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Existence of Black Hole Solutions for the Einstein-Yang/Mills Equations

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Abstract. This paper provides a rigorous proof of the existence of an infinite number of black hole solutions to the Einstein-Yang/Mills equations with gauge group $SU(2)$, for any event horizon. It is also demonstrated that the ADM mass of each solutions is finite, and that the corresponding Einstein metric tends to the associated Schwarzschild metric at a rate $1/r^2$, as r tends to infinity.

1. Introduction

In this paper we prove that the Einstein-Yang/Mills (EYM) equations, with $SU(2)$ gauge group, admit an infinite family of "black-hole" solutions having a regular event horizon, for every choice of the radius r_H of the event horizon. The solutions obtained are indexed by a "winding number". Moreover, we prove that the ADM mass, [2] of each solution is finite, and the corresponding Einstein metric tends to the associated Schwarzschild solution in the far field. Some of our results were observed numerically in [4, 5]; see also [3]. Numerical discussions of the stability properties of some of these solutions can be found in [6, 9].

The existence problem reduces to finding solutions of the following system of ordinary differential equations in the region $r \geq r_H$:

$$\begin{gathered} r^2 A w'' + \left[r(1-A) - \frac{(1-w^2)^2}{r} \right] w' + w(1-w^2) = 0\,, \\ rA' + (1+2w'^2)A = 1 - \frac{(1-w^2)^2}{r^2}\,, \end{gathered} \tag{1.1}$$

subject to certain boundary conditions. These equations were studied in [7, 8], where the existence of globally defined regular (i.e., non-black-hole) solutions was proved.

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In the case considered here, we are given that Eqs. (1.1) are singular at $r = r_H$, in the sense that $A(r_H) = 0$. If $w'(r)$ is bounded near r_H, then $\lim_{r\to r_H} w'(r)$ exists, and this constrains the data pair $(w(r_H), w'(r_H))$ to lie on a curve $\mathscr{C}(r_H)$ in the $w - w'$ plane. We prove that for every $r_H > 0$, there exists a sequence of initial values (γ_n, β_n) on $\mathscr{C}(r_H)$ for which the corresponding solutions $(w_n(r), w_n'(r), A_n(r))$ of (1.1), are both non-singular ($A_n(r) > 0$), and bounded, for all $r > r_H$. Furthermore, we show that each solution has finite ADM mass, $\bar{\mu}_n = \lim_{r\to\infty} r(1 - A_n(r))$, and since $\bar{\mu}_n > r_H$, these masses can be arbitrarily large. However, in a future publication it will be shown that $(\bar{\mu}_n - r_H)$ is uniformly bounded.

Our results disprove the "no hair" conjecture for non-abelian black holes [4, 10]. This conjecture states that a stationary non-abelian black hole is uniquely determined by the following quantities at infinity: its mass, its angular momentum, and its Yang/Mills charge, [10]. We demonstrate the existence of infinitely-many black hole solutions. Each of our solutions agrees at infinity with a Schwarzschild solution having as usual, constant mass function, zero angular momentum, and zero Yang-Mills charge. That is, for any event horizon $r_H > 0$, we produce an infinite number of counter examples to the no hair conjecture. (These solutions were shown to be numerically unstable in the papers, [4, 6].) We note that the singularity at $r = r_H$ in the Einstein metric for our solutions can be "transformed away" exactly as in the familiar Schwarzschild case, see [1].

We now describe the contents of this paper. In the next section we discuss the equations and the boundary conditions, and state the local existence theorem. We also recall some results proved in [8]. In Sect. 3, we prove the main theorem: Given any $r_H > 0$ and $n \in \mathbb{Z}_+$, there exists a smooth, bounded solution $(w_n(r), w_n'(r), A_n(r))$ of (1.1), defined for all $r > r_H$, satisfying $A_n(r) > 0$, $\lim_{r \searrow r_H} A_n(r) = 0$, and $\lim_{r\to\infty} \mathrm{Tan}^{-1}(w_n'(r)/w_n(r)) = -n\pi$. It is a curious fact that the proofs in the cases $r_H > 1$, $r_H = 1$, and $r_H < 1$ are quite different, but the results are the same.

In Sect. 4 we shall obtain some properties of our solutions; in particular we show that each of our solutions has finite total mass $\bar{\mu}_n$, and that the corresponding Einstein metric coefficients tend to the Schwarzschild metric coefficients

$$ds^2 = -\left(1 - \frac{\bar{\mu}_n}{r}\right)dt^2 + \left(1 - \frac{\bar{\mu}_n}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\,,$$

as $r \to \infty$, at a rate $\dfrac{1}{r^2}$.

Finally, in the appendix, we prove the local existence theorem. (The existence and continuous dependence issues are somewhat delicate here because the equations are singular at r_H.)

2. Preliminaries

As has been discussed elsewhere, [3, 7, 8], the EYM equations with gauge group G can be written as

$$R_{ij} - \tfrac{1}{2}\,Rg_{ij} = \sigma T_{ij}\,, \qquad d^* F_{ij} = 0\,. \tag{2.1}$$

where T_{ij} is the stress-energy tensor associated to the $\mathfrak{g}$-valued Yang-Mills curvature 2-form F_{ij}, and $\mathfrak{g}$ is the Lie-algebra of G. If we consider static solutions; i.e., solutions depending only on r, and $G = SU(2)$, then we may write the metric as

$$ds^2 = -T(r)^{-2}dt^2 + A(r)^{-1}dr^2 + r^2 d\Omega^2 ,$$

(where $d\Omega^2$ is the standard metric on the 2-sphere), and the curvature 2-form as

$$F = w'\tau_1 dr \wedge d\theta + w'\tau_2 dr \wedge (\sin\theta d\phi) - (1 - w^2)\tau_3 d\theta \wedge (\sin\theta d\phi) .$$

Here T, A and w are the unknown functions, and τ_1, τ_2, τ_3 form a basis for the Lie algebra $su(2)$. The EYM equations are given by

$$rA' + (1 + 2w'^2)A = 1 - \frac{(1 - w^2)^2}{r^2} , \tag{2.2}$$

$$r^2 A w'' + \left[r(1 - A) - \frac{(1 - w^2)^2}{r} \right] w' + w(1 - w^2) = 0 , \tag{2.3}$$

$$2rA(T'/T) = \frac{(1 - w^2)^2}{r^2} + (1 - 2w'^2)A - 1 . \tag{2.4}$$

Since (2.2) and (2.3) de-couple from (2.4), we solve them first, and then use (2.4) to obtain T. [We shall return to (2.4) in Sect. 4.]

As in [7, 8], we define the function Φ by

$$\Phi(r, w, A) = r(1 - A) - \frac{(1 - w^2)^2}{r} , \tag{2.5}$$

and then (2.2) and (2.3) become

$$rA' + 2w'^2 A = \Phi/r , \tag{2.2'}$$

$$r^2 A w'' + \Phi w' + w(1 - w^2) = 0 . \tag{2.3'}$$

Now let $\bar{r} > 0$; in order to obtain "black hole" solutions of radius $\bar{r}$, the initial data is required to satisfy the following conditions:

$$A(\bar{r}) = 0, \;\; w^2(\bar{r}) < 1, \;\text{ and }\; |w'(\bar{r})| < \infty .$$

We then seek a smooth solution of (2.2), (2.3) defined for all $r > \bar{r}$ which satisfies the condition

$$\lim_{r \to \infty} (A(r), w(r), w'(r)) \text{ is finite} .$$

The following result gives some necessary conditions in order that (2.2), (2.3) admits a $C^{2+\alpha}$ solution satisfying (2.6) on some interval $|r - \bar{r}| < \varepsilon$; we use the notation $\bar{w} = w(\bar{r})$.

Proposition 2.1. *Let* $\bar{r} > 0$ *be given, and let* $(w(r), w'(r), A(r))$ *be a* $C^{2+\alpha}$ *solution of* (2.2), (2.3) *on an interval* $\bar{r} < r < \bar{r} + \varepsilon$, *which satisfies:* $\lim_{r \to \bar{r}} A(r) = 0$, *and* $\lim_{r \to \bar{r}} w^2(r) \le 1$, *and* $\lim_{r \to \bar{r}} (w(r), w'(r)) \ne (0, 0)$. *Then the following conditions hold:*

$$\Phi(\bar{r}, \bar{w}, 0) w'(\bar{r}) + w(\bar{r})(1 - w^2(\bar{r})) = 0 , \tag{2.1}$$

$$\Phi(\bar{r}, \bar{w}, 0) \ne 0 , \tag{2.8}$$

and

$$\begin{aligned} w''(\bar{r}) &= \lim_{r \to \bar{r}} w''(r) \\ &= \frac{-w'(\bar{r})}{2\Phi(\bar{r}, \bar{w}, 0)} \left[\frac{2(1 - \bar{w}^2)^2}{\bar{r}^2} + \frac{4\bar{w}(1 - \bar{w}^2) w'(\bar{r})}{\bar{r}} + (1 - 3\bar{w}^2) \right] . \end{aligned} \tag{2.9}$$

