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On the Natural Line Bundle on the Moduli Space of Stable Parabolic Bundles

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Abstract. We construct the natural holomorphic line bundle on the moduli space of stable parabolic bundles on a compact marked Riemann surface, which is the prequantum line bundle for the Chern–Simons gauge theory. The fusion rule in the Chern–Simons gauge theory can be viewed as the existence condition of this line bundle.

0. Introduction

In 1988 Witten introduced a new topological invariant for 3-manifolds based on the Chern–Simons gauge theory [12]. Since he used the Feynman integral to define his invariant, Atiyah formulated a framework of topological quantum field theories to understand Witten's invariant in a mathematical sense [1]. Roughly speaking, 2 + 1 dimensional topological quantum field theory is the following:

(1) To each closed oriented surface Σ, a finite dimensional vector space $Z(\Sigma)$ is assigned, and
(2) To each compact oriented 3-manifold M with boundary Σ, a vector $Z(M) \in Z(\Sigma)$ is assigned, and they satisfy certain axioms.

Moreover Witten extended his invariant for 3-manifolds to an invariant for a colored link in 3-manifolds and showed that it is a generalization of Jones polynomials. Recall that a colored link is a link, which has a representation of the fixed simple Lie group G assigned to each of its connected components. In this case the framework of topological quantum field theory is adjusted so that (1) should be replaced by the following:

(1′) When a finite set of points $P_1, \ldots, P_n$ in a closed oriented surface Σ is given and to each point P_i a representation λ_i of G is also given, a finite dimensional vector space $Z(\Sigma; P_1, \ldots, P_n; \lambda_1, \ldots, \lambda_n)$ is assigned.

It is believed that $Z_k(\Sigma; P_1, \ldots, P_n; \lambda_1, \ldots, \lambda_n)$ (Witten's invariant has a parameter k, which is a positive integer called a level) is realized as a space of holomorphic sections of a certain line bundle on the moduli space of stable parabolic bundles with fixed weights on a marked Riemann surface [1]. Recall that a parabolic bundle on a compact marked Riemann surface $(\Sigma; P_1, \ldots, P_n)$ is a following data: a holomorphic vector bundle E on Σ, and at each P_i a flag of E_{P_i} and an increasing sequence of real numbers called weights. See Sect. 1 for the precise definition. Up to now, to realize $Z_k(\Sigma; P_1, \ldots, P_n; \lambda_1, \ldots, \lambda_n)$, the following is not known:

(a) Which weights should we fix?
(b) Which line bundle on the moduli space should we take?

The purpose of this paper is to answer the above question and to construct the line bundle on the moduli space explicitly in the case of $G = SU(2)$. To do this, we consider the lift of the action of the gauge group on the connection space to the line bundle on the connection space. When there are no marked points in the Riemann surface Σ, this is treated by several authors [10, 4]. We consider the case where there are some marked points. As a result, only when the weights of parabolic bundles take special values, the action of the gauge group turn out to be liftable to the line bundle. This gives the answer to (a), (b).

Moreover we study this line bundle when $\Sigma = \mathbb{CP}^1$, $n = 3$ and obtain a geometric interpretation of the fusion rule in the level k $SU(2)$ Chern–Simons gauge theory or the level k $SU(2)$ Wess–Zumino–Witten conformal field theory. In fact the fusion rule turns out to be the existence condition of the line bundle on the moduli space, which we constructed above, when $\Sigma = \mathbb{CP}^1$, $n = 3$.

The moduli space of stable parabolic bundles is a subset of the moduli space of stable parabolic Higgs bundles [8]. In this paper we show that the line bundle on the moduli space of stable parabolic bundles naturally extends to the moduli space of stable parabolic Higgs bundles. This gives the appropriate setting of Hitchin's abelianization procedure [1], which is studied in the forthcoming paper.

As a concluding remark, it is interesting to compare our interpretation of the fusion rule with Gawedzki and Kupiainen's [6]. They consider only usual vector bundles, that is, not parabolic bundles, and observed the relation between the fusion rules and unstable vector bundles. On the other hand we consider only stable parabolic bundles and observed the relation between the fusion rules and the moduli space of stable parabolic bundles. Our geometric interpretation of the fusion rule avoids the use of unstable vector bundles.

This paper is organized as follows. In Sect. 1 we review the construction of stable parabolic bundles and fix our notation. In Sect. 2 we give some lemmas on the lift of a symplectic action of a Lie group to the line bundle. In Sect. 3 we discuss the lift of the action of the gauge group. In Sect. 4 we construct the line bundle on the moduli space of stable parabolic bundles. In Sect. 5 we give a geometric interpretation of the fusion rule in the level k $SU(2)$ Chern–Simons gauge theory. In Sect. 6 we discuss the extension of the line bundle to the moduli space of stable parabolic Higgs bundles.

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1. The Moduli Space of Stable Parabolic Bundles

In this section we review the construction of the moduli space of stable parabolic bundles and fix our notations. Let Σ be a compact Riemann surface. Fix a finite set of points $P_1, \ldots, P_n \in \Sigma$. Recall the definition of parabolic bundles due to Mehta–Seshadri [9].

