 1,2,3$ with Lorentz (greek) indices and Lie algebra (latin) indices (the group is noted SU(2) and the algebra su(2)). We have also often to distinguish between the index $\mu = 0$, called the time, and the three other indices, called the spatial indices, usually noted $m, n, \ldots$, $m, n, \ldots = 1,2,3$. Geometrically A is a connection on the considered principal bundle; again in the case of the trivial SU(2) bundle one can consider that each A_μ is simply a function with values in su(2). Our conventions are those of [IZ], which we recall briefly; later to simplify the notations we will forget indices most of the time. We write $A = \sum_{a=1}^{3} A^a t_a$, with $t_a = (i\sigma_a/2)$, where the σ's are the three usual hermitian Pauli matrices. With this convention the covariant derivative is $D_\mu = \partial_\mu - \lambda[A_\mu, .]$ We have $\mathrm{Tr}\, t_a t_b = -\frac{\delta_{ab}}{2}$. The field curvature is:

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - \lambda[A_\mu, A_\nu] = (\partial \wedge A - \lambda[A, A]) , \tag{II.1}$$

λ being the coupling constant; the second notation is a condensed one in which indices are omitted (and $\partial\wedge$ is the exterior derivative). Remark that in the three dimensional su(2) space, the commutator is a wedge product: $[A^a_\mu, A^b_\nu] = \varepsilon^c_{ab} A^a_\mu A^b_\nu$. The pure Yang-Mills action is:

$$-\frac{1}{2}\int_\Lambda d^4x \,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{4}\int_\Lambda d^4x \sum_a F^a_{\mu\nu}F^{\mu\nu a} \tag{II.2}$$

(for e.g. Euclidean canonical metric on the flat torus the raising of "Lorentz" indices is trivial so that $F_{\mu\nu} = F^{\mu\nu}$). To simplify, we define a scalar product $\langle A, B\rangle$ on space time tensors A and B of the same type with values in the Lie algebra, by the convention that a trace is taken over all correspondent space time indices and minus a trace over group indices, so that it is positive definite with a factor 1/2 in component notation. We also write simply A^2 for $\langle A, A\rangle$, and with this convention we can write the action as $\frac{1}{2}\int_\Lambda F^2$. We distinguish between the quadratic, trilinear and quartic pieces of F^2, writing:

$$F^2 = F_2 + \lambda F_3 + \lambda^2 F_4 \,. \tag{II.3}$$

This action is invariant under the gauge transformations:

$$A \to A^g\,; \qquad (A^g)_\mu = gA_\mu g^{-1} + \frac{1}{\lambda}\,\partial_\mu g \cdot g^{-1}\,. \tag{II.4}$$

In what follows these gauge transformations are limited to a particular topological sector, for instance the functions from the compact space to G. It is often useful to consider the infinitesimal gauge transformations γ with values in the Lie algebra, which are tangent to the gauge transformations, such that $g = \varepsilon^{\lambda\gamma}$; the corresponding formula is:

$$A \to A^\gamma\,; \qquad (A^\gamma)_\mu = A_\mu + D_\mu \gamma\,, \tag{II.5}$$

where $D = \partial - \lambda[A, .]$ is the covariant derivative. Finally for technical reasons it is also useful to introduce infinitesimal gauge transformations which correspond to expanding to a finite order in γ the exponential in (II.4). For instance we are interested in the regime where $A \cong \lambda^{-1/2-\varepsilon_1}$ and $\gamma \cong \lambda^{-1/2-\varepsilon_2}$, where ε_1 and ε_2 are very small and we want to keep all terms not small as $\lambda \to 0$. Then we should define

$$A_\mu^{\gamma,2} = A_\mu + D_\mu\gamma + \lambda/2[\gamma, \partial_\mu \gamma]\,. \tag{II.6}$$

This "truncated" gauge transformed configuration $A^{\gamma,2}$ is a polynomial of second order in γ and its derivatives. We could define further expansions of the gauge transformations; with these notations, if $g = \varepsilon^{\lambda\gamma}$, we have $A^g = A^{\gamma,\infty}$ and $A^\gamma = A^{\gamma,1}$.

Our starting point is the Yang-Mills theory in the axial gauge.

This gauge is defined by the condition

$$A_0 = 0\,. \tag{II.7}$$

This is a gauge condition that can be imposed in the sense that for any field configuration A there is a gauge transformation such that A^g in (II.4) satisfies it; indeed we can take

$$g = P\exp\left(-\int_{0,\vec{x}}^{x} A_\mu dx^\mu\right), \tag{II.8}$$

where the P means a path ordered exponential (limit of a Trotter product of exponentials along the path), and the path goes from $\{O, \vec{x}\}$, the point on the hyperplane $x_0 = 0$ to x, hence this path is parallel to the direction 0.

Remark that such an axial gauge condition a priori is not complete, in the sense that even after imposing it there remains a subgroup of the gauge group which acts still on the configurations satisfying (II.7), namely the gauge transformations independent of x_0, the "time coordinate." We do not fix this remaining invariance yet. Remark also that (II.7) is not Euclidean invariant, and the corresponding correlation functions are therefore not Euclidean invariant. However in principle physical quantities (which are gauge invariant and Euclidean covariant observables) can be recovered from the gauge fixed theory. Since these observables involve composite operators, they have to be renormalized and we do not provide the corresponding constructions in this paper, although the task seems accessible to us with our methods.

The main advantage of the axial gauge condition is that it provides some definite positivity. Another advantage (perhaps related...) is that there is no Fadeev-Popov determinant in the axial gauge (more precisely it is a constant absorbed in the normalization), which is a big simplification.

Indeed in the axial gauge we have

$$F^2 = \langle A p_0^2 A \rangle + F_{\mathrm{sp}}^2 , \tag{II.9}$$

where the spatial part of F^2 is by definition

$$F_{\mathrm{sp}} \equiv -\frac{1}{2} \int_\Lambda d^4x \,\mathrm{Tr}\, F_{mn} F^{mn} = \frac{1}{4} \int_\Lambda d^4x \sum_a F^a_{mn} F^{mn,a} , \tag{II.10}$$

and both pieces in (II.9) are obviously positive. Also the piece $\langle A p_0^2 A \rangle$ is quadratic, hence (II.9) looks almost like the usual case of a positive quadratic measure and a positive interaction. This is not the case in e.g. the Feynman gauge where the interaction is not positive in itself but only when combined to the Gaussian measure.

We want to have a well defined functional integral to start with. The scale of our ultraviolet cutoff is called M^ϱ, and the ultraviolet limit is when $\varrho \to \infty$.

From standard renormalization group analysis we learn that in order to get a finite non-trivial renormalized theory at the unit scale of our finite box, we should use a bare coupling constant which has the usual asymptotic behavior with ϱ implied by asymptotic freedom. Hence a good ansatz for the bare coupling λ_ϱ should be:

$$\lambda_\varrho^2 = \frac{1}{-\beta_2 (\mathrm{Log}\, M)\, \varrho + \beta_3/\beta_2 \log \varrho + C} , \tag{II.11}$$

where C is a large constant, and β_2 and β_3 are the usual first non-vanishing coefficients of the β function, whose numerical value is given in standard textbooks like [IZ]. Then one hopes that the renormalized coupling constant λ_{ren}, which should be defined as the last one in a sequence of effective constants, is finite and arbitrarily small as C becomes arbitrarily large (if perturbative renormalization group analysis turns out to be correct). Let us define first the tentative effective coupling at scale i as

$$(\lambda_i^t)^2 \equiv \frac{1}{-\beta_2 (\ln M) i + \beta_3/\beta_2 \ln i + C} . \tag{II.12}$$

Later (in Sect. V) we will have performed the necessary expansions to compute the flow of exact effective coupling constants λ_i which will be very close to the tentative ones.

The class of ultraviolet cutoffs we consider is defined as follows. τ is a fixed function which is between 0 and 1, is 1 near 0 and decreases at infinity. For instance we take a one variable C_0^∞ function, monotone decreasing, which is 0 for $x \geq 2$ and is 1 for $x \leq 1$ (the monotone decreasing and C_0^∞ character are perhaps not essential but it is important that the slices built out of this cutoff by the scaling process defined below have good spatial decay; it is also useful (although perhaps not absolutely necessary) that they vanish identically at zero momentum, a property which we call "good momentum conservation": this property will be used several times in our construction when we need to bound contributions which violate momentum conservation rules. We will usually not provide the corresponding arguments, referring the reader to the last section of [FMRS1] for an example treated in detail.

Then we define our scaled momentum cutoff κ_ϱ to be:

$$\kappa_\varrho(p) = \kappa(pM^{-\varrho}), \qquad \text{(II.13)}$$

where κ is the following function:

$$\begin{aligned}
&\kappa(p) \equiv 1 \quad \text{if} \quad |p| \le 1, \\
&\kappa(p) \equiv \frac{1+\tau(|p|)}{2} \quad \text{if} \quad 1 < |p| \le 2, \\
&\kappa(p) \equiv 1/2 \quad \text{if} \quad 2 < |p| \le 2 + \eta^{-1}, \\
&\kappa(p) \equiv (1/2)\tau(|p| - 1 - \eta^{-1}) \quad \text{if} \quad 2 + \eta^{-1} < |p|,
\end{aligned} \qquad \text{(II.14)}$$

where η is a small constant. This unusual form, shown in Fig. II.1 leads to a stabilizing A^4 counterterm whose strength can be made as large as desired and to a stabilizing functional integral for large background fields; both effects are obtained by taking η sufficiently small as shown in Sect. III.

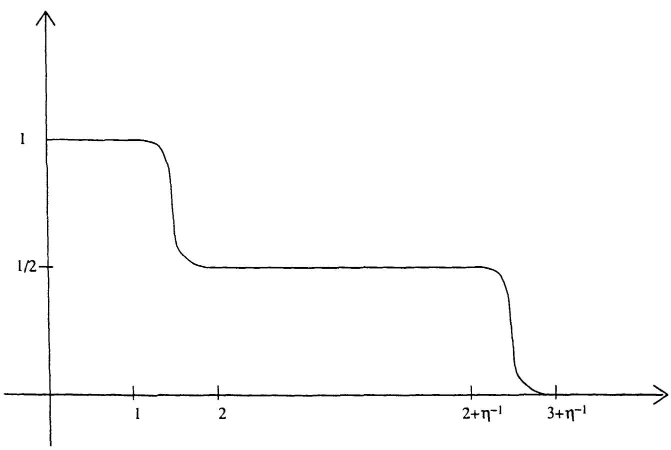

Fig. II.1. The ultraviolet cutoff

We write also

$$\kappa^i = \kappa_i - \kappa_{i-1} \quad \text{if} \quad i \ge 1; \qquad \kappa^0 = \kappa_0. \qquad \text{(II.15)}$$

The quadratic form p_0^2 is not invertible when $p_0 = 0$ and in order to have a good propagator we add and subtract $\sum_i (\lambda_i^t)^2 \langle A, p^2 \kappa^i(p) A\rangle$ to the action [1]. We warn the reader that below we usually write $\lambda^2 p^2$ instead of $\sum_i (\lambda_i^t)^2 p^2 \kappa^i(p)$ which is quite heavy.

The term added is used to create a well defined positive quadratic form which is useful for generating a well defined starting ansatz; also together with the p_0^2 term it

[1] This small term which makes the propagator well defined is harmless since as shown below the cutoffs that we later use will generate a term $-c \cdot \lambda^4 A^4$ in the action which can be used to bound the bad interaction term $\lambda^2 \langle A, p^2 A\rangle$

will be used to prove that the field cannot be much larger than $\lambda^{-1/2}$ in probability. The subtracted piece is treated as an interaction. We define

$$\sum_i (\lambda_i^t)^2 \langle A, p^2 \kappa^i(p) A\rangle = \langle A, C_0^{-1} A\rangle . \tag{II.16}$$

For the moment our formal functional integral in the axial gauge without cutoff is:

$$e^{(1/2)\left(-F_{\text{sp}}^2 - \langle A, p_0^2 A\rangle + \sum_i (\lambda_i^t)^2 \langle A, (p^2\kappa^i(p))\, A\rangle\right)} d\mu_0 , \tag{II.17}$$

where $d\mu_0$ is the normalized Gaussian measure with propagator C_0.

The support of a Gaussian measure such as $d\mu_0$ in (II.17) is made of distributions, as is well known, and since the multiplication of distributions is illegal, (II.17) is still formal. To make sense out of it we have to introduce now a first "fake" ultraviolet cutoff of the theory, at a scale M^{ϱ_1}, with $\varrho_1 \gg \varrho$. This will not be the true cutoff of the theory but is useful in order to manipulate as soon as possible well defined quantities.

We could write instead of (II.17) the functional measure of the theory as:

$$d\mu_{0,\varrho_1}(A)\, e^{(1/2)\left(-F_{\text{sp}}^2 - \langle A, p_0^2 A\rangle + \sum_i (\lambda_i^t)^2 \langle A, (p^2\kappa^i(p))\, A\rangle\right)}$$

which is proportional to

$$d\mu_{\text{axial},\varrho_1}(A)\, e^{(1/2)\left(-F_{\text{sp}}^2 + \sum_i (\lambda_i^t)^2 \langle A, (p^2\kappa^i(p))\, A\rangle\right)} , \tag{II.18}$$

where $d\mu_{0,\varrho_1}$ is the Gaussian measure with propagator $C_0(p)\kappa_{\varrho_1}(p)$, and the sum over i in (II.18) stops at ϱ_1, and the Gaussian measure $d\mu_{\text{axial}}$ and propagators C_{axial} are obtained by joining to C_0 the quadratic piece $\langle A, p_0^2 A\rangle$:

$$\langle A, p_0^2 A\rangle + \sum_i (\lambda_i^t)^2 \langle A p^2 \kappa^i(p) A\rangle = \langle A, C_{\text{axial}}^{-1} A\rangle . \tag{II.19}$$

This formula is still formal, because the positive exponential cannot be integrated simply with the Gaussian measure $d\mu_{0,\varrho_1}$ (this would give back the ill-defined Lebesgue measure). Fortunately this is also not the correct starting point because any continuous ultraviolet cutoff really breaks gauge invariance and to check ultimately Slavnov identities we have to introduce gauge-variant counterterms to compensate these gauge breaking effects of the ultraviolet cutoff. All our construction relies on the use of the additional positivity given by these counterterms.

However these counterterms cannot be computed in perturbation theory in the axial gauge (II.18) because the axial gauge is still incompletely fixed in perturbation theory. In particular our trick of introducing $\lambda^2 p^2$ to create a well defined propagator does not allow perturbative computations. It is only a technical trick to extract easily a small factor for the large field regions at a later stage where the true ultraviolet cutoff and the stabilizing counterterms have been introduced.

All our perturbative computations will be done in the small field region, in which we pass to a particular gauge well suited for perturbation theory, which we call the *homothetic* gauge. It is defined exactly as the Feynman or Landau gauge but with a parameter, called λ in [IZ] and ζ in this paper, which takes a value close to 3/13 [2]. This value is chosen so that there is no infinite wave function renormalization; indeed

[2] More precisely we pass to a background dependent homothetic gauge as discussed below

the one loop wave function renormalization is proportional to $10/3+(1-1/\zeta)$, hence vanishes for $\zeta = 3/13$ [IZ]. Taking a value close to 3/13 we can ensure vanishing at any given order, hence a finite total wave function renormalization either exactly 0 or as small as we want (if we want an explicit formula for ζ).

We want to have an ultraviolet cutoff that gives us simple gauge-breaking effects, computable in perturbation theory in the homothetic gauge. An explicit and still relatively simple cutoff in the axial gauge will transform in a complicated cutoff in this homothetic gauge. Therefore we prefer to impose as our true cutoff a second cutoff which has a simple form in the homothetic gauge.

It is therefore in the way our true ultraviolet cutoff is defined that we incorporate the missing piece of information that we are going to use the homothetic gauge when the field A is small. This piece of information is critical because we actually compute the stabilizing A^4 term generated by the ultraviolet cutoff by a one-loop perturbative computation made in the homothetic gauge.

Remark that the stabilizing term which is part of our initial ansatz is used to stabilize the theory when the field is large although its value is given by a perturbative computation, which seems to require small fields. The ultimate justification of this apparent contradiction lies in the fact that it allows to construct the model with correct Slavnov identities; but we can add a further comment. Stabilization could not be achieved in the large field regions by artificial means such as irrelevant operators (A^6 and so on) because these operators would not be enhanced correctly at lower momenta. In contrast the A^4 term has a flow governed by the small field perturbative regions, which keeps it in tune with the increasing coupling constant at lower scales, and we think that this is the deep reason why its value, computed in the small field region, can also be used to stabilize the large field regions.

The formal formula for passing from the axial gauge to the homothetic gauge is obtained by writing

$$1 = \det[K(A)] \int d\gamma \, e^{-(\zeta/2)(\partial_\mu A_\mu^{\gamma,\infty})^2} , \tag{II.20}$$

where the determinant is the usual determinant of the Fadeev-Popov operator $K(A) = \partial_\mu D_\mu$, with D_μ as in (II.5) (see [IZ]).

This formula in itself cannot contain any new information. But we will use an approximation to (II.20) which amounts no longer to insert 1 but to insert cutoffs, hence there is no contradiction.

Remark that we have written (II.20) in terms of an integration variable γ which lies in the Lie algebra rather than in the Lie group. Indeed it will be easier for us to give a well defined analogue of this functional integration on a flat Lie algebra variable, using standard techniques of constructive field theory such as Gaussian measures perturbed by polynomial interactions.

First we will modify (II.20) by using an approximate gauge transformation $A^{\gamma,2}$ instead of $A^{\gamma,\infty}$ [3]. Also a well defined Gaussian measure with cutoff will be used on γ together with a polynomial which ensures that γ is small compared to λ^{-1} (so that small fields after gauge transformations remain small) but large compared to $\lambda^{-1/2}$ (so that for small fields the formula performs its usual job of integrating out gauge degrees of freedom and changing the gauge at the price of a Fadeev-Popov determinant).

[3] In fact to have correct renormalization group flows to third order in later sections we have to be more cautious and would need something like $A^{\gamma,10}$. But the corresponding formulas are just more complicated and the use of $A^{\gamma,2}$ should make clear how they work in a more general case

Then we will introduce the true ultraviolet cutoff on the transformed field $A^{\gamma,2}$, which as we said is effectively put in the homothetic gauge in the small field region. The gauge restoring counterterms can be therefore perturbatively computed in the homothetic gauge as we desired. These counterterms are also written in terms of $A^{\gamma,2}$. The combination of cutoff and counterterms is balanced so as to restore Slavnov identities.

There is a problem with the use of an ordinary homothetic gauge, which is that some couplings of high momentum fields to low momentum fields are not dominable. This and the meaning of dominable is explained in [R], to which we refer the reader not familiar with this terminology. This problem can be tackled by using covariant derivatives with respect to the low momentum field instead of ordinary derivatives. The price to pay is that the homothetic gauge condition has also to be written with a covariant derivative in a background field instead of an ordinary one. This makes formulas more complicated. The total field is written as the sum of two fields, the one associated with the large field regions and the one associated with small field regions. The gauge transformation of the total field is divided into a gauge transformation on the small field and a rotation on the large field. The background field at a given scale is then made of the large field of lower scales. For this reason it is convenient to introduce the small/large field decomposition before to give the precise form of the ultraviolet cutoff, although it is possible to proceed also in the reverse order, but this would require slightly correcting the formulas by an expansion which suppresses the unwanted small fields from the background.

B) The Small Field and Large Field Decomposition

We want to decide, for a sequence of frequencies M^i, $i = 1, \dots, \varrho$ and a sequence of adapted boxes whether the corresponding fields are smaller or larger than $\lambda^{-1/2-\varepsilon_1}$. This is done by a first expansion. When the field is large, the boxes will be put in the so-called large field region and the axial gauge positivity together with the stabilizing counterterms, which restore gauge invariance after imposition of the cutoff, will provide an associated small factor. This factor is so small that it can be used to finance the creation of protection corridors around the large field regions.

In each box of the small field region the sum of the gradients of the fields of smaller frequencies localized in the box is small because of the A^4 term and the protection corridors around the initial large field region. However for technical reasons it is convenient to increase the strength of this effect. This is done by a second expansion; the boxes which do not satisfy the strengthened condition give small factors and are rejected in the large field region.

At the end of all these tests, in the remaining small field region where all these conditions are satisfied, it will be at last possible to perform the change of gauge which brings us to the homothetic gauge.

We start with the first main test, whether the field A is large or small. The positivity will come from the axial propagator C_{axial}. This propagator is very anisotropic, hence we need to introduce a corresponding anisotropic momentum decomposition.

For every value of $i = 1, \dots, \varrho_1$ we introduce an index α with integer values between N_i and $i+1$, where N_i is the integer part of $i - |\ln(\lambda_i^t)/\ln M|$ (this rule seems obscure but is introduced because when $|p|$ is of order M^i we want to decompose p_0 between λM^i and M^{i+1}). The full set of ordered pairs (i, α) is called $\mathbf{P}$, and the letter j is used for a typical pair (i, α) of $\mathbf{P}$. On this set we introduce an ordering relation, namely we say that $j' = (i', \alpha') < j = (i, \alpha)$ iff $i' < i$ or $i' = i$ and $\alpha' < \alpha$.

For $\alpha \neq N_i$, and $j = (i, \alpha)$ we define

$$\kappa^j(p) = \kappa^{i,\alpha}(p, p_0) = \kappa^i(p)\kappa^{N_i}(p_0), \tag{II.21a}$$

and for $\alpha = N_i$ we put

$$\kappa^j = \kappa^{i,\alpha}(p, p_0) = \kappa^i(p)\kappa_\alpha(p_0). \tag{II.21b}$$

In this way we have allowed all values of p_0 down to $p_0 = 0$.

We extend also formula (II.15) to pair of indices $j = (i, \alpha)$ by setting

$$\kappa_j(p) = \sum_{j'<j} \kappa^{j'}(p), \tag{II.22}$$

and the frequencies appearing in (II.22) are called the low or background frequencies (relative to the pair j).

We decompose the field in direct space as follows:

$$A = \sum_{\mathbf{P}} \tilde{\kappa}^{i,\alpha} * A \equiv \sum_{j\in\mathbf{P}} A^j \tag{II.23}$$

(where the tilde means the Fourier transform).

We introduce also anisotropic lattices $\mathbf{D}_{i,\alpha}$ for $(i, \alpha) \in \mathbf{P}$. The union of these lattices is called $\mathbf{D}$. $\mathbf{D}_{i,\alpha}$ is the lattice of boxes of side M^{-i} in the directions $1, 2, 3$ and of side $M^{-\alpha}$ in the direction 0. It is convenient to take M an integer and these boxes as refinements of a fixed lattice at the unit scale.

In each box $\Delta \in \mathbf{D}$ we write the expansion:

$$1 = e^{-E_\Delta} + \int_0^1 ds E_\Delta \, e^{-(1-s)E_\Delta}, \tag{II.25}$$

where:

$$E_\Delta = \frac{1}{\Delta} \int_\Delta ((\lambda_i^t)^{\frac{1}{2}+\varepsilon_1} M^{-i} \tilde{\kappa}^{i,\alpha} * A)^{P_i}, \tag{II.26}$$

where $P_i = (\lambda_i^t)^{-\varepsilon_1/2}$.

The set of boxes in which the error term is chosen in (II.25) is called the kernel of the large field region (KLFR). This region will be surrounded by protection corridors and enlarged several times, so to have an idea of what is possible, let us announce in advance a lemma whose full proof will be completed later in the paper, and give a sketch of its proof:

Lemma II.1. *To each box of* KLFR $\cap\, \mathbf{D}^{i,\alpha}$ *we can associate a small factor in the functional integral which is* $e^{-(\lambda_i^t)^{-\varepsilon}}$, *for some* $\varepsilon = 0$.

Sketch of proof. Let us consider the propagator C_{axial} obtained by combining $e^{-(1/2)\langle A, p_0^2 A\rangle}$ to $d\mu_0$ as in (II.18). This propagator is multiplied in (II.18) by a positive interaction. If we slice the propagator $C_{\text{axial}}(p)$ according to the partition of unity given by the functions $\kappa^j(p)$, we obtain pieces $C^j_{\text{axial}}(p) \equiv \kappa^j(p) C_{\text{axial}}(p)$ which satisfy, for any fixed large integer q,

$$C^j_{\text{axial}}(x - y) \le \frac{K_q M^{2i}}{\lambda} \left(\frac{1}{1 + |x_0 - y_0| M^\alpha} \cdot \frac{1}{1 + |\vec{x} - \vec{y}| M^i} \right)^q, \tag{II.27}$$

where K_q is some constant depending on q. This bound is immediate if we use integration by parts and the bound

$$\frac{M^{3i}M^{\alpha}}{M^{2\alpha}+\lambda^2 M^{2i}} \le M^{3i}M^{\alpha}M^{-2r\alpha}\lambda^{-2i(1-r)}M^{-2(1-r)} \le \frac{M^{2i}}{\lambda} \quad \text{if} \quad r=1/2\,.$$

Remark that the anisotropic nature of C_{axial} leads to different rates of spatial decay in the zero component and the spatial component of $x-y$. This is the reason for which we must use rectangular boxes with a double index. Remark also that the factor $1/\lambda$ in (II.27) means, as announced, that the Gaussian measure corresponding to C_{axial} gives for a field A^j a typical size $M^i\lambda^{-1/2}$, which is large compared to the size M^i corresponding to Gaussian integration with the propagator of the homothetic gauge, but small compared to the size $M^i\lambda^{-1}$ where perturbation theory becomes meaningless.