Proof. Since $A(\bar{r}) = 0$, (2.7) follows from (2.3). Now if $\Phi(\bar{r}, \bar{w}, 0) = 0$, then $w(\bar{r}) = \pm 1$ or $w(\bar{r}) = 0$. If $w(\bar{r}) = \pm 1$, then $0 = \Phi(\bar{r}, \bar{w}, 0) = \bar{r}$, contrary to our assumption. Thus we may assume $w(\bar{r}) = 0$. In this case $0 = \Phi(\bar{r}, \bar{w}, 0) = \bar{r} = \frac{1}{\bar{r}}$ so $\bar{r} = 1$. Now suppose $w'(1) < 0$; if $w'(1) > 0$ a similar argument will work. Since

$$\Phi' = 2Aw'^2 + \frac{2(1-w^2)^2}{r^2} + \frac{4ww'(1-w^2)}{r}, \tag{2.10}$$

we see that $\Phi'(1) = 2$ so $(2.2)'$ implies that $A'(1) = 0$, and $A''(1) > 0$. Thus $A(r) > 0$ for $r > 1$, r near 1. Now set $v = Aw'$; then $v(1) = 0$, and

$$v' + \frac{2w'^2 v}{r} + \frac{w(1-w^2)}{r^2} = 0, \tag{2.11}$$

so that $v'(1) = 0$, and $v''(1) > 0$. Thus $v(r) > 0$ for $r > 1$, r near 1. On the other hand $v(r) = A(r)w'(r) < 0$ for r near 1, $r > 1$. This contradiction shows that $\Phi(\bar{r}, \bar{w}, 0) \neq 0$. Finally $(2.2)'$, $(2.3)'$, and (2.10) give, using L'Hospital's rule,

$$w''(\bar{r}) = \lim_{r \to \bar{r}} \frac{-\Phi w' - w(1-w^2)}{r^2 A} = -\frac{1}{\bar{r}^2} \lim_{r \to \bar{r}} \frac{\Phi w'' + \Phi' w' + (1-3w^2)w'}{\left(\frac{1}{r} - \frac{(1-w^2)^2}{r^3}\right)}$$

$$= \frac{-\left[\bar{r} - \frac{(1-\bar{w}^2)^2}{\bar{r}}\right] w''(\bar{r}) - w'(\bar{r})\left[\frac{2(1-\bar{w}^2)^2}{\bar{r}^2} + \frac{4\bar{w}(1-\bar{w})}{\bar{r}} w'(\bar{r})\right] - w'(\bar{r})(1-3\bar{w}^2)}{\bar{r} - \frac{(1-\bar{w}^2)^2}{\bar{r}}}.$$

Thus solving for $w''(\bar{r})$ gives (2.9). □

We remark that if $\bar{w} \neq 0$ and $w'(r)$ is bounded for $r > \bar{r}$, r near $\bar{r}$, then $\lim_{r \searrow \bar{r}} w'(r)$ exists, (see [8, Lemma 4.4]), and thus (2.7) and (2.8) are satisfied even if the solution is defined only for $r > \bar{r}$. Thus w is a C^1-function. If $\lim_{r \searrow \bar{r}} w''(r)$ exists then (2.9) holds, and w is then a C^2-function.

There is a converse to the last result; i.e., the condition (2.8) is also sufficient for the existence of a smooth solution. We need the following definition, [8].

Definition 2.1. A one-parameter family $(w(r,\gamma), w'(r,\gamma), A(r,\gamma), r)$ of solutions of (2.2), (2.3) defined for $\bar{r} < r < \bar{r} + s(\gamma)$, is called *continuous*, provided that it satisfies the following. If $\Gamma_1 < \Gamma_2$, then there exists a number $S(\Gamma_1, \Gamma_2) > 0$ such that for all $\gamma \in [\Gamma_1, \Gamma_2]$,
(a) $s(\gamma) \geq S(\Gamma_1, \Gamma_2)$, and
(b) $(w(r,\gamma), w'(r,\gamma), A(r,\gamma), r)$ depends continuously on (r,γ) for $\bar{r} \leq r < \bar{r} + S(\Gamma_1, \Gamma_2)$.

We now have the following converse to Proposition 2.1.

Proposition 2.2. *Given $\bar{r}$ and γ such that $\bar{r} > 0$, $\gamma^2 \leq 1$, and such that*

$$\Phi(\bar{r}, \gamma, 0) \equiv \bar{r} - \frac{(1-\gamma^2)^2}{\bar{r}} \neq 0, \tag{2.12}$$

then there exists a unique $\beta = \beta(\bar{r}, \gamma)$ such that

$$\gamma(1-\gamma^2) + \Phi(\bar{r}, \gamma, 0)\beta = 0, \tag{2.13}$$

and β depends continuously on γ. Moreover, there exists a unique solution

$$(w(r,\gamma), w'(r,\gamma), A(r,\gamma), r)$$

of (2.2), (2.3) *satisfying* (2.9) *and the initial conditions $w(\bar{r},\gamma) = \gamma$, $w'(\bar{r},\gamma) = \beta$, $A(\bar{r},\gamma) = 0$, defined on some interval $\bar{r} \le r < \bar{r} + s(\gamma)$. The solutions form a continuous* 1*-parameter family, and are analytic on $|r - \bar{r}| < s(\gamma)$.*

Proof. Given in the appendix.

In the rest of this paper, for any fixed $\bar{r} > 0$, we shall denote by $(w(r,\gamma), w'(r,\gamma), A(r,\gamma), r)$ the $C^{2+\alpha}$-solution of (2.2), (2.3), satisfying $w(\bar{r},\gamma) = \gamma$, $w'(\bar{r},\gamma) = \beta$, [where β is determined by (2.12), and $\Phi(\bar{r},\gamma) \neq 0$], and $A(\bar{r},\gamma) = 0$; we shall call this solution the "γ-orbit."

We shall next recall a few results from [8]. Before stating them, we must introduce some notation. First we define the region $\Gamma \subset \mathbb{R}^4$ by

$$\Gamma = \{(w, w', A, r)\colon w^2 < 1,\ A > 0,\ r > \bar{r},\ (w, w') \neq (0,0)\}\,.$$

Next, we let $r_e(\gamma)$ be the smallest $r > \bar{r}$ for which the γ-orbit exists Γ; $r_e(\gamma) = +\infty$ if the γ-orbit stays in Γ for all $r > 0$. If the γ-orbit exits Γ via $A = 0$, we say that the γ-orbit *crashes*. [A priori it is possible for $w'(r)$ to become unbounded as $r \to r_1 > \bar{r}$, say. But in fact this cannot happen unless $A(r) \to 0$ as $r \to r_1$; this follows from (2.3)$'$.] Next, if $\theta(r,\gamma)$ is defined by $\theta(r,\gamma) = \mathrm{Tan}^{-1}(w'(r,\gamma)/w(r,\gamma))$, $-\frac{\pi}{2} < \theta(\bar{r},\gamma) \le 0$, we set

$$\Omega(\gamma) = -\frac{1}{\pi}\left[\theta(r_e(\gamma),\gamma) - \theta(\bar{r},\gamma)\right].$$

Theorem 2.3 [8, Theorem 3.1]. *Let $\bar{r} > 0$ be a fixed positive number. Suppose that $\gamma_n \to \bar{\gamma}$, and let*

$$\Lambda_n = \{(w(r,\gamma_n), w'(r,\gamma_n), A(r,\gamma_n), r)\colon \bar{r} < r < r_e(\gamma_n)\}$$

be a sequence of orbit segments in Γ, satisfying $\Omega(\gamma_n) \le N$ for all n. Then the $\bar{\gamma}$-orbit lies in Γ for $\bar{r} < r < r_e(\bar{\gamma})$, and $\Omega(\bar{\gamma}) \le N$.

Note that the proof in [8] had $\bar{r} = 0$ but the proof for $\bar{r} > 0$ requires no essential modifications.

Corollary 2.4 [8, Proposition 2.10]. *Suppose that for some γ, and some integer $k > 0$, $(k-1) < \Omega(\gamma) \le k$, the γ-orbit doesn't crash and $w^2(r,\gamma) < 1$ for all $r > 0$. Then $\Omega(\gamma) = k$.*

We next have

Proposition 2.5 [8, Proposition 4.8]. *Let the orbit segments Λ_n be as in Theorem 2.3. Suppose $\gamma_n \to \bar{\gamma}$ and $\Omega(\bar{\gamma}) = k$. Then for sufficiently large n, $\Omega(\gamma_n) < k + 1$.*

In the rest of this paper, $\theta(r,\gamma)$, $r_e(\gamma)$, and $\Omega(\gamma)$ will denote the quantities defined above. If k is a non-negative integer and $\Omega(\gamma) = k$, we shall refer to the γ-orbit as a *k-connecter*. Finally if $\sigma^2 < 1$, we define $r_\sigma(\gamma)$ by

$$w(r_\sigma(\gamma),\gamma) = \sigma\,. \tag{2.14}$$

[More precise notation would be $r_{\sigma,n}(\gamma)$ to denote the n^{th} time that the γ-orbit meets the hyperplane $w = \sigma$. However in each instance, the n will be clear and we shall thus omit the dependence on n.]

3. The Main Theorem

In this section we shall prove the following theorem.

Theorem 3.1. *Given any $\bar{r} > 0$, there exists a sequence $(\gamma_n, \beta_n, \bar{r})$, where $(\gamma_n, \beta_n, \bar{r})$ satisfies* (2.12) *and* (2.13), *such that the corresponding solution $(w(r, \gamma_n), w'(r, \gamma_n), A(r, \gamma_n), r)$ of* (2.2), (2.3) *with this data lies in Γ for all $r > \bar{r}$, and $\Omega(\gamma_n) = n$.*

Proof. The proof breaks up into three distinct cases, depending on whether $\bar{r} < 1$, $\bar{r} = 1$, or $\bar{r} > 1$. These cases are different because the "initial data" sets $\mathscr{C}(\bar{r})$ defined by[1]

$$\mathscr{C}(\bar{r}) = \{(w, w')\colon \Phi(\bar{r}, \bar{w}, 0)w' + w(1 - w^2) = 0,\ w' \le 0,\ w > 0\}, \tag{3.1}$$

are different in each of the three cases. We begin with the case $\bar{r} > 1$ [see Fig. 1, where the "dashed" curve represents $\mathscr{C}(r)'$].