Definition 1.1. *Let E be a holomorphic vector bundle over Σ. The parabolic structure at P_i is a pair of a flag of E_{P_i} and an increasing sequence of real numbers called weights*:

$$E_{P_i} = F_1 E_{P_i} \supsetneqq \cdots \supsetneqq F_{r_i} E_{P_i} \supsetneqq F_{r_i+1} E_{P_i} = \{0\}, \quad \alpha_1^{(i)} < \cdots < \alpha_{r_i}^{(i)}.$$

We call $(\Sigma, E, P_i, F_l E_{P_i}, \alpha_l^{(i)})$ a parabolic bundle. Define its parabolic degree by

$$\operatorname{pardeg} E = \deg E + \sum_{i=1}^{n} \sum_{l=1}^{r_i} (\dim F_l E_{P_i} - \dim F_{l+1} E_{P_i}) \alpha_l^{(i)} .$$

Let E be a trivial smooth Hermitian vector bundle over Σ. We fix a unitary trivialization of E and we always write $E = \Sigma \times \mathbb{C}$ with respect to this unitary trivialization. There is a notion of stability for parabolic bundles, which is introduced in [9]. See Sect. 5 for the definition of stability for rank 2 parabolic bundles. Now we contruct the moduli space of rank 2 stable parabolic bundles with pardeg $= 0$ and the weights $-\alpha_i < \alpha_i$ $(0 < \alpha_i < \frac{1}{2})$ at each P_i. Fix a $SU(2)$ connection d_0 on E with a singularity at P_i as follows. Fix a local holomorphic coordinate $z_i = \rho_i e^{\sqrt{-1}\theta_i}$ on a neighbourhood U_i of P_i with $z_i(P_i) = 0$. Then

$$d_0 = d + \sqrt{-1} \begin{pmatrix} -\alpha_i & 0 \\ 0 & \alpha_i \end{pmatrix} d\theta_i \quad \text{on } U_i .$$

Define the $SU(2)$ connection space $\mathscr{A}$, which is compatible to the parabolic structure, by

$$\mathscr{A} = d_0 + \Omega^1(\operatorname{End}^0_{sk} E) ,$$

where $\operatorname{End}^0_{sk} E$ denotes the vector bundle of trace free skew adjoint endomorphisms of E and $\Omega^1(\operatorname{End}^0_{sk} E)$ denotes the space of smooth $\operatorname{End}^0_{sk} E$ valued 1-forms. Strictly speaking, we need appropriate Sobolev completion of $\mathscr{A}$ to construct the moduli space. See [3] for details. However, to construct the line bundles on the moduli space, the Sobolev completion is not essential. So we omit the Sobolev completion for simplicity.

Define the gauge group $\mathscr{G}$, which is compatible with the parabolic structure, by

$$\mathscr{G} = \{s\colon \Sigma \to SU(2) \mid s(P_i) \in T \text{ for each } P_i\} ,$$

where $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$. Define

$$\mathscr{A}_F = \{d_A \in \mathscr{A} \mid d_A \text{ is flat on } \Sigma \backslash \{P_1, \ldots, P_n\}\} ,$$

$$\mathscr{A}_F^{\text{irr}} = \{d_A \in \mathscr{A}_F \mid d_A \text{ is irreducible}\} .$$

Then Mehta–Seshadri's theorem implies

$$\mathscr{M}_{(\Sigma; P_1, \ldots, P_n; \alpha_1, \ldots, \alpha_n)} = \mathscr{A}_F^{\text{irr}} / \mathscr{G}$$

is the moduli space of stable parabolic bundles with pardeg $= 0$ and the weights $-\alpha_i < \alpha_i$ at each P_i.

2. The Lift of the Symplectic Group Action

In this section we study the following situation. Let (M, ω) be a symplectic manifold. Assume that there exists a 1-form $\eta \in \Omega^1(M)$ such that $\omega = d\eta$. A Lie group G acts on M from the right preserving the symplectic structure ω. Let $L = M \times \mathbb{C}$ be a Hermitian line bundle with a fixed unitary trivialization. Define a unitary connection ∇ on L by

$$\nabla = d - 2\pi\sqrt{-1}\eta .$$

Then we have

$$\frac{\sqrt{-1}}{2\pi} R^{\nabla} = d\eta = \omega ,$$

where R^{∇} is the curvature of ∇.

In this section we discuss the lift of the action of G to the line bundle L. We represent the action of G on $L = M \times \mathbb{C}$ by

$$(x, v) \mapsto (xg, e^{-2\pi\sqrt{-1}m(x,g)}v) ,$$

for $x \in M$, $v \in \mathbb{C}$ and $g \in G$, where

$$m \colon M \times G \to \mathbb{R}/\mathbb{Z} .$$

Then we have the following lemma by direct computations.

Lemma 2.1. *Assume the above.*
(1) *The action of G is well defined if and only if*

$$m(x, gh) = m(x, g) + m(xg, h) \quad \textit{for any } x \in M \textit{ and } g, h \in G .$$

(2) *Assume that the action of G on L is well defined. Then G preserves the connection ∇ if and only if*

$$\eta_{xg}\left(\frac{d}{dt}c(t)g\bigg|_{t=0}\right) + \frac{d}{dt}m(c(t), g)|_{t=0} = \eta_x\left(\frac{d}{dt}c(t)\bigg|_{t=0}\right),$$

for any $x \in M$, $g \in G$ and any curve $c \colon (-\varepsilon, \varepsilon) \to M$ with $c(0) = x$.