In theory it might be sufficient to take q in (II.27) equal to 4, so that the propagator is summable, but in practice we will take it to be large, e.g. 100, in order to have some margin for the convergence of cluster expansions.

Using the bound (II.27) we can perform a cluster expansion between the rectangular boxes of the large field region [4]. Each factor E_Δ contains P_i fields which are integrated with respect to C_{axial}. As is usual when the spatial decrease of the propagator is matched to the shape of the boxes in which the cluster expansion is performed, we obtain a product of local factorials in the number of the fields in each box [R].

Therefore for each box Δ we have a factor

$$(\lambda_i^t)^{\varepsilon_1 P_i} K^{P_i}(P_i/2)! \le e^{-(\lambda_i^t)^{-\varepsilon_1/2}} \tag{II.28}$$

if λ_i^t is small enough (such that $\sqrt{K}(\lambda_i^t)^{3\varepsilon_1/4} \le 1/e$, recalling that $P_i = (\lambda_i^t)^{-\varepsilon_1/2}$). This is the small factor announced in Lemma II.1. However the complete proof of Lemma II.1 is of course more complicated than this sketch; indeed the functional integral (II.18) contains the negative factor $-F_{\text{sp}}^2$ which helps in reducing the value of the functional integral, but also the positive factor $\lambda\langle A, p^2 A\rangle$; it is also incomplete because we should add to it both the counterterms required to restore gauge invariance and a term coming from the functional integrals over a certain set of small fields associated to the large field box Δ. This last term, a normalizing determinant announced in the introduction, comes from the dependence in the background field of the Gaussian measure used in the small field regions.

The stability estimates of Sect. VI then prove that the total weight of the positive factor $\lambda\langle A, p^2 A\rangle$, the counterterms CT [see (II.40)] and the normalizing determinant coming form the functional integrals over small field regions associated to any large field box is bounded by 1. This really achieves the proof of Lemma II.1, but we think that to state it here may help the reader understand the choice of the factors (II.25–26) and the definition of protection corridors which we introduce later.

The morale of Lemma II.1 is that we can associate a small factor not only to any box of KLFR but also to a lot of neighboring boxes; first the ordinary neighboring boxes in the same slices $\mathbf{D}^j$ up to a certain distance, but also boxes which are included into or contain a box of KLFR and have neighboring values of their index j. The small factor of Lemma II.1 finances the creation of all these corridors later in this paper

[4] The large field frequency splitting is in fact performed on the field, not on the axial propagator [see (II.23)] so that the covariance is not diagonal; this complicates slightly the cluster expansion, but the conclusion is the same

provided we respect to golden rule that their width both in space and momentum (index) directions be bounded so that this small factor in the (II.28) divided into the total number p of boxes in the corridors around a single large field box is still small as $\lambda \to 0$ (i.e. $\lambda^{-\varepsilon_1/2} \gg p$). This rule is necessary for the cluster and Mayer expansions to converge (see e.g. [DMR] for a simple example of such a situation).

Let us return to the complete definition of the large field region.

We need further to know in each box of $\mathbf{D}$ whether the sum of the gradient of the fields of lower frequencies localized in the box is large or small. To gain a small factor we need to create a gap between the scale of the box and the frequencies tested.

In every box of $\mathbf{D}$ we write:

$$1 = \tau(H_\Delta) - \int_0^1 ds H_\Delta \tau'((1-s)H_\Delta), \tag{II.29a}$$

where τ is our reference C_0^∞ function and

$$H_\Delta = \frac{1}{\Delta}\int_\Delta ((\lambda_i^t)^{1-(\varepsilon_1/64)} M^{-2i} \nabla B(\Delta, x))^{P_{1,i}}, \tag{II.29b}$$

where $P_{1,i}$ is an even integer close to $(\lambda_i^t)^{-\varepsilon_1/32}$,

$$B(\Delta, x) = \sum_{\Delta' \in D_{i',\alpha'}, r(\Delta) > r(\Delta') - k(\Delta)} \chi_{\Delta'}(x) \tilde{\kappa}_{i',\alpha'} * A \tag{II.29c}$$

with $r(\Delta) = (3i+\alpha)/4$, and if $\Delta \in \mathbf{D}_{i,\alpha}$:

$$M^{k(\Delta)} = (\lambda_i^t)^{-\varepsilon_1/16}. \tag{II.30}$$

The large field region is now defined as the set $\mathbf{D}_1$ of boxes in which the error term of (II.25) is chosen, plus their protection corridors, i.e. the boxes Δ' which intersect a box Δ of $\mathbf{D}_1$ and satisfy to $(\lambda_i^t)^{1/16} < M^{r(\Delta)-r(\Delta')} < (\lambda_i^t)^{-1/16}$, to which we add the set $\mathbf{D}_2$ of boxes in which the error term in (II.29a) is chosen, plus a protection corridor around them of the same type but with smaller width, i.e. the boxes Δ' which intersect some Δ which belongs to $\mathbf{D}_2$ and satisfy

$$(\lambda_i^t)^{1/128} < M^{r(\Delta)-r(\Delta')} < (\lambda_i^t)^{-1/128}. \tag{II.31}$$

Before the final bounds are derived, let us again explain in anticipation why the boxes with the error terms have a small factor attached to them. The reasoning is similar to the sketch of proof of Lemma II.1. This is because a ∇B field of scale i_2 produced at level $i_1 > i_2$ is evaluated by a factor $\lambda_{i_2}^{-1/2} M^{2i_2} \leq \lambda_{i_1}^{-1/2} M^{2i_1} M^{-2(i_1-i_2)}$. The local factorials created by accumulation of many ∇B factors coming from the many boxes of scale i_1 in the same box of scale i_2 are then compensated by the $M^{-2(i_1-i_2)}$ factors. The rest of the argument is as in Lemma II.1, the value of $P_{1,i}$ being adapted for it to work.

Finally we want to prepare the formulas better for the small field change of gauges. We want that the background field is reduced to the field of the low momentum large field regions. Recall that the rationale for introducing this background field and a modified homothetic gauge in background field, was that some interaction terms between high and low momentum field were not dominable; hence it is necessary to absorb them into the propagator. However fields in the small field region are always dominable (by the small field condition). Therefore we do not need to put them in the background. Furthermore it would be bad to leave them in the background, because we want to perform a gauge transformation on the full small field, region and to decompose the field into a gauge-transformed small field plus a rotated background field.

The large field region is called LFR $= \bigcup_{i,\alpha\in\mathbf{P}} L_{i,\alpha}$. Its complement is the small field region SFR $= \bigcup_{i,\alpha\in\mathbf{P}} S_{i,\alpha}$. We are going to introduce relations between the rectangular boxes of SFR and LFR. A box $\Delta \in S_{i,\alpha}$ is called *relevant* if there exists a box $\Delta' \in L_{i,\alpha'}$ such that $\Delta \subset \Delta'$. In this case we call the smallest such box Δ' the ancestor of Δ. If this is not the case the box Δ, called irrelevant, is divided into $M^{i+1-\alpha}$ boxes of the standard lattice $\mathbf{D}_i$, and we forget about the corresponding division of frequencies on p_0. This is justified because in these regions the frequencies on p_0 do not have any cutoff imposed by the presence of large field boxes (recall that there cannot be any ultraviolet limitation on p_0 because of our definition of protection corridors). Since in the small field region an Euclidean invariant propagator is going to be introduced, there is therefore no need to keep the decomposition over α and over rectangular boxes. From now on, when we consider a small field region it is therefore made of relevant rectangular boxes associated to specific ancestors boxes of large field regions with *same* index i but lower index α, and of ordinary *cubes* of $\mathbf{D}_i$. For these cubes Δ we also define a notion of ancestor. We consider in turn all indices $i' < i$, starting with $i' = i - 1$, then $i' = i - 2$ and so on, and for each such value of i' we search for the smallest rectangular box of $L_{i',\alpha'}$ containing the cubic box Δ. The first rectangular box found in this way is called the ancestor of Δ. The cubes which have no ancestor are said to be in the main small field region. All the small field boxes which have as common ancestor the large field box Δ' are said to form the small field region SFR(Δ') associated to Δ', or in short the Δ'-small field region.

Because of the rarity of large field boxes (see Lemma II.1), the reader should imagine that most boxes are small field boxes in the main region, i.e. without ancestors. In Sect. IV–VI, it will be shown how the functional integrals corresponding to a Δ-small field region give a non-trivial contribution associated to the large field box Δ which has to be bounded in a non-perturbative way. In the next subsection we will indeed prepare a background-dependent gauge in the small field region which is responsible for this non-trivial effect.

C) The Modified Homothetic Gauge in the Background Field

A gauge transformation which acts on the sum of two fields can be decomposed into a gauge transformation on the first and a rotation on the second:

$$(A+B)^{\gamma,\infty} = A^{\gamma,\infty} + B^{\mathrm{rot}\,\gamma,\infty}\,. \tag{II.32a}$$

This is also true for the truncated versions of the gauge transformations introduced above:

$$(A+B)^{\gamma,n} = A^{\gamma,n} + B^{\mathrm{rot}\,\gamma,n}\,, \tag{II.32b}$$

where the index n means that the gauge transformation for A and the rotation for B are truncated at order n. The term $B^{\mathrm{rot}\,\gamma}$ is the linear part in B of the gauge transformation B^γ. For example

$$B^{\mathrm{rot}\,\gamma,2} = B - \lambda[B,\gamma]\,. \tag{II.32c}$$

We decompose the full field A as the sum of the small field A_s, and the large field B_l:

$$A_s(x) = \sum_{(\imath,\alpha),\Delta\in S_{\imath,\alpha},x\in\Delta} A^{\imath,\alpha}(x)\,, \tag{II.33}$$

$$B_l(x) = \sum_{(i,\alpha),\Delta\in L_{i,\alpha},x\in\Delta} A^{i,\alpha}(x)\,, \tag{II.34}$$

hence $A = A_s + B_l$.

We put together the factors associated to the small field and large field conditions as $\chi_{\mathrm{LFR}}(A)$: SFR is then automatically the complement of LFR.

The functional measure of the theory is now written as:

$$\sum_{\mathrm{LFR}} \int d\mu_{0,\varrho_1}(A)\,\chi_{\mathrm{LFR}}(A)\,e^{(1/2)\left(-F_{\mathrm{sp}}^2-\langle A,p_0^2A\rangle+\sum_\imath(\lambda_\imath^t)^2\langle A,(p^2\kappa^\imath(p))\,A\rangle\right)} \tag{II.35}$$

and we insert now our analogue of the Fadeev-Popov formula (II.20).

We want that the gauge transformations γ in (II.20) cover all the small field regions. As explained in the outline the scale given by the axial gauge positivity plus the A^4 counterterm is $A' \cong \lambda^{-1/2}M^i$. Since we need a small margin to gain some small factor in the large field region, the small field region was chosen in (II.26) to be of the type $A^i < (\lambda_i^t)^{-1/2-\varepsilon_1}M^i$. Therefore we use as a measure over γ a quadratic form which gives to γ at scale i a typical size $(\lambda_i^t)^{-(1/2+\varepsilon_2)}$, where $1 \gg \varepsilon_2 > \varepsilon_1$. Hence we choose as propagator

$$\Gamma_{\varrho_2}(p) = \sum_{i=1}^{\varrho_2}\Gamma^\imath_{\varrho_2}(p)\,,\qquad \Gamma^i_{\varrho_2}(p) = \sum_{\imath=1}^{\varrho_2}(\lambda_i^t)^{-(1+2\varepsilon_2)}\,\frac{\kappa^i(p)}{p^4}\,, \tag{II.36}$$

where $\varrho_2 \ll \varrho_1$. The Gaussian measure on γ with covariance Γ_{ϱ_2} is called $d\nu_{\varrho_2}(\gamma)$. According to the decomposition (II.15) we can also split in Fourier space γ as $\sum_{i=0}^{\varrho_2}\gamma^i(p)$, $\gamma^i(p) \equiv \kappa^\imath(p)\gamma(p)$ (we do not need the anisotropic indices at this stage because the homothetic gauge and the corresponding Fadeev-Popov formula that we are going to introduce are perfectly isotropic).

A Gaussian measure to bound the size of γ is however not sufficient; for technical reasons we need to reinforce its strength by a polynomial of high degree. In order for this polynomial to behave at small γ as a small perturbation of a Gaussian measure so that perturbative analysis remains all right, we take this polynomial to give a slightly larger size, $(\lambda_i^t)^{-(1/2+2\varepsilon_2)}$, to γ, (still much smaller than λ^{-1}). Therefore we define

$$K_{\varrho_2}(A_s,B_l) = \int d\nu_{\varrho_2}(\gamma)\,e^{-(\zeta/2)(\nabla_B(A_s,B_l,\gamma,2))^2}\,e^{-\sum_\imath((\lambda_\imath^t)^{1/2+\varepsilon_2}\gamma^\imath)^N}\,, \tag{II.37}$$

where N is some large integer, ζ is the number close to 3/13 defining the homothetic gauge, and ∇_B is the covariant derivative in the background field, which is defined in the case of our truncated transformations by

$$\nabla_B(A_s,B_l,\gamma,2) \equiv \partial_\mu(A_s^{\gamma,2})_\mu - \sum_j \lambda[\kappa_\jmath * (B_l^{\mathrm{rot}\,\gamma,2})_\mu, \kappa^\jmath * (A_s^{\gamma,2})_\mu]\,, \tag{II.38}$$

where the star is a convolution in x-space, and for simplification the Fourier transform of $\kappa^j, \kappa_j \ldots$ is from now on also noted $\kappa^j, \kappa_j \ldots$. The rotation $A^{\mathrm{rot}\, n}$ is defined in (II.32a–c).

From now on we warn the reader that most of the time we use simply the notation λ instead of λ_i^t and leave to the reader to reconstruct the correct value according to the frequency of the fields concerned; for instance in (II.38), λ should be understood as λ_j^t. This wil simplify the rather complicated formulas of this section. Similarly we omit from now on the explicit dependence in λ_i^t in (II.35–37) etc.

$K_{\varrho_2}(A)$ is well defined since it is a functional integral of a bounded polynomial interaction with a Gaussian measure with ultraviolet cutoff.

Then we write instead of (II.18) the functional measure of the theory as:

$$\sum_{\mathrm{LFR}} \int d\mu_{0,\varrho_1}(A)\, d\nu_{\varrho_2}(\gamma)\, \chi_{\mathrm{LFR}}[K_{\varrho_2}(A_s, B_l)]^{-1} \times e^{(1/2)\left(-F_{\mathrm{sp}}^2 - \langle A, p_0^2 A\rangle + \sum_i \lambda^2 \langle A, (p^2 \kappa^i(p))\, A\rangle\right)}\, e^{-(\zeta/2)(\nabla_B(A_s, B_l, \gamma, 2))^2}\,. \tag{II.39}$$

We impose now the true cutoff at a scale M^{ϱ} with $\varrho \ll \varrho_2 \ll \varrho_1$. To compensate the gauge breaking effects of this true cutoff will require some well defined counterterms which are computed in Sect. III. For the moment these counterterms are written simply as CT_{ϱ}.

This true cutoff changes the formula (II.39) into

$$\sum_{\mathrm{LFR}} \int d\mu_{0,\varrho_1}(A)\, d\nu_{\varrho_2}(\gamma)\, \chi_{\mathrm{LFR}}(A)\, e^{-\sum_i (\lambda^{1/2+2\varepsilon_2}\gamma^i)^N} [K_{\varrho,\varrho_2}(A_s, B_l, \gamma)]^{-1} \times e^{-(1/2)\langle A^{\gamma,2}, [(\kappa_{\varrho})^{-1} - 1](p^2) A^{\gamma,2}\rangle}\, e^{(1/2)\left(-F_{\mathrm{sp}}^2 - \langle A, p_0^2 A\rangle + \sum_i \lambda^2 \langle A, (p^2\kappa^i(p))\, A\rangle\right)} \times e^{-(\zeta/2)(\nabla_B(A_s, B_l, \gamma, 2))^2}\, e^{\mathrm{CT}_{\varrho}(A^{\gamma,2})}\,. \tag{II.40}$$

It remains to explain what is K_{ϱ,ϱ_2} in this formula. It is an analogue of K_{ϱ_2} in (II.36) but takes into account the addition of the true cutoff at scale ϱ to the fake protecting cutoff at scale ϱ_2. However its precise definition is somewhat complicated and we want to postpone it for a while, but let us explain at least the guiding idea here. We want to have a cutoff of the same scale and shape for the propagator of the A field in the homothetic gauge and the propagator of the Fadeev-Popov ghosts. Therefore we do not want to use directly the functional integral $K_{\varrho_2}(A)$, which is our analogue of the Fadeev-Popov determinant; it would not have the correct cutoff on the ghost field propagator. We use the fact that this functional integral tends to the usual Fadeev-Popov determinant in the limit $A \to 0$, and we replace it by a different functional integral K_{ϱ,ϱ_2} in which the propagator of the ghost field is cut again at scale ϱ, this time in the way we want for nice perturbative computations. The reader might object that therefore we have not inserted really the value 1 as in (II.20), hence the formula does not correspond really to the axial gauge starting point. This remark applies also to the imposition of the cutoff on $A^{\gamma,2}$, and indeed we explained already that our ansatz contains more than simply the axial gauge ansatz. However the difference between K_{ϱ,ϱ_2} and K_{ϱ_2} will be made of terms with momenta between M^{ϱ} and M^{ϱ_2}. They will not affect the validity of Slavnov identities for the final theory, since their effect on any fixed scale vanishes in the limit $\varrho \to \infty$, hence the replacement of K_{ϱ_2} by K_{ϱ,ϱ_2} can be also considered as part of the definition of our ultraviolet cutoff. Let us stress that this point is technical; it would be presumably possible to use the cutoff given by K_{ϱ_2} but the computation of the ghost contribution to the gauge breaking counterterms, in particular the graph G_4 in Sect. III would be more difficult. Since the contribution of this graph is much smaller than the contributions of G_1, G_2 or G_3 in Sect. III, the conclusions concerning the stability of the ultraviolet cutoff would be presumably almost the same. However we prefer to use a complicated redefinition of the initial cutoff on the γ field which is our substitute for the Fadeev-Popov ghosts, in order to allow in the next section a simpler perturbative computation of CT_ϱ.

We are going to give later in this section the precise definition of K_{ϱ,ϱ_2}. This definition like most of the ansatz (II.40) is best expressed in terms of $A^{\gamma,2}$, the correct variable in (II.40) for a small field. We explain therefore first the change of variables which consists in using as a new variable for the main functional integration $A' = A^{\gamma,2}$ instead of A.

Remark that this change of variables is one to one, namely it is possible to compute directly the initial field A in terms of A', since the transformation $A \to A^{\gamma,2}$ is invertible. The inversion formula exists (also for higher orders approximations to true gauge transformations) and is a rational function of γ. (There are also good polynomial approximations to the inverse transformation, namely the transformations $A \to A^{-\gamma,n}$). For instance, if we write $A' = A^{\gamma,2} = T(\gamma).A + U(\gamma)$, with $T.A = A - \lambda[A,\gamma]$ and $U = \partial\gamma + 1/2[\gamma,\partial\gamma]$, in su(2) space the matrix of T is

$$T_{ab}(\gamma) = \delta_{ab} - \varepsilon_{abc}\lambda\gamma_c\,. \tag{II.41}$$

Its inverse is

$$T^{-1}_{ab} = \frac{1}{1+\lambda^2\sum_d \gamma_d^2}\,(\delta_{ab} + \varepsilon_{abc}\lambda\gamma_c + \lambda^2\gamma_a\gamma_b) = \delta_{ab} + H_{ab}\,, \tag{II.42}$$

where H is a small matrix; the transformation $A \to A'$ is therefore inverted by $A = T^{-1}(A' - U)$.

Furthermore the Jacobian of the change of variables associated to a true gauge transformation is one, since the linear piece is an inner automorphism. For the truncated gauge transformation this is no longer exactly true. For instance for the truncated transformation $A \to A^{\gamma,2}$ the linear piece is $A \to TA$, and the Jacobian is $J(\gamma) \equiv (1+\lambda^2\gamma^2)^{-1}$.

The formal Lebesgue measure changes therefore, if $A' = A^{\gamma,2}$ as:

$$\prod_x dA(x) \to \prod_x dA'(x) \prod_x (1+\lambda^2\gamma^2(x))^{-1}\,. \tag{II.43}$$

Since we use Gaussian measures, we have a well defined analogue of this formal formula. Let us consider again $d\mu_{0,\varrho_1}(A)$ which is the initial normalized Gaussian measure with propagator C_{0,ϱ_1} used to define our functional integral over A. This measure can be recomputed exactly in terms of $A' = A^{\gamma,2}$ using as a guide the following formal manipulations: $A = T^{-1}(A' - U(\gamma))$,

$$\begin{aligned} d\mu_{0,\varrho_1}(A) &= \frac{e^{-(1/2)AC^{-1}_{0,\varrho_1}A}dA}{\int e^{-(1/2)AC^{-1}_{0,\varrho_1}A}dA} = \frac{e^{-(1/2)(A'-U)(T^{\mathrm{tr}})^{-1}C^{-1}_{0,\varrho_1}T^{-1}(A'-U)}J(\gamma)\,dA'}{\int e^{-(1/2)(A'-U)(T^{\mathrm{tr}})^{-1}C^{-1}_{0,\varrho_1}T^{-1}(A'-U)}J(\gamma)\,dA'} \\ &= d\mu_{0,\varrho_1}(A')\,\frac{G(A',\gamma)}{\int G(A',\gamma)\,d\mu_{0,\varrho_1(A')}}\,, \end{aligned} \tag{II.44}$$

where G contains correction terms in the difference H between T^{-1} and Id, and terms in U:

$$G(A',\gamma) \equiv e^{+A'^t H^{\text{tr}} C^{-1}_{0,\varrho_1} T^{-1} A' + A'^t (T^{\text{tr}})^{-1} C^{-1}_{0,\varrho_1} H A' - A'^t H^{\text{tr}} C^{-1}_{0,\varrho_1} H A'} \times e^{+A'^t (T^{\text{tr}})^{-1} C^{-1}_{0,\varrho_1} T^{-1} U + U^{\text{tr}} (T^{\text{tr}})^{-1} C^{-1}_{0,\varrho_1} T^{-1} A' - U^{\text{tr}} (T^{\text{tr}})^{-1} C^{-1}_{0,\varrho_1} T^{-1} U} . \quad \text{(II.45)}$$

Our initial axial field A has only nine scalar components since A_0 was identically 0. We want that the special-gauge field A' contains the usual twelve components. In fact one should have $A'_0 \cong 0^{\gamma,2} = \partial_0\gamma + (\lambda/2)[\gamma, \partial_0\gamma]$. But since the change of variables $\gamma \to \partial_0\gamma$ is not invertible and we need to keep in our formulas a functional integration over γ, it is convenient (although not necessary) to create the A'_0 field ex nihilo by a functional formula which peaks it automatically around the desired value. This formula is

$$1 = L_{0,\varrho_1}(\gamma) \int d\mu_{C_{0,\varrho_1}}(A'_0) F(A'_0,\gamma), \quad \text{(II.46)}$$

$$F(A'_0,\gamma) \equiv e^{-\sum_{\Delta \in \mathbf{D}_{\varrho_1}} |\Delta|^{-1} \int_\Delta (A'_0 - \partial_0\gamma - 1/2\lambda[\gamma,\partial_0\gamma])^{N'}}, \quad \text{(II.47)}$$

where N' is some large integer and L_{0,ϱ_1} is the inverse of the integral in (II.46) so that (II.46) is true; it is a slowly varying function of γ which can be integrated with the measure on γ in (II.37). In this way, since the frequency ϱ_2 is much smaller than ϱ_1 the field A'_0 coincides very accurately at scale ϱ_2 with the desired expression $\partial_0\gamma + (\lambda/2)[\gamma, \partial_0\gamma]$.

We write

$$A'_s \equiv A_s^{\gamma,2}; \qquad B'_l \equiv B_l^{\text{rot}\,\gamma,2}; \qquad (A'_s)_0 \equiv A'_0; \qquad (B'_l)_0 \equiv 0 \quad \text{(II.48)}$$

so that $A' = A'_s + B'_l$.