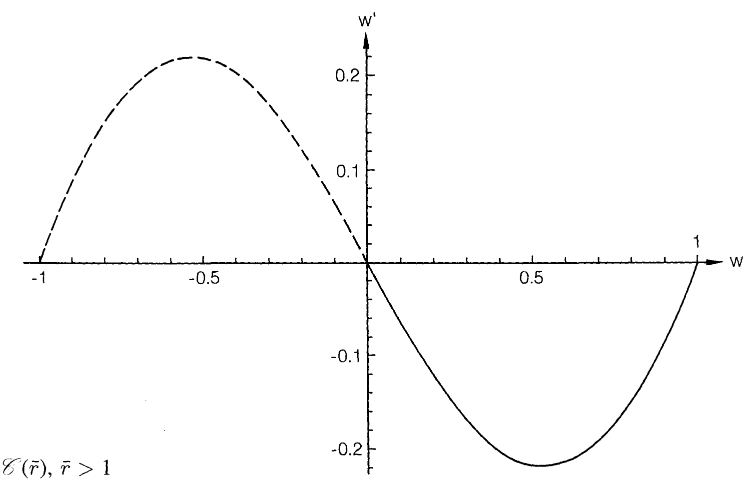

Fig. 1. $\mathscr{C}(\bar{r})$, $\bar{r} > 1$

Proposition 3.2 *Theorem 3.1 holds if $r > 1$.*

We shall first show that orbits of (2.2), (2.3) starting on $\mathscr{C}(\bar{r})$ with $\bar{r} \ge 1$ don't crash.

Lemma 3.3. *Let $\bar{r} \ge 1$; then no orbit $(w(r, \gamma), w'(r, \gamma), A(r, \gamma), r)$ crashes. Moreover, there are constants $\delta > 0$, $\tau > 0$, independent of γ such that $A(r, \gamma) \ge \delta$ and $|w'(r, \gamma)| < \tau$, for $r \ge \bar{r} + 1$.*

Proof. For ease in notation, we shall suppress the dependence on γ. Consider first the case $\bar{r} > 1$. Define $\mu(r)$ by[2]

$$\mu(r) = r(1 - A(r)). \tag{3.2}$$

[1] In what follows, we shall only be concerned with $\mathscr{C}(\bar{r})$; all results we give concerning data $(w(\bar{r}), w'(\bar{r}))$ in $\mathscr{C}(\bar{r})$ have analogous statements for data in the symmetric branch $\mathscr{C}(\bar{r})' = \{(w, w')\colon \Phi(\bar{r}, \bar{w}, 0)w' + w(1 - w^2) = 0,\ w' \ge 0,\ w < 0\}$, lying in the region $w' \ge 0$

[2] $\mu(r) = 2m(r)$, where $m(r)$ is the ADM mass [2]

Then $\mu' = 2Aw'^2 + (1-w^2)^2/r^2 > 0$. From (2.5), for $r > \bar{r}$,

$$\Phi(r,w,A) = \mu(r) - \frac{(1-w(r)^2)^2}{r} \geq \mu(\bar{r}) - \frac{1}{r}$$
$$= \bar{r} - \frac{1}{\bar{r}} \equiv \sigma > 0\,. \qquad (3.1)$$

The results now follow from [8, Propositions 2.5 and 2.7]; i.e., the positivity of Φ enables us to bound A from below, and to keep w' bounded. If $\bar{r} = 1$, then by our local existence theorem, Proposition 2.2, the solution exists for $\bar{r} < r < \bar{r} + \varepsilon$, for some $\varepsilon > 0$, so this case reduces to the previous one.

Lemma 3.4. *Let $k \in \mathbb{Z}_+$; if there exists a γ_1 for which $\Omega(\gamma_1) > k$, then there is a $\bar{\gamma}$ for which $\Omega(\bar{\gamma}) = k$.*

Proof. Let $S = \{\gamma\colon \gamma \geq \gamma_1$, and $\Omega(\gamma) \leq k\}$; $S \neq \phi$ since $1 \in S$. [The $\gamma = 1$ orbit is $w(r) \equiv 1$, $A(r) = 1 - \bar{r}/r$.] Let $\bar{\gamma} = \inf S$; then by the previous lemma the $\bar{\gamma}$-orbit cannot crash. The $\bar{\gamma}$-orbit cannot exit Γ via $w^2 = 1$ (if so, $\Omega(\bar{\gamma}) < k$ and hence $\Omega(\gamma) < k$ for some $\gamma \in S$, $\gamma < \bar{\gamma}$, γ near $\bar{\gamma}$). The $\bar{\gamma}$-orbit must then be a j-connecter for some integer $j \leq k$, by Proposition 2.5. If $j < k$, then by Corollary 2.4, $\Omega(\gamma) < j + 1 \leq k$ for γ near $\bar{\gamma}$; this violates the definition of $\bar{\gamma}$ and thus $\Omega(\bar{\gamma}) = k$. □

In view of this last result, in order to prove Proposition 3.2, we may assume that for some $\bar{r} > 1$,

$$\Omega(\gamma) \leq M \quad \text{for all } \gamma\,, \qquad (3.4)$$

and show that this leads to a contradiction.

From [8, Proposition 2.11], the assumption (3.4) implies that we can find

$$\bar{\mu} > \max\left(\bar{r}, \frac{1}{\sigma}\right) \qquad (3.5)$$

such that for all γ,

$$\mu(r,\gamma) \leq \bar{\mu}/2\,, \quad \text{if } \bar{r} \leq r \leq r_e(\gamma)\,. \qquad (3.6)$$

We shall obtain the desired contradiction by showing that orbits which start sufficiently near $(w, w') = (0,0)$ have arbitrarily high rotation; that is, we will prove the following proposition, (which will complete the proof of Proposition 3.2).

Proposition 3.5. *Let $\bar{r} > 1$ be given, and assume that* (3.6) *holds. Then given any $M > 0$, there is an $\varepsilon > 0$ such that if $0 < \gamma < \varepsilon$, then $\Omega(\gamma) > M$.*

Proof. We define a distance function $\varrho(r,\gamma)$, by

$$\varrho(r,\gamma)^2 = w(r,\gamma)^2 + r^2 w'(r,\gamma)^2\,, \qquad (3.1)$$

and an "angle" $\psi(r,\gamma)$ defined for $\bar{r} \leq r \leq r_e(\gamma)$ by

$$\operatorname{Tan}\psi(r,\gamma) = rw'(r,\gamma)/w(r,\gamma)\,, \qquad -\pi/2 < \psi(\bar{r},\gamma) < 0\,. \qquad (3.8)$$

(Notice that if $\theta(r,\gamma)$ is defined as usual by $\operatorname{Tan}\theta(r,\gamma) = w'(r,\gamma)/w(r,\gamma)$ for $r > \bar{r}$, and $-\pi/2 < \theta(\bar{r},\gamma) \leq 0$, then $\psi(r,\gamma) > \theta(r,\gamma)$ for all $r \geq \bar{r} > 1$.)

The proof will be broken up into several technical steps. First define $r_0(\gamma)$ by

$$w(r_0(\gamma),\gamma) = 0\,,$$

$r_0(\gamma)$ being minimal with respect to this property.

Step 1. If γ is sufficiently close to 0, then $\varrho(\hat{r}(\gamma), \gamma)$ is small, where

$$\hat{r}(\gamma) = \min[r_0(\gamma), \bar{\mu}] . \tag{3.9}$$

Step 2. Given any $T > 0$, if $\varrho(\hat{r}(\gamma), \gamma)$ is sufficiently close to zero, then $\varrho(r, \gamma)$ is near zero for $\hat{r}(\gamma) \leq r \leq \hat{r}(\gamma) + T$.

Step 3. If $\varrho(r, \gamma)$ is near zero, then

$$\psi'(r, \gamma) < -\frac{1}{3r}$$

if $T + \hat{r}(\gamma) > r > 3\bar{\mu}$.

The proof of Proposition 3.5 then follows by integrating ψ' from $3\bar{\mu}$ to T for an appropriately chosen T.

We proceed now with the details. First, $A(r, \gamma) = 1 - \frac{\mu(r, \gamma)}{r} \geq 1 - \frac{\bar{\mu}}{2r}$, and hence

$$A(r, \gamma) \geq \tfrac{1}{2}, \quad \text{if } r > \bar{\mu} .$$

Moreover, each γ-orbit must cross $w = 0$; see [7, Proposition 6.1]. The important step is to show that we can keep an orbit close to the origin, $(w, w') = (0, 0)$, for bounded time, provided that we take γ sufficiently close to zero. Recall $\hat{r}(\gamma)$ is defined in (3.9).

The proof of Steps 1–3 will follow from a few "facts." In what follows, we shall often drop the dependence of w, A, etc., on γ when there is no chance of confusion.

Let σ be as in (3.3); i.e.,

$$\Phi(r, w, A) \geq \bar{r} - \frac{1}{\bar{r}} \equiv \sigma > 0, \quad \text{if } r > \bar{r} .$$

Fact 1. $-w'(r, \gamma) \leq \gamma/\sigma$ if $\bar{r} \leq r \leq r_0(\gamma)$, and for all r satisfying $\bar{r} < r < r_e(\gamma)$, $|w'(r, \gamma)| \leq 1/\sigma$.

Proof. From (2.3)′, and (3.3), $w''(r_0) > 0$; thus on $\bar{r} \leq r \leq r_0(\gamma)$, $-w'$ has its maximum either at $\bar{r}$ or when $w''(r) = 0$. From (2.3)′

$$-w'(\bar{r}, \gamma) = -w(r, \gamma)(1 - w^2(r, \gamma))/\Phi(r, \gamma) \leq \gamma/\sigma .$$

If $w''(r) = 0$, then again $-w''(r) \leq \gamma/\sigma$, and this proves the first statement. If $\bar{r} < r < r_e(\gamma)$, then $w'(r)$ has its maximum when $w''(r) = 0$, and at such an r, $|w'(r) \leq \frac{1}{\sigma}$. □

Fact 2. $r_0(\gamma) - \bar{r} > \sigma$.

Proof. $r_0(\gamma) - \bar{r} = \frac{-\gamma}{w'(\xi, \gamma)} < \sigma$, from Fact 1 and the mean-value theorem. □

Fact. 3. Suppose $r_0(\gamma) \leq \bar{\mu}$ and $0 < \gamma^2 < \sigma^3/4\bar{\mu}$. Then $r^2 A'(r, \gamma) \geq \sigma/2$ if $\bar{r} \leq r \leq r_0(\gamma)$.