For later use we establish the following lemma.

Lemma 2.2. *Assume that a Lie group G on $L_i = M_i \times \mathbb{C}$ from the right preserving the connection $\nabla_i = d - 2\pi\sqrt{-1}\eta_i$ as follows ($i = 1, 2$):*

$$(x_i, v) \mapsto (x_i g, e^{-2\pi\sqrt{-1}m_i(x_i,g)}v) ,$$

for any $x_i \in M_i$, $g \in G$. Let $p_i \colon M_1 \times M_2 \to M_i$ be the projection. Define

$$(L, \nabla) = p_1^*(L_1, \nabla_1) \otimes p_2^*(L_2, \nabla_2) .$$

We write $L = M_1 \times M_2 \times \mathbb{C}$ naturally and define the action of G on L by

$$(x_1, x_2, v) \mapsto (x_1 g, x_2 g, e^{-2\pi\sqrt{-1}(m_1(x_1,g)+m_2(x_2,g))}v) ,$$

where $x_i \in M_i$ ($i = 1, 2$), $g \in G$. Then G preserves the connection ∇.

3. The Lift of the Action of the Gauge Group

In this section we construct the Hermitian line bundle with the natural Hermitian connection on the connection space $\mathscr{A}$ and discuss the lift of the action of the guage group $\mathscr{G}$ on $\mathscr{A}$ to this line bundle.

Fix a positive integer k, which we call a level. Define a symplectic form θ_k on $\mathscr{A}$ by

$$\theta_k(\alpha, \beta) = -\frac{k}{4\pi^2}\int_\Sigma \mathrm{Tr}(\alpha \wedge \beta) \quad \text{for } \alpha, \beta \in \Omega^1(\mathrm{End}^0_{sk}E) .$$

If we define a 1-form η_k on $\mathscr{A}$ by

$$\eta_k(\alpha)_{\mathrm{at}\, d_A} = -\frac{k}{8\pi^2}\int_\Sigma \mathrm{Tr}\{(d_A - d_0) \wedge \alpha\} \quad \text{for } \alpha \in \Omega^1(\mathrm{End}^0_{sk}E) ,$$

then we have

$$d\eta_k = \theta_k .$$

Let $\mathscr{L}_k = \mathscr{A} \times \mathbb{C}$ be a Hermitian line bundle with a fixed trivialization. Define a unitary connection ∇^k on $\mathscr{L}_k$ by

$$\nabla^k = d - 2\pi\sqrt{-1}\eta_k .$$

Then we have

$$\frac{\sqrt{-1}}{2\pi} R^{\nabla^k} = d\eta_k = \theta_k ,$$

where R^{∇^k} is the curvature of ∇^k.

The gauge group $\mathscr{G}$ acts on the connection space $\mathscr{A}$ from the right by

$$d_A \mapsto s^*d_A = s^{-1} \circ d_A \circ s \quad \text{for } d_A \in \mathscr{A}, s \in \mathscr{G} .$$

In this section we discuss the lift of the action of $\mathscr{G}$ to $\mathscr{L}_k = \mathscr{A} \times \mathbb{C}$. As in the last section we represent the lift by

$$(d_A, v) \mapsto (s^*d_A, e^{-2\pi\sqrt{-1}m_k(d_A, s)}v) ,$$

where

$$m_k \colon \mathscr{A} \times \mathscr{G} \to \mathbb{R}/\mathbb{Z} .$$

The following proposition is the key to define m_k.

Proposition 3.1. *Let $t\colon \Sigma \times [0, 2] \to SU(2)$ be a map satisfying*

$$t(\Sigma \times \{0, 2\}) = \mathrm{id}_{SU(2)} ,$$
$$t(\{P_1, \ldots, P_n\} \times [0, 2]) \subset T .$$

Define δ_i to be the degree of the map $t|_{\{P_i\} \times [0,2]}\colon [0, 2] \to T$. Assume $d_0 \in \mathscr{A}_F$. Let $p\colon \Sigma \times [0, 2] \to \Sigma$ be a projection and $\tilde{d}_0 = p^*d_0$ be a connection on $(\Sigma \times [0, 2]) \times \mathbb{C}^2$. Then we have

$$\frac{-1}{24\pi^2}\int_{\Sigma \times [0,2]} \mathrm{Tr}(t^{-1}\tilde{d}_0 t)^3 = \deg t + 2\sum_{i=1}^{n} \delta_i\alpha_i ,$$

where $\deg t$ is the degree of the map $t\colon \Sigma \times [0, 2] \to SU(2)$.

To show Proposition 3.1 we need the following two lemmas.