We obtain:

$$\sum_{\text{LFR}} \int d\mu_{0,\varrho_1}(A') d\nu_{\varrho_2}(\gamma) \chi_{\text{LFR}}(A) G(A',\gamma) e^{-\sum_i (\lambda^{1/2+2\varepsilon_2}\gamma^i)^N}$$
$$\times e^{-(1/2)\langle A',[(\kappa_\varrho)^{-1}-1](p^2)A'\rangle} e^{\text{CT}_\varrho(A')}$$
$$\times e^{(1/2)\left(-F^2_{\text{sp}}(A) - \langle A, p_0^2 A\rangle + \sum_i \lambda^2 \langle A,(p^2\kappa^i(p))A\rangle\right)} [K_{\varrho,\varrho_2}(A_s,B_l,\gamma)]^{-1}$$
$$\times e^{-(\zeta/2)(\nabla_{B'_l}\cdot A'_s)^2} L_{0,\varrho_1}(\gamma) F(A'_0,\gamma), \quad \text{(II.49)}$$

where now:

$$\nabla_{B'_l} \cdot A'_s = \partial_\mu (A'_s)_\mu - \sum_j \lambda[\kappa_j * (B'_l)_\mu, \kappa^j * (A'_s)_\mu]. \quad \text{(II.50)}$$

The operator $\nabla_{B'_l}$ is very important in what follows, because the covariance of A'_s in the small field regions where we will perform most of our analysis is built out of it. This operator has of course a spatial index μ which is omitted in (II.49–50); we hope that the scalar product in (II.49) is clear; the rôle played by the factor $\partial_\mu A^\mu$ in the Landau, Feynman or homothetic gauge condition is now played by the factor $(\nabla_{B'_l})_\mu (A'_s)_\mu$.

In this way we have both a functional integral over a twelve component field A' and a three component functional integral over γ. In (II.49) it is now the old field A which should be considered a function of A' and γ through the formula $A(A',\gamma) = T^{-1}(A' - U)$.

Let us turn now to the precise definition of K_{ϱ,ϱ_2}. It is given by an integral over a variable which we will call γ' to distinguish it from the variable γ in (II.50). We have first to reexpress K_{ϱ_2} in terms of the new variable A'.

Instead of using the inverse transformation T^{-1} we will use the approximate inverse transformation so that we have polynomial error terms. More precisely we write:

$$A_\mu = (A_\mu^{+\gamma,2})^{-\gamma,2} + R_\mu(A,\gamma)\,, \tag{II.51}$$

$$B_\mu = (B^{\mathrm{rot}\,\gamma,2})^{\mathrm{rot}(-\gamma),2} + R'_\mu(B,\gamma)\,, \tag{II.52}$$

$$R_\mu(A,\gamma) = \lambda^2([[A_\mu,\gamma],\gamma] + (1/2)\,[[\partial_\mu\gamma,\gamma],\gamma])\,, \tag{II.53}$$

$$R'_\mu(A,\gamma) = \lambda^2[[A_\mu,\gamma],\gamma]\,. \tag{II.54}$$

We have first to express $K_{\varrho_2}(A)$ in terms of A' and γ:

$$K_{\varrho_2}(A',\gamma) = \int d\nu_{\varrho_2}(\gamma')\, e^{-\sum_i((\lambda_i^t)^{1/2+\varepsilon_2/4}\gamma'^i)^N}$$
$$\times\, e^{-(\zeta/2)(\nabla_B((A'_s)^{-\gamma,2}+R_\mu(A_s,\gamma),(B'_l)^{-\,\mathrm{rot}\,\gamma,2}+R'_\mu(B_l,\gamma),\gamma',2))^2} \tag{II.55}$$

(see (II.37) for definition of our notation $\nabla_B(A,B,\gamma,2)$).

We can now compute:

$$((A'_s)_\mu^{-\gamma,2} + R_\mu(A,\gamma))^{\gamma',2} = (A'_s)_\mu + D_\mu(A'_s)\cdot(\gamma'-\gamma) + S_\mu(A'_s,\gamma,\gamma')\,, \tag{II.56}$$

$$S_\mu(A',\gamma,\gamma') = +(\lambda/2)\,([\partial_\mu(\gamma-\gamma'),\gamma'] + [\gamma-\gamma',\partial_\mu\gamma] + O(\lambda^2)\,, \tag{II.57}$$

$$O(\lambda^2) = R - \lambda[R,\gamma'] - \lambda^2([[A'_\mu,\gamma],\gamma'] + (1/2)\,[[\gamma,\partial_\mu\gamma],\gamma'])\,. \tag{II.58}$$

Similarly:

$$((B'_l)^{\mathrm{rot}\,-\gamma,2} + R')^{\mathrm{rot}\,\gamma',2} = (B'_l)^{\mathrm{rot}(\gamma'-\gamma),2} + S'\,, \tag{II.59}$$

$$S' = R' - \lambda[R',\gamma'] - \lambda^2[[B',\gamma],\gamma']\,. \tag{II.60}$$

We substitute (II.56–60) into (II.55). We find:

$$\begin{aligned}
&\nabla_B((A'_s)^{-\gamma,2} + R_\mu(A_s,\gamma),(B'_l)^{-\,\mathrm{rot}\,\gamma,2} + R'_\mu(B_l,\gamma),\gamma',2)\\
&= \partial_\mu[(A'_s)_\mu + D_\mu(A'_s)\,(\gamma'-\gamma) + S_\mu(A'_s,\gamma,\gamma')]\\
&\quad - \sum_j \lambda[\kappa_j * ((B'_l)^{\mathrm{rot}(\gamma'-\gamma),2} + S')_\mu, \kappa^j * ((A'_s)_\mu\\
&\quad + D_\mu(A'_s)\,(\gamma'-\gamma) + S_\mu(A'_s,\gamma,\gamma'))]\\
&= (\nabla_{RB,\gamma'-\gamma})_\mu((A'_s)_\mu + D_\mu(A'_s)\,(\gamma'-\gamma)) + S''\,,
\end{aligned} \tag{II.61}$$

where the operators $\nabla_{RB,\gamma'-\gamma}$ is defined by

$$(\nabla_{RB,\gamma'-\gamma})_\mu A' = \partial_\mu A' - \sum_j \lambda[\kappa_j * (B'_l)_\mu^{\mathrm{rot}(\gamma'-\gamma),2}, \kappa^j * A'] \tag{II.62}$$

and

$$S'' = \partial S - \sum_j \lambda[\kappa_j * S', \kappa^j * (A'_s + D(A'_s)(\gamma' - \gamma) + S(A'_s, \gamma, \gamma'))] - \sum_j \lambda[\kappa_j * ((B'_l)^{\mathrm{rot}(\gamma'-\gamma)} + S'), \kappa^j * S(A'_s, \gamma, \gamma')] . \tag{II.63}$$

Therefore

$$((\nabla_{RB,\gamma'-\gamma})_\mu (A'_s + D(A'_s)(\gamma' - \gamma))_\mu + S'')^2 = (\nabla_{RB,\gamma'-\gamma} \cdot (A'_s + D(A'_s)(\gamma' - \gamma)))^2 + \Sigma(A', \gamma, \gamma') , \tag{II.64}$$

where

$$\Sigma(A', \gamma, \gamma') \equiv 2S''(\nabla_{RB,\gamma'-\gamma} \cdot (A'_s + D(A'_s)(\gamma' - \gamma))) + (S'')^2 , \tag{II.65}$$

is a small correction term which will be treated as an interaction. (There are some implicit summations over μ in the formulas above.)

With these notations:

$$K_{\varrho_2}(A', \gamma) = \int d\nu_{\varrho_2}(\gamma')\, e^{-\sum_i (\lambda^{1/2+2\varepsilon_2}\gamma'^i)^N} \times e^{-(\zeta/2)(\nabla_{RB,\gamma'-\gamma} \cdot (A'_s + D(A'_s)(\gamma'-\gamma)))^2}\, e^{-(\zeta/2)\,\Sigma(A',\gamma,\gamma')} . \tag{II.66}$$

Let us develop $\nabla_{RB,\gamma'-\gamma} \cdot D$. We have

$$\nabla_{RB,\gamma'-\gamma} \cdot D(A'_s) = U(A'_s, B'_l) + V(A'_s, B'_l, \gamma' - \gamma) , \tag{II.67}$$

$$U(A'_s, B'_l) \equiv \partial^2 - \lambda\partial[A'_s, .] - \lambda \sum_i [\kappa_j * B'_l, \kappa^j * \partial .] , \tag{II.68}$$

$$V(A'_s, B'_l, \gamma' - \gamma) = +\lambda^2 \sum_j [\kappa_j * [B'_l, \gamma' - \gamma], \kappa^j * \partial .] + \lambda^2 \sum_j [\kappa_j * (B'_l)^{\mathrm{rot}\,\gamma'-\gamma}, \kappa^j *] A'_s, .]] . \tag{II.69}$$

Again we write

$$\begin{aligned} (U + V + \nabla_{RB,\gamma'-\gamma} \cdot A'_s)^2 &= (U + \nabla_{RB,\gamma'-\gamma} \cdot A'_s)^2 + W , \\ W &\equiv 2V \cdot (U + \nabla_{RB,\gamma'-\gamma} \cdot A'_s) + V^2 \end{aligned} \tag{II.70}$$

(to simplify these formulas we write them like squares instead of scalar products, and we omit the necessary transpositions of operators, which are straightforward). Since V is small (with a factor λ^2), we can treat $\Sigma + W$ in the integral over γ' as a complicated interaction, and we group together the measure $d\nu_{\varrho_2}(\gamma')$ with the main quadratic piece $(\zeta/2)\langle\gamma' - \gamma, U^{\mathrm{tr}}U(\gamma' - \gamma)\rangle$. Again we write U^2 for $U^{\mathrm{tr}}U$, etc.

If the measure $d\nu_{\varrho_2}(\gamma')$ had been translation invariant we would have as a main piece a Gaussian integral over $\gamma' - \gamma$. This is not exactly the case, but by our condition $\varepsilon_2 \gg \varepsilon_1$ it will be approximately true in the small field region and the correction terms will be the ones containing powers of Γ^{-1}, Γ being the propagator (II.35) for $d\nu_{\varrho_2}$.

Therefore we define a new Gaussian variable γ'' which has propagator $(\zeta U^2+\Gamma^{-1})^{-1}$ and which is defined by:

$$\gamma'' = \gamma' - \gamma + \frac{\zeta U(\nabla_{RB,\gamma'-\gamma} \cdot A'_s) + \Gamma^{-1}\gamma}{\zeta U^2 + \Gamma^{-1}} . \tag{II.71}$$

We define also

$$\Omega \equiv \frac{(\zeta U(\nabla_{RB,\gamma'-\gamma} \cdot A'_s) + \Gamma^{-1}\gamma)^2}{\zeta U^2 + \Gamma^{-1}} - \zeta(\nabla_{RB,\gamma'-\gamma} \cdot A'_2)^2 - \gamma\Gamma^{-1}\gamma . \tag{II.72}$$

We see that Ω is small as Γ^{-1} when $\Gamma^{-1} \to 0$. This is the reason for which we can treat it as an interaction, and it is here that enters in a key way the fact that our explicit Fadeev-Popov averaging formula effectively covers all the small field region, where it performs therefore correctly its gauge fixing job.

Now we obtain, rewriting everything in terms of γ'',

$$K_{\varrho_2}(A',\gamma) = \left(\det \frac{\zeta U^2 + \Gamma^{-1}}{\Gamma^{-1}}\right)^{-1/2} \int d\pi_{\varrho_2}(\gamma'') \times e^{-\sum_i (\lambda^{1/2+2\varepsilon_2}\kappa^i(\gamma'(\gamma'',\gamma)))^N} e^{-(\zeta/2)(\Sigma+W+\Omega)(A',\gamma,\gamma'(\gamma,\gamma''))} \tag{II.73}$$

by completing the square.

The determinant $\left(\det \dfrac{\zeta U^2 + \Gamma^{-1}}{\Gamma^{-1}}\right)^{1/2} = (\det(1+\zeta\Gamma U^2))^{1/2}$ which appears in $K_{\varrho_2}(A)^{-1}$ is the analogue of the Fadeev-Popov determinant (up to the constant normalization $\det\Gamma^{-1}$). In particular if we neglect Γ^{-1} (which is small as $\lambda^{1/2\varepsilon_2}$) and rewrite $\det|U|$ as a fermionic integral over ghosts, we recover the ordinary ghost-ghost propagator p^2 and the complete ghost-ghost-field coupling at least for fields of lower momentum than the momentum of the ghosts. Indeed in this case the total field $A'_s + B'_l$ appears in U. This justifies the perturbative computations of the next section.

Indeed since $\kappa_j * B'_l$ has only low frequencies, we have $\lambda[\kappa_j * B'_l, \kappa^j * \partial.] \cong \lambda\partial[\kappa_j * B'_l, \kappa^j * .]$, and therefore $U(A'_s + B'_l) \cong \partial_\mu \sum_j D_\mu(\kappa_j * (A'_s + B'_l) + (1-\kappa_j) * A'_s)\kappa^j *$.

We are at last in the position to define K_{ϱ,ϱ_2}. We write Δ for the ordinary Laplacian and

$$\begin{aligned}
&\zeta U^{\mathrm{tr}} U + \Gamma^{-1} \\
&= (\zeta\Delta^2 + \Gamma^{-1})\Bigg(1 - \lambda \frac{\zeta\Delta}{(\zeta\Delta^2 + \Gamma^{-1})} \\
&\quad \times \Big(\partial[A'_s, .] + \sum_j [\kappa_j * B'_l, \kappa^j * \partial.]\Big) \\
&\quad - \lambda \frac{1}{(\zeta\Delta^2 + \Gamma^{-1})} \Big(\partial[A'_s, .] + \sum_j [\kappa_j * B'_l, \kappa^j * \partial.]\Big)\zeta\Delta \\
&\quad + \lambda^2 \frac{\zeta}{(\zeta\Delta^2 + \Gamma^{-1})} \Big(\partial[A'_s, .] + \sum_j [\kappa_j * B'_l, \kappa^j * \partial.]\Big)^2\Bigg) ,
\end{aligned} \tag{II.74}$$

and we change this operator into:

$$\begin{aligned}
&(\zeta U^{\mathrm{tr}} U + \Gamma^{-1})_\varrho \\
&\quad \equiv (\zeta \Delta + \Gamma^{-1}) \Bigg(1 - \lambda \kappa_\varrho(p) \frac{\zeta \Delta}{(\zeta \Delta^2 + \Gamma^{-1})} \\
&\quad \times \Big(\partial[A'_s, .] + \sum_j [\kappa_j * B'_l, \kappa^j * \partial .] \Big) \\
&\quad - \lambda \frac{\kappa_\varrho(p)}{(\zeta \Delta^2 + \Gamma^{-1})} \Big(\partial[A'_s, .] + \sum_j [\kappa_j * B'_l, \kappa^j * \partial .] \Big) \zeta \Delta \\
&\quad + \lambda^2 \frac{\zeta \kappa_\varrho^2(p)}{(\zeta \Delta^2 + \Gamma^{-1})} \Big(\partial[A'_s, .] + \sum_j [\kappa_j * B'_l, \kappa^j * \partial .] \Big)^2 \Bigg) . \qquad \text{(II.75)}
\end{aligned}$$

We define

$$\begin{aligned}
K_{\varrho,\varrho_2}(A', \gamma) = &\left(\det \frac{(\zeta U^{\mathrm{tr}} U + \Gamma^{-1})_\varrho}{\Gamma^{-1}} \right)^{-1/2} \int d\pi_{\varrho_2}(\gamma'') \\
&\times e^{-\sum_i ((\lambda_i^t)^{1/2+2\varepsilon_2} \kappa^i(\gamma'(\gamma'', \gamma)))^N} e^{-(\zeta/2)[\Sigma + W + \Omega]} . \qquad \text{(II.76)}
\end{aligned}$$

In this formula the remaining functional integral is close to one since $\Sigma + W + \Omega$ is a small interaction, as explained above. The main piece is the determinant which is nothing but the ordinary Fadeev-Popov term. The important fact about this way to reimpose a cutoff at scale ϱ is that in the small field regime and at zero external momenta for A', the only contribution of K_{ϱ,ϱ_2} to the counterterm $\lambda^4 A'^4$ (see below) comes from the ordinary Fadeev-Popov determinant. At this order we obtain therefore as only contribution (taking out constant factors, in particular a global power of ζ)

$$\begin{aligned}
&\Bigg(\det \Delta^2 \Bigg(1 - \lambda \kappa_\varrho(p) \Delta^{-1} \Big(\partial[A'_s, .] + \sum_j [\kappa_j * B'_l, \kappa^j * \partial .] \Big) \\
&\qquad - \lambda \kappa_\varrho(p) \Delta^{-2} \Big(\partial[A'_s, .] + \sum_j [\kappa_j * B'_l, \kappa^j * \partial .] \Big) \Delta \\
&\qquad + \lambda \frac{\kappa_\varrho^2(p)}{\Delta^2} \Big(\partial[A'_s, .] + \sum_j [\kappa_j * B'_l, \kappa^j * \partial .] \Big)^2 \Bigg) \Bigg)^{1/2} \\
&\quad = \det \Big| \Delta - \lambda \kappa_\varrho(p) \Big(\partial[A'_s, .] + \sum_j [\kappa_j * B'_l, \kappa^j * \partial .] \Big) \Big| \qquad \text{(II.77)}
\end{aligned}$$

which is the same as the ordinary Fadeev-Popov determinant with a cutoff on the ghosts propagator of the desired simple form, and a ghost-ghost-field vertex which is the ordinary coupling to the full field A' at least at zero momentum external field A'. This proves that the gauge breaking effect associated to this cutoff can be computed in the way this is done in the next section.

Let us recapitulate our starting point:

$$\sum_{\mathrm{LFR}} \int d\mu_{0,\varrho_1}(A') d\nu_{\varrho_2}(\gamma) \chi_{\mathrm{LFR}}(A) G(A',\gamma) e^{-\sum_i (\lambda^{1/2+2\varepsilon_2}\gamma^i)^N}$$
$$\times e^{-(1/2)\langle A',[(\kappa_\varrho)^{-1}-1](p^2)A'\rangle} e^{\mathrm{CT}_\varrho(A')}$$
$$\times e^{(1/2)\left(-F^2_{\mathrm{sp}}(A)-\langle A,p_0^2 A\rangle+\sum_i \lambda^2\langle A,(p^2\kappa^i(p))A\rangle\right)} [K_{\varrho,\varrho_2}(A',\gamma)]^{-1}$$
$$\times e^{-(\zeta/2)(\nabla_{B'_l}\cdot A'_s)^2} L_{0,\varrho_1}(\gamma) F(A'_0,\gamma)\,, \qquad \text{(II.78)}$$

where K_{ϱ,ϱ_2} is defined by (II.76), and we can fix e.g. $N = 100$ in what follows.

This functional integral is now similar in the first orders of perturbation theory to the ordinary functional integral with ultraviolet cutoff $\kappa_\varrho(p)$ both on the field and ghosts propagator. In these conditions it is easy to compute the counterterms $\mathrm{CT}_\varrho(A')$ which restore Slavnov identities. We show now how to perform this task.

III. Computation of the Counterterms due to the Ultraviolet Cutoff

In all this section the collaboration of Feldman is gratefully acknowledged. The main result on the stability of certain types of cutoffs was derived with him around 1986; there is also an exposition of this result in [S] and [R].

The computation of the gauge variant counterterms which restore Ward identities is made in terms of the field A'. For this computation we can assume that $A' = A'_s$ and $B'_l = 0$. Furthermore *in this section* we write A for simplicity instead of A'.

Our ultraviolet cutoff does not break global SU(2) or Euclidean invariance (small Euclidean breaking effects nevertheless occur due to the infrared cutoff; for instance in the case of a torus there exist such effects due to the lattice structure of Λ^*, but they are tied to the unit scale and do not need counterterms). Therefore the only new relevant or marginal operators that we should consider are $-\mathrm{Tr}\, A_\mu A_\mu$, $(-\mathrm{Tr}\, A_\mu A_\mu)^2$, $(-\mathrm{Tr}\, A_\mu(-\Delta)A_\mu)$ and $-\mathrm{Tr}(\partial_\mu A_\mu)^2$ which we abbreviate respectively as A^2, A^4, $A(-\Delta)A$ and $(\partial A)^2$ (recall the convention that traces are definite negative). This is only true for the SU(2) theory, for an SU(N) theory there would be a longer list of operators to consider and the analysis would be more complicated.

In fact our gauge breaking cutoff also disturbs the magic relation $Z_2 Z_4 = Z_3^2$ which relates the multiplicative renormalization of F_2, F_3 and F_4 in F^2 and expresses the fact that up to a rescaling of A only the coupling constant λ is renormalized [IZ]. To correct this problem, using the possibility of rescaling A, we need only to introduce a single counterterm, for instance of the type F_4.

Therefore the counterterms that we introduce are:

$$e^{\mathrm{CT}} = e^{-a_\varrho \int_\Lambda (A^4/4!) - b_\varrho \int_\Lambda (A^2/2!) - c_\varrho \int_\Lambda (A(-\Delta)A) - d_\varrho \int_\Lambda (\partial A)^2 - e_\varrho \int_\Lambda F_4}\,. \qquad \text{(III.1)}$$

The relevant counterterm $b_\varrho \int_\Lambda (A^2/2)$ must be fine tuned exactly to have a renormalized mass which is zero. This is the same problem as fixing the critical bare mass in infrared ϕ^4_4 [FMRS1], [R] and should be solved by a fixed point argument as in [R] or using a full renormalization of the two point function (and a one particle irreducible analysis) as in [FMRS1]. For the marginal counterterms, an analysis to lowest order in perturbation theory is in fact enough for our purpose (because of asymptotic freedom, further orders again should give no contributions to finite scales in the limit $\varrho \to \infty$). We obtain:

Lemma III.1.

$$a_\varrho \cong a\lambda_\varrho^4, \quad b_\varrho \cong bM^{2\varrho}\lambda_\varrho^2, \quad c_\varrho \cong c\lambda_\varrho^2, \quad d_\varrho \cong d\lambda_\varrho^2, \quad e_\varrho \cong e\lambda_\varrho^4. \quad \text{(III.2)}$$

Furthermore by choosing the cutoff of the form (II.14) with η small enough (depending on the shape of τ), the coefficient a is strictly positive [5].

Proof. We recall the Feynman rules for the pure SU(2) gauge theory in a general gauge with parameter ζ (the case $\zeta = 1$ corresponds to the Feynman gauge, and $\zeta = \infty$ corresponds to the Landau gauge) [IZ].

The propagators for the Yang-Mills fields and the ghost fields are respectively:

$$\delta_{ab}\left(\frac{\delta_{\mu\nu}}{p^2} + (1/\zeta - 1)\frac{p_\mu p_\nu}{p^4}\right); \quad \frac{\delta_{ab}}{p^2}. \quad \text{(III.3)}$$

The interaction vertices are of three kinds. For simplicity we always forget to write the overall multiplication factor (of 2π) and the δ function which expresses momentum conservation which equips them. These three kinds of vertices are then pictured in Fig. III.1.

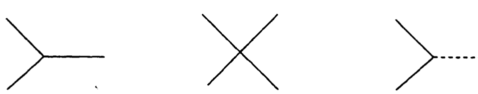

Fig. III.1. The vertices of pure Yang-Mills. —— A field propagator; ------ a ghost propagator

We concentrate on the computation of the A^4 counterterms, which is the most interesting, and include also the computation of the A^2 counterterm. The other ones are less interesting and left to the reader.

At one loop, which also means at order λ^4 in perturbation theory, there are 4 graphs which may contribute to the A^4 term. They are pictured in Fig. III.2 and called G_1, G_2, G_3 and G_4. To compute their contribution to the coefficient a, we may assume by symmetry that in all four external legs, both the space time and group indices are equal to 1.

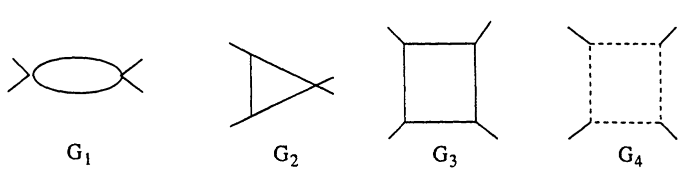

Fig. III.2. The graphs contributing to A^4

a) Computation of G_1. The graph is obtained by applying 4 derivatives $\dfrac{\partial}{\partial A_1^1}$ on $(1/2!)(-F^2/4)^2$. The result is $3(\partial^2 F^2/4)^2$ where derivatives are taken with respect

[5] It is not clear whether a cutoff for which a would be negative (or zero) is strictly forbidden for a constructive analysis. The answer may indeed depend on considering irrelevant counterterms of higher order generated by the cutoff, which may stabilize the theory. The analysis woul certainly be much more complicated and we will therefore not try to explore this possibility here

to A_1^1. The only non-vanishing pieces come from the derivatives acting on the commutator in F, hence $\partial^2 F^2/4$ gives $(1/2)(\partial F)^2$.