Proof. If $r^2 A'(r) < \sigma/2$ for some r in the above interval, then from (2.2)′ and Fact 1,

$$A(r) \geq \frac{\Phi(r) - \sigma/2}{2rw'(r)^2} \geq \frac{\sigma}{4rw'(r)^2} \geq \frac{\sigma^3}{4\gamma^2 r} \geq \frac{\sigma^3}{4\gamma^2 \bar{\mu}} > 1 .$$

On the other hand, $A(r) = 1 - \frac{\mu(r)}{r} < 1$; this contradiction establishes Fact 3. $\square$

Now armed with these facts, we complete the proof of Step 1; namely we have

$$\varrho(\hat{r}(\gamma), \gamma) \le c\gamma\,, \quad \text{where } c = [1 + \bar{\mu}^2/\sigma^2]^{1/2}\,. \tag{3.11}$$

Proof. If $\bar{\mu} \le r_0(\gamma)$, then from Fact 1, $w'^2(\bar{\mu}, \gamma) < \gamma^2/\sigma^2$ so $\varrho(\bar{\mu}, \gamma)^2 \le \gamma^2 + \bar{\mu}^2\gamma^2/\sigma^2 = c^2\gamma^2$. If $r_0(\gamma) < \bar{\mu}$, then again from Fact 1, $\varrho(r_0(\gamma), \gamma)^2 \le \bar{\mu}^2\gamma^2/\sigma^2 < c^2\gamma^2$. $\square$

To complete the proof of Step 2 we need one last fact.

Fact 4. There is an $\eta > 0$ such that $A(r, \gamma) \ge \eta$ if $\gamma^2 \le \sigma^3/4\bar{\mu}$, and $r > \hat{r}(\gamma)$.

Proof. If $r \le \bar{\mu}$, then $A(r, \gamma) \ge \frac{1}{2}$, by (3.10). Thus we may assume that $\bar{\mu} > r$. If $r_0(\gamma) \ge \bar{\mu}$, then $A(r, \gamma) \ge \frac{1}{2}$ if $r \ge \bar{\mu} = \hat{r}(\gamma)$. Thus we may assume $\bar{\mu} > r_0(\gamma)$, and consider only those r for which $\bar{\mu} > r \ge r_0(\gamma)$. Since $\gamma^2 \le \sigma^3/4\bar{\mu}$, Facts 2 and 3 give

$$A(r_0(\gamma), \gamma) = A(r_0(\gamma), \gamma) - A(\bar{r}, \gamma) = \int_{\bar{r}}^{r_0(\gamma)} A'(s)ds$$

$$\ge \frac{\sigma}{2} \int_{\bar{r}}^{r_0(\gamma)} \frac{ds}{s^2} \ge \frac{\sigma}{2} \int_{\bar{r}}^{r_0(\gamma)} \frac{ds}{\bar{\mu}^2} \ge \frac{\sigma}{2\bar{\mu}^2}(r_0(\gamma) - \bar{r}) > \frac{\sigma^2}{2\bar{\mu}^2} \equiv \tau_1\,.$$

Now if there were a smallest $\tilde{r} > r_0(\delta)$, $\tilde{r} < \bar{\mu}$, for which $A(\tilde{r}, \gamma) = \tau_1$, then from (2.2)$'$, and Fact 3,

$$\begin{aligned}\tilde{r}^2 A'(\tilde{r}, \gamma) &= \Phi(\tilde{r}) - 2w'^2(\tilde{r})A(\tilde{r})\tilde{r} \\ &\ge \sigma - 2w'^2(\tilde{r})\,\frac{\sigma^2}{2\bar{\mu}^2}\,\tilde{r} \ge \sigma - 2\,\frac{1}{\sigma^2}\,\frac{\sigma^2}{2\bar{\mu}^2}\,\bar{\mu} = \sigma - \frac{1}{\bar{\mu}} > 0\,,\end{aligned}$$

by (3.5). Thus no such $\tilde{r}$ exists, and we conclude that $A(r, \gamma) \ge \tau$, if $r_0(\gamma) < r \le \bar{\mu}$. Thus Fact 4 holds with $\eta = \min\left[\tau_1, \frac{1}{2}\right]$.

We can now give the

Proof of Step 2. We have, from (2.3)$'$, for $r > \bar{r}$,

$$\begin{aligned}\varrho\varrho' &= ww' + \frac{r^2w'^2}{r} - \frac{\Phi w'^2}{A} - \frac{w(1-w^2)w'}{A} \\ &\le \frac{\varrho^2}{\bar{r}} + \frac{\varrho^2}{\bar{r}} - \frac{w(1-w^2)w'}{A} \le \frac{2\varrho^2}{\bar{r}} + \frac{\varrho^2}{\bar{r}A}\,,\end{aligned}$$

because $|w| \le \varrho$, $|rw'| \le \varrho$, $|ww'| \le \varrho^2/r$, and $\Phi > 0$ from (3.3). Thus since $\bar{\mu} > \bar{r}$ [by (3.5)], and $r_0(\gamma) > \bar{r}$, we have, for $r \ge \hat{r}(\gamma)$, from Fact 4,

$$\frac{\varrho'(r, \gamma)}{\varrho(r, \gamma)} \le \frac{2}{\bar{r}} + \frac{2}{\bar{r}\eta} \equiv J\,,$$

if $\gamma^2 \le \sigma^2/r\bar{\mu}$. Thus for such r and γ, $(\ln \varrho(r,\gamma))' \le J$. Now if $r_0(\gamma) < \bar{\mu}$, and $0 \le \tilde{r} \le T$, $\ln[\varrho(r_0(\gamma)+\tilde{r},\gamma)/\varrho(r_0(\gamma),\gamma)] \le J\tilde{r} \le JT$. Thus from Fact 4, for $\gamma^2 \le \sigma^2/4\bar{\mu}$,

$$\varrho(r_0(\gamma)+r,\gamma) \le \varrho(r_0(\gamma),\gamma)e^{JT} \le ce^{JT}\gamma\,, \quad \text{if } 0 \le r \le T\,. \tag{3.12}$$

On the other hand, if $\bar{\mu} < r_0(\gamma)$, a similar argument shows that for $\gamma^2 \le \sigma^2/4\bar{\mu}$,

$$\varrho(\bar{\mu}+r,\gamma) \le ce^{JT}\gamma, \qquad \text{if } 0 \le r \le T\,.$$

This inequality, together with (3.12) implies

$$\varrho(\hat{r}(\gamma)+r,\gamma) \le ce^{JT}\gamma \qquad 0 \le r \le t\,,$$

and this proves Step 2. □

We have done Step 2, which we state as

Lemma 3.6. *Given any $T > 0$ and any $\varepsilon > 0$, there exists a $\bar{\gamma} > 0$ such that if $0 < \gamma < \bar{\gamma}$, then $\varrho(r,\gamma) < \varepsilon$ if $\hat{r}(\gamma) \le r \le \hat{r}(\gamma)+T$.*

Notice that this lemma does not follow trivially from "continuous dependence on initial conditions", since the point $(w, w', A, \bar{t}) = (0,0,0,\bar{r})$ is a singular point for the system (2.2), (2.3).

Completion of the Proof of Step 3. Recall that $\psi = \psi(r,\gamma)$ is defined in (3.8).

We have, from (3.2)′,

$$\begin{aligned}\psi' &= \frac{ww' - \dfrac{w}{rA}\Phi w' - \dfrac{w(1-w^2)}{rA} - rw'^2}{w^2 + r^2w'^2}\\ &= \left(1 - \frac{\Phi}{rA}\right)\frac{\sin 2\psi}{2r} - \frac{(1-w^2)}{rA}\cos^2\psi - \frac{\sin^2\psi}{r}\,.\end{aligned} \tag{3.13}$$

Now choose ε such that, $0 < \varepsilon^2 < \frac{1}{2}$, and choose $T > 3\bar{\mu}$ so large that

$$\int_{3\bar{\mu}}^{T} -\frac{1}{3s}\,ds < -M\pi\,.$$

Next, take γ so close to zero that $\gamma^2 \le \sigma^2/r\bar{\mu}$, and

$$w(r,\gamma)^2 \le \varrho(r,\gamma)^2 < \varepsilon^2 \quad \text{if } \hat{r}(\gamma) \le r \le \hat{r}(\gamma)+T\,; \tag{3.14}$$

this is possible because of Lemma 3.6. Now since $\Phi \le \mu < \bar{\mu}$, we see that for $r > 3\bar{\mu}$, $rA = r - \mu \ge r - \bar{\mu} \ge 2\bar{\mu}$, so $\Phi/rA < \mu/2\bar{\mu} < \frac{1}{2}$, and thus $1 > (1 - \Phi/rA) > \frac{1}{2}$. Using this and (3.14) in (3.13) gives, for $\hat{r}(\gamma) \le r \le \hat{r}(\gamma)+T$, (suppressing the γ),

$$\begin{aligned}\psi' &\le (1-\Phi/rA)\,\frac{\sin 2\psi}{2r} - \frac{(1-\varepsilon^2)}{r}\cos^2\psi - \frac{(1-\varepsilon^2)\sin^2\psi}{r} - \frac{\varepsilon^2\sin^2\psi}{r}\\ &\le (1-\Phi/rA)\frac{\sin 2\psi}{2r} - \frac{(1-\varepsilon^2)}{r}\\ &\le \frac{1}{4r} - \frac{(1-\varepsilon^2)}{r} = \frac{-(3/4-\varepsilon^2)}{r} < -\frac{1}{3r}\,.\end{aligned}$$

This completes the proof of Step 3.

Now $\hat{r}(\gamma) \le \bar{\mu} < 3\bar{\mu} < T < T + \hat{r}(\gamma)$, so for those γ satisfying $\gamma^2 \le \sigma/4\bar{\mu}$, (3.13) gives

$$\psi(T) \le \psi(T) - \psi(3\bar{\mu}) = \int_{3\bar{\mu}}^{T} \psi'(s)ds \le \int_{3\bar{\mu}}^{T} -\frac{ds}{3s} < -M\pi\,.$$

This contradicts (3.4) and completes the proof of Proposition 3.2 □

We have thus proved Theorem 3.1 in the case $\bar{r} > 1$. We turn now to the case $\bar{r} = 1$; cf. Fig. 2, where we depict $\mathscr{C}(1)$ [and $\mathscr{C}(1)'$].