Lemma 3.2. *Let* $u_i\colon \Sigma\times[0,2]\to SU(2)$ $(i=1,2)$ *be a map satisfying*

$$u_i(\Sigma\times\{0,2\})=\mathrm{id}_{SU(2)}\,.$$

Set

$$\operatorname{spt} u=\text{the closure of }\{x\in\Sigma\times[0,2]\,|\,u(x)\neq\mathrm{id}_{SU(2)}\}\,.$$

Assume $d_0\in\mathscr{A}_F$ *and*

$$\operatorname{spt} u_1\cap\operatorname{spt} u_2\cap\{P_1,\ldots,P_n\}\times[0,2]=\emptyset\,.$$

Then we have

$$\int_{\Sigma\times[0,2]}\mathrm{Tr}\{(u_1u_2)^{-1}\tilde{d}_0(u_1u_2)\}^3=\int_{\Sigma\times[0,2]}\mathrm{Tr}(u_1^{-1}\tilde{d}_0u_1)^3+\int_{\Sigma\times[0,2]}\mathrm{Tr}(u_2^{-1}\tilde{d}_0u_2)^3\,.$$

Lemma 3.3. *Let* $u\colon\Sigma\times[0,2]\to SU(2)$ *be a map satisfying*

$$\operatorname{spt} u\subset U_i\times(0,2),\quad u(\{P_i\}\times[0,2])\subset T\,,$$

where U_i *is a neighborhood of* P_i, *which is fixed in Sect.* 1. *Then we have*

$$\frac{-1}{24\pi^2}\int_{\Sigma\times[0,2]}\mathrm{Tr}(u^{-1}\tilde{d}_0u)^3=\deg u+2\delta_i\alpha_i\,.$$

Proof. Set $B=U_i\times[0,2]$. Then we have

$$\tilde{d}_0=d+\sqrt{-1}\,\alpha\,d\theta\quad\text{on } B\,,$$

where $\alpha=\begin{pmatrix}-\alpha_i & 0\\ 0 & \alpha_i\end{pmatrix}$, $\theta=\theta_i$. Set

$$\text{l.h.s.}=\frac{-1}{24\pi^2}\int_B\mathrm{Tr}(u^{-1}\tilde{d}_0u)^3\,.$$

Then we have

$$\begin{aligned}\text{l.h.s.}&=\frac{-1}{24\pi^2}\int_B\mathrm{Tr}(u^{-1}du+\sqrt{-1}\,u^{-1}[\alpha,u]\,d\theta)^3\\&=\deg u-\frac{\sqrt{-1}}{8\pi^2}\int_B\mathrm{Tr}(duu^{-1}duu^{-1}\alpha d\theta)\\&\quad+\frac{\sqrt{-1}}{8\pi^2}\int_B\mathrm{Tr}(u^{-1}duu^{-1}du\alpha d\theta)\,.\end{aligned}$$

Note that

$$\mathrm{Tr}(duu^{-1}duu^{-1}\alpha d\theta)=d\,\mathrm{Tr}(duu^{-1}\alpha d\theta)\,,$$

$$-\mathrm{Tr}(u^{-1}duu^{-1}du\alpha d\theta)=d\,\mathrm{Tr}(u^{-1}du\alpha d\theta)\,.$$

If we write $D_\varepsilon = \{r_i e^{\sqrt{-1}\theta_i} \in U_i \mid r_i < \varepsilon\}$, then we have

$$\text{l.h.s.} = \deg u - \frac{\sqrt{-1}}{8\pi^2} \lim_{\varepsilon\to 0} \int_{B\setminus D_\varepsilon \times [0,2]} d\,\mathrm{Tr}\{(duu^{-1} + u^{-1}du)\alpha d\theta\}$$

$$= \deg u + \frac{\sqrt{-1}}{8\pi^2} \lim_{\varepsilon\to 0} \int_{\partial D_\varepsilon \times [0,2]} \mathrm{Tr}\{(duu^{-1} + u^{-1}du)\alpha d\theta\}$$

$$= \deg u + \frac{\sqrt{-1}}{4\pi} \int_{[0,2]} \mathrm{Tr}\left\{\left(\frac{\partial u(P_i,t)}{\partial t} u(P_i,t)^{-1} + u(P_i,t)^{-1}\frac{\partial u(P_i,t)}{\partial t}\right)\alpha\right\} dt\,.$$

By the assumption we have

$$u(P_i,t) = \begin{pmatrix} e^{2\pi\sqrt{-1}f(t)} & 0 \\ 0 & e^{-2\pi\sqrt{-1}f(t)} \end{pmatrix}.$$

So we have

$$\text{l.h.s.} = \deg u + 2\int_{[0,2]} f'(t)\alpha_i dt$$

$$= \deg u + 2\delta_i\alpha_i\,. \quad \blacksquare$$

Proof of Proposition 3.1. Using Lemma 3.2 and Lemma 3.3, we can easily prove Proposition 3.1. ∎

Now we can show the main result of this section.

Theorem 3.4. *Assume $d_0 \in \mathscr{A}_F$. Define $m_k\colon \mathscr{A}\times\mathscr{G} \to \mathbb{R}/\mathbb{Z}$* by

$$m_k(d_A, s) = \frac{k}{8\pi^2}\int_\Sigma \mathrm{Tr}\{d_0 ss^{-1} \wedge (d_A - d_0)\} + \frac{k}{24\pi^2}\int_{\Sigma\times[0,1]} \mathrm{Tr}(\tilde{s}^{-1}\tilde{d}_0\tilde{s})^3\,,$$

where $\tilde{s}\colon \Sigma\times[0,1] \to SU(2)$ is a map satisfying

$$\tilde{s}|_{\Sigma\times\{1\}} = s, \quad \tilde{s}(\Sigma\times\{0\}) = \mathrm{id}_{SU(2)}\,,$$

$$\tilde{s}(\{P_i\}\times[0,1]) \subset T\,.$$

Assume that $2k\alpha_i \in \mathbb{Z}$ for $i = 1,\dots,n$. Then the map $m_k\colon \mathscr{A}\times\mathscr{G} \to \mathbb{R}/\mathbb{Z}$ is well defined and the action of the gauge group $\mathscr{G}$ on *$\mathscr{L}_k$ is also well defined. Moreover this action preserves the connection ∇^k on $\mathscr{L}_k$.*