Moreover we have $\partial F^c_{\alpha\beta} = \varepsilon^{c1b}[A^b_\delta \delta_{\alpha 1} - A^b_\alpha \delta_{\beta 1}]$, where ε is the usual antisymmetric tensor. But remark that if $\alpha = \beta = 1$ the term vanishes. Hence when developing the square $(1/2)(\partial F)^2$ the cross terms vanish. Therefore this square gives $(\varepsilon^{c1b})^2(A^b_\beta)^2\delta_{\alpha 1}$, $\beta \neq 1$. There are now two possible Wick contractions, a sum over three values (2, 3 and 4) for β and a sum over 2 values (2 and 3) for b. Collecting all factors we obtain a positive coefficient $3.4(3+3(1/\zeta-1)/2+5(1/\zeta-1)^2/8) = 36+18(1/\zeta-1)+15(1/\zeta-1)^2/2$ in front of the integration $\int \frac{d^4k}{k^4}$ over the loop momentum of G_1.

b) Computation of G_2. We apply 4 derivatives on $(1/3!)(-F^2/4)^3$. The result is $-6(\partial^2/F^2/4)(\partial F^2/4)^2$, where derivatives are again with respect to A_1^1. The term in $\partial^2 F^2/4$ is the same as before, hence gives $(\varepsilon^{c1b})^2(A^b_\beta)^2\delta_{\alpha 1}$, $\beta \neq 1$. But we have now two trilinear vertices in $\partial F^2/4$, hence terms with derivative couplings; remark that a partical derivative ∂_μ can be replaced by $-ik_\mu$. The computation of this term leads to two identical vertices, one which gives $\varepsilon^{1mn}A^n_\mu[\partial_1 A^m_\mu - \partial_\mu A^m_1]$, and the other with m, n, μ respectively replaced by p, q, λ. In the Wick contraction schemes we can first contract to form the line between these two trilinear vertices. Since the two half legs of the remaining vertex bear the same index $\beta \neq 1$, a tedious computation gives that the only term compatible with future contractions is $(\varepsilon^{1mn})^2(A^m_\mu)^2[4k_1^2+k_\mu^2]$. Using Euclidean symmetry, this is equivalent to $(\varepsilon^{1mn})^2(A^m_\mu)^2[5k_1^2]$. Contracting with the remaining vertex, we have now as before two possible Wick contractions, a sum over three values (2, 3 and 4) for β and a sum over 2 values (2 and 3) for b. Collecting all factors we obtain a negative coefficient

$$-6\cdot 2\cdot 2\Bigg[15k_1^2/k^2+(1/\zeta-1)\Bigg(\sum_{\mu\neq 1} k_1^2k_\mu^2/k^4+2\Bigg(3\sum_{\mu\neq 1} k_1^2k_\mu^2 + \sum_{\mu\neq 1,\mu'\neq 1} k_\mu^2 k_{\mu'}^2\Bigg)\Bigg/k^4+(1/\zeta-1)^2\sum_{\mu\neq 1,\mu'\neq 1} k_1^2k_\mu^2k_{\mu'}^2/k^6\Bigg)\Bigg]$$

which is equivalent by Euclidean symmetry to $-90-45(1/\zeta-1)-15(1/\zeta-1)^2$ in front of the integration $\int \frac{d^4k}{k^4}$ over the loop momentum of G_2.

c) Computation of G_3. We apply 4 derivatives on $(1/4!)(-F^2/4)^4$. The result is $+(\partial F^2/4)^4$, where derivatives are again with respect to A_1^1. The term in $\partial F^2/4$ gives the same trilinear vertex as before, hence gives $\varepsilon^{1mn}A^n_\mu[\partial_1 A^m_\mu - \partial_\mu A^m_1]$. In the Wick contraction schemes we can first choose one particular leg of vertex 1 to form a first line between two trilinear vertices. To choose the vertex (2, 3 and 4) to which this leg contracts gives a factor 3. After this contraction has been performed, the line equipped with two not yet contracted fields gives a term $(\varepsilon^{1mn})^2[2k_1^2(A^m_\mu)^2 + k_\mu^2(A^m_1)^2 - 3k_1k_\mu A^m_1 A^m_\mu]$. Here we can assume $\mu \neq 1$. We can now contract once more to create one line between the two remaining vertices, and this can be done in all possible ways, hence gives a different term, which is $(\varepsilon^{1mn})^2[4k_1^2(A^m_\mu)^2 + k_\mu^2(A^m_1)^2 - 6k_1k_\mu A^m_1A^m_\mu + k_\mu k_\lambda A^m_\mu A^m_\lambda]$. We can assume that $\mu \neq 1$ in the first three terms and that $\mu = \lambda = 1$ is excluded in the last one. It remains to contract together both expressions. We have as before two possible Wick contractions, a sum over three values (2, 3 and 4) for μ and a sum over 2 values (2 and 3) for m. After collecting all factors and taking into account Euclidean symmetry to convert it into units of $\int \frac{d^4k}{k^4}$, we find a final factor in front of the integration over the loop momentum of $G_3 6(9+1/4+9(1/\zeta-1)/2+5(1/\zeta-1)^2/4) = 55.5+27(1/\zeta-1)+7.5(1/\zeta-1)^2$.

d) Computation of G_4. We apply 4 derivatives on (1/4!) (F.P.)^4, where F.P. means the Fadeev-Popov term $\partial_\mu \bar{\eta}_a (D_\mu \eta)_a$, with D the covariant derivative. The result is $(\partial_1 \bar{\eta}_a \varepsilon_{ab1} \eta_b)^4$. The combinatoric is easier. We obtain a factor 6 for the Wick contractions, a factor 2 for summations over latin indices and a minus sign corresponding to the fermionic loops, which comes from reordering correctly the anticommuting fields η and $\bar{\eta}$. Hence the contribution is $-12 \cdot k_1^4$ in front of the integration over the loop momentum of G_4. Applying the same conversion rate, we obtain in units of $(k^2)^2$ a final combinatoric factor of -1.5.

Remark that when the cutoff is 1 we can all add the terms together and the 4 coefficients add up to 0. This is a particular case of the famous miracle of renormalizability (at one loop...) of four dimensional gauge theories.

Let us perform now a similar analysis for the A^2 counterterm. There are three graphs contributing at order λ^2, pictured in Fig. III.3.

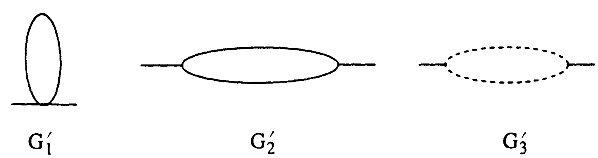

Fig. III.3. The graphs contributing to A^2

The first graph, G_1', gives a computation quite similar to that of G_1. We have $\partial\partial(-F^2/4) = -(1/2)(\partial F)^2$. Again $\partial F^c_{\alpha\beta} = \varepsilon^{c1b}[A^b_\beta \delta_{\alpha 1} - A^b_\alpha \delta_{\beta 1}]$ which is non-zero only for $\alpha, \beta \neq 1$. The cross terms therefore again vanish and we find $(\varepsilon^{c1b})^2 (A^b_\beta)^2 \delta_{\alpha 1}$, $\beta \neq 1$. There are two values for b and three for β. Hence the contribution is $-6(1+(1/\zeta-1)/4)$ in front of the integration over the loop momentum (in units of $1/k^2$).

The second graph, G_2', is given by $\partial\partial(1/2)(F^2/4)^2 = (\varepsilon^{1mn} A^n_\mu [\partial_1 A^m_\mu - \partial_\mu A^m_1])^2$ (which is non-zero only for $\mu \neq 1$). The contribution is $2(9k_1^2/k^4 + (1/\zeta-1)(k_\mu^2 k_1^2 + k_\mu^2 k_{\mu'}^2)/k^6) = (9/2+6(1/\zeta-1)/4)$ in front of the integration over the loop momentum. The last graph, with ghosts, G_3', gives $\partial\partial(1/2)(\text{F.P.})^2 = (\partial\text{F.P.})^2 = (\partial_1 \bar{\eta}_a \varepsilon_{ab1} \eta_b)^2$. There is a minus sign due to the fermion loop (two minus signs due to the rule $\partial_\mu \to -ik_\mu$ compensate; beware that there is a sign mistake in the corresponding computation in [R]). The contributions is therefore $-2k_1^2/k^4 = -(1/2)/k^2$ in front of the integration over the loop momentum.

The result for the $(A^2/2)$ term in the region where the ultraviolet cutoff is one is obtained by adding all the terms and is $-6-6/4(1/\zeta-1)-1/2+9/2+6/4(1/\zeta-1) = -2$ times the loop integration. Remark that this result is independent of ζ.

To complete the lemma, we want to study the sign of the A^4 counterterm. Let us explain why it is important to us. Our strategy is to cancel explicitly the A^4 and A^2 contributions due to the gauge breaking character of our ultraviolet cutoff by appropriate counterterms. Remark that strictly speaking, only the A^2 contribution diverges as $\varrho \to \infty$ and requires a counterterm (for the A^4 term the coefficient of the divergent piece is 0, as computed above). However this A^2 counterterm is positive (since the contribution is negative, see the -2 above). This is dangerous for stability estimates. We will use the (finite) A^4 counterterm to control this dangerous A^2 term and stabilize the theory. But this requires that we use an ultraviolet cutoff such that the A^4 counterterm is negative, hence such that the total A^4 contribution induced by the cutoff is positive. As a consequence of our expansion the leading contribution is the one-loop contribution; we want its sign to be positive. We show now that this is possible if we start with a cutoff function of a particular shape such as (II.14) η being a small constant. This explains at last the curious definition (II.14) of our ultraviolet cutoff. Later we will show that this particular shape also leads to a stabilizing functional integral associated to a large background field.

Let $\kappa(p)$ be the ultraviolet cutoff function in momentum space. Up to now we did not take it into account. Remark that since there is one cutoff per propagator the cutoff acts differently G_1, G_2, G_3 and G_4. More precisely using the coefficients computed in the preceding section, the one loop contribution to the $(A^4/24)$ term is, for a single cutoff $\kappa_\varrho(p) = \kappa(pM^{-\varrho})$ (all our integrals are infrared regularized and "finite" means finite as $\varrho \to \infty$):

$$\begin{aligned}\int \frac{d^4p}{p^4}\, &[(36 + 18(1/\zeta - 1) + 7.5(1/\zeta - 1)^2)\kappa^2(pM^{-\varrho}) - (90 + 45)1/\zeta - 1)\\ &+ 15(1/\zeta - 1)^2)\kappa^3(pM^{-\varrho}) + (54 + 27)1/\zeta - 1) + 7.5(1/\zeta - 1)^2)\kappa^4(pM^{-\varrho})]\\ &= 0 \cdot \varrho + \text{finite terms},\end{aligned} \tag{III.4}$$

where the finite terms are finite functions of the particular shape of κ and are therefore difficult to compute in the general case. However we are going to use a shape such as (II.14) in which there is a free parameter η that we can vary, and we will study the finite terms in the limit $\eta \to 0$. In this case it is easy to analyze the asymptotic behavior of the finite terms in (III.4)

For κ_ϱ defined as in (II.13–14), the corresponding contribution is indeed:

$$\begin{aligned}&\int_{1\le|p|M^{-\varrho}<2} \frac{d^4p}{p^4}\, [(36+18(1/\zeta-1)+7.5(1/\zeta-1)^2)\kappa^2(pM^{-\varrho}) - (90+45)1/\zeta-1)\\ &+ 15(1/\zeta-1)^2)\kappa^3(pM^{-\varrho}) + (54+27)1/\zeta-1)+7.5(1/\zeta-1)^2)\kappa^4(pM^{-\varrho})]\\ &+ \int_{2+\eta^{-1}<|p|M^{-\varrho}\le 3+\eta^{-1}} \frac{d^4p}{p^4}\, [(36+18(1/\zeta-1)+7.5(1/\zeta-1)^2)\kappa^2(pM^{-\varrho})\\ &- (90+45)1/\zeta-1)+15(1/\zeta-1)^2)\kappa^3(pM^{-\varrho})\\ &+ (54+27)1/\zeta-1)+7.5(1/\zeta-1)^2)\kappa^4(pM^{-\varrho})]\\ &+ \int_{2\le|p|M^{-\varrho}<2+\eta^{-1}} \frac{d^4p}{p^4}\left[\frac{36+18(1/\zeta-1)+7.5(1/\zeta-1)^2}{4}\right.\\ &\left. - \frac{90+45(1/\zeta-1)+15(1/\zeta-1)^2}{8} + \frac{54+27(1/\zeta-1)+7.5(1/\zeta-1)^2}{16}\right].\end{aligned} \tag{III.5}$$

As a consequence the one loop $(A^2/24)$ contribution behaves as

$$\left(\frac{36+18(1/\zeta-1)+7.5(1/\zeta-1)^2)}{4}-\frac{90+45(1/\zeta-1)+15(1/\zeta-1)^2}{8}\right.$$
$$\left.+\frac{54+27(1/\zeta-1)+7.5(1/\zeta-1)^2}{16}\right)(-\ln\eta)+\text{finite terms}$$
$$=9/8(1+(1/\zeta-1/2+5/12(1/\zeta-1)^2)\,|\ln\eta|+\text{finite terms}, \qquad \text{(III.6)}$$

where "finite terms" now means terms which are uniformly bounded both as ϱ tends to $+\infty$ and η tends to 0. The polynomial $1+(1/\zeta-1)2+(5/12)(1/\zeta-1)^2$ is always positive and greater than 17/20. Since $(17/20)\cdot(9/8)\geq 1/2$, taking η small enough (depending on the details of our cutoff, which are responsible for the particular value of the finite terms) we can always achieve our goal of a positive total A^4 contribution, hence of a negative stabilizing counterterm, with value at least

$$e^{-(1/2)\int_\Lambda (A^4/24)\,|\ln\eta|}. \qquad \text{(III.7)}$$

Remark that the coefficient of this stabilizing term can be made as large as we want, if η is small enough.

IV. The Propagators for Large and Small Fields

Let us recall our starting point:

$$\sum_{\text{SFR}}\int d\mu_0(A')\,d\nu_{\varrho_2}(\gamma)\chi_{\text{LFR}}(A)G(A',\gamma)e^{-\sum_i((\lambda_i^t)^{1/2+2\varepsilon_2}\gamma^i)^{100}}$$
$$\times e^{-(1/2)\langle A',[(\kappa_\varrho)^{-1}-1](p^2)A'\rangle}\,e^{\text{CT}_\varrho(A')}$$
$$\times e^{(1/2)\left(-F^2_{\text{sp}}(A)-\langle A,p_0^2A\rangle+\sum_i(\lambda_i^t)^2\langle A,(p^2\kappa^i(p))A\rangle\right)}[K_{\varrho,\varrho_2}(A',\gamma)]^{-1}$$
$$\times e^{-(\zeta/2)(\nabla_{B_l'}\cdot A_s')^2}L_{0,\varrho_1}(\gamma)F(A_0',\gamma)\,. \qquad \text{(IV.1)}$$

This starting point is clearly well defined because we have both finite volume and ultraviolet cutoff on each of the fields involved. Hence the sample fields are smooth. Furthermore for large fields A' the leading terms are the F_4 term and the $(A')^4$ term in CT_ϱ, which are respectively positive and positive definite. The γ integrals are also convergent at large γ thanks to the protecting term in γ^{100}. Remark however that it is only for fields of order λ^{-1} that the $(A')^4$ term provides convergence, so this term alone does not confine the field in the true perturbative region ($A'\ll\lambda^{-1}$). It is only the combination of this term with the axial gauge positivity which does this. Our goal in this section is to manipulate the complicated expression (IV.1) in order to extract the Gaussian pieces which are essential for our analysis and to combine them with the (fake) measure $d\mu_0$ (which is used mainly as a substitute for the non-existence of a continuum Lebesgue functional measure). These essential pieces are all contained in the Yang-Mills action. We use a rather complicated symmetric way to extract them in order to preserve positivity as much as possible (positivity is indeed essential for constructive estimates).

The Yang-Mills action is invariant under exact gauge transformations. However if we use truncated transformations, i.e. such as $A' \equiv A^{\gamma,2}$ the action is not exactly invariant, but the difference is a complicated polynomial with at least two powers of λ:

$$F^2_{\rm sp}(A) + \langle A, p_0^2 A\rangle = F^2(A') + M(A,\gamma)\,, \qquad M(A,\gamma) = O(\lambda^2)\,. \tag{IV.2}$$

Our goal is to perform a multiscale analysis of the theory and we stop at this point to explain further why we need to pay some special attention to some vertices in (IV.1) which are called non-dominable. The main problem when one tries such a multiscale expansion is that some low momentum fields derived by cluster expansions at a certain scale have to be bounded using the stability of an effective potential in the interaction, otherwise (for instance if they are integrated with respect to the Gaussian measure) they give rise to divergent factorials which are a remnant of the divergence of perturbation theory [R]. The interaction vertices created by the various error terms of Sect. II [or by formula (IV.2)] correspond to factors such that, when the low momentum fields A'_s are bounded using the small field condition, the low momentum fields B'_l are bounded using the $(B'_l)^4$ counterterm, and the low momentum γ fields are bounded using the γ^{100} term in (IV.1), a small factor remains. We have therefore to examine the vertices which come from $F^2(A')$ in (IV.2). If we simplify the situation by considering that we have two fields, A and B, where A is high momentum and B low momentum, the vertices with only one high momentum field can be eliminated because they violate momentum conservation; the other vertices which couple A to B have at most two A fields. When the B field is of the small field type A'_s, there is never any domination problem, because the small field condition itself can be used to dominate the field, and a small factor remains (because the size of the field at which no small factor remains is λ^{-1} and the small field condition acts well before that size).

Hence we conclude that only couplings with low momentum large fields can be non-dominable. Such fields are called background fields. (This is the reason for which we use the same generic letter B (as in background) both for low momentum fields and for large fields.) Let us consider two such background fields; if they occur in the form of a commutator, there is no problem because the decoupled effective action for the B field contains a commutator squared [in $F_4(B)$] and the situation is therefore analogous to that of a positive polynomial coupling such as ϕ^4 (see e.g. [R]). If there is a single B field with a partial derivative acting on it, there is still no problem. The small factor then comes from the fact that B is of a much lower frequency than A, hence the derivative gives a small factor compared to the initial scale of A (also called the localization scale). This small factor is in turn related to the gap between the frequencies of A and B, hence related to the creation of protection corridors (see Sect. II.B).

Therefore we can conclude that the only vertices which are not dominable are the ones with one or two large B fields coming both from a commutator of the type $[A, B]$. These are the only remaining possibilities as far as the vertices of F^2 are concerned. Since such vertices cannot be treated as interaction, the only other possibility that remains is to put them in the measure for A with respect to which the cluster expansion is performed. It is very fortunate indeed that this operation gives a Gaussian measure, albeit a B-dependent one; were it not the case we could not do anything because up to now Gaussian functional integrals are the only ones that we know how to perform explicitly. In fact the corresponding measure on A is just similar to $F_2(A)$ [see (II.3)], but with ordinary derivatives replaced by covariant derivatives in the background field.

Furthermore if we return to (IV.1) we remark that it contains also the analogue of the Fadeev-Popov determinant (the term $K^{-1}_{\varrho,\varrho_2}$), which up to small correction terms is equal to the determinant (II.76). This determinant can be written in the usual way as an integral over anticommuting ghosts; this is only a formal trick which is useful to summarize the rules of perturbation theory. We realize then that there are two types of vertices coupling the ghosts to the field, namely the ordinary vertex coupling ghosts to A'_s, which is the small field, and new vertices which couple the ghosts fields of a given frequency to the sum of the large fields B'_l of lower frequencies; these new vertices are the direct remnant of the fact that we used a gauge condition which depends on the large background field. If we consider the usual multiscale analysis of the theory we have to give special attention to the vertices which couple different scales and are not dominable. By Pauli principle, low momentum ghosts fields are dominable [R]; their functional integration gives a determinant which can be evaluated without any factorial effect. Low momentum fields of the type A'_s can be dominated using the small field condition; hence we conclude that the only non-dominable vertices coming from the Fadeev-Popov determinant are the ones which contain two high momentum ghosts and one B'_l field. Together with the free measure on the ghosts which is the Laplacian in (II.76) these vertices form an object which cannot be expanded in perturbation theory. The corresponding functional integral compared to the functional integral when the low momentum field B'_l is absent gives a quotient of determinants. This quotient for a constant background field B is exactly the same as the normalized functional integral over ghosts of the ordinary Fadeev-Popov determinant of this constant background field B (indeed the position of the ∂ operator in (II.76) relative to B is then irrelevant since $\partial B \cong 0$ for a low momentum field).

The conclusion of this discussion is that we have to use background dependent propagators both for the field A' and for the ghosts. Only large low momentum fields need to be considered as background. The normalization of the Gaussian measures with background field gives a factor which can be associated to the large field regions. This factor will be called the large field dressing factor. It must correspond in [Ba9] to the problem of renormalizing the large field regions. A nonperturbative evaluation of this factor is crucial in proving that the total weight of the functional integrals over these large field regions is small compared to the weight of the small field regions, hence to complete the rigorous version of the sketchy Lemma II.1.

We start now to implement this program of extracting the desired Gaussian measures with background fields from the functional integral (IV.1).

We want to group together the pieces which involve $\nabla_{B'_l}$ in the Yang-Mills action with the gauge condition in (IV.1) in order to obtain a Gaussian factor

$$e^{-(\zeta/2)\langle \nabla_{B'_l} A'_s \cdot \nabla_{B'_l} A'_s \rangle} . \tag{IV.3}$$

We write for simplicity $-\Delta_{B'_l}$ instead of $\nabla_{B'_l} \cdot \nabla_{B'_l}$. This Gaussian piece is exactly the analogue of the homothetic gauge Gaussian measure on A'_s but with background field B'_l.

The Yang-Mills action is therefore decomposed as:

$$\begin{aligned} F_{\mu\nu}(A'_s + B'_l) &= D_\mu(B'_l) \cdot (A'_s)_\nu - D_\nu(B'_l) \cdot (A'_s)_\mu - \lambda[(A'_s)_\mu, (A'_s)_\nu] + F_{\mu\nu}(B'_l) \\ &= (\nabla_{B'_l})_\mu (A'_s)_\nu - (\nabla_{B'_l})_\nu (A'_s)_\mu \\ &\quad - \lambda[(A'_s)_\mu, (A'_s)_\nu] + F_{\mu\nu}(B'_l) + G_{\mu\nu}(B'_l, A'_s) , \end{aligned} \tag{IV.4}$$

$$G_{\mu,\nu}(A'_s, B'_l) \equiv -\left(\sum_i \lambda[(1-\kappa_i) * (B'_l)_\mu, \kappa^i * (A'_s)_\nu] - (\mu \to \nu)\right). \tag{IV.5}$$

We introduce also a protection corridor around LFR and call ELFR (extended large field region) the region LFR plus its corridor. The region complementary to ELFR, called the core small field region CSFR is contained in SFR and protected from LFR by the corridor (see Fig. IV.1). The region ELFR–LFR is called the boundary region BR. Returning to the definitions of Sect. II we see that roughly speaking A'_s lives on SFR and B'_l leaves on LFR; in particular B'_l is heavily suppressed in CSFR. This allows us to extract in these regions the correct pieces that we want to join to $d\mu_0$, namely in CSFR the piece to create the propagator with covariance $(\Delta_{B'_l})^{-1}$ on A'_s and in LFR the piece $\langle B'_l, p_0^2 B'_l\rangle$ to create the propagator C_{axial}.

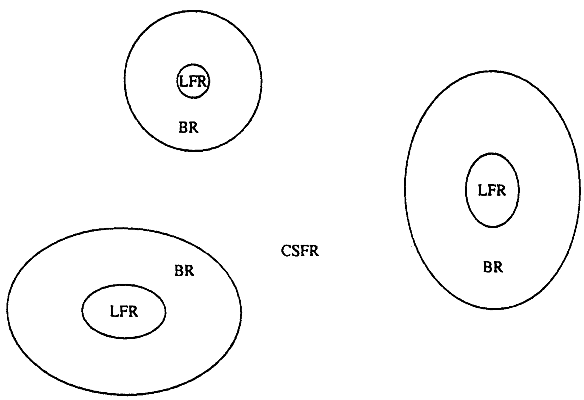

Fig. IV.1. The small and large field regions

In the boundary region BR it is enough to keep the initial measure $d\mu_0$ and to remark that F^2, which is treated as an interaction, remains positive. This gives a bad normalization to the boxes of this boundary (of the order of $\lambda^{-O(1)}$ per such box), which is compensated by the excellent small factor for the boxes of LFR (see Lemma II.1) if the width of the protection corridors is in $\lambda^{-\varepsilon}$ with ε very small. For a simple example of how to treat such normalization effects we refer e.g. to [DMR].