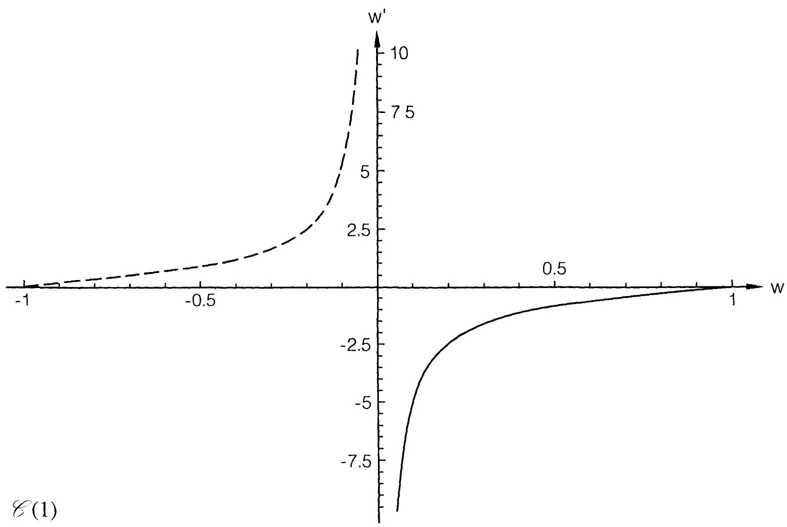

Fig. 2. $\mathscr{C}(1)$

Note that if $\bar{r} = 1$, then $w = 0$ satisfies (2.7) but not (2.8) and thus it is not included in $\mathscr{C}(1)$.

Proposition 3.7. *Theorem 3.1 holds if* $\bar{r} = 1$.

Proof. By induction on n. First of all, for $n = 0$ the solution of (2.2), (2.3) given by

$$(w(r), w'(r), A(r), r) = \left(1, 0, 1 - \frac{1}{r}, r\right),$$

defined for $r \ge 1$, satisfies $A(1) = 0$, and $(w(1), w'(1)) = (1, 0) \in \mathscr{C}(1)$. Since this solution has zero rotation, we see that the case $n = 0$ is proved.

Assume now that the theorem is true for all $n < k$; we prove that it is true for $n = k$. Note that from Proposition 2.2 and Lemma 3.3, no γ-orbit crashes. We need the following "compactness" result.

Lemma 3.8. *Let* $\bar{\gamma} = \inf\{\gamma\colon 0 < \gamma < 1,\ \Omega(\gamma) \le k\}$*; then* $\bar{\gamma} > 0$.

Proof. Suppose that the lemma were false. Then we could find a sequence $\gamma_n \to 0$, $0 < \gamma_n < 1$, and $\Omega(\gamma_n) \le k$. Define orbit segments Λ_n by

$$\Lambda_n = \{(w(r, \gamma_n), w'(r, \gamma_n), A(r, \gamma_n), r)\colon \bar{r} < r \le r_e(\gamma_n)\}\,; \qquad (3.15)$$

then $\Lambda_n \subset \Gamma$ by our earlier remark. From [8, Proposition 3,2], we can find sub-orbit segments $\Lambda'_n \subset \Lambda_n$,

$$\Lambda'_n = \{(w(r,\gamma_n), w'(r,\gamma_n), A(r,\gamma_n), r) \colon \bar{r} \le r \le r_n\},$$

$r_n < r_e(\gamma_n)$, such that the right-hand endpoints converge to a point $P \in \Gamma$; i.e.,

$$(w(r_n,\gamma_n), w'(r_n,\gamma_n), A(r_n,\gamma_n), r_n) \to P = (\tilde{w}, \tilde{w}', \tilde{A}, \tilde{r}) \in \Gamma .$$

Now $\theta(r_n,\gamma_n) \to \theta(P) \equiv \mathrm{Tan}^{-1}(\tilde{w}'/\tilde{w}), \bmod 2\pi$. Since $\theta(r_n,\gamma_n) \ge -k\pi$ for every n, then by passing to a subsequence, we may define $\theta(P)$ in $\mathbb{R}$ such that $\theta(r_n,\gamma_n) \to \theta(P)$. By the methods of [8, Proposition 3.1], the backwards orbit $(w(r), w'(r), A(r), r)$, through P, (defined for $r < \tilde{r}$) stays in Γ until $r = r_0$, where $w(r_0) = 0$, $A(r_0) > 0$, and $\theta(r_0) = -\pi/2$. In fact, this orbit can be continued further to $r = r_0 - \delta$, where $w(r_0 - \delta) = \varepsilon > 0$, and $-\pi/2 < \theta(r_0 - \delta) < 0$. We note in addition that $A(r) > 0$ for $\tilde{r} \ge r \ge r_0 - \delta$. Thus for n sufficiently large, $A(r,\gamma_n) > 0$ for $\tilde{r} \ge r \ge r_0 - \delta$. Since $A(\bar{r},\gamma_n) = 0$, we have that $r_0 - \delta > \bar{r}$. Now note that $w(\bar{r},\gamma_n) = \gamma_n$, and if $\theta(r,\gamma_n) > -\pi/2$, then since $w' < 0$ in this quadrant,

$$w(r,\gamma_n) < \gamma_n .$$

We have $\theta(r_0 - \delta, \gamma_n) \to \theta(r_0 - \delta) < -\pi/2$, and hence $w(r_0 - \delta, \gamma_n) < \gamma_n$, for large n. But $0 = \lim \gamma_n \ge \lim w(r_0 - \delta, \gamma_n) = w(r_0 - \delta) = \varepsilon > 0$. This contradiction completes the proof of Lemma 3.8. □

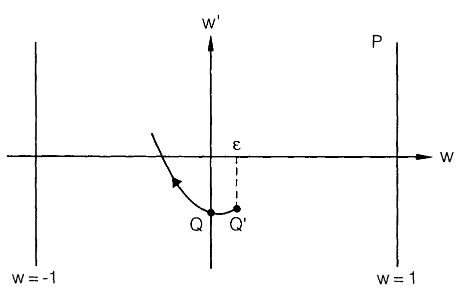

Fig. 3

We can now complete the proof of Proposition 3.7. With $\bar{\gamma}$ defined as in the statement of the last lemma, we choose $\gamma_n \to \bar{\gamma}$, where $0 < \gamma_n < 1$, the γ_n-orbits lie in Γ, and $\Omega(\gamma_n) \le k$. From Theorem 2.3, the $\bar{\gamma}$-orbit lies in Γ and $\Omega(\bar{\gamma}) \le k$. The $\bar{\gamma}$-orbit cannot exit Γ via $w^2 = 1$ for otherwise the same would be true for nearby γ-orbits, with $\gamma < \bar{\gamma}$, thus violating the definition of $\bar{\gamma}$. Thus the $\bar{\gamma}$-orbit stays in Γ for all $r > \bar{r}$, so from [8, Proposition 2.10], the $\bar{\gamma}$-orbit is a connecting orbit; i.e. $\Omega(\bar{\gamma})$ is an integer. If $\Omega(\bar{\gamma}) < k$, then from Corollary 2.4 for γ near $\bar{\gamma}$, $\gamma < \bar{\gamma}$, the γ-orbit exists Γ via $w^2 = 1$ with $\Omega(\gamma) \le \Omega(\bar{\gamma}) + 1 \le k$. This again violates the definition of $\bar{\gamma}$, and this contradiction completes the proof of Proposition 3.7. □

We turn now to the final case; namely $\bar{r} < 1$; cf. Fig. 4. Note that $\Phi(\bar{r}, \bar{w}, 0) = 0$ precisely when $\bar{r} = 1 - \gamma^2$. Define the constant α, $0 < \alpha < 1$

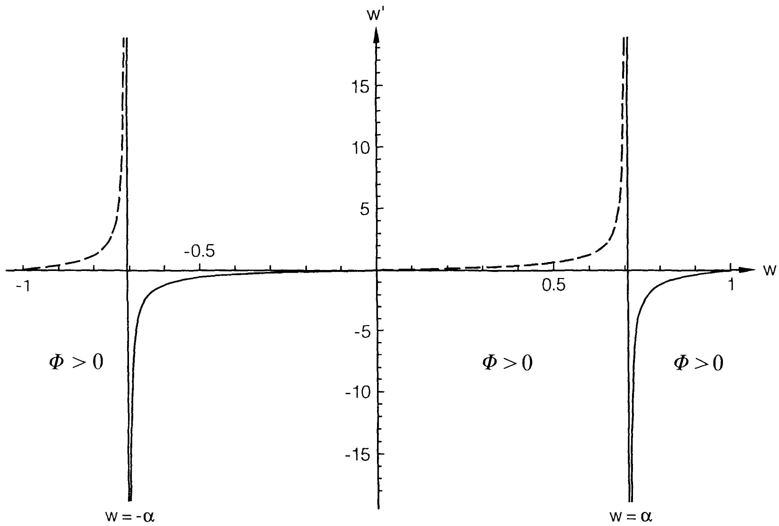

Fig. 4. $\mathscr{C}(\bar{r})$, $\bar{r} < 1$

by

$$\alpha = \sqrt{1 - \bar{r}}\,. \tag{3.16}$$

Notice that along the branch of $\mathscr{C}(\bar{r})$ between the lines $w = \alpha$ and $w = -\alpha$, $\Phi(\bar{r}, \bar{w}, 0)$ is negative, so if $A(\bar{r}) = 0$ and $(w(\bar{r}), w'(\bar{r}))$ lies on this branch, (2.2)$'$ implies that $A'(\bar{r}) < 0$. Thus from Proposition 2.2, it follows that $A(r) < 0$ for $r > \bar{r}$, r near $\bar{r}$. Thus there are no solutions to (2.2), (2.3) starting on this branch. Hence if $\bar{r} < 1$, we consider points $(w(\bar{r}), w'(\bar{r}))$, which lie on the set

$$\mathscr{C}(\bar{r}) \cap \{\alpha < w < 1\}\,.$$

Proposition 3.9. *Theorem 3.1 holds if* $\bar{r} < 1$.