Proof. The well definedness of the map $m_k\colon \mathscr{A}\times\mathscr{G} \to \mathbb{R}/\mathbb{Z}$ follows from Proposition 3.1. For the rest we have to show that m_k satisfies the conditions of Lemma 2.1(1) and (2). In fact it can be shown by direct computations. ∎

So far we have fixed a flat connection $d_0 \in \mathscr{A}_F$ and the action of the gauge group $\mathscr{G}$ on $\mathscr{L}_k$ depends on d_0 at first sight. We show that the action of $\mathscr{G}$ on $\mathscr{L}_k$ does not depend on the choice of a connection $d_0 \in \mathscr{A}_F$.

Take $d_0, d_1 \in \mathscr{A}_F$. Define a connection on $\mathscr{L}_k = \mathscr{A} \times \mathbb{C}$ by

$$\nabla^{k,i} = d - 2\pi\sqrt{-1}\eta_k^i \quad (i = 0, 1),$$

where η_k^i is a 1-form on $\mathscr{A}$ defined by

$$\eta_k^i(\alpha)_{\text{at } d_A} = -\frac{k}{8\pi^2}\int_\Sigma \operatorname{Tr}\{(d_A - d_i) \wedge \alpha\} \quad \text{for } \alpha \in \Omega^1(\operatorname{End}^0_{sk}E).$$

Define a gauge transformation G on $\mathscr{L}_k = \mathscr{A} \times \mathbb{C}$ by

$$G(d_A, v) = (d_A, e^{-2\pi\sqrt{-1}g(d_A)}v),$$

where $g\colon \mathscr{A} \to \mathbb{R}$ is defined by

$$g(d_A) = \frac{k}{8\pi^2}\int_\Sigma \operatorname{Tr}\{(d_1 - d_0) \wedge (d_A - d_0)\}.$$

When a base point $d_i \in \mathscr{A}_F$ $(i = 0, 1)$ is fixed, we write s^{*i} for the right action of $s \in \mathscr{G}$ on $\mathscr{L}_k = \mathscr{A} \times \mathbb{C}$. Recall

$$s^{*i}(d_A, v) = (s^* d_A, e^{-2\pi\sqrt{-1}m_k^i(d_A, s)}v),$$

where

$$m_k^i(d_A, s) = \frac{k}{8\pi^2}\int_\Sigma \operatorname{Tr}\{d_i s s^{-1} \wedge (d_A - d_i)\} + \frac{k}{24\pi^2}\int_{\Sigma\times[0,1]} \operatorname{Tr}(\tilde{s}^{-1}\tilde{d}_i\tilde{s})^3$$

as Theorem 3.4. The following proposition means that the action of $\mathscr{G}$ on $\mathscr{L}_k$ does not depend on the choice of $d_0 \in \mathscr{A}_F$.

Proposition 3.5. *Assume $d_0, d_1 \in \mathscr{A}_F$, connections $\nabla^{k,i}$ $(i = 0, 1)$ on $\mathscr{L}_k$, G, $s^{*i}\colon \mathscr{L}_k \to \mathscr{L}_k$ as above. Then we have*

(1)
$$G^{-1} \circ \nabla^{k,0} \circ G = \nabla^{k,1}.$$

(2) For any $s \in \mathscr{G}$, *the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{L}_k & \xrightarrow{s^{*0}} & \mathscr{L}_k \\ \uparrow G & & \uparrow G \\ \mathscr{L}_k & \xrightarrow{s^{*1}} & \mathscr{L}_k \end{array}$$

Proof. We can show this proposition by direct computations. ∎

4. The Natural Line Bundle on the Moduli Space

In the last section we discussed the action of the gauge group $\mathscr{G}$ on the line bundle $\mathscr{L}_k$. In this section we study when this action defines the line bundle on the moduli space $\mathscr{M}_{(\Sigma; P_1, \ldots, P_n; \alpha_1, \ldots, \alpha_n)} = \mathscr{A}_F^{\text{irr}}/\mathscr{G}$. To do this, we have only to decide when the isotropy subgroup at $d_A \in \mathscr{A}_F^{\text{irr}}$ acts on $\mathscr{L}_k$ trivially. For any $d_A \in \mathscr{A}_F^{\text{irr}}$, if $s^* d_A = d_A$, then $s = \pm \operatorname{id}_{SU(2)}$. So we have only to show the following lemma.