Now we decompose F^2 in three pieces in a way which respects the positivity of each piece:

$$F^2(A') = F^2(A')_{\text{CSFR}} + F^2(A')_{\text{LFR}} + F^2(A')_{\text{BR}}, \tag{IV.6}$$

$$F^2(A')_{\text{CSFR}} \equiv \sum_j \sum_{\Delta \in \text{CSFR}_j} F_{\mu\nu}(\kappa^j)^{1/2}\chi_\Delta(\kappa^j)^{1/2}F_{\mu\nu}, \tag{IV.7}$$

$$F^2(A')_{\text{LFR}} \equiv \sum_j \sum_{\Delta \in \text{LFR}_j} F_{\mu\nu}(\kappa^j)^{1/2}\chi_\Delta(\kappa^j)^{1/2}F_{\mu\nu}, \tag{IV.8}$$

$$F^2(A')_{\text{BR}} \equiv \sum_j \sum_{\Delta \in \text{BR}_j} F_{\mu\nu}(\kappa^j)^{1/2}\chi_\Delta(\kappa^j)^{1/2}F_{\mu\nu}. \tag{IV.9}$$

More generally an index like CSFR, LFR, BR etc. is a short notation for a decomposition of the type (IV.6–9).

In this decomposition we substitute the value (IV.4) of $F_{\mu\nu}$ in terms of A'_s, B'_l. Furthermore we want to replace the background field B'_l by a background field $\bar{B}'_l$ which is piecewise constant and corresponds to the average of B'_l on the box Δ appearing in the decomposition (IV.7–9). The constant value $\bar{B}'_l$ in such a box Δ is noted $\bar{B}'_{l,\Delta}$ when necessary. A difference term such as $B'_l - \frac{1}{|\Delta|}\int_\Delta B'_l$, where the box Δ is the one appearing in (IV.7–9) is noted $\delta B'_l$. Such terms can be treated as interaction and are dominable, since we can rewrite them as integrals of gradients applied on B'_l; using (II.29a–c) these gradients are bounded. Therefore we write:

$$F^2(A')_{\mathrm{CSFR}} = ((\nabla_{B'_l})_\mu (A'_s)_\nu - (\nabla_{B'_l})_\nu (A'_s)_\mu)^2_{\mathrm{CSFR}} + H(A'_s, B'_l)\,, \tag{IV.10}$$

$$F^2(A')_{\mathrm{LFR}} = (\langle B'_l, p_0^2 B'_l\rangle)_{\mathrm{LFR}} + K(A'_s, B'_l)\,, \tag{IV.11}$$

where H and K, which are localized respectively in CSFR and LFR will be treated as small interactions. One has:

$$\begin{aligned} H = {} & (\text{terms with at least one } G, \text{ one } F_{\mu\nu}(A'_s), \\ & \text{one commutator } [A'_s, A'_s] \text{ or one difference } \delta B'_l)_{\mathrm{CSFR}}\,, \end{aligned} \tag{IV.12}$$

$$\begin{aligned} K = {} & (F_{\mathrm{sp}}(B'_l))^2_{\mathrm{CLFR}} + (\text{terms with at least one } (\nabla_{B'_l})_\mu (A'_s)_\nu - (\nabla_{B'_l})_\nu (A'_s)_\mu\,, \\ & \text{one } G \text{ or one commutator } [A'_s, A'_s])_{\mathrm{LFR}}\,. \end{aligned} \tag{IV.13}$$

(We used the fact that $(B'_l)_0 = 0$ to replace $F^2_{0,\mu}(B'_l)$ by $\langle B'_l, p_0^2 B'_l\rangle$.)

We want now to extract the fact that the gauge condition in $(\nabla_{B'_l} \cdot A'_s)^2$ is almost equal to the desired one $(\nabla_{\bar{B}'_l} \cdot A'_s)^2_{\mathrm{CSFR}}$. Again to respect positivity we decompose the gauge condition as:

$$(\nabla_{B'_l} \cdot A'_s)^2 = (\nabla_{\bar{B}'_l} \cdot A'_s)^2_{\mathrm{CSFR}} + (\nabla_{B'_l} \cdot A'_s)^2_{\mathrm{ELFR}} + J\,, \tag{IV.14}$$

where J is a term localized in CSFR containing at least one difference $\delta B'_l$. Finally we use the fraction of the gauge condition localized in CSFR to write:

$$\begin{aligned} & ((\nabla_{\bar{B}'_l})_\mu (A'_s)_\nu - (\nabla_{\bar{B}'_l})_\nu (A'_s)_\mu)^2_{\mathrm{CSFR}} + \zeta(\nabla_{\bar{B}'_l} \cdot A'_s)^2_{\mathrm{CSFR}} \\ & = \langle A'_s \Delta_B A'_s\rangle + L(B'_l, A'_s, \gamma)\,. \end{aligned} \tag{IV.15}$$

In this formula let us explain what are Δ_B and L. Let us introduce the "homothetic" Laplace operator $-\Delta^{\mathrm{homothetic}}$ which in Fourier space is simply $p^2\delta_{\mu\nu} - (1-\zeta)p_\mu p_\nu$; its inverse is $1/p^2(\delta_{\mu\nu} - (1-\zeta^{-1})p_\mu p_\nu/p^2)$. Then the operator Δ_B in (IV.15) can be thought of as the analogue of $-\Delta^{\mathrm{homothetic}}$ but with covariant derivatives in the background field instead of ordinary ones. More precisely it is defined by

$$\begin{aligned} \Delta_B &= \sum_j \sum_{\Delta\in\mathrm{CSFR}_j} \Delta_{B,J,\Delta}\,, \\ \Delta_{B,J,\Delta} &\equiv ((\nabla_{\bar{B}'_l})_\sigma (\kappa^j)^{1/2}\chi_\Delta(\kappa^j)^{1/2}(\nabla_{\bar{B}'_l})_\sigma \delta_{\mu\nu} \\ &\quad + (\zeta-1)(\nabla_{\bar{B}'_l})_\mu (\kappa^j)^{1/2}\chi_\Delta(\kappa^j)^{1/2}(\nabla_{\bar{B}'_l})_\nu)\,. \end{aligned} \tag{IV.16}$$

This operator is clearly positive but not strictly positive (in the Appendix, bounds are given in the case of a constant background field).

Finally L in (IV.15) is a correction term which is treated as an interaction. This term contains indeed either derivatives acting on B_l' or commutators of the background field $[A'_{l,\mu}, A'_{l,\nu}]$ which are dominable as explained above. Indeed usually when one combines the quadratic piece $\sum_\mu \sum_\nu (\partial_\mu A_\nu)^2 - (\partial_\mu A_\nu)(\partial_\nu A_\mu)$ coming from F_2 with the homothetic gauge condition $\zeta\left(\sum_\mu \partial_\mu A_\mu\right)\left(\sum_\nu \partial_\nu A_\nu\right)$ one needs an integration by parts, so that the gauge condition combines with the term with the minus sign, leaving the term $\sum_\mu \sum_\nu (\partial_\mu A_\nu)^2 + (\zeta - 1)\sum_\mu \sum_\nu (\partial_\mu A_\nu)(\partial_\nu A_\mu)$ which corresponds to the homothetic propagator $1/p^2(\delta_{\mu\nu} - (1-\zeta^{-1})p_\mu p_\nu/p^2)$. In our case this integration by parts is no longer exact for two reasons. First the partial derivatives are replaced by covariant derivatives. However if the background field is constant the reader can check that at least for su(2) the formula of integration by parts is still true up to a term proportional to $[A'_{l,\mu}, A'_{l,\nu}]\,[A'_{s,\mu}, A'_{s,\nu}]$. The fact that the field $\bar{B}_l'$ is piecewise constant then gives an error term containing derivatives of this field.

We want to use this Gaussian measure to perform a multiscale cluster expansion. The units corresponding to this expansion are roughly speading small field cubes and blocks of large field cubes. The fact that the propagator corresponding to joining the quadratic form (IV.16) to the "fake" measure $d\mu_0$ is not translation invariant forces us to use an expansion more complicated than usual, inspired by random path expansions used for propagators with boundary conditions such as Dirichlet, in which one writes the propagator as a product of a regular translation invariant operator for which spatial decay is easy to prove, a "first hitting time" to the boundary and then a messy non-translation invariant piece.

V. The Expansion

A) The Preparation of the Propagator

We want to perform a multiscale cluster expansion, i.e. starting from the propagator Δ_B^{-1} we have to distinguish momentum slices with index $j = \varrho, \varrho - 1, \ldots, 1$. Recall that by our convention operators such as Δ_B are the analogues of minus Laplacians, so that they are of positive type. (This convention saves a lot of minus signs). The main problem is the fact that Δ_B^{-1} is not translation invariant, due to the presence of large field regions and their associated background fields. We shall introduce a modified version of this propagator Δ_B^{-1} which is better suited for a cluster expansion. The large field region ELFR is first divided into connected components $E_1, E_2, \ldots, E_n$, where a connected component means a maximal set of boxes of LFR belonging to a connected component (in the ordinary sense!) of ELFR. Therefore two boxes of E_i are connected if they are close enough, and between the E_i's there are wide separation corridors. Our goal is to decompose the field into an orthogonal sum of fields, $A = A_0 + \sum_{i=1}^n A_i$. The general field A_0 extends in the full space and has a good propagator. Each field A_i is localized in or near the connected component $E_i = \bigcup_j E_i^j$, where E_i^j is the subset of the i^{th} large field region made of its boxes of scale j. Such a field A_i has a non-translation invariant, hence poorly decreasing propagator, but this propagator has no longer any memory of the existence of the other large field regions, so this formalism is suited for the factorization of these regions. This is the general outline. Before to proceed, we suggest eventually to read reference [DMR] for a more detailed account of such a scenario in a simpler but similar case.

More precisely we define an inductive resolvent expansion. An ordinary resolvent expansion is of the type

$$\frac{1}{\Delta+\delta}=\frac{1}{\Delta}-\frac{1}{\Delta}\delta\frac{1}{\Delta+\delta}\,. \tag{V.1}$$

In our case we imagine Δ to be a translation invariant propagator suited for a cluster expansion in the small field region such as $(-\Delta^{\text{homothetic}})^{-1}$, and the perturbation δ contains the background field, hence it is variable. Even inside the small field region we cannot iterate formula (V.1) infinitely many times because the background fields produced in δ could lead to factorials when bounded. Also in the large field region we must certainly keep this expansion in a resummed form, since the true Gaussian measure there, which has propagator C_{axial} is very far from the small field region propagator. This is the source of many technical difficulties.

First because large fields cannot be bounded effectively we must forget about using a background independent propagator for $\frac{1}{\Delta}$. But derivatives of background fields can be dominated quite effectively. This suggests that we should first compare in the small field region the general propagator to be propagator built with constant background fields.

The interest of using propagators with constant background field is that they are translation invariant and have obviously good spatial decrease. But even when the background field is constant these propagators still have a defect; the Laplacian with background field can have a zero mode if all spatial components of the background field are aligned in su(2) space. As a consequence the bounds on the inverse Laplacian with covariant derivatives are not the same as for the ordinary Laplacian. We need a further decomposition of the momentum around the dangerous zero mode which corresponds to $p_\mu=\lambda(\bar{B}'_{l,\Delta})_\mu.e$, where the scalar product is in su(2) and the vector e is the unit vector of su(2) which is aligned with, say $(\bar{B}'_{l,\Delta})_1$. Since this decomposition is only necessary when all $(\bar{B}'_{l,\Delta})_\mu$, $\mu=1,2,3$ are approximately aligned, the particular choice of $\mu=1$ for e is unimportant.

This decomposition is done in the following way. Let us consider some large field box Δ of scale l, and the corresponding set of boxes in the small field region which have it as ancestor, with scales $m>l$. We redefine only the cutoffs corresponding to the scales m between l and l', where $M^{l'}$ is the order of magnitude of the modulus of $\lambda(\bar{B}'_{l,\Delta})$. If we introduce the corresponding sum of slices $\kappa_l^{l'}=\sum_{l<m\le l'}\kappa^m$ we redecompose the function $\kappa_{l'}^{l}$ as

$$\kappa_l^{l'}=\sum_{m=l+1}^{l'}\kappa^m_{\bar{B}'_{l,\Delta}}+E_{l,B'_l(\Delta)}\,, \tag{V.2}$$

where $\kappa^m_{\bar{B}'_{l,\Delta}}$ restricts $|p_\mu-\lambda(\bar{B}'_{l,\Delta})_\mu.e|$ to be of order M^m, i.e. we simply translate the cutoff without changing its shape (using the same C_0^∞ function), and $E_{l,\bar{B}'_{l,\Delta}}$ is an error term which is the difference between the cutoff κ_l and κ_l translated at $\lambda(\bar{B}'_{l,\Delta})_\mu.e$.

The error term $E_{l,\bar{B}'_{l,\Delta}}$ corresponds to a field which can be associated to the large field box Δ.

In other words the set of slices is no longer cut around the point $p = 0$ but around the point $p_\mu = \lambda B_\mu.e$. The important fact is that we have now for each slice of the propagator in the background field the same scaling and, using integration by parts, the same spatial decrease as for the ordinary slices with the ordinary propagator (see the Appendix). (The reader can think of the background field as a kind of mass so that when the momentum is not almost aligned with it, there is good spatial decay.) In order not to obscure too much the notations, we forget in most of what follows the dependence in B'_l of the cutoffs κ.

Combining the quadratic form (IV.16) with the "fake" measure $d\mu_0$ we have a Gaussian measure on A'_s whose propagator is the inverse of

$$\Delta^0_B = \sum_j \sum_{\Delta \in \mathbf{D}_j} \Delta^0_{B,j,\Delta} ,$$

$$\begin{aligned} \Delta^0_{B,j,\Delta} = & ((\nabla_{\bar{B}'_{l,\Delta}})_\sigma (\kappa^j)^{1/2} \chi_\Delta (\kappa^j)^{1/2} (\nabla_{\bar{B}'_{l,\Delta}})\sigma\, \delta_{\mu\nu} \\ & + (\zeta - 1)(\nabla_{\bar{B}'_{l,\Delta}})_\mu (\kappa^j)^{1/2} \chi_\Delta (\kappa^j)^{1/2} (\nabla_{\bar{B}'_{l,\Delta}})_\nu) \\ & + \lambda^2 \nabla_\sigma (\kappa^j)^{1/2} \chi_\Delta (\kappa^j)^{1/2} \nabla_\sigma \delta_{\mu\nu} \quad \text{if} \quad \Delta \in \mathrm{CSFR}_j , \end{aligned} \tag{V.3a}$$

$$\Delta^0_{B,j,\Delta} \equiv \lambda^2 \nabla_\sigma (\kappa^j)^{1/2} \chi_\Delta (\kappa^j)^{1/2} \nabla_\sigma \delta_{\mu\nu} \quad \text{if} \quad \Delta \in \mathrm{ELFR}_j . \tag{V.3b}$$

We write

$$\frac{1}{\Delta^0_B}(x,y) = \sum_j \sum_{\Delta \in \mathbf{D}_j} \sum_{j'} \sum_{\Delta' \in \mathbf{D}_{j'}} \chi_\Delta(x) \kappa^j \frac{1}{\Delta^0_B} \Delta^0_B \frac{1}{\Delta^0_B} \kappa^{j'} \chi_{\Delta'}(y) . \tag{V.4}$$

Now we can prepare the theory in order to use a propagator but with constant background field, for an horizontal (i.e. slice by slice) cluster expansion. To reach this goal we perform a rewriting of the covariance which replaces the theory with variable background field by a theory with constant background field.

Let us introduce for $\Delta \in \mathrm{CSFR}_j$,

$$\begin{aligned} \Delta^0_\Delta \equiv & ((\nabla_{\bar{B}'_{l,\Delta}})_\sigma (\bar{\kappa}^j)(\nabla_{\bar{B}'_{l,\Delta}})_\sigma \delta_{\mu\nu} + (\zeta - 1)(\nabla_{\bar{B}'_{l,\Delta}})_\mu (\bar{\kappa}^j)(\nabla_{\bar{B}'_{l,\Delta}})_\nu) \\ & + \lambda^2 \nabla_\sigma (\bar{\kappa}^j)^{1/2} \chi_\Delta (\bar{\kappa}^j)^{1/2} \nabla_\sigma \delta_{\mu\nu} , \end{aligned} \tag{V.5a}$$

where $(\bar{\kappa}^j) = \sum_{j' \le j+10} \bar{\kappa}^{j'}$; for $\Delta \in \mathrm{ELFR}_j$ we write simply

$$\Delta^0_\Delta \equiv \lambda^2 \nabla_\sigma (\bar{\kappa}^j)^{1/2} \chi_\Delta (\bar{\kappa}^j)^{1/2} \nabla_\sigma \delta_{\mu\nu} , \tag{V.5b}$$

One expansion step on the propagator consists in writing

$$\begin{aligned} \frac{1}{\Delta^0_B} = & \sum_{j,\Delta \in \mathbf{D}_j} \sum_{j',\Delta' \in \mathbf{D}_{j'}} \chi_\Delta(x) \kappa^j \frac{1}{\Delta^0_B} \Delta^0_B \frac{1}{\Delta^0_B} \kappa^{j'} \chi_{\Delta'}(y) \\ = & \sum_{j,\Delta \in \mathbf{D}_j} \sum_{j',\Delta' \in \mathbf{D}_{j'}} \chi_\Delta(x) \kappa^j \left(\frac{1}{\Delta^0_\Delta} \left[1 + (\Delta^0_\Delta - \Delta^0_B) \frac{1}{\Delta^0_B} \right] \right) \Delta^0_B \\ & \times \left(\left[\frac{1}{\Delta^0_B} (\Delta^0_{\Delta'} - \Delta^0_B) + 1 \right] \frac{1}{\Delta^0_{\Delta'}} \right) \kappa^{j'} \chi_{\Delta'}(y) . \end{aligned} \tag{V.6}$$

Each difference of the type $(\Delta^0_\Delta - \Delta^0_B)$ is then rewritten as (here for simplification we consider only the diagonal terms in $\delta_{\mu\nu}$ and neglect the terms proportional to $\zeta - 1$ which are exactly similar):

$$\Bigg(\sum_{j'' \le j+10, \Delta'' \in \mathbf{D}_{j''}} (\nabla_{\bar{B}'_{l,\Delta}})_\sigma (\kappa^{j''})^{1/2} \chi_{\Delta''} (\kappa^{j''})^{1/2} (\nabla_{\bar{B}'_{l,\Delta}})_\sigma \\ - \sum_{j'', \Delta'' \in \mathbf{D}_{j''}} (\nabla_{\bar{B}'_{l,\Delta''}})_\sigma (\kappa^{j''})^{1/2} \chi_{\Delta''} (\kappa^{j''})^{1/2} (\nabla_{\bar{B}'_{l,\Delta''}})_\sigma \Bigg) . \tag{V.7}$$

For each box $\Delta'' \in \mathrm{CSFR}_{j''}$, $j'' \le j + 10$ we get a difference $\bar{B}'_{l,\Delta} - \bar{B}'_{l,\Delta''}$ which we can rewrite in terms of a gradient acting on B'_l times a length bounded by the distance between Δ and Δ''. This term will deliver a small factor because the background field with a gradient can be bounded using (II.29a–c). This bound delivers a small factor because either the path between the ends of the propagator remain in the small field region and the coupling constant is not completely consumed in the bound [see (II.29a–c)], or it crosses some large field region and the small factor comes from the width of the corridor BR, combined with the good decrease of the propagator.

Finally we might worry that repeating this argument might generate a large number of gradients of background fields; but this concern is taken care of by a rule below which stops the expansion as soon as five error terms have been produced.

In the difference (V.7) there is also an error term

$$-(\nabla_{\bar{B}'_{l,\Delta''}})_\sigma (\kappa^{j''})^{1/2} \chi_{\Delta''} (\kappa^{j''})^{1/2} (\nabla_{\bar{B}'_{l,\Delta''}})_\sigma ,$$

with $j'' > j + 10$. This term will also deliver a small factor through momentum conservation corresponding to integration of x in the box Δ.

In the case where either one of these two terms is chosen in (V.7) the expansion step (V.6) is reiterated on $\dfrac{1}{\Delta^0_B}$ with Δ replaced by Δ''.

There remains terms such as $(\nabla_{\bar{B}'_{l,\Delta}})_\sigma (\kappa^{j''})^{1/2} \chi_{\Delta''} (\kappa^{j''})^{1/2} (\nabla_{\bar{B}'_{l,\Delta}})_\sigma$, $j'' \le j+10$, $\Delta'' \in \mathrm{ELFR}_{j''}$ or $-\lambda^2 \nabla_\sigma (\kappa^{j''})^{1/2} \chi_{\Delta''} (\kappa^{j''})^{1/2} \nabla_\sigma$, $\Delta'' \in \mathrm{ELFR}_{j''}$.

These error terms couple Δ to a box Δ'' of ELFR. Remark that this coupling arises through a propagator with constant background field. When any of these terms is chosen we stop the expansion.

An important additional rule is the following one; when more than five low momentum background fields have been produced we stop the expansion and consider that the boxes of the corresponding string of propagators are attached to the large field region of the corresponding lower scale; therefore we do not need to consider it as a part of the small field region any longer. This rule is necessary even when both ends x and y of our propagators are localized in the small field region, where we have by (II.29a–c) a good compact support restriction on the size of these gradients, because the path of integration from x to y can cross large field regions where the gradient of the field is no longer bounded in a C_0^∞ way and factorials of accumulation could occur.

If we apply this process symmetrically on Δ and Δ', i.e. at both ends of (V.6), we obtain the covariance in the form

$$C = \frac{1}{\Delta_B^0} = C_{11} + C_{12} + C_{21} + C_{22} ,$$
$$C_{11} = \chi_{\mathrm{CSFR}} \Gamma \Delta_B^0 \Gamma \chi_{\mathrm{CSFR}} ; \qquad C_{12} = \chi_{\mathrm{CSFR}} \Gamma [\Gamma' \chi_{\mathrm{CSFR}} + \chi_{\mathrm{ELFR}}] ,$$
$$C_{21} = [\chi_{\mathrm{CSFR}} \Gamma' + \chi_{\mathrm{ELFR}}] \Gamma \chi_{\mathrm{CSFR}} , \tag{V.8}$$
$$C_{22} = [\chi_{\mathrm{CSFR}} \Gamma' + \chi_{\mathrm{ELFR}}] \frac{1}{\Delta_B^0} [\Gamma' \chi_{\mathrm{CSFR}} + \chi_{\mathrm{ELFR}}] ,$$

where Γ is some string of propagators each corresponding to a constant background field (with insertions of ∇B_l^j or of momentum violating terms) and $\Gamma' = \Gamma D_{\mathrm{ELFR}}$, where D_{ELFR} is an insertion explicitly localized in some box of ELFR of scale j.

Then we introduce an interpolation parameter $t \in [0, 1]$ which at $t = 0$ suppresses the coupling pieces C_{12} and C_{21}. Hence we write $C(t) = C_{11} + tC_{12} + tC_{21} + C_{22}$. This is still a positive operator since C_{11} and C_{22} are positive. Then we perform a first order Taylor expansion in this parameter.

The interpolating terms contain an explicit C_{12} or C_{21} link which connects one or two boxes $\Delta, \Delta' \dots$ to one large field box of scale j in the middle (either in the form of a D_{ELFR} insertion or simply by a χ_{ELFR} factor). For this error term we add $\Delta, \dots, \Delta'$ to the large field region ELFR. The process is then reiterated on the remaining $\frac{1}{\Delta_B^0}$ factor with this new definition of ELFR, until finally it stops by exhaustion of all boxes in CSFR.

The decoupled term at $t = 0$ corresponds to a new covariance $C^{11} + C^{22}$. If we introduce the simpler covariance $C^{11} + \bar{C}^{22}$ with $\bar{C}^{22} = \chi_{\mathrm{ELFR}} \frac{1}{\Delta_B} \chi_{\mathrm{ELFR}}$, then we can perform the change of variables $A \to (1 + \chi_{\mathrm{CSFR}} \Gamma')A$ and obtain the same theory with the simplified covariance but a more complicated interaction. The C_{11} piece links boxed of CSFR through strings of propagators in constant background fields, which have both good power counting and good spatial decay. The $\bar{C}^{22}$ lives purely in ELFR. However it is not true that at $t = 0$ the CSFR and ELFR regions have been factorized. Indeed the field is now non-local, so the interaction still couples both regions. This coupling however is easy to control since it occurs through the well controlled Γ' operator.[6]

B) Decoupling of the Different Connected Components of the Large Field Regions

We have not yet a satisfying propagator for performing cluster expansions, because the distant large field regions still interact together through the C_{22} piece of the propagator in (V.8). In this subsection we should describe a general method for removing this interaction. We return to our decomposition of the large field region into connected

[6] In all this discussion we have considered that a D_{ELFR} insertion is equivalent to a χ_{ELFR} characteristic function. Strictly speaking this is not true; the D_{ELFR} insertion contains a χ_{ELFR} term but followed by a controlled non-local operator $(\kappa^j)^{1/2}$. The necessary modifications to take this into account are inessential but painful. They require the use of the corridor BR and some modifications of the formulas. We do not include them in order not to distract the reader from the main argument

components $E_1, E_2, \ldots, E_n$ (we recall our rule that two regions close together are in fact connected, so that the distance between two large field regions is at least a fixed number of boxes of the scale considered). Recall that we want to decompose the field into an orthogonal sum of fields, $A = A_0 + \sum_{i=1}^{n} A_i$, each A_i being associated with E_i, with a poorly decreasing propagator, but this propagator has no longer any memory of the existence of the other large field regions, so that these regions factorize. Instead of that we have at the end of the preceding section, V, A the sum of two fields, one in the small field region and the other in the large field region, but not factorized over its connected components.