Proof. Let $k \in \mathbb{Z}_+$ be given, and set

$$\gamma_k = \inf\{\gamma > \alpha\colon \Omega(\gamma) \le k \text{ and the } \gamma\text{-orbit doesn't crash}\}\,.$$

(Notice that the above set contains 1 and is thus non-void.) Now suppose that $\gamma_k > \alpha$. By Theorem 2.3, the γ_k-orbit doesn't crash and $\Omega(\gamma_k) \le k$. If the γ_k-orbit exits Γ via $w^2 = 1$, i.e., $\Omega(\gamma_k) < k$, then for γ near γ_k, $\gamma < \gamma_k$, the γ-orbit must also exit Γ via $w^2 = 1$ with $\Omega(\gamma) < k$; this is impossible as it violates the definition of γ_k. Thus the γ_k-orbit stays in Γ for all $r > \bar{r}$, and hence is a connecting orbit by [8, Proposition 2.10]. If $\Omega(\gamma_k)$ is an integer $< k$, then for $\gamma < \gamma_k$, γ near γ_k, $\Omega(\gamma) < \Omega(\gamma_k) + 1 \le k$, by Corollary 2.4. This again violates the definition of γ_k. Thus $\Omega(\gamma_k) = k$. It only remains to show that $\gamma_k > \alpha$, and this is the content of the following "compactness" lemma.

Lemma 3.10. *If* $\bar{r} < 1$, *then there is a* γ_1, $\alpha < \gamma_1 < 1$, *such that if* $\alpha < \gamma < \gamma_1$, *then the* γ*-orbit crashes.*

Proof. The proof of Lemma 3.10 relies on the following result from [7, Lemma 5.13]:

Lemma 3.11. *If $\Phi(r,w,A) \le -\eta < 0$ for $0 < a \le w \le b$, $r_b < 1$, and $w'(r_b)$ is sufficiently negative, then the orbit $(w(r), w'(r), A(r), r)$ crashes.*

Thus to prove Lemma 3.10, we need only verify the conditions of Lemma 3.11. We choose constants as follows:

(i) Let $\sigma > 0$ be a lower bound for $w(1-w^2)$ on $\frac{\alpha}{2} \le w \le \frac{3\alpha}{4}$; i.e.

$$w(1-w^2) \ge \sigma\,, \quad \text{if } \frac{\alpha}{2} \le w \le \frac{3\alpha}{4}\,. \tag{3.17}$$

(ii) Let L be an upper bound for Aw'^2 (cf. [7, Proposition 5.1], i.e., if $\alpha < \gamma < 1$,

$$(Aw'^2)(r,\gamma) \le L \quad \text{if } \bar{r} \le r \le r_e(\gamma)\,. \tag{3.18}$$

(iii) Choose $N > 1$ such that the following hold:

$$\frac{1}{\bar{r}^2} + L - \frac{2\sigma}{\bar{r}} N < -1\,, \tag{3.19}$$

and

$$2\left(\frac{1}{\bar{r}^2} + L\right)\frac{1}{N} < \frac{\sigma\alpha}{\bar{r}+1}\,. \tag{3.20}$$

(iv) By (3.20) we can choose $\varepsilon > 0$ such that

$$\varepsilon + 2\left(\frac{1}{\bar{r}^2} + L\right)\frac{1}{N} < \frac{\sigma\alpha}{\bar{r}+1}\,. \tag{3.21}$$

Now again write $\Phi = \Phi(r,A,w)$, and $\xi(r,w) = \Phi(r,0,w) = r - (1-w^2)^2/r$. From (3.16), $\xi(\bar{r},\alpha) = 0$; thus if w is near α, $\xi(r,w)$ is small. It follows that we can find γ_1, $\alpha < \gamma_1 < 1$ such that if $\alpha < \gamma < \gamma_1$, the following three statements hold:

$$\Phi(\bar{r},\gamma) \le \varepsilon\,, \tag{3.22}$$

$$w'(\bar{r},\gamma) < -N\,, \tag{3.23}$$

and

$$w''(\bar{r},\gamma) < 0\,. \tag{3.24}$$

[To obtain (3.24), we have as in Proposition 2.1, (dropping the dependence on γ),

$$2w''(\bar{r}) = \lim_{r \searrow \bar{r}} \frac{(-\Phi' - 1 + 3w^2)w'}{\Phi}\,. \tag{3.25}$$

An easy calculation using (3.15) gives, for $r > \bar{r}$,

$$\begin{aligned}\Phi'(r) &= \frac{2(1-w^2)^2}{r^2} + 2Aw'^2 + \frac{4w(1-w^2)w'}{r} \\ &\le \frac{2}{\bar{r}^2} + 2L - \frac{4\sigma N}{\bar{r}} < -2\,,\end{aligned}$$

Using this in (3.25) yields $w''(\bar{r}) < 0$.]

We now show that w'' stays negative if $w > \frac{\alpha}{2}$; i.e., for $\alpha < \gamma < \gamma_1$,

$$w''(r,\gamma) < 0 \quad \text{if } w(r,\gamma) \geq \frac{\alpha}{2}, \tag{3.26}$$

and thus for $\alpha < \gamma < \gamma_1$,

$$w'(r,\gamma) < -N \quad \text{if } w(r,\gamma) \geq \frac{\alpha}{2}. \tag{3.27}$$

To see that (3.26) holds, we assume that there is a (first) $\tilde{r}$, $\bar{r} < \tilde{r} < r_{\alpha/2}(\gamma)$ such that $w''(\tilde{r},\gamma) = 0$. Then from (2.3)$'$, we have at $\tilde{r}$, (suppressing the γ),

$$\tilde{r}^2 A w''' = -\Phi' w' - (1-3w^2)w' = (-\Phi + 3w^2 - 1)w' < 0$$

because $w'(\tilde{r},\gamma) < -N$. Thus no such $\tilde{r}$ can exist and so (3.26) holds, as does (3.27).

Now if $b = \frac{3\alpha}{4}$, and $a = \frac{\alpha}{2}$, then $r_b - \bar{r} = \frac{\gamma - 3\alpha/4}{-w'(\xi)} < \frac{1}{N}$, where ξ is an intermediate point. Thus $r_b < 1$ if N is sufficiently large. Also, if γ is sufficiently close to α, then $\beta(\gamma) \to -\infty$; i.e., $w'(\bar{r},\gamma) \to -\infty$ as $\gamma \to \alpha$, and by (3.26), since $w'(r_{3\alpha/4},\gamma) < w'(\bar{r},\gamma)$, we see that $w'(r_b(\gamma),\gamma) \to -\infty$ as $\gamma \to \alpha$.

We next claim that we can find an $\eta > 0$ such that $\alpha < \gamma < \gamma_1$ implies that

$$\Phi(r,\gamma) \leq -\eta, \quad \text{if } \frac{\alpha}{2} < w(r,\gamma) < \frac{3\alpha}{4}. \tag{3.28}$$

To see this, we first note that for $\alpha < \gamma < \gamma_1$, if $\frac{\alpha}{2} \leq w(r,\gamma) \leq \gamma_1$ then as above, $r - \bar{r} = \frac{w(r,\gamma) - \gamma}{w'(\xi,\gamma)} < \frac{1}{N} < 1$, where ξ is an intermediate point. Thus

$$r < 1 + \bar{r}, \quad \text{if } \frac{\alpha}{2} \leq w(r,\gamma) \leq \gamma_1.$$

Moreover, if $\frac{\alpha}{2} \leq w(r,\gamma) \leq \gamma_1$, then, (dropping the γ),

$$\begin{aligned}\Phi'(r) &= \frac{2(1-w(r)^2)^2}{r^2} + 2Aw'^2 + \frac{4w(1-w^2)w'}{r} \\ &\leq \frac{2}{\bar{r}^2} + 2L - \frac{4N\sigma}{\bar{r}+1} < 0,\end{aligned}$$

from (3.20). Thus, if we show that $\Phi(r,\gamma)$ is bounded away from zero (uniformly in γ), at $r = r_{3\alpha/4}(\gamma)$, [cf. (2.14)] then $\Phi(r,\gamma)$ will be uniformly negative if $\frac{\alpha}{2} \leq w(r,\gamma) \leq \frac{3\alpha}{4}$. Thus all the hypotheses of Lemma 3.10 will be verified, and as we have noted above, this will complete the proof of the lemma.

We have, for $\alpha < \gamma < \gamma_1$, (dropping the γ's),

$$\begin{aligned}\Phi(r_{3\alpha/4}) &= \Phi(\bar r) + \int_{\bar r}^{r_{3\alpha/4}} \Phi'(r)dr \\ &\le \varepsilon + \int_{\bar t}^{r_{3\alpha/4}} \left[\frac{2(1-w^2)^2}{r^2} + 2Aw'^2 + \frac{4(1-w^2)ww'}{r}\right]dr \\ &\le \varepsilon + \int_{\bar r}^{r_{3\alpha/4}} \left(\frac{2}{\bar r^2} + 2L\right)dr + 4\int_{\gamma}^{3\alpha/4} \frac{w(1-w^2)}{\bar r + 1}\,dw \\ &\le \varepsilon + (r_{3\alpha/4} - \bar r)\left(\frac{2}{\bar r^2} + 2L\right) - \frac{4\sigma}{\bar r + 1}\left(\gamma - \frac{3}{4}\alpha\right),\end{aligned}$$

where we have used (3.22) and (3.29). But as $(r_{3\alpha/4} - \bar r) \le \dfrac{\gamma - \frac{3}{4}\alpha}{w'(\xi)} \le \dfrac{1}{N}$, and $\left(\gamma - \dfrac{3}{4}\alpha\right) \ge \alpha - \dfrac{3}{4}\alpha = \dfrac{\alpha}{4}$, we have, from (3.21),

$$\Phi(r_{3\alpha/4}) \le \varepsilon + 2\left(\frac{1}{\bar r^2} + L\right)\frac{1}{N} - \frac{\alpha\sigma}{\bar r + 1} \equiv \eta < 0;$$

that is, $\Phi(r_{2\alpha/4}(\gamma))$ is uniformly negative if $\alpha < \gamma < \gamma_1$. This completes the proof of Lemma 3.10. □

As we have remarked above, this completes the proof of Proposition 3.9, and hence of Theorem 3.1. □

4. Far Field Behavior of the Metric

We now examine the metric in the "far field"; i.e., when $r \gg 1$. For this, we first note that for each $\bar r > 0$, if $(w(r,\gamma_n), w'(r,\gamma_n), A(r,\gamma_n), r) \equiv (w_n(r), w_n'(r), A_n(r), r)$ is a connecting orbit solution, then if $\mu(r,\gamma_n) = r(1 - A(r,\gamma_n))$, then as in [7, 8] we can show that the following limit exists and is finite:

$$\lim_{r\to\infty} \mu(r,\gamma_n) = \bar\mu_n\,; \tag{4.1}$$

that is, as noted earlier, $\mu(r) = 2m(r)$, so that each solution has finite ADM mass.