Lemma 4.1. *Assume that $2k\alpha_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. Set $s \equiv -\mathrm{id}_{SU(2)}$. Then, for any $d_A \in \mathscr{A}$, we have*

$$m_k(d_A, s) = -k \sum_{i=1}^{n} \alpha_i \in \mathbb{R}/\mathbb{Z} .$$

Proof. Recall

$$m_k(d_A, s) = \frac{k}{24\pi^2} \int_{\Sigma \times [0,1]} \mathrm{Tr}(\tilde{s}^{-1}\tilde{d}_0\tilde{s})^3 ,$$

where the map $\tilde{s}: \Sigma \times [0, 1] \to SU(2)$ is defined by

$$\tilde{s}(x, t) = \begin{pmatrix} e^{\pi\sqrt{-1}t} & 0 \\ 0 & e^{-\pi\sqrt{-1}t} \end{pmatrix} .$$

We extend $\tilde{s}$ naturally to the map from $\Sigma \times [0, 2]$ to $SU(2)$. By Proposition 3.1 we have

$$\frac{1}{24\pi^2} \int_{\Sigma \times [0,2]} \mathrm{Tr}(\tilde{s}^{-1}\tilde{d}_0\tilde{s})^3 = -2 \sum_{i=1}^{n} \alpha_i .$$

So we have

$$m_k(d_A, s) = k \times \frac{1}{2} \times \frac{1}{24\pi^2} \int_{\Sigma \times [0,2]} \mathrm{Tr}(\tilde{s}^{-1}\tilde{d}_0\tilde{s})^3 = -k \sum_{i=1}^{n} \alpha_i . \quad \blacksquare$$

So we can obtain the following theorem.

Theorem 4.2. *Assume*

$$2k\alpha_i \in \mathbb{Z} \quad \text{for } i = 1, \ldots, n, \quad k \sum_{i=1}^{n} \alpha_i \in \mathbb{Z}.$$

Then $L_k = (\mathscr{L}_k|_{\mathscr{A}_F^{\mathrm{irr}}})/\mathscr{G}$ is well defined as a line bundle on the moduli space $\mathscr{M}_{(\Sigma; P_1, \ldots, P_n; \alpha_1, \ldots, \alpha_n)} = \mathscr{A}_F^{\mathrm{irr}}/\mathscr{G}$. Moreover L_k has a natural holomorphic structure.

Proof. By Theorem 3.4 and Lemma 4.1 the first part of the theorem is clear. To see the second part, we note that this line bundle has a Hermitian connection, whose curvature form is a (1, 1) form on the moduli space. So by Theorem 5.1 in [2] this line bundle has a natural holomorphic structure. $\blacksquare$

Remark. By Proposition 3.5 the holomorphic structure of L_k is independent of the choice of $d_0 \in \mathscr{A}_F$. In this sense this holomorphic structure of L_k is natural and canonical.

5. The Relation to the Fusion Rules

In the last section we have constructed the line bundle on the moduli space of stable parabolic bundles $\mathscr{M}_{(\Sigma; P_1, \ldots, P_n; \alpha_1, \ldots, \alpha_n)}$. In this section we study this line bundle when $\Sigma = \mathbb{CP}^1$, $n = 3$ and the relation to the fusion rules in the level k $SU(2)$ Chern–Simons gauge theory or the level k $SU(2)$ Wess–Zumino–Witten conformal field theory.

First we recall the definition of stability for rank 2 parabolic bundles. Let E be a rank 2 holomorphic vector bundle on a compact Riemann surface Σ. Let $P_1, \ldots, P_n \in \Sigma$ be parabolic points in Σ. At P_i, fix a parabolic structure:

$$E_{P_i} \supsetneqq L_{P_i} \supsetneqq \{0\}, \quad \alpha_i < \beta_i .$$

Definition 5.1. *Let $(\Sigma, E, P_i, L_{P_i}, \alpha_i, \beta_i)$ be a parabolic bundle as above. Let V be a holomorphic sub-line bundle of E. Define its parabolic degree by*

$$\operatorname{pardeg} V = \deg V + \sum_{i=1}^{n} \varepsilon_i ,$$

where

$$\varepsilon_i = \begin{cases} \alpha_i & \text{if } V_{P_i} \neq L_{P_i} \\ \beta_i & \text{if } V_{P_i} = L_{P_i} . \end{cases}$$

We say a parabolic bundle $(\Sigma, E, P_i, L_{P_i}, \alpha_i, \beta_i)$ is stable, if for any holomorphic sub-line bundle V,

$$\operatorname{pardeg} V < \frac{\operatorname{pardeg} E}{\operatorname{rank} E} = \frac{1}{2}\left\{\deg E + \sum_{i=1}^{n} (\alpha_i + \beta_i)\right\}$$

holds.

Now we study the moduli space $\mathscr{M}_{(\mathbb{CP}^1;\, P_1, P_2, P_3;\, \alpha_1, \alpha_2, \alpha_3)}$.

Theorem 5.2. *Assume* $0 < \alpha_1 \leqq \alpha_2 \leqq \alpha_3 < \frac{1}{2}$. *Then we have*

$$\mathscr{M}_{(\mathbb{CP}^1;\, P_1, P_2, P_3;\, \alpha_1, \alpha_2, \alpha_3)} = \begin{cases} 1 \text{ point} & \text{if } \begin{cases} \alpha_3 < \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 < 1 \end{cases} \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Let $(\mathbb{CP}^1, E, P_i, L_{P_i}, -\alpha_i, \alpha_i)$ be a rank 2 parabolic bundle with pardeg $= 0$. Suppose that $(\mathbb{CP}^1, E, P_i, L_{P_i}, -\alpha_i, \alpha_i)$ is stable.