The construction of the fields A_i and of their measure is performed as follows. We introduce for the i^{th} region E_i the operator $\Delta_B^{0,i}$ which is roughly speaking the same as Δ_B^0 but in which the other regions E_j, $j \neq i$ are now treated as small field regions. More precisely the formula (V.3b) for Δ_B^0 is changed into formula (V.3a) for $\Delta \in E_j$, $j \neq i$, where the background field B_l' is now introduced also for the boxes of E_j. Finally we can introduce also the operator $\Delta_B^{0,\emptyset}$ in which formula (V.3b) is replaced by (V.3a) for *every* $\Delta \in E_i$, $i = 1, \ldots, n$.

We introduce also $\chi_i \equiv \chi_{E_i}$ for the characteristic function of E_i. In addition to the background field each insertion $\Delta_B^{0,i} - \Delta_B^{0,\emptyset}$ contains a characteristic function χ_i and each insertion $\Delta_B^{0,i} - \Delta_B^0$ contains a characteristic function χ_j, $j \neq i$. Let us for a moment forget the background fields and consider the structure of the expansion according to the localizations.

We start with $\chi_{\text{ELFR}} \dfrac{1}{\Delta_B^0} \chi_{\text{ELFR}}$ [see (V.8)]. We want to decouple a first large field region, say E_1, from the rest. We insert a first resolvent step which is

$$\chi_{\text{ELFR}} \frac{1}{\Delta_B^0} = \chi_{\text{ELFR}} \left[\frac{1}{\Delta_B^{0,\emptyset}} + \frac{1}{\Delta_B^{0,\emptyset}} (\Delta_B^{0,\emptyset} - \Delta_B^0) \frac{1}{\Delta_B^0} \right]. \tag{V.9}$$

Then we decompose the difference $(\Delta_B^{0,\emptyset} - \Delta_B^0)$ as a sum over insertions of χ_i, $i = 1, \ldots, n$.

Iterating this formula at infinity we obtain chains of arbitrary length. In these chains we formally resum every series of insertions of at least *two* consecutive identical functions χ_i. This reconstructs the operator $\dfrac{1}{\Delta_B^{0,i}}$ sandwiched by characteristic functions χ_i on *both* sides; furthermore each $\dfrac{1}{\Delta_B^{0,\emptyset}}$ is sandwiched by χ_i on one side and χ_j on the other side with $i \neq j$.

In order not to use too heavy notations, let us call C_0 the kernel for $\dfrac{1}{\Delta_B^{0,\emptyset}}$. Then formally our expansion has the structure:

$$\chi_{\text{ELFR}} \frac{1}{\Delta_B^0} \chi_{\text{ELFR}} = \sum_{p \geq 0} \sum_{i_0, i_1, \ldots, i_{p+1}} \chi_{i_0} C_0 \chi_{i_1} C_0 \ldots C_0 \chi_{i_p} C_0 \chi_{i_{p+1}} . \tag{V.10}$$

This theory is therefore equivalent to a theory with a substitution rule for the field corresponding to the C_{22} covariance: $A \to \sum_{i=1}^{n} A_i$, in which the A_i's form an independent set of orthogonal random variables, each A_i being distributed with a

Gaussian measure of covariance $\chi_i \frac{1}{\Delta_B^{0,i}} \chi_i$, plus a quadratic interaction of the form

$$e^{\sum_{i,j} \sum_{\substack{i_1,\dots,i_p, i_1\neq i, i_p\neq j \\ i_k\neq i_{k+1}, k=1,\dots,p-1}} A_i C_0 \chi_{i_1} \dots C_0 \chi_{i_p} C_0 A_j} . \tag{V.11}$$

This interaction and the propagator $\chi_i \frac{1}{\Delta_B^{0,i}} \chi_i$ for A_i generates precisely the chains (V.10) (see [DMR]). The fact that we can consider the transition terms $A_i \dots A_j$ in (V.11) as interactions is due to the fact that they are indeed small because of our rule that two disjoint regions E_i, E_j are separated by a corridor of some finite (large) width.

The covariance $C_0 \equiv \frac{1}{\Delta_B^{0,\emptyset}}$ can indeed be now controlled by the same method as in part A) and has therefore good decrease properties.

In this way the remaining covariances $\frac{1}{\Delta_B^{0,i}}$ are now factorized over each connected large field regions. To decouple truly the large field regions there are two equivalent possibilities. The first is to use (V.11) and to expand the corresponding quadratic interaction up to infinity. There is no factorial associated to this expansion, since it is Gaussian. On the chains developed in this way we can read *algebraically* the connections between large field regions. We do not need any interpolation parameters. Positivity of the Gaussian measure is therefore automatically respected. The only drawback of this approach is that one has to be careful that the insertions χ_i in (V.10) really are a short notation for an insertion of $\Delta_B^{0,\emptyset} - \Delta_B^{0,i}$, which contains a background field localized in E_i. There can be arbitrary accumulations of such background fields in the same region. If we dominate them naively (e.g. by the $e^{-\lambda^4 B^4}$ term) we would generate local factorials and the series corresponding to the expansion at infinity of the exponential in (V.11) would not converge. But this is a problem only for B large and we can use the fact that each insertion is paired with a new $C_0 = (\Delta_B^{0,\emptyset})^{-1}$ propagator which precisely decays exactly in the same way at large B.

A second possibility, instead of expanding (V.10–11) to infinity, is to test inductively the coupling of E_1 to $E_2 \cup \dots \cup E_n$ and to iterate. This generates only one link at a time, hence prevents the accumulation of background fields. But this process requires interpolation parameters á la Brydges-Battle-Federbush, and we need to do it symmetrically from both sides of C_{22} in order to preserve positivity. This is done by applying (V.9) for $i = 1$ on both sides of $\frac{1}{\Delta_B^0}$ as in (V.6). We give the corresponding result on one side for simplicity:

$$\begin{aligned} \chi_1 \frac{1}{\Delta_B^0} = \chi_1 \Bigg[& \frac{1}{\Delta_B^{0,\emptyset}} + \frac{1}{\Delta_B^{0,1}} (\Delta_B^{0,\emptyset} - \Delta_B^{0,1}) \frac{1}{\Delta_B^{0,\emptyset}} + \frac{1}{\Delta_B^{0,\emptyset}} (\Delta_B^{0,1} - \Delta_B^0) \frac{1}{\Delta_B^0} \\ & + \frac{1}{\Delta_B^{0,1}} (\Delta_B^{0,\emptyset} - \Delta_B^{0,1}) \frac{1}{\Delta_B^{0,\emptyset}} (\Delta_B^{0,i} - \Delta_B^0) \frac{1}{\Delta_B^0} \Bigg] . \end{aligned} \tag{V.12}$$

Then we multiply the differences $\Delta_B^{0,1} - \Delta_B^0$ by an interpolation parameter s_1, Taylor expand to first order, decompose the remainder term as a sum over χ_j, $j \neq 1$ and iterating, with E_1 and E_j joined together, and iterate.

In both strategies expansion (V.6) (iterated at most five times) has to be used on C_0 to complete the argument, together with the good spatial decrease (A.29) of the homothetic propagator in a fixed background field.

Remark that we could have used more complicated formulas which at once perform the decoupling of the small field regions and the large field regions, hence gathered subsections A) and B) into a single step, but we think it is easier to understand this complicated construction in successive stages.

C) Horizontal Decoupling

At this stage we have factorized the connected large field regions from one another and from the small field region. For each slice $i = \varrho, \ldots, 1$, it remains to perform an ordinary cluster expansion between all boxes in the small field region with respect to the Gaussian measures with propagators C_0 as prepared in the previous subsections. This is done á la Brydges-Battle-Federbush [R]. A key requirement for such an expansion is to preserve positivity of the underlying quadratic form. We have to perform both an ordinary cluster expansion in the small field region, using the propagator C_{11} in (V.8), for which expansion (V.6), iterated at most five times, allows good spatial decrease [see (A.29)]. The positivity requirements are satisfied because of the symmetry of the expansion step (V.6).

Recall also that in each large field region E_i we must combine the measure $C_i = (\Delta_B^i)^{-1}$ on the field A_i with the p_0^2 piece in (IV.11) to reconstruct a propagator with which we can contract the fields produced in the large field expansion (II.25), as required in Lemma II.1.

D) Vertical Decoupling

We need to perform an inductive decoupling of the various momentum slices. There are several facts to consider. First there are interactions which are dominable, but couple different momentum slices. A typical interaction of such a type is a term like $[A, A]\,[B, B]$, where A is a high momentum field of slice i and B a low momentum field of slice j. These interactions are treated as regular dominable interactions of the ϕ_4^4 type. This means that a parameter t_Δ is introduced for each box Δ and an expansion to fifth order is performed in each of these parameters in the standard manner explained in [R]. We should not repeat the corresponding details here.

Some new features however require more explanations. The existence of rectangular anisotropic boxes (the double valuedness of the momentum indices $j = (i, \alpha)$) should not confuse the reader. As explained in Sect. II, the small field boxes, in which the propagator is isotropic, are themselves isotropic, except when they have a large field ancestor of same index i and lower index α. In this case we perform the vertical expansion also for the p_0 slices, which means that a parameter is introduced which tests the coupling, at fixed i, between the p_0 frequencies larger and smaller than α. This vertical piece of the expansion should be thought of really as an auxiliary one, however, because we show in Sect. VII that it is superrenormalizable.

The vertical expansion between large field boxes is similar, except that here like in the horizontal expansion we must first consider that neighboring boxes in index space, one contained into the other, are automatically linked. In this way the necessary vertical expansion to decouple large field regions one from another will give summable links plus small factors for the same reason as in the horizontal case, namely because the vertical distance is at least some large constant.

The background dependence of the Gaussian measure, which still couples small field regions to large field regions of smaller frequency, is also a new feature and requires a vertical decoupling.

The main remark here is to consider that in the vertical decoupling, the parameter t_Δ is introduced in the Gaussian measure with propagator $\frac{1}{\Delta_B}$ which we constructed in the previous subsections, but not in the determinant which corresponds to the normalization of this measure, and also not in the non-dominable part of the Fadeev-Popov determinant. These determinants (or more precisely the effective potential extracted from them) are associated to the normalization of the corresponding large field regions. This is because if we were to develop these determinants which precisely contain the non-dominable piece of the interaction, domination by the $\lambda^4 B^4$ in term in the small field boxes would give a large product of terms of order 1, not bounded by the single small factors associated to the large field box; alternatively domination in the large field box would cost a factorial not summable.

This problem forces us to attribute the corresponding normalization to the large field regions considered and to bound it with a special argument in Sect. VI. For the normalized background dependent Gaussian measure, there is no such problem, because the corresponding propagators hit ordinary dominable vertices which provide the necessary coupling constants, hence the necessary small factors.

We have to define the t_Δ interpolation that we introduce on the explicit propagators created by the previous expansion steps (in particular the horizontal cluster expansion).

Finally we should explain the modifications introduced by the use of the slices (V.2) around $\bar{B}'_l$ (the decomposition with the cutoffs (V.2)). The reader should not consider that these slices are new momentum slices but rather that they replace former slices. In the vertical expansion we introduce therefore also a dependence in $t_{\Delta'}$ in the corresponding cutoffs, that is we write, if the ancestor of Δ' is the large field box Δ:

$$\kappa^l_{l'} = \sum_{m=l+1}^{l'} \kappa^m_{t_{\Delta'}\bar{B}'_{l,\Delta}}|_{t_{\Delta'}=1}, \tag{V.13}$$

where $\kappa^m_{t_{\Delta'}\bar{B}'_{l,\Delta}}$ restricts $|p_\mu - \lambda t_{\Delta'}(\bar{B}'_{l,\Delta})_\mu, e|$ to be of order M^m. In this way the parameter $t_{\Delta'}$ interpolates smoothly between the translated slices at $t = 1$ and the ordinary slices at $t = 0$.

In the usual way we perform for each box Δ a fifth order Taylor expansion in t_Δ. Each derivative creates at least one badly localized low momentum field. The connected objects which have at most four such low momentum legs, hence which need renormalization are in this way necessarily decoupled terms at $t_\Delta = 0$.

E) The Renormalization and the Flow of the Effective Coupling Constant

To simplify the notations we now call $B(\Delta)$ or sometimes simply B the averaged field called previously $B'_{l,\Delta}$.

After the horizontal and vertical decoupling has been performed at a given scale, one is in a position to renormalize the divergent two and four point contributions. This requires first a Mayer expansion in order to render these contributions translation-invariant. For the details of such a Mayer expansion we refer to [R]. Once the divergent contributions are free from hardcore constraints they can be cancelled by counterterms of the desired form. These counterterms in turn generate a flow for the coupling constant of the theory which is the one of an asymptotically free theory such as the Gross-Neveu model in two dimensions and treated in the same way [FMRS2, R]. In this way the ansatz (II.11) is justified. The effective constants λ_i at scale i are shown in the standard way to be very close to the simplified effective couplings given by formula (II.12), since the true flow corresponds really to a bounded flow for the constant C in (II.11). Remark that to control the flow we are not forced to construct the correct β function (which would contain renormalons); as explained e.g. in [R] it is enough to get the two first terms correctly, plus a uniform bound of the correct order on the rest, which remains expressed in terms of the effective couplings.

Remark also that we have to distinguish from the rest the case of the normalization determinant corresponding to the background dependent Gaussian measure and the non-dominable part of the Fadeev-Popov determinant. Because these determinants are not expanded in the vertical expansion but directly associated to the corresponding large field regions, we do not have a complete cancellation between the background field counterterms and the corresponding polymers with two or four background field external legs. One could believe that the not-cancelled counterterms together with the determinants simply form subtracted determinants of the $\det_4$ type. We recall that

$$\det_4(1+K) \leq \det(1+K)\, e^{\mathrm{Tr}\, -K+\frac{K^2}{2}-\frac{K^3}{3}+\frac{K^4}{4}} \,. \tag{V.14}$$

But to our surprise this turned out not to be true. The non-dominable part of the Fadeev-Popov determinant plus the corresponding uncancelled counterterms gives indeed a $\det_4$. But because counterterms are gauge dependent and the background field gauge does not coincide with the ordinary gauge where the small field flows (hence the counterterms) are computed, it appears a difference between the normalization determinant plus the counterterms, $\det^{-1/2} \Delta^{\text{homothetic}}_{B(\Delta)}\, e^{\mathrm{CT}(B)}$ and $\det_4^{-1/2} \Delta^{\text{homothetic}}_{B(\Delta)}$. This difference is computed in detail in the next section, and plays a crucial rôle in the final bound on the large field regions.

The only special difficulty in this case has to do with the fact that the flow keeps the structure of the theory unchanged. In the standard textbooks about perturbative renormalization of gauge theories such as [IZ] one proves that the renormalization group flow keeps the effective interaction of the theory of the Yang-Mills form but with a running coupling constant. This is usually done in perturbation theory using dimensional regularization. In our case this fact will also be true, provided we take into acount the flow of our gauge-restoring counterterms in $\lambda^4 A^4$, because our cutoff is not gauge invariant and is used at every scale with the same shape to construct the new cutoff of the effective theory. This has really to be done only at the one loop order for the $\lambda^4 A^4$ term, but concerning the relevant mass operator, the value of the mass counterterm has to be fixed exactly by a fixed point method (as is done e.g. for the critical value of the mass in infrared ϕ^4_4 [R]).

Remark however that if we consider the renormalization group flow of non-abelian gauge theories in an ordinary gauge, there is usually both coupling constant and wave function renormalization. This makes the discussion of the normalization of the Gaussian measure in the background field much less transparent. We prefer to use the homothetic gauge in which there is almost no or no wave function renormalization (depending on whether one wants a completely explicit formula for this gauge, or one is satisfied with the solution of an implicit function theorem). This is a technical trick which could presumably be circumvented by a more complicated analysis but it is very convenient in this respect.

F) The Effective Potential

It remains now to prepare the theory in order to compute effectively the normalization of the background dependent Gaussian measure and of the Fadeev-Popov determinant which have to be attributed to the large field regions. In order to do this we use the effective potential method [BG, MS, dCMSdV]. This means that for a given large field box we want to introduce specific boundary conditions so that we can compute explicitly the functional integral over the associated small field region.

This in turn requires to create some gap between the size of the large field cube Δ which we suppose of index $j = (i, \alpha)$ and the frequencies of the fields in the small field region whose functional integration gives the dressing factor. We have to recall that the boxes are rectangular rather than cubic. We have to distinguish two cases:
– if both p_0 and $\vec{p}$ are large compared to the respective scales M^α and M^i of the large field box Δ (i.e. $|\vec{p}| \geq M^{i+100}$ and $|p_0| \geq M^{\alpha+100}$), there is no problem with the boundary conditions. As in the usual effective potential method, the boundary effects are negligible compared to the functional integral in the whole volume Δ.
– if p_0 or $\vec{p}$ are small, we can use a rough bound such as

$$|\det(1+K)| \leq e^{(1/2)\,\mathrm{Tr}(K+K^*+K,K^*)} \tag{V.15}$$

on the corresponding determinants; since the corresponding momentum integrals are unbounded either in three or one dimensions, the resulting bound in $\lambda^2 B^2$ is either linearly divergent or convergent. We can add it to the quadratically divergent $\lambda^2 B^2$ term computed in Sect. III, and bound the total by increasing slightly the coefficient of this term. In Sect. VI it is shown that this term is controlled by the effective potential generated by the dressing factor.

This effective potential method (up to small error terms) yields in each large field box Δ a determinant of the operator $\Delta^{\text{homotehtic}}_{B(\Delta)}$, combined with the non-dominable part of the Fadeev-Popov determinant; we have to compare it to the normalization of a small field box in which the Gaussian measure is the one obtained with the usual homothetic gauge covariance; this is exactly the same determinant but in which $B(\Delta) = 0$. Similarly the Fadeev-Popov determinant has to be divided by the Fadeev-Popov determinant at $B(\Delta) = 0$. The reader should indeed keep in mind that the typical case is (a posteriori, after all estimates have been performed) ELFR $= \emptyset$, in which case the covariance of the A'_s field (which is then equal to the full A' field) is simply $1/p^2(\delta_{\mu\nu} - (1-\zeta^{-1})p_\mu p_\nu/p^2)$, the homothetic gauge covariance (up to the small "fake" term $\lambda^2 p^2$ which can be recombined with the corresponding counterterm in (VI.1).

To this determinant we have to add the effective action for a constant B field which reduces to the $e^{-F_4} = e^{-\lambda^2[B,B]^2}$ term in e^{-F^2}. Finally as mentioned in the previous subsection we have to add to this term the uncancelled pieces of the B-dependent counterterms. Returning to Sect. III, Eq. (III.1), we remark that the terms with derivatives of B cancel in the effective potential computation. We have therefore to consider only the terms in B^2, B^4 and $[B,B]^2$. The uncancelled piece of the $\lambda^4[B,B]^2$ counterterm is a small correction to the action term in $\lambda^2[B,B]^2$, and we do not need to compute it precisely. But the B^2 and B^4 terms are crucial. Using Sect. III the reader can check that at one loop the totality of these counterterms is uncancelled because the graphs $G_1, G_2, G_3, G_4, G'_1, G'_2$, and G'_3 only contain vertices of the non-dominable type $[A,B]$. Therefore in the effective potential the full value of these counterterms has to be included.

The conclusion of this analysis is that we have to bound the product of these counterterms, of an $e^{-\lambda^2[B,B]^2}$, and of a quotient of properly normalized determinants in which the background field is a constant and there are e.g. periodic boundary conditions on the operator in the determinant; the proper normalization means simply that at $B(\Delta) = 0$ this quotient of determinants is simply 1. The precise form of this factor is written down in the next section [Eq. (VI.1)].

To obtain a non-perturbative bound on this object is one of the main points of this paper and is explained in detail in Sect. VI. Again the solution ultimately depends of a correct choice of the ultraviolet cutoff function.

VI. The Main Stability Estimate for a Large Field Region

In this section we prove that the non-trivial factor associated to the large field regions which come from the B dependent gauge fixing and functional integration in the associated small field regions can be bounded uniformly in B by 1, if the ultraviolet cutoff is of a certain stabilizing shape. This result (a kind of non-perturbative stability) is at the core of our whole analysis.

We have to consider the small field functional integral associated to the normalization of a unit of the large field domain such as a fixed cube Δ of $\mathbf{D}^{\mathbf{i},\alpha}$. As explained in the preceding section, we can compute this normalization in the case of a constant background field $B(\Delta) = B$ and of e.g. periodic boundary conditions (the type of boundary conditions being inessential).

This functional integration is (taking into account the action $F_{\mu\nu}(B)$ which for a constant B reduces to $(\lambda^2/2) \int_\Delta [B, B]^2$):

$$g_{i,\alpha,\Delta}(B) = \frac{\det_{i,\alpha,\Delta}(\mathrm{FP})}{\det_{i,\alpha,\Delta}(BF)^{1/2}} \, e^{\mathrm{CT}_{i,\alpha,\Delta} - \lambda^2/2 \int_\Delta [B,B]^2} , \qquad \text{(VI.1)}$$

where $\Delta \in \mathbf{D}^{\mathbf{i},\alpha}$ is a cube of scale i, α, hence of volume $|\Delta| = M^{-3i-\alpha}$ (at most $\lambda^{-1}M^{-4i}$); $\det_{i,\alpha,\Delta}(BF)$ means the determinant with the appropriate cutoffs corresponding to our background dependent measure on the small field region associated to Δ; it has infrared cutoff of the type $\kappa^i(p)\kappa^\alpha(p_0)$ and ultraviolet cutoff given by the exact shape of the small field domain associated to Δ; this domain in the typical case of an isolated large field cube corresponds to an integral over momenta with the ultraviolet cutoff $\kappa^\varrho(p)$ of scale ϱ. As explained in Sect. V, the case of more complicated large field domains reduce to this case. Finally $\mathrm{CT}_{i,\alpha,\Delta}$ is the associated set of counterterms (which contains a positive, potentially dangerous B^2 counterterm and a B^4 counterterm, which thanks to our choice of ultraviolet cutoff is negative and stabilizing); the exact value of these counterterms depends on the precise shape of the ultraviolet cutoff, and in particular of the parameter η in (II.14).

Similarly $\det_{i,\alpha,\Delta}(\mathrm{FP})$ is the background dependent determinant corresponding to the integration over the ghosts in the small field region associated to Δ; it is equal to an ordinary Fadeev-Popov determinant.

More precisely, if we rewrite the cutoff simply as $\kappa(p^2)$ and take into account our form of the cutoff and the normalization at $B = 0$, we have a twelve by twelve matrix

BF in su(2) $\otimes \mathbb{R}^4$ space (we forget to put the su(2) indices):

$$BF = \delta_{\mu\nu} + \frac{\kappa(p)}{p^2} \times \left[-(D^2\delta_{\mu\sigma} - (1-\zeta)G_\mu D_\sigma)\left(\delta_{\sigma\nu} + (\zeta^{-1}-1)\frac{p_\sigma p_\nu}{p^2}\right) - p^2\delta_{\mu\nu}\right], \quad \text{(VI.2)}$$

where repeated indices are summed, and

$$-D^2 = -\sum_\mu D_\mu D_\mu = [p^2\delta_{ab} + 2ip_\mu\varepsilon_{abc}\lambda B^c_\mu + \lambda^2 B^c_\mu B^c_\mu \delta_{ab}(1-\delta_{ac}) - \lambda^2 B^a_\mu B^b_\mu(1-\delta_{ab})], \quad \text{(VI.3)}$$

Similarly the Fadeev-Popov operator (normalized at $B = 0$) is a three by three matrix in su(2) space:

$$\mathrm{FP} = 1 + \frac{\kappa(p)}{p^2}\left[-\sum_\mu \partial_\mu D_\mu - p^2\right]. \quad \text{(VI.4)}$$

Using the Euclidean and global SU(2) symmetry and the fact that $B_0 = 0$ (because of the axial gauge), we can explore completely the function $g_{\imath,\alpha,\Delta}$ by considering a field B with only two non-zero components $B^1_1 = x/\lambda$ and $B^2_2 = y/\lambda$. Then $\lambda^2 B^2 = x^2 + y^2$ and $\lambda^4[B,B]^2 = x^2y^2$. The function $g_{\imath,\alpha,\Delta}$ becomes a symmetric function of x and y. We are going to prove the following uniform estimate:

Lemma VI.1. *For a sufficiently wide ultraviolet cutoff (in the sense of the parameter η in (II.14) being small) we have:*

$$g_{\imath,\alpha,\Delta}(x,y) \le 1. \quad \text{(VI.5)}$$

We are going to compute this function $g_{\imath,\alpha,\Delta}$ as the exponential of an action integrated over Δ, and prove that this action is always negative. Let us stress that a constant bound such as $O(1)M^{4i}$ on this action would not be sufficient because the volume of the cube can be as large as $\lambda^{-1}M^{-4\imath}$ if α is quite small compared to i, and the small factor gained from the positivity of the axial gauge for such a cube is in $e^{\lambda^{-\varepsilon}}$, *not* in $e^{\lambda^{-1}}$.