We shall now show that for each n, the Einstein metric

$$ds_n^2 = -T_n(r)^{-2}dt^2 + A_n^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\phi\theta^2)\,, \tag{4.2}$$

tends, as $r \to \infty$ to the corresponding Schwarzschild metric

$$dS_n^2 = -\left(1 - \frac{\bar\mu_n}{r}\right)dt^2 + \left(1 - \frac{\bar\mu_n}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.3}$$

at a rate $O\left(\dfrac{1}{r^2}\right)$.

For this, we fix n and γ_n and consider the behavior of the γ_n-orbit. Since n is fixed, we shall suppress it.

We begin with the following lemma.

Lemma 4.1. (i) $\lim_{r\to\infty} rw'(r) = 0$,

(ii) *If* $\bar{\mu} = \lim_{r\to\infty} \mu(r)$, *then* $|\mu(r) - \bar{\mu}| = O\left(\frac{1}{r}\right)$, $r \to \infty$.

Proof. For (i), we have, from l'Hospital's rule and (2.3)′,

$$\begin{aligned}\lim_{r\to\infty} rw'(r) &= \lim_{r\to\infty} \frac{w'(r)}{1/r} = \lim_{r\to\infty} \frac{w''(r)}{-1/r^2} = \lim_{r\to\infty}(-r^2 w''(r))\\ &= \lim_{r\to\infty} [\Phi(r)w'(r) + w(r)(1 - w^2(r))]A(r)^{-1}\\ &= 0\,,\end{aligned}$$

because $\Phi(r) \to \bar{\mu}$, $w'(r) \to 0$, $w^2(r) \to 1$ as $r \to \infty$, and $A(r) \to 1$ since $\mu(r)$ is bounded.

To prove (ii), we have

$$\bar{\mu} - \mu(r) = \int_r^\infty \mu'(s)ds = \int_r^\infty \left(2Aw'^2 + \frac{(1-w^2)^2}{s^2}\right)ds = \int_r^\infty O\left(\frac{1}{s^2}\right)ds = O\left(\frac{1}{r}\right),$$

where we have used $rw'(r) \to 0$ as $r \to \infty$. □

Theorem 4.2. *Fix* $\bar{r} > 0$; *then for any* n,

$$\left|A_n(r) - \left(1 - \frac{\bar{\mu}_n}{r}\right)\right| = O\left(\frac{1}{r^2}\right) \quad \textit{as } r \to \infty\,, \tag{4.4}$$

and

$$\left|T_n^{-2}(r) - \left(1 - \frac{\bar{\mu}}{r}\right)\right| = O\left(\frac{1}{r^2}\right) \quad \textit{as } r \to \infty\,, \tag{4.5}$$

Proof. We begin with (4.4); thus,

$$\left|A_n(r) - \left(1 - \frac{\bar{\mu}_n}{r}\right)\right| = \left|\frac{r(A_n(r) - 1) + \bar{\mu}_n}{r}\right| = \frac{\bar{\mu}_n - \mu_n(r)}{r} = O\left(\frac{1}{r^2}\right)$$

as $r \to \infty$, in view of part (ii) of our lemma.

To prove (4.5), we define functions P and Q by

$$P'(r) = \frac{\Phi(r, w(r), A(r))}{r^2 A(r)}\,, \qquad P(\bar{r} + 1) = 0\,,$$

$$Q'(r) = \frac{2w'^2(r)}{r}\,, \qquad Q(\bar{r} + 1) = 0\,,$$

and then notice that we have, from (2.2)′, and (2.4),

$$A'/A = P' - Q'\,,$$

and

$$T'/T = -\frac{Q'}{2} - \frac{P'}{2}\,.$$

Thus we may write $(\ln A)' = P' - Q'$, $(\ln T^2)' = -P' - Q'$, so that $(\ln AT^2)' = -2Q'$. Thus $\ln AT^2 = -2Q + k$, or

$$AT^2 = e^k e^{-2Q}\,, \tag{4.6}$$

where k is a constant. From Lemma 4.1, part (i), it is easy to see that

$$\lim_{r\to\infty} Q(r) = C$$

exists and is finite. Choosing $k = 2C^2$, we see that (4.5) follows from (4.4) and (4.6). This completes the proof of Theorem 4.2. □

5. Appendix

In this section we shall prove the following local existence theorem.

Theorem 5.1. *Let $\bar{r} > 0$ be given. Assume $A(\bar{r}) = 0$, and that $(\bar{w}, \beta)$ satisfies*

$$\Phi(\bar{r})\beta + \bar{w}(1 - \bar{w}^2) = 0\,, \tag{5.1}$$

where

$$\Phi(\bar{r}) = \bar{r} - \frac{(1-\bar{w}^2)^2}{\bar{r}} \neq 0\,. \tag{5.2}$$

Then there exists a unique $C^{2+\alpha}$ solution $(w(r,\bar{w}), w'(r,\bar{w}), A(r,\bar{w}))$ of (2.2), (2.3) *defined on some interval $\bar{r} < r < \bar{r} + s(\bar{w})$, satisfying the initial conditions*

$$(A(\bar{r},\bar{w}), w(\bar{r},\bar{w}), w'(\bar{r},\bar{w})) = (0, \bar{w}, \beta)\,.$$

The solution is analytic on $|r - \bar{r}| < s(\bar{w})$, and the one-parameter family $(A(r,\bar{w}), w(r,\bar{w}), w'(r,\bar{w}))$ is continuous in the sense of Definition 2.1.

Remark. Note that we do not require $\Phi(\bar{r}) > 0$, nor do we rule out the case $(\bar{w}, \bar{w}) = (1, 0)$.

The proof is by iteration.

Proof. For notational convenience, we set

$$c = \Phi(\bar{r})/\bar{r}^2\,,$$
$$d = -\frac{\beta}{2}\left[\frac{2(1-\bar{w}^2)^2/\bar{r}^2 + 4\bar{w}\beta(1-\bar{w}^2)/\bar{r} - (1-3\bar{w}^2)}{\bar{r} - (1-\bar{w}^2)^2/\bar{r}}\right],$$

[cf. (2.9)].

We first rewrite our differential equations (2.2), (2.3)′ as a first-order system:

$$w' = z\,,$$
$$z' = \frac{-\Phi z - w(1-w^2)}{r^2 A}\,,$$
$$A' = \frac{1 - \dfrac{(1-w^2)^2}{r^2}}{r} - \frac{(1+2z^2)A}{r}\,.$$

Now for a given $\varepsilon > 0$, we consider the following Hölder spaces with slight variances on the usual norms:

$$C^{\alpha}(\bar{r},\bar{r}+\varepsilon)\colon \|f\|_{\alpha} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} + |f(\bar{r})|\,,$$
$$C^{1+\alpha}(\bar{r},\bar{r}+\varepsilon)\colon \|f\|_{1+\alpha} = \|f'\|_{\alpha} + |f(\bar{r})|\,,$$
$$C^{2+\alpha}(\bar{r},\bar{r}+\varepsilon)\colon \|f\|_{2+\alpha} = \frac{1}{2}\|f'\|_{1+\alpha} + |f(\bar{r})|\,.$$

Next, we define the sets D_i, $i = 1,2,3$ by

$$D_1 = D_1(\varepsilon) = \{w \in C^{2+\alpha}(\bar{r},\bar{r}+\varepsilon)\colon w(\bar{r}) = \bar{w},\ w'(\bar{r}) = \beta,\ w''(\bar{r}) = d\}\,, \quad (5.6)$$
$$D_2 = D_2(\varepsilon) = \{z \in C^{1+\alpha}(\bar{r},\bar{r}+\varepsilon)\colon z(\bar{r}) = \beta, z'(\bar{r}) = d\}\,. \quad (5.7)$$

Note that for any solution of (2.2), (2.3), $A''(\bar{r})$ is determined by differentiating (2.2); $A''(\bar{r}) = e(\bar{w},\beta) \equiv e$. We use this fact in defining D_3:

$$D_3 = D_3(\varepsilon) = \{A \in C^{2+\alpha}(\bar{r},\bar{r}+\varepsilon)\colon A(\bar{r}) = 0,\ A'(\bar{r}) = c,\ A''(\bar{r}) = e\}\,. \quad (5.8)$$

Since D_1 is a closed subset of $C^{2+\alpha}(\bar{r},\bar{r}+\varepsilon)$, D_1 is a complete metric space. Similarly, D_2 and D_3 are complete metric spaces. Let X be defined by

$$X = D_1 \times D_2 \times D_3\,,$$

where we denote points in X by $\theta = (w,z,A)$. We put the following metric on X:

$$d(\theta_1,\theta_2) = \max\{\|w_1 - w_2\|_{2+\alpha},\ \|z_1 - z_2\|_{1+\alpha},\ \|A_1 - A_2\|_{2+\alpha}\}\,.$$

Remark. If w_1 and $w_2 \in D_1$, then

$$\|w_2 - w_1\|_{2+\alpha} = \frac{1}{2}\sup_{x\neq y} \frac{|w_2''(x) - w_1''(y)|}{|x-y|^{\alpha}}\,,$$

because $w_1(\bar{r}) = w_2(\bar{r}) = \bar{w}$, and $w_1'(\bar{r}) = w_2'(\bar{r}) = \beta$. Similar remarks apply to z and A.