First we show that E is holomorphically trivial. By Grothendieck's theorem [5] we have

$$E = H^l \oplus H^{-1} \quad \text{for some } l \in \mathbb{Z}_{\geqq 0},$$

where H is the hyperplane bundle on $\mathbb{CP}^1$. We show $l = 0$.

Suppose $l \geqq 2$. Then we have

$$\operatorname{pardeg} H^l \geqq l - (\alpha_1 + \alpha_2 + \alpha_3) > 0 .$$

This contradicts to the stability of $(\mathbb{CP}^1, E, P_i, L_{P_i}, -\alpha_i, \alpha_i)$. So we have $l = 0$ or 1.

Suppose $l = 1$. Since

$$1 - (\alpha_1 + \alpha_2 + \alpha_3) \leqq \operatorname{pardeg} H < 0 ,$$

we have

$$1 < \alpha_1 + \alpha_2 + \alpha_3 .$$

Moreover we have

$$H_{P_i} \neq L_{P_i} \quad \text{for } i = 1, 2, 3 .$$

Then there exists a holomorphic sub-line bundle V such that

$$V_{P_i} = L_{P_i} \quad \text{for } i = 1, 2, 3 ,$$

$$\deg V = -1 .$$

Since

$$\text{pardeg } V = -1 + \alpha_1 + \alpha_2 + \alpha_3 < 0 ,$$

we have

$$\alpha_1 + \alpha_2 + \alpha_3 < 1 .$$

This is a contradiction. So we have $l = 0$.

Thus we may write $E = \mathbb{CP}^1 \times \mathbb{C}^2$ as a holomorphic vector bundle. So we can write $[L_{P_i}] \in \mathbb{CP}^1$, which is the projective space of the fiber of E. Then it is easy to see that

$$[L_{P_i}] \neq [L_{P_j}] \quad \text{if } i \neq j ,$$

$$\alpha_1 + \alpha_2 + \alpha_3 < 1, \quad \alpha_3 < \alpha_1 + \alpha_2 .$$

On the other hand, it is easy to see that, if $\alpha_1 + \alpha_2 + \alpha_3 < 1$, $\alpha_3 < \alpha_1 + \alpha_2$, then

$$\mathcal{M}_{(\mathbb{CP}^1; P_1, P_2, P_3; \alpha_1, \alpha_2, \alpha_3)} = 1 \text{ point.} \quad \blacksquare$$

Therefore the condition that the moduli space $\mathcal{M}_{(\mathbb{CP}^1; P_1, P_2, P_3; \alpha_1, \alpha_2, \alpha_3)}$ is 1 point and there exists the natural line bundle on the moduli space in Theorem 4.2 is the following:

$$k\alpha_3 < k\alpha_1 + k\alpha_2 ,$$

$$k\alpha_1 + k\alpha_2 + k\alpha_3 < k ,$$

$$k\alpha_1 + k\alpha_2 + k\alpha_3 \in \mathbb{Z} ,$$

$$k\alpha_1, k\alpha_2, k\alpha_3 \in \frac{1}{2}\mathbb{Z} ,$$

where we assume $0 < k\alpha_1 \leqq k\alpha_2 \leqq k\alpha_3 < \frac{k}{2}$.

Let $Z_k(\Sigma; P_1, \dots, P_n; j_1, \dots, j_n)$ be the quantum Hilbert space of the level k $SU(2)$ Chern–Simons gauge theory, where $j_i \in \frac{1}{2}\mathbb{Z}_{\geqq 0}$, which is the set of all irreducible representations of $SU(2)$. In [12] Witten observed that $Z_k(\Sigma; P_1, \dots, P_n; j_1, \dots, j_n)$ is isomorphic to the space of the corresponding conformal block $CB_k(\Sigma; P_1, \dots, P_n; j_1, \dots, j_n)$ in the level k $SU(2)$ Wess–Zumino–Witten conformal field theory, where $j_i \in \left\{0, \frac{1}{2}, \dots, \frac{k}{2}\right\}$, which is the set of all irreducible level k integrable highest weight representations of the affine Lie algebra $\widehat{sl(2, \mathbb{C})}$. The dimension of $CB_k(\mathbb{CP}^1, P_1, P_2, P_3; j_1, j_2, j_3)$ is known as the fusion rule as follows [11].

Fact. *Assume* $j_1 \leqq j_2 \leqq j_3$. *Then we have*

$$\dim CB_k(\Sigma; P_1, P_2, P_3; j_1, j_2, j_3) = \begin{cases} 1 & \text{if } \begin{cases} j_3 \leqq j_1 + j_2 \\ j_1 + j_2 + j_3 \in \mathbb{Z} \\ j_1 + j_2 + j_3 \leqq k \end{cases} \\ 0 & \text{otherwise}. \end{cases}$$

If we set $k\alpha_i = j_i$, we conclude that the fusion rule is almost the same as the condition that the corresponding moduli space of stable parabolic bundles is a point and the line bundle exists on it. The exceptional cases are the cases where equalities hold. This is because we don't treat semi-stable parabolic bundles.

Thus we obtain a geometric interpretation of the fusion rule. It is interesting to compare our interpretation of the fusion rule with Gawedzki and Kupiainen's [6]. They consider only usual vector bundles, that is, not parabolic bundles, and observed the relation between the fusion rules and unstable vector bundles. On the other hand we consider only stable parabolic bundles and observed the relation between the fusion rules and the moduli space of stable parabolic bundles. This suggests that the moduli space of stable parabolic bundles has deep information, which includes the fusion rules.