We write $\psi = \kappa_{\imath,\alpha}(p^2)/p^2$. It is convenient to define P_μ such that $D_\mu = iP_\mu$ (in Fourier space). With these conventions we have in su(2) space:

$$P_0 = \begin{pmatrix} p_0 & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & p_0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} p_3 & 0 & 0 \\ 0 & p_3 & 0 \\ 0 & 0 & p_3 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_1 & -ix \\ 0 & ix & p_1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} p_2 & 0 & iy \\ 0 & p_2 & 0 \\ -iy & 0 & p_2 \end{pmatrix}. \quad \text{(VI.6)}$$

The Fadeev-Popov operator $K_{\mathrm{FP}} = -\partial \cdot D = \sum_\mu p_\mu P_\mu$ is

$$K_{\mathrm{FP}} = \begin{pmatrix} p^2 & 0 & ip_2y \\ 0 & p^2 & -ip_1x \\ -ip_2y & ip_1x & p^2 \end{pmatrix}. \quad \text{(VI.7)}$$

The hermitian matrix $-D^2 = P^2$ is

$$-D^2 = P^2 = \begin{pmatrix} p^2+y^2 & 0 & 2ip_2y \\ 0 & p^2+x^2 & -2ip_1x \\ -2ip_2y & 2ip_1x & p^2+x^2+y^2 \end{pmatrix}. \tag{VI.8}$$

The twelve by twelve matrix BF that has to be computed in the general case $\zeta \neq 1$ is decomposed into a four by four matrix of three by there su(2) blocks:

$$\begin{aligned} BF &= \delta_{\mu\nu} + \psi\Big[(P^2\delta_{\mu\sigma} - (1-\zeta)P_\mu P_\sigma) \\ &\quad \times \left(\delta_{\sigma\nu} + (\zeta^{-1}-1)\frac{p_\sigma p_\nu}{p^2}\right) - p^2\delta_{\mu\nu}\Big] \\ &= \delta_{\mu\nu}(1+\psi U) + \psi\frac{p_\mu p_\nu}{p^2}V + \psi W_{\mu\nu}, \end{aligned} \tag{VI.9}$$

where

$$U = P^2 - p^2 = \begin{pmatrix} y^2 & 0 & 2ip_2y \\ 0 & x^2 & -2ip_1x \\ -2ip_2y & 2ip_1x & x^2+y^2 \end{pmatrix}, \tag{VI.10}$$

$$V = \begin{pmatrix} (1/\zeta-1)y^2 & 0 & (1/\zeta-\zeta)ip_2y \\ 0 & (1/\zeta-1)x^2 & -(1/\zeta-\zeta)ip_1x \\ -(1/\zeta-\zeta)ip_2y & (1/\zeta-\zeta)ip_1x & (1/\zeta-1)(x^2+y^2) \end{pmatrix}, \tag{VI.11}$$

and

$$W_{00} = W_{03} = W_{30} = W_{33} = 0, \tag{VI.12a}$$

$$W_{01} = W_{10} = -(1-\zeta)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -ip_0x \\ 0 & ip_0x & 0 \end{pmatrix}, \tag{VI.12b}$$

$$W_{02} = W_{20} = -(1-\zeta)\begin{pmatrix} 0 & 0 & ip_0y \\ 0 & 0 & 0 \\ -ip_0y & 0 & 0 \end{pmatrix}, \tag{VI.12c}$$

$$W_{13} = W_{31} = -(1-\zeta)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -ip_3x \\ 0 & ip_3x & 0 \end{pmatrix}, \tag{VI.12d}$$

$$W_{23} = W_{32} = -(1-\zeta)\begin{pmatrix} 0 & 0 & ip_3y \\ 0 & 0 & 0 \\ -ip_3y & 0 & 0 \end{pmatrix}, \tag{VI.12e}$$

$$W_{11} = \begin{pmatrix} 0 & 0 & 0 \\ xy\,\frac{(1-\zeta)^2}{\zeta}\,\frac{p_1p_2}{p^2} & -x^2\left[\frac{(1-\zeta)^2}{\zeta}\,\frac{p_1^2}{p^2}+(1-\zeta)\right] & -ip_1x(\zeta-1/\zeta) \\ 0 & ip_1x(\zeta-1/\zeta) & -x^2\left[\frac{(1-\zeta)^2}{\zeta}\,\frac{p_1^2}{p^2}+(1-\zeta)\right] \end{pmatrix}, \quad \text{(VI.12f)}$$

$$W_{12} = \begin{pmatrix} 0 & 0 & -ip_1y(1-\zeta) \\ xy\left[\frac{(1-\zeta)^2}{\zeta}\,\frac{p_2^2}{p^2}-(1-\zeta)\right] & -x^2\,\frac{(1-\zeta)^2}{\zeta}\,\frac{p_1p_2}{p^2} & -ip_2x(1-1/\zeta) \\ ip_1y(1-\zeta) & ip_2x(1-1/\zeta) & -x^2\,\frac{(1-\zeta)^2}{\zeta}\,\frac{p_1p_2}{p^2} \end{pmatrix}, \quad \text{(VI.12g)}$$

$$W_{21} = \begin{pmatrix} -y^2\,\frac{(1-\zeta)^2}{\zeta}\,\frac{p_1p_2}{p^2} & xy\left[\frac{(1-\zeta)^2}{\zeta}\,\frac{p_1^2}{p^2}+(1-\zeta)\right] & ip_1y(1-1/\zeta) \\ 0 & 0 & ip_2x(1-\zeta) \\ -\,ip_1y(1-1/\zeta) & -ip_2x(1-\zeta) & -y^2\,\frac{(1-\zeta)^2}{\zeta}\,\frac{p_1p_2}{p^2} \end{pmatrix}, \quad \text{(VI.12h)}$$

$$W_{22} = \begin{pmatrix} -y^2\left[\frac{(1-\zeta)^2}{\zeta}\,\frac{p_2^2}{p^2}+(1-\zeta)\right] & xy\,\frac{(1-\zeta)^2}{\zeta}\,\frac{p_1p_2}{p^2} & ip_2y(\zeta-1/\zeta) \\ 0 & 0 & 0 \\ -\,ip_2y(\zeta-1/\zeta) & 0 & -y^2\left[\frac{(1-\zeta)^2}{\zeta}\,\frac{p_2^2}{p^2}+(1-\zeta)\right] \end{pmatrix}, \quad \text{(VI.12i)}$$

Similarly we can compute the three by three matrix FP:

$$\mathrm{FP} = 1 + \psi(K_{\mathrm{FP}} - p^2) = \begin{pmatrix} 1 & 0 & i\psi p_2 y \\ 0 & 1 & -i\psi p_1 x \\ -i\psi p_2 y & i\psi p_1 x & 1 \end{pmatrix} . \tag{VI.13}$$

Remark that all these matrices are homogeneous. We introduce the variables $u \equiv p^2 M^{-2i}$, θ and ϕ such that $p_1^2 = uM^{2i}\cos^2\theta$ and $p_2^2 = uM^{2i}\sin^2\theta\cos^2\phi$. Then d^4p is proportional to $M^{4i}udu\sin^2\theta\sin\phi d\theta d\phi$. We represent the effect of the infrared and ultraviolet cutoffs by the cutoff $\kappa_{i,\alpha}(p) = \kappa_{\varrho-i}(p^2M^{-2i}) \equiv \kappa_k(u)$, $k = \varrho - i$. Indeed we can limit ourselves to the case where α takes its minimum value, in which case there is no particular cutoff on the value of p_0; as explained in Sect. V, other cases are similar.

We can suppose that $y \le x$, since the function $g_{i,\alpha,\Delta}$ is symmetric in x and y. We put $v = p^2/x^2 = uM^{2i}/x^2$, $\beta = \kappa_k(u)/v = x^2M^{-2i}\kappa_k(u)/u$ (β is a function of k, u and x), $\kappa = \kappa_k(u)$ and $y = tx$, $t \in [0,1]$.

Remark that $|\Delta|M^{+4i} = M^{i-\alpha}$. With these notations we can compute explicitly the three by three determinant (VI.13) and in principle the twelve by twelve determinant (VI.9) and we find (forgetting the commutator $[B,B]^2$ in (VI.1), which is positive and plays no rôle anyway):

$$\begin{aligned} g_{i,\alpha,\Delta}(x,y) = g_{k,\alpha}(x,t) = \exp\Bigg(|\Delta|M^{4i}\Bigg(\Bigg[\int_0^{+\infty} & udu(2/\pi)\int_0^{+\pi}\sin^2\theta d\theta(1/2)\int_0^{+\pi}\sin\phi d\phi \\ & \times \ln|1 - \beta\kappa[\cos^2\theta + t^2\sin^2\theta\cos^2\phi]| \\ & - (1/2)\ln(1 + \beta P(\beta,\kappa,t,\cos\theta,\sin\phi,\zeta,1/\zeta))\Bigg] \\ & + \Bigg[\int_0^{+\infty} udu(\beta/2)[6(1 + (1/\zeta - 1)/4) - \kappa(4 + 3(1/\zeta - 1)/2)](1 + t^2)\Bigg] \\ & - \Bigg[\int_0^{+\infty} udu(\beta^2/24)[(36 + 18(1/\zeta - 1) + 7.5(1/\zeta - 1)^2) \\ & - \kappa(90 + 45(1/\zeta - 1) + 15(1/\zeta - 1)^2 + \kappa^2(54 + 27(1/\zeta - 1) \\ & + 7.5(1/\zeta - 1)^2)](1 + t^2)^2\Bigg]\Bigg)\Bigg) , \end{aligned} \tag{VI.14}$$

where P is a polynomial in all the variables listed, whose explicit computation requires the evaluation of the twelve by twelve determinant (VI.9). In the case of the Feynman gauge $\zeta = 1$, this determinant simplifies into a three by three determinant to the fourth power, which is easily computed, and one finds:

$$\begin{aligned} &(1 + \beta P(\beta,\kappa,t,\cos\theta,\sin\phi,\zeta,1/\zeta)) \\ &\quad = [[1 + 2\beta(1 - 2\kappa(\cos^2\theta + t^2\sin^2\theta\cos^2\phi)) + \beta^2] \\ &\qquad + \beta t^2[2 - \beta(t^2 + 3 - 4\kappa(\cos^2\theta + \sin^2\theta\cos^2\phi)) + \beta^2(1 + t^2)]]^4 . \end{aligned} \tag{VI.15}$$

Remark that the relative normalization of the terms and the counterterms in (VI.14) is crucial for what follows. To check that this relative normalization is correct the reader should check that the mass term (in x^2) which comes from the Fadeev-Popov determinant in (VI.14) agrees with the mass counterterm from the same Fadeev-Popov determinant computed in Sect. III, which is the piece $e^{\int_0^{+\infty} udu(\beta/2)(\kappa/2)(1+t^2)}$ in the total mass counterterm

$$e^{\int_0^{+\infty} udu(\beta/2)[6(1+(1/\zeta-1)/4)-\kappa((9/2)+6(1/\zeta-1)/4)+\kappa/2](1+t^2)} \tag{VI.16}$$

as computed in Sect. III. Also the quartic term coming from the Fadeev-Popov determinant matches the corresponding computation of the graph G_4 in Sect. III (recall that the natural normalization in Sect. III of the quadratic counterterm was $A^2/2$ and of the quartic counterterm was $A^4/24$, see (III.1)).

In the general case we do not attempt the computation of P, but we compute only the integral over θ and ϕ of its first order term in β which is much more accessible. It can be done by tracing the twelve by twelve matrix (VI.9) and its square, which is much easier than computing the determinant. Equivalently one can perform a graphical computation; one finds the same graphs as in Sect. III (remember that they have the opposite sign of the counterterms in (VI.14)), plus additional pieces which come from the new vertex $(\zeta/2)[(P\cdot A)^2-(p\cdot A)^2]$ which is due to our background dependent gauge fixing. This new vertex is $(\zeta/2)(2\varepsilon_{abc}B^b_\mu A^c_\mu\partial_\nu A^a_\nu+\varepsilon_{abc}\varepsilon_{a'b'c'}B^b_\mu A^c_\mu B^{b'}_\nu A^{c'}_\nu)$, hence we obtain additional graphs G''_1 and G''_2 analogous to G'_1 and G'_2. After a straightforward computation, the contribution of G''_1 is $-2\zeta(1+(1/\zeta-1)/4)$ and the contribution of G''_2 is $\kappa(-2+(3/2)\zeta)$ (beware that in the computation of G''_2 one has to add the case with two "new" vertices to the case with one old and one "new" vertex). Hence adding the contribution of the ordinary graphs G'_1 and G'_2 we have

$$\begin{aligned}&\left[(2/\pi)\int_0^{+\pi}\sin^2\theta d\theta(1/2)\int_0^{+\pi}\sin\phi d\phi-(1/2)\ln(1+\beta P,\kappa,t,\cos\theta,\sin\phi,\zeta,1/\zeta))\right]\\&\quad=(\beta/2)(1+t^2)(-6(1+(1/\zeta-1)/4)-2\zeta(1+(1/\zeta-1)/4)\\&\qquad+\kappa[(9/2)+(3/2)(1/\zeta-1)-2+(3/2)\zeta])+O(\beta^2)\,.\end{aligned} \tag{VI.17}$$

We will prove Lemma VI.1 using a crude bound which follows form (VI.17) and the fact that the polynomial P has a fixed number of (in principle) computable coefficients. But in Appendix A we provide also an explicit computation of (VI.14) in the tractable case of the Feynman gauge $\zeta=1$ which the reader might find enlightening.

Using the asymptotic expansion (VI.17) near $\beta=0$, we find

$$\begin{aligned}&(2/\pi)\int_0^{+\pi}\sin^2\theta d\theta(1/2)\int_0^{+\pi}\sin\phi d\phi\ln|1-\beta\kappa[\cos^2\theta+t^2\sin^2\theta\cos^2\phi]|\\&\quad-(1/2)\ln(1+\beta P(\beta,\kappa,t,\cos\theta,\sin\phi,\zeta,1/\zeta))\\&\quad+[(\beta/2)[6(1+(1/\zeta-1)/4)-\kappa(4+3(1/\zeta-1)/2)](1+t^2)]\\&\cong_{\beta-0}(\beta/2)(1+t^2)[-3\zeta/2-1/2-\kappa(2-(3/2)\zeta)]+O(\beta^2)\,.\end{aligned} \tag{VI.18}$$

If we restrict us to the region $0 \le \zeta \le 1$, we have

$$(\beta/2)(1+t^2)[-3\zeta/2 - 1/2 - \kappa(2-(3/2)\zeta)] \le -(\beta/4)(1+\kappa) \le -(\beta/4)\,. \quad \text{(VI.19)}$$

Now using the fact that $\kappa, t, \cos\theta, \sin\phi, \zeta$ and ζ^{-1} all vary in compact intervals (for ζ and ζ^{-1} this is because we can restrict us to a small interval centered respectively around 3/13 or 13/3), and the fact that the logarithms of explicit polynomials in β such as those of (VI.14) are bounded by a constant times β at large β (uniform in $\kappa, t, \cos\theta, \sin\phi, \zeta$ and ζ^{-1} by compactness) it is easy to check the following lemma:

Lemma VI.2. *If* $0 \le \zeta \le 1$ *there exists two (large...) constants* K_1 *and* K_2 *such that*

$$\begin{aligned}
&(2/\pi)\int_0^{+\pi} \sin^2\theta d\theta (1/2)\int_0^{+\pi} \sin\phi d\phi \ln|1 - \beta\kappa[\cos^2\theta + t^2\sin^2\theta\cos^2\phi]| \\
&\quad - (1/2)\ln(1+\beta P(\beta,\kappa,t,\cos\theta,\sin\phi,\zeta,1/\zeta)) \\
&\quad + \left[\int udu(\beta/2)[6(1+(1/\zeta-1)/4) - \kappa(4+3(1/\zeta-1)/2)](1+t^2)\right] \\
&\le K_2\beta \quad \text{if} \quad \beta \ge (K_1)^{-1} &\text{(VI.20a)}\\
&\le -(\beta/8) \quad \text{if} \quad \beta \le (K_1)^{-1}\,. &\text{(VI.20b)}
\end{aligned}$$

We can then complete the proof of Lemma VI.1. Indeed we write (using the fact that $\beta = \kappa/v$ and $0 \le \kappa \le 1$ and using (III.7)):

$$\begin{aligned}
g_k(x) &\le \exp\Bigg(|\Delta|\Bigg(x^4\Bigg[\int_{v<K_1} K_2\beta v dv - \Bigg[\int vdv(\beta^2/24)[(36+18(1/\zeta-1) \\
&\quad + 7.5(1/\zeta-1)^2) - \kappa(90+45(1/\zeta-1)+15(1/\zeta-1)^2) \\
&\quad + \kappa^2(54+27(1/\zeta-1)+7.5(1/\zeta-1)^2)](1+t^2)^2\Bigg]\Bigg)\Bigg) \\
&\le \exp(|\Delta|x^4[K_1K_2 - |\log\eta|]) \le 1 &\text{(VI.35)}
\end{aligned}$$

if, again, we choose the parameter η in Sect. III sufficiently small so that $|\log\eta| \ge K_1K_2$. This achieves the proof of the lemma.

VII. The Convergence of the Expansion: Bounds on Error Terms

In this section we summarize the reasons for convergence of our expansion.

We have to give a rough description of the polymers which, at a given stage of the expansion, have to be summed for the Mayer expansion to converge. We know that in this type of expansion convergence follows if in the amplitudes of the polymers there is an adjustable small constant per box (such as λ), and if furthermore we can resum all the polymers containing a fixed box.

The small constant per box is the easiest part to check. The case of empty small field boxes is taken into account by the normalization. A small field box which is not empty must contain some explicit vertex or error term, which by inspection is small. The large field boxes are small because of the Lemma II.1 corrected by the computation to the previous section which justifies the existence of an associated small factor. But we must check that given some fixed box we can perform the sum over all the boxes linked to it by previous horizontal, vertical or Mayer expansions using the decay of the corresponding links. The structure of these sums is basically similar to the case considered in [R] with one most notable exception, the existence of various lattices with anisotropic shapes. Therefore we focus first on explaining the convergence of this new feature. More precisely let us explain why at fixed value of i the vertical expansion in α, the index labeling the slices for p_0, is convergent.

Let us consider the small field propagator in a slice of index $j = \{i, \alpha\}$. From the point of view of power counting the homothetic gauge is similar to the Feynman gauge, hence to simplify notations let us pretend the small field propagator to be simply $1/p^2$. The same propagator in the slice j would then be $C^j = \kappa^j(p)/p^2$. It satisfies the estimate

$$C^j(x-y) \le K_q M^i M^\alpha \left(\frac{1}{1+|x_0-y_0|M^\alpha} \cdot \frac{1}{1+|\vec{x}-\vec{y}|M^i} \right)^q \tag{VII.1}$$

for some large integer q.

The power counting of the worse vertex (which is a trilinear vertex with derivative coupling $A^2\partial A$ rather than a quartic A^4 vertex) integrated in a box of $\mathbf{D^j}$ is $M^i M^{(3/2)(i+\alpha)} M^{-3i-\alpha} = M^{-(i-\alpha)/2}$. This vertex is equipped with one factor λ. However by parity a single such vertex vanishes. Therefore we have at least two such vertices, which means the same power counting as a single quartic vertex (of power counting $M^{2(i+\alpha)} M^{-3i-\alpha} = M^{-(i-\alpha)}$) with coupling λ^2.

Since the smallest value of the index α at i fixed is $\alpha_{\min}$ such that $M^{-(i-\alpha_{\min})} \le \lambda^{-1}$ (see II.21a–b), we conclude that in the vertical expansion any contribution of scale α to attached to a box of scale α', with $\alpha_{\min} < \alpha' < \alpha$ can be resummed in the box of scale α' (which contains $M^{\alpha-\alpha'}$ boxes of scale α) using one coupling constant $\lambda \le M^{-(\alpha-\alpha')}$; furthermore there remains a small factor of size at least λ. Finally there remains a factor at least $M^{-(i-\alpha)}$ which means that the sum over α can be performed and that in fact most of the sum comes from the case $\alpha = i$. This confirms the auxiliary nature of this expansion. We have to perform it because we had to keep the decomposition of the isotropic small field propagator into anisotropic scales in the case of relevant small field boxes in the sense of Sect. IIb. Otherwise we would not have the right spatial decay in the x_0 direction (this is due to the fact that in the domain of a large field box the small field cutoffs is limited by the ancestor of the small field boxes). However we see that the main small field contribution comes really from the case $i = \alpha$, so this decomposition is not very important.

The other types of links are either horizontal between small field boxes, in which case we have enough spatial decay to resum these links, or vertical links; in the case of five or more legs ($t_\Delta \ne 0$) power counting provides the necessary factor $M^{-5|i-i'|}$ to resum a box of scale i among the $M^{4|i-i'|}$ boxes of scale i contained in a box of scale i'. In the case of two and four point functions, renormalization performs the same task in the usual way.

We have also a new type of links if we compare to the multiscale expansion of [R] which are the "proximity" links, both in the vertical and horizontal directions between large field boxes (and the associated "protection corridors" introduced previously). All these links extend only to a bounded distance, independent of λ. It is possible to resum over such links (which costs only a factor independent of λ) using a piece of the small factors $O(e^{-\lambda^{-\varepsilon}})$ associated to the large field regions (see II.28).

We have to check that the functional integral over γ can be performed. In particular, one problem which might worry the reader is that our use of polynomial approximations to the true gauge transformations might forbid us from doing exact computations at all orders in perturbations theory. Strictly speaking this is correct, but as explained already we need only to perform, e.g. for the flow of the coupling constant, one and two loop computations, and to show that the remainders are of the next order. Since $A^{\gamma,2}$ gives vertices small only as $\lambda^{1/2}$, it is not, strictly speaking, adapted to this problem. But we can replace everywhere in Sect. II $A^{\gamma,2}$ by e.g. $A^{\gamma,10}$, and still control all the integrals in the same way, using the fact that the factor N in (II.36–40) can be made arbitrarily large; the resulting formulas are simply more complicated [7].

In the small field region the bounds we use are similar to the case of the infrared ϕ_4^4 critical theory (see [FMRS1, R]); remark however that for the "domination" process, we have to use the small field conditio e^{-E_Δ} in (II.25), which costs a factor $\lambda^{1/2+\varepsilon_1}$ per field, rather than the $\lambda^4 A^4$ term (which would cost one full λ). In this way a vertex such as $\lambda^2[A,A]^2$ with the worst case of three badly localized legs has still a small factor of order $\lambda^{1/2+\varepsilon_1}$, because it has at least one well localized leg, and if this well localized leg is a small field A'_s, its Gaussian integration does not cost any fraction of λ.

At this point the attentive reader may ask what happens if this well localized leg is of the large field type. More generally why do the vertical couplings between high momentum large field regions and lower momentum small field regions also lead to small factors? For instance a vertex such as $\lambda[B^i, A^{i'}]$ with $i' < i$, using the $\lambda^4 B^4$ term for domination of the B field seems to eat up the factor λ, hence to lead to no small factor. This is not true because the vertical corridor that we decided to include can be made much bigger in the direction of lower frequencies than of higher frequencies. Indeed there are M^4 boxes of the next higher frequency in a typical cubic box, which in practice limit us to consider corridor of bounded width (independent of λ) in the vertical direction upwards. But there is only one box which contains a box in the next lower frequency. If we take into account the fact that we have a small factor of order $e^{-\lambda_i^{-\varepsilon}}$ in a large field box of scale i, we can include all the boxes which contain it until frequency $i' = i - K|\log\lambda|$ in the protection corridor (K being a large constant), and still attribute a small factor of similar order (with ε slightly smaller) to all these boxes. It is then true that the integration of the well localized B' field using e.g. the $\lambda^4 B^4$ terms eats up the coupling constants, but this is more than compensated by the fact that if we dominate the $A^{i'}$ field using the small field condition we gain a small factor $M^{-K|\log\lambda|}$ which comes from writing $M^{i'} \le M^i M^{-K|\log\lambda|}$. This factor is by itself a very large power of λ so the corresponding terms are indeed extremely small.