6. The Moduli Space of Stable Parabolic Higgs Bundles

In Sect. 4 we constructed the line bundle L_k on the moduli space of stable parabolic bundles. If we define a parabolic Higgs bundle appropriately, we can construct the moduli space of stable parabolic Higgs bundles as a hyperkähler quotient by the gauge group and it contains the cotangent bundle of the moduli space of stable parabolic bundles [8]. In this section we show that the line bundle on the moduli space of stable parabolic bundles extends naturally to the moduli space of stable parabolic Higgs bundles.

First of all we recall the definition of a parabolic Higgs bundle. Let $(\Sigma, E, P_i, F_l E_{P_i}, \alpha_l^{(i)})$ be a parabolic bundle. We say ψ is a *Higgs field* of $(\Sigma, E, P_i, F_l E_{P_i}, \alpha_l^{(i)})$ if the following conditions (a), (b) hold:

(a) ψ is a $\mathrm{End}^0 E$-valued meromorphic $(1,0)$ form on Σ and holomorphic on $\Sigma \backslash \{P_1, \ldots, P_n\}$.
(b) At P_i, ψ has a pole of at most 1st order with a nilpotent residue with respect to $\{F_l E_{P_i}\}$.

We call a *parabolic Higgs bundle* a pair of a parabolic bundle and its Higgs field.

Take a compact Riemann surface Σ, a finite set of points $P_1, \ldots, P_n \in \Sigma$, a smooth Hermitian vector bundle with a unitary trivialization $E = \Sigma \times \mathbb{C}^2$ as in Sect. 1. Now we construct the moduli space of stable parabolic Higgs bundles with pardeg $= 0$ and the weights $-\alpha_i < \alpha_i$ $(0 < \alpha_i < \frac{1}{2})$ at each P_i. As in Sect. 1 let $\mathscr{A} = d_0 + \Omega^1(\mathrm{End}^0_{sk} E)$ be a $SU(2)$ connection space, which is compatible with the parabolic structure. There is a hyperkähler structure on $\mathscr{A} \times \Omega^{1,0}(\mathrm{End}^0 E)$ and the gauge group acts on $\mathscr{A} \times \Omega^{1,0}(\mathrm{End}^0 E)$, preserving the hyperkähler structure. The moduli space can be constructed as a hyperkähler quotient by the gauge group. Strictly speaking, we need the appropriate Sobolev completion to construct the moduli space. See [8] for details.

To construct the line bundle on the moduli space of stable parabolic Higgs bundles, we have to construct the line bundle on $\mathscr{A} \times \Omega^{1,0}(\mathrm{End}^0 E)$ and define the action of the gauge group on this line bundle. However, thanks to Lemma 2.2, since we have already obtained the line bundle on $\mathscr{A}$ and the action of the gauge group $\mathscr{G}$ on it, we have only to construct the line bundle on $\Omega^{1,0}(\mathrm{End}^0 E)$ and define the action of the gauge group on it.

Now we construct the line bundle $\mathscr{L}'_k$ on $\Omega^{1,0}(\mathrm{End}^0 E)$. Fix a level $k \in \mathbb{Z}_{>0}$ as in Sect. 1. Define a symplectic form θ'_k on $\Omega^{1,0}(\mathrm{End}^0 E)$ by

$$\theta'_k(p, q) = \frac{k}{4\pi^2} \int_\Sigma \mathrm{Tr}\{(p - p^*) \wedge (q - q^*)\} .$$

for $p, q \in \Omega^{1,0}(\mathrm{End}^0 E)$. Define a 1-form η'_k on $\Omega^{1,0}(\mathrm{End}^0 E)$ by

$$\eta'_k(p)_{\mathrm{at}\,\psi} = \frac{k}{8\pi^2} \int_\Sigma \mathrm{Tr}\{(\psi - \psi^*) \wedge (p - p^*)\} .$$

Let $\mathscr{L}'_k = \Omega^{1,0}(\mathrm{End}^0 E) \times \mathbb{C}$ be a Hermitian line bundle with a unitary trivialization. Define a unitary connection $\nabla^{k'}$ on $\mathscr{L}'_k$ by

$$\nabla^{k'} = d - 2\pi\sqrt{-1}\eta'_k .$$

Then we have

$$\frac{\sqrt{-1}}{2\pi} R^{\nabla^{k'}} = d\eta'_k = \theta'_k .$$

Define the action of the gauge group $\mathscr{G}$ on $\mathscr{L}'_k = \Omega^{1,0}(\mathrm{End}^0 E) \times \mathbb{C}$ by

$$(\psi, v) \mapsto (s^{-1}\psi s, v) \quad \text{for } s \in \mathscr{G}, \psi \in \Omega^{1,0}(\mathrm{End}^0 E) .$$

Then we have the following proposition.

Proposition 6.1. *The above action of the gauge group $\mathscr{G}$ on $\mathscr{L}'_k$ preserves the connection $\nabla^{k'}$.*

Proof. We can easily check the condition of Lemma 2.1. ∎

So applying Lemma 2.2, we obtain the following theorem.

Theorem 6.2. *The line bundle constructed in Theorem* 4.2 *extends naturally to the moduli space of stable parabolic Higgs bundles.*

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