VIII. Slavnov Identities

Slavnov identities can be used in a perturbative gauge such as the Feynman or Landau gauge to check the necessary relations which ensure order by order that up to a rescaling of A (wave function renormalization) there is only a coupling constant

[7] It is presumably even possible to use the true gauge transformations $A^{\gamma,\infty}$ in our change of gauge formulas (II.36–40) but it is less clear how to define an adequate analogue of the protection term in this case

renormalization (perturbative renormalizability of the model). These identities can be expressed in terms of the Schwinger functions of the theory, but in this form they were of course up to now true only in the sense of formal power series.

Our bare ansatz is written for a theory with field A satisfying an axial gauge condition, hence the Slavnov identities that we have to check are the identities adapted to this condition. However one can also reexpress these identities in terms of the small field A'; one would then prove the usual identities in the perturbative homothetic gauge, up to a remainder which is not 0 but which vanishes to any order in perturbation theory, and which corresponds to the large field regions in which the field has been kept in the axial gauge. These identities are those used in the preceding sections at order 3 in perturbation theory to control the renormalization group flow of the coupling constant.

To derive the exact form of Slavnov identities in the axial gauge, one introduces the generating functional for the theory with gauge condition $A_0 = 0$. Formally we can write this functional as:

$$W(J) = \langle e^{-F^2/4+J\cdot A}\delta(A_0)\rangle = \langle e^{J\cdot A}\rangle_{ax}\,, \tag{VIII.1}$$

where the first expectation value is with respect to the formal Lebesgue measure, and the second one is the expectation value in the axial gauge constructed by the limit process described above. The A field can be though of as the corresponding $\frac{\delta}{\delta J}$ functional derivation. Then in (VIII.1) we perform a change of variables $A \to A+D\gamma$; by (infinitesimal) gauge invariance, there is no first order dependence in γ, which gives the "Ward" or "Slavnov" equation:

$$\left\langle\left(J^a_m(x)\,D^{ab}_m(x) - \left[\frac{\partial}{\partial x^0}\,A^a_m\cdot D^{ab}_m\right]\frac{\partial}{\partial x^0}\right)e^{J\cdot A}\right\rangle_{ax} = 0\,. \tag{VIII.2}$$

Integrating by parts and taking into account the fact that $A_m D_m = A_m\,\frac{\partial}{\partial x^m}$ we can rewrite this identity simply as:

$$\left\langle\left(J^a_m(x)\,D^{ab}_m(x) - \left[\frac{\partial}{\partial x^m}\,A^b_m(x)\right]\left(\frac{\partial}{\partial x^0}\right)^2\right)e^{J\cdot A}\right\rangle_{ax} = 0\,. \tag{VIII.3}$$

This identity gives rise to a hierarchy of identities with any number N of external sources. For instance the two point function identity is obtained by applying one functional derivative $\frac{\delta}{\delta J}(y)$ to (VIII.3) and is simply (exchanging the names of x and y);

$$\left\langle A^a_m(x)\left[\frac{\partial}{\partial y^n}\,A^b_n(y)\right]\left(\frac{\partial}{\partial y^0}\right)^2\right\rangle_{ax} = \delta_{ab}\delta(x-y)\,\frac{\partial}{\partial y^m}\,. \tag{VIII.4}$$

We should of course understand this identity as applied to two test functions of x and y.

Similarly we can write e.g. an identity involving N point functions:

$$\Bigg\langle\sum_{i=1}^{N-1}\Bigg(\prod_{j=1,j\neq i}^{N-1} A^{a_j}_{m_j}(x_j)\Bigg)\,D^{a_i b}_{m_i}\delta(x_i - y) - \prod_{j=1}^{N-1} A^{a_j}_{m_j}(x_j)\left[\frac{\partial}{\partial y^n}\,A^b_n(y)\right]\left(\frac{\partial}{\partial y^0}\right)^2\Bigg\rangle_{ax} = 0\,. \tag{VIII.5}$$

For a theory with a fixed infrared-cutoff of a given type, the linear term in γ for a gauge transformation $A \to A + D\gamma$ receives a contribution from the presence in (VIII.1) of the cutoff. This leads to correction terms in the Slavnov identities. For example Eq. (VIII.5) takes the form

$$\Big\langle \sum_{i=1}^{N-1} \Big(\prod_{j=1, j\neq i}^{N-1} A_{m_j}^{a_j}(x_j) \Big) D_{m_i}^{a_i b} \delta(x_i - y) \\ - \prod_{j=1}^{N-1} A_{m_j}^{a_j}(x_j) \left[\frac{\partial}{\partial y^n} A_n^b(y) \right] \left(\frac{\partial}{\partial y^0} \right)^2 \Big\rangle_{ax} = E_N(\{x_j\}), \qquad \text{(VIII.6)}$$

where E_N can be computed for any given infrared cutoff. If this cutoff is a finite compact box with some kind of boundary conditions, we see that E_N can be interpreted as a boundary term, because the gauge invariance is exact inside the box.

The identities (VIII.6) are those that we are going to check in the limit $\varrho \to \infty$. In order to prove them we write first approximate identities which are satisfied for the theory with cutoffs (both infrared and ultraviolet) and gauge restoring counterterms. These identities take the form of equality between the left-hand side of (VIII.6) where $\langle\,.\,\rangle_{ax}$ is replaced by $\langle\,.\,\rangle_{ax,\varrho}$, the normalized functional integral of our theory with cutoff and a right-hand side which is no longer E_N, but $E_N + \delta_N(\varrho)$ because of the effects δ_N of the gauge transformation $A \to A + D\gamma$ on the ultraviolet cutoff and on the gauge restoring counterterms. When $\varrho \to \infty$, the left-hand side, made of normalized Schwinger functions with cutoff ϱ, by definition tends to the same Schwinger functions without ultraviolet cutoff that we have constructed. Our expansion proves that in the right-hand side the error term $\delta_N(\varrho)$ tends to zero, because it is made of contributions tied to the ultraviolet cutoff.

This achieves our sketch of proof of the main statement in the introduction.

Appendix 1

In this appendix we provide some explicit bounds and computations of the determinants considered in Sect. VI in the simpler case of the Feynman gauge $\zeta = 1$.

We start by a warm up: the case $t = 0$. The integral over ϕ is then trivial and we have to compute:

$$g_k(x) = \exp(1/2) \int_{\Delta} \Bigg(x^4 \int v dv \Bigg[(4/\pi) \int_0^{\pi} \sin^2\theta d\theta \\ \times \ln|1 - \beta\kappa\cos^2\theta| - 2\ln(1 + 2\beta - 4\beta\kappa\cos^2\theta + \beta^2) \Bigg] \\ - cx^4 + \left[x^4 \int \beta[6 - 4\kappa] v dv \right] \Bigg) . \qquad \text{(A.1)}$$

We perform first at fixed β and κ, hence fixed v, the angular integral over θ. In order to simplify slightly the computation we remark first that we have the rigorous inequality (since $0 \le \kappa \le 1$):

$$-2\ln(1 + 2\beta - 4\beta\kappa\cos^2\theta + \beta^2) \le -2\ln(1 - 2\beta\cos 2\theta + \beta^2) . \qquad \text{(A.2)}$$

Using this inequality, we can simply compute:

$$G(\beta) = (4/\pi) \int_0^\pi \sin^2\theta d\theta \ln|1 - \beta\kappa\cos^2\theta| - 2\ln(1 - 2\beta\cos 2\theta + \beta^2). \qquad \text{(A.3)}$$

To study G when β varies we can e.g. differentiate once again, so that the θ integration can be performed by elementary contour integrals. Then we integrate the result. The outcome is:

$$\begin{aligned}
G(\beta) &= -2(1+\ln 4) - 4\beta + 4/(\beta\kappa) + 2\ln(\beta\kappa) + 2\ln\frac{1+\sqrt{1-\beta\kappa}}{1-\sqrt{1-\beta\kappa}} \\
&\quad + \frac{2}{1+\sqrt{1-\beta\kappa}} - \frac{2}{1-\sqrt{1-\beta\kappa}} \quad \text{if} \quad \beta \le 1, \\
G(\beta) &= -2(1+\ln(4/\kappa)) - 6\ln\beta + \frac{4(1-\kappa)}{\beta\kappa} + 2\ln\frac{1+\sqrt{1-\beta\kappa}}{1-\sqrt{1-\beta\kappa}} \\
&\quad + \frac{2}{1+\sqrt{1-\beta\kappa}} - \frac{2}{1-\sqrt{1-\beta\kappa}} \quad \text{if} \quad 1 \le \beta \le 1/\kappa, \\
G(\beta) &= -2(1+\ln(4/\kappa)) - 6\ln\beta + \frac{4(1-\kappa)}{\beta\kappa} \quad \text{if} \quad \beta \ge 1/\kappa.
\end{aligned} \qquad \text{(A.4)}$$

The function G' is always negative (remark that $G'(0) = -4 - \kappa/2$; this value is critical for the rest of our analysis). Therefore G is always negative (this can also be checked directly on (A.4)); moreover for $\beta \le 1$ we have $G(\beta) \le (-4 - \kappa/2)\beta$. Therefore, taking into account the fact that $\beta v = \kappa \le 1$, hence that $\beta > 1$ implies $v < 1$, and the shape of κ chosen in Sect. III:

$$\begin{aligned}
g_k(x) &\le \exp\int_\Delta \Bigg(x^4\Bigg(-c + \int_{\kappa\ge 1/2} 4\beta v dv + \int_{\kappa<1/2} 6\beta v dv \\
&\qquad - \int (4+\kappa/2)\beta v dv + \int_{\beta>1} (4+\kappa/2)\beta v dv\Bigg)\Bigg) \\
&\le \exp\int_\Delta \Bigg(x^4\Bigg(-c + 9/2 + \int_{\kappa>1/2} 4\beta v dv + \int_{\kappa<1/2} 6\beta v dv - \int_{\kappa=1/2} \frac{\beta v}{4} dv\Bigg)\Bigg) \\
&\le \exp\int_\Delta (x^4(-c+9/2) + x^2 M^{2k}(8 + 6/2 - (1/8\eta))) \le 1
\end{aligned} \qquad \text{(A.5)}$$

if we take the parameter η in (II.14) sufficiently small, so that the constant c (which diverges like $|\log\eta|$) is bigger than 9/2 and such that η is smaller than 1/88; of course in (A.5) we assumed a cutoff with the particular shape of Fig. II.1. Many other shapes will work as well, but it seems that cutoffs which tend very slowly to zero spending a lot of time between 1/2 and 0 are ruled out by this method.

Let us return to the general case $\zeta = 1$ but $t \ne 0$.

The third order polynomial in β in (VI.15) becomes if we put $\tau = t^2$:

$$\begin{aligned}
&[1 + 2\beta(1 - 2\kappa(\cos^2\theta + \tau\sin^2\theta\cos^2\phi) + \beta^2] \\
&\quad + \beta\tau[2 + \beta(\tau + 3 - 4\kappa(\cos^2\theta + \sin^2\theta\cos^2\phi)) + \beta^2(1+\tau)] \\
&\ge [1 + 2\beta(1 - 2(\cos^2\theta + \tau\sin^2\theta\cos^2\phi)) + \beta^2] + 7\beta\tau/4.
\end{aligned} \qquad \text{(A.6)}$$

If we put $s = \sqrt{\tau}\cos\phi$ and $w = s^2$, we have therefore to bound the integral

$$H(\beta) = (1/2)\int_{-\sqrt{\tau}}^{\sqrt{\tau}} \frac{ds}{\sqrt{\tau}} \int_0^{\pi} (4/\pi)\sin^2\theta d\theta \ln|1 - \beta\kappa(\cos^2\theta + w\sin^2\theta)|$$
$$- 2\ln(1 + 2\beta(1 + 7/8\tau - 2\cos^2\theta - 2w\sin^2\theta) + \beta^2)\,. \tag{A.7}$$

The singularity in the logarithm is integrable hence the result is obviously well defined and real. However to compute it we decide to regularize the singularity in the logarithm by adding $+i\varepsilon$ and taking the real part (there are non-trivial imaginary parts corresponding to the way we avoid the singularity but we can just throw them away since $\ln|f| = \mathbf{Re}\ln f$). Then we derive with respect to β and changing to the variable $z = e^{2i\theta}$ we have a rational contour integral to compute. It is convenient to define $\beta' = \kappa\beta$. Then one finds:

$$H(\beta) = (1/2)\int_{-\sqrt{\tau}}^{\sqrt{\tau}} \frac{ds}{\sqrt{\tau}} (1/2i\pi) \oint_{|z|=1} \frac{dz}{z}(2 - z - 1/z)$$
$$\times \lim_{\varepsilon\to 0} \mathbf{Re}\ln(4 - 4\beta\kappa w - \beta\kappa(1-w)(2 + z + 1/z) + 4i\varepsilon)$$
$$- 2\ln(1 + 2\beta(1 + 7\tau/8) - \beta(1-w)$$
$$\times (2 + z + 1/z) - 4\beta w + \beta^2)\,, \tag{A.8}$$

$$\frac{d}{d\beta}H(\beta) = (1/2)\int_{-\sqrt{\tau}}^{\sqrt{\tau}} \frac{ds}{\sqrt{\tau}}[\kappa A + B]\,, \tag{A.9}$$

$$A = (1/2i\pi)\oint_{|z|=1} \frac{dz}{z^2}(2z - z^2 - 1)$$
$$\times \lim_{\varepsilon\to 0}\mathbf{Re}\frac{-4wz - (1-w)(2z + z^2 + 1)}{(4 - 4\beta' w)z - \beta'(1-w)(2z + z^2 + 1) + 4i\varepsilon z}\,, \tag{A.10}$$

$$B = (1/2i\pi)\oint_{|z|=1} \frac{dz}{z^2}(2z - z^2 - 1)$$
$$\times \left[(-2)\frac{-2wz - (1-w)(z^2+1) + 2\beta z + 7\tau z/4}{(1 - 2\beta w)z - \beta(1-w)(z^2+1) + \beta^2 z + 7\beta\tau z/4}\right]. \tag{A.11}$$

The first integral, A, has poles at $z = 0$ and at

$$z_{\pm} = \frac{2 - \beta'(1+w) + 2i\varepsilon \pm 2\sqrt{1 - \beta' - \beta' w + (\beta')^2 w - \varepsilon^2 + i\varepsilon(2 - \beta' - \beta' w)}}{\beta'(1-w)}\,. \tag{A.12}$$

The pole at $z = 0$ for the first piece of the integrand gives a real contribution equal to

$$\frac{-2}{(\beta')^2(1-w)}(2 - \beta' + \beta' w)\,. \tag{A.13}$$

When $1 < \beta' < 1/w$ the two poles at $z_\pm$ are approximately on the contour of integration and are approximately complex conjugate; if ε is small positive one of the two poles $z_\pm$ is inside the unit circle and the other outside (the one inside depends on the convention for the square root, but with the most natural convention it is z_+ for $2/(1+w) < \beta' < 1/w$ and z_- for $1 < \beta' < 1/w$). However we do not need to take these residues into account since they become purely imaginary when ε goes to 0, hence when we take the real part they disappear.

For $\beta' < 1$ there is the contribution of one real pole inside the unit disk, with residue

$$\frac{4(1-\beta')}{(\beta')^2(1-w)\sqrt{1-\beta'-\beta' w+(\beta')^2 w}}\,. \tag{A.14}$$

For $\beta' > 1/w$ there is another pole inside the unit circle with residue:

$$\frac{4(\beta'-1)}{(\beta')^2(1-w)\sqrt{1-\beta'-\beta' w+(\beta')^2 w}}\,. \tag{A.15}$$

Hence

$$A = \frac{2}{(\beta')^2(1-w)} \times \left[-2+\beta'(1-w)+\frac{2(1-\beta')}{\sqrt{(1-\beta')(1-\beta' w)}}\right] \quad \text{if} \quad \beta' < 1\,, \tag{A.16}$$

$$A = \frac{2}{(\beta')^2(1-w)}\left[-2+\beta'(1-w)\right] \quad \text{if} \quad 1 < \beta' < 1/w\,, \tag{A.17}$$

$$A = \frac{2}{(\beta')^2(1-w)} \times \left[-2+\beta'(1-w)-\frac{2(1-\beta')}{\sqrt{(1-\beta')(1-\beta' w)}}\right] \quad \text{if} \quad \beta' > 1/w\,. \tag{A.18}$$

The term B has a pole at $z=0$ giving the contribution

$$\frac{2}{\beta^2(1-w)}(1-\beta^2-2\beta+2\beta w)\,. \tag{A.19}$$

Finally there is a pole inside the unit disk at the location

$$\frac{(1-\beta)^2+\beta\tau'-\sqrt{(1-2\beta w+\beta^2+\beta\tau')^2-4\beta^2(1+w^2)}}{2\beta(1-w)}\,, \tag{A.20}$$

where $\tau' \equiv 7\tau/4$, which gives the contribution

$$-2\,\frac{((1-\beta)^2+\beta\tau')(1-\beta^2)}{\beta^2(1-w)\sqrt{(1-\beta)^2\left[(1+\beta)^2-4\beta w\right]+\beta\tau'(2-4\beta w+2\beta^2+\beta\tau')}}\,. \tag{A.21}$$

Therefore we have

$$B = \frac{2}{\beta^2(1-w)}\left[1-\beta^2-2\beta+2\beta w - \frac{((1-\beta)^2+\beta\tau')(1-\beta^2)}{\sqrt{(1-\beta)^2\left[(1+\beta)^2-4\beta w\right]+\beta\tau'(2-4\beta w+2\beta^2+\beta\tau')}}\right]. \tag{A.22}$$

Adding κA and B we find

$$\kappa A + B = \frac{2}{\beta^2(1-w)} \times \left[1 - 2/\kappa - \beta(1-w) - \beta^2 + \frac{2(1-\beta\kappa)}{\kappa\sqrt{(1-\beta\kappa)(1-\beta\kappa w)}} - \frac{((1-\beta)^2+\beta\tau')(1-\beta^2)}{\sqrt{(1-\beta)^2[(1+\beta)^2-4\beta w]+\beta\tau'(2-4\beta w+2\beta^2+\beta\tau')}}\right] \quad \text{if} \quad \beta\kappa < 1\,, \tag{A.23}$$

$$\kappa A + B = \frac{2}{\beta^2(1-w)} \times \left[1 - 2/\kappa - \beta(1-w) - \beta^2 - \frac{((1-\beta)^2+\beta\tau')(1-\beta^2)}{\sqrt{(1-\beta)^2[(1+\beta)^2-4\beta w]+\beta\tau'(2-4\beta w+2\beta^2+\beta\tau')}}\right] \quad \text{if} \quad 1 < \beta\kappa < 1/w\,, \tag{A.24}$$

$$\kappa A + B = \frac{2}{\beta^2(1-w)} \times \left[1 - 2/\kappa - \beta(1-w) - \beta^2 - \frac{2(1-\beta\kappa)}{\kappa\sqrt{(1-\beta\kappa)(1-\beta\kappa w)}} - \frac{((1-\beta)^2+\beta\tau')(1-\beta^2)}{\sqrt{(1-\beta)^2[(1+\beta)^2-4\beta w]+\beta\tau'(1-4\beta w+2\beta^2+\beta\tau')}}\right] \quad \text{if} \quad \beta\kappa > 1/w\,. \tag{A.25}$$

If we use this explicit computation to expand the function $\frac{d}{d\beta}H(\beta)$ near $\beta = 0$ we find that it behaves as

$$(1/2)\int_{-\sqrt{\tau}}^{+\sqrt{\tau}} \frac{ds}{\sqrt{\tau}} \frac{2}{1-w} \times ((\kappa/4)(-1-2w+3w^2) - 2 + 8w - 6w^2 - 2\tau'(1-w)) = -4 - 3\tau - (\kappa/2)(1+\tau)\,, \tag{A.26}$$

using the fact that $w = s^2$.

Hence the slope of the function H near $\beta = 0$ is more negative than when $\tau = 0$. It is possible to integrate explicitly the formula for H, but we do not give the expression here. Using the asymptotic expansion (A.26) near $\beta = 0$, the fact that τ varies in the compact interval $[0, 1]$ and the fact that the derivative of H is bounded by a constant at large β we can of course always achieve a uniform bound as in Sect. VI. We want also to derive a bound showing the strict positivity of $-\Delta_B^{\text{homothetic}}$ in a constant field B, unless all the components of B are in the same direction in su(2) space, and p is then exactly aligned with the corresponding (unique) vector λB. For this we use the fact that

$$\delta_{\mu\nu}D^2 - 10/13\nabla_\mu\nabla_\nu \geq 3/13 D^2\,; \qquad \delta_{\mu\nu}p^2 - 10/13 p_\mu p_\nu \leq 23/13 p^2 \tag{A.27}$$

in order to show that the normalized operator $(-\Delta_B^{\text{homothetic}})(-\Delta^{\text{homothetic}})^{-1}$ is bounded up to a factor $(3/23)^{12}$ exactly by the same bound as in the Feynman case of ordinary Laplacians. In that case we can use the explicit computations above to establish the necessary bounds. In particular this proves that the determinant of $(-\Delta_B^{\text{homothetic}})(-\Delta^{\text{homothetic}})^{-1}$ is bounded away from 0 up to a constant factor by the bound (A.6). The only zeroes of the right-hand side of (A.6)

$$([1 + 2\beta(1 - 2(\cos^2\theta + \tau \sin^2\theta \cos^2\phi)) + \beta^2] + 7\beta\tau/4) \qquad \text{(A.28)}$$

occur for $\tau = 0$, $\beta = 1$ and $\theta = 0$, which correspond to the announced case of all components of B aligned in su(2) space (since $\tau = 0$) and B aligned with the momentum p ($\beta = 1, \theta = 0$). We can consider that for a fixed B (with approximate alignment of all components in su(2) space) the zero at $p = 0$ of the ordinary Laplace operator p^2 is simply translated. If we use a cutoff function κ_B^m as in (V.2) with correct scaling around this translated zero of the operator $(-\Delta_B^{\text{homothetic}})$ with *constant* background, we obtain the correct polynomial bounds on the spatial decay using integration by parts on the cutoff function:

$$\kappa_B^m * (-\Delta_B^{\text{homothetic}})^{-1}(x, y) \leq K_q M^{2m} \left(\frac{1}{1 + M^m|x - y|}\right)^q . \qquad \text{(A.29)}$$

It is this decrease which is finally used in the horizontal cluster expansion.

Acknowledgements. We thank warmly J. Feldman for his collaboration at an early stage of this work. V. Rivasseau, who by himself alone would have abandoned this difficult technical problem, thanks particularly J. Magnen for his tenacity and for all his ideas.

References

[B] Balaban, T.: Commun. Math. Phys. **95**, 17 (1984); **96**, 223 (1984); **98**, 17 (1985); **99**, 75 (1985); **99**, 389 (1985); **102**, 277 (1985); **109**, 249 (1987); **116**, 1 (1988); **119**, 243 (1988); **122**, 175 (1989); **122**, 355 (1989)

[BG] Balaban, T., Gawedzki, K.: Ann. Inst. Henri Poincaré **36**, 271 (1982)

[dCMSdV] de Calan, C., Magnen, J., Sénéor, R., da Veiga, P.: In preparation

[DMR] Dunlop, F., Magnen, J., Rivasseau, V.: Preprint Ecole Polytechnique (February 1992)

[F] Federbush, P.: A phase cell approach to Yang-Mills theory. I–VII. Commun. Math. Phys. **107**, 319 (1986); **110**, 293 (1987); **114**, 317 (1988); Ann. Inst. Henri Poincaré **47**, 17 (1987)

[FMRS1] Feldman, J., Magnen, J., Rivasseau, V., Sénéor, R.: Construction of infrared ϕ_4^4 by a phase space expansion. Commun. Math. Phys. **109**, 437 (1987)

[FMRS2] Feldman, J., Magnen, J., Rivasseau, V., Sénéor, R.: A renormalizable field theory; the massive Gross-Neveu model in two dimensions. Commun. Math. Phys. **103**, 67 (1986)

[IZ] Itzykson, C., Zuber, J.B.: Quantum field theory. New York: McGraw-Hill 1980

[L] Lüscher, M.: Construction of a self-adjoint, strictly positive transfer matrix for the Euclidean lattice gauge theories. Commun. Math. Phys. **54**, 283 (1977)

[MS] Magnen, J., Sénéor, R.: In Third International Conference on Collective Phenomena. New York: The New York Academy of Sciences 1980

[OS] Osterwalder, K., Seiler, E.: Gauge field theories on the lattice. Ann. Phys. **110**, 440 (1978)

[R] Rivasseau, V.: From perturbative to constructive renormalization. Princeton, NJ: Princeton University Press 1991

[S] Sénéor, R.: In: Renormalization of quantum field theories with non-linear field transformations, Breitenlohner, P., Maison, D., Sibold, K. (eds.), Lect. Notes Physics, Vol. 303. Berlin, Heidelberg, New York: Springer 1988

Communicated by A. Jaffe