We define the mapping T on X by

$$T(w,z,A) = (\tilde{w},\tilde{z},\tilde{A})\,, \quad (5.9)$$

where

$$\tilde{w} = T_1(w,z,A) \equiv \bar{w} + \int_{\bar{r}}^{r} z(s)ds\,,$$
$$\tilde{z} = T_2(w,z,A) = \beta - \int_{\bar{r}}^{r} \frac{uw + \Phi z}{s^2 A} ds\,, \quad (5.10)$$
$$\tilde{A} = T_3(w,z,A) = \int_{\bar{r}}^{r} \frac{\left(1 - \frac{u^2}{s^2}\right) - (1+2z^2)A}{s} ds\,.$$

Here we are using the notation

$$u = (1 - w^2)\,.$$

Note that the integral in T_2 is not improper since $(uw + \phi z)(\bar{r}) = 0$, and $A'(\bar{r}) \neq 0$. We leave it to the reader to verify that $\tilde{w} \in D_1$, $\tilde{z} \in D_2$, and $\tilde{A} \in D_3$.

For any real number $\sigma > 0$, we must show that there exists an $\varepsilon > 0$ such that both of the following hold:

$$T(B_\sigma) \subset B_\sigma ,$$

and

$$T \text{ is a contraction}.$$

(Here B_σ is the ball of radius σ about the point

$$(w_0(r), z_0(r), A_0(r)) = \left(\bar{w} + \beta(r-\bar{r}) + \tfrac{1}{2}\, d(r-\bar{r})^2, \beta + d(r-\bar{r}), c(r-\bar{r}) + \tfrac{1}{2}\, e(r-\bar{r})^2\right).$$

Since it is straightforward to show $T(B_\sigma) \subset B_\sigma$ if ε is sufficiently small, we shall omit the details. We now show that T is a contraction; this will imply the existence of a local solution in X. Because we have used the maximum in Definition (5.8), it suffices to show that each T_i is a norm-decreasing, (with a uniform constant). We begin with T_1. Set $\theta_i = (w_i, z_i, A_i)$, $i = 1, 2$; then

$$T_1(\theta_1) - T_1(\theta_2) = \int_{\bar{r}}^{r} (z_1 - z_2)ds ,$$

so

$$\|T_1(\theta_1) - T_1(\theta_2)\|_{2+\alpha} = \tfrac{1}{2}\, \|z_1 - z_2\|_{1+\alpha} \leq \tfrac{1}{2}\, \|\theta_1 - \theta_2\| . \tag{5.11}$$

In order to show that T_2 and T_3 are norm-decreasing, we shall need the two following lemmas.

Lemma 5.2. (a) *If $v \in C^{1+\alpha}(\bar{r}, \bar{r} + \varepsilon)$, and $v(\bar{r}) = 0$, then $\|v/(r - \bar{r})\|_\alpha \leq \dfrac{1}{(1+\alpha)} \|v\|_{1+\alpha}$.*
(b) *If $v \in C^{1+\alpha}(\bar{r}, \bar{r} + \varepsilon)$, and $v(\bar{r}) = v'(\bar{r}) = 0$, then $\|v\|_\alpha \leq \varepsilon \|v\|_{1+\alpha}$.*
(c) *If $v \in C^{\alpha}(\bar{r}, \bar{r} + \varepsilon)$, and $v(\bar{r}) = 0$, then $\|v\|_\infty \leq \varepsilon^\alpha \|v\|_\alpha$.*
(d) *If $v, w \in C^{\alpha}(\bar{r}, \bar{r} + \varepsilon)$, then $\|vw\|_\alpha \leq \|v\|_\alpha \|w\|_\infty + \|v\|_\infty \|w\|_\alpha$.*
(e) *If $v \in C^{2+\alpha}(\bar{r}, \bar{r} + \varepsilon)$, and $v(\bar{r}) = v'(\bar{r}) = v''(\bar{r}) = 0$, then*

$$\|v\|_{1+\alpha} \leq 2\varepsilon \|v\|_{2+\alpha} .$$

Proof. Parts a) through d) were proved in [7, Lemma 7.1]. To prove e), we have, from b)

$$\|v\|_\alpha \leq \frac{\varepsilon}{1+\alpha} \|v\|_{1+\alpha} ,$$

and since $v(0) = 0$, $\|v\|_{1+\alpha} = \|v'\|_\alpha$, and $\|v\|_{2+\alpha} = \frac{1}{2}\, \|v'\|_{1+\alpha}$. But $v'(0) = 0 = v''(0)$ implies that

$$\|v'\|_\alpha \leq \varepsilon \|v'\|_{1+\alpha} , \quad \text{[by (b)]} ,$$

and thus

$$\|v\|_{1+\alpha} = \|v'\|_\alpha \leq \varepsilon \|v'\|_{\alpha+1} = 2\varepsilon \|v\|_{2+\alpha} . \quad \square$$

Next, we have, for $A \in D_3$, $A(\bar{r}) = 0$, $A'(\bar{r}) = c \neq 0$, and $A''(\bar{r}) = e$. Thus if we set

$$\varrho = r - \bar{r} ,$$

we may write

$$A(r) = c\varrho B(r), \quad \text{where } B(\bar{r}) = 1.$$

Note that $B \in C^{1+\alpha}(\bar{r}, \bar{r}+\varepsilon)$, and since $B(\bar{r}) = 1$, it follows that $1/B$ lies in $C^{1+\alpha}(\bar{r}, \bar{r}+\varepsilon)$, and $B'(\bar{r}) = e/2c$. (This last fact follows from differentiating the equation $A(r) = c\varrho B(r)$:

$$\frac{A'(r) - c}{\varrho} = \frac{cB(r) - c}{\varrho} + cB'(r),$$

and letting $\varrho \to 0$.)

We shall need one more lemma. In what follows ϱ will be defined as above.

Lemma 5.3. *Suppose that $f_i \in C^{2+\alpha}(\bar{r}, \bar{r}+\varepsilon)$, $i = 1, 2$ and that $f_1^{(i)}(\bar{r}) = f_2^{(i)}(\bar{r})$, $i = 0, 1, 2$. Then*

$$\left\| \frac{f_1 - f_2}{\varrho} \right\| \to 0, \quad \text{and} \quad \left\| \frac{f_1 - f_2}{\varrho} \right\|_\infty \to 0$$

as $\varepsilon \to 0$. If in addition $f_1(\bar{r}) = f_2(\bar{r}) = 0$, and $g_i \in C^{1+\alpha}(\bar{r}, \bar{r}+\varepsilon)$, where $g_1^{(i)}(\bar{r}) = g_2^{(i)}(\bar{r})$, $i = 1, 2$, then

$$\left\| \frac{g_1 f_1}{\varrho} - \frac{g_2 f_2}{\varrho} \right\|_\alpha \to 0$$

as $\varepsilon \to 0$.

Proof. Let $\Delta f = f_1 - f_2$; then $\Delta f/\varrho \in C^{1+\alpha}$, $(\Delta f/\varrho)(\bar{r}) = 0$, $(\Delta f/\varrho)'(\bar{r}) = 0$, so that from Lemma 5.2b, c,

$$\|\Delta f/\varrho\|_\alpha \to 0 \quad \text{and} \quad \|\Delta f/\varrho\|_\infty \to 0,$$

as $\varepsilon \to 0$. For the second part, we have

$$\frac{g_1 f_1}{\varrho} - \frac{g_2 f_2}{\varrho} = g_1\left(\frac{f_1 - f_2}{\varrho}\right) + f_2\left(\frac{g_1 - g_2}{\varrho}\right).$$

Thus

$$\left\| \frac{g_1 f_1}{\varrho} - \frac{g_2 f_2}{\varrho} \right\|_\alpha \le \left\| g_1\left(\frac{f_1 - f_2}{\varrho}\right) \right\|_\alpha + \left\| f_2\left(\frac{g_1 - g_2}{\varrho}\right) \right\|_\alpha.$$

Now from Lemma 5.2d,

$$\left\| g_1\left(\frac{f_1 - f_2}{\varrho}\right) \right\|_\alpha \le \|g_1\|_\infty \left\| \frac{f_1 - f_2}{\varrho} \right\|_\alpha + \|g_1\|_\alpha \left\| \frac{f_1 - f_2}{\varrho} \right\|_\infty.$$

But as $\|g_1\|_\infty$ and $\|g_1\|_\alpha$ are finite, we see that

$$\left\| g_1\left(\frac{f_1 - f_2}{\varrho}\right) \right\|_\alpha \to 0 \quad \text{as } \varepsilon \to 0,$$

in view of the part already proved. Next,

$$\left\| f_2\left(\frac{g_1 - g_2}{\varrho}\right) \right\|_\alpha \le \|f_2\|_\infty \left\| \frac{g_1 - g_2}{\varrho} \right\|_\alpha + \|f_2\|_\alpha \left\| \frac{g_1 - g_2}{\varrho} \right\|_\infty.$$

Now $f_2 \in C^{2+\alpha}$, $f_2(\bar{r}) = 0$ so Lemma 5.2c implies $\|f_2\|_\infty \to 0$ as $\varepsilon \to 0$. Since $(g_1 - g_2)/\varrho$ is in C^α, we see that $\|f_2\|_\infty \left\| \frac{g_1 - g_2}{\varrho} \right\|_\alpha \to 0$ as $\varrho \to 0$. Also, Lemma 5.2c implies that $\left\| \frac{g_1 - g_2}{\varrho} \right\|_\infty \to 0$, as $\varepsilon \to 0$, and as $\|f_2\|_\infty$ is finite, we conclude that $\left\| f_2 \left(\frac{g_1 - g_2}{\varrho} \right) \right\|_\alpha \to 0$ as $\varepsilon \to 0$. □

Now we consider T_3; we have

$$T_3(\theta_1) - T_3(\theta_2) = \int_{\bar{r}}^{r} \left[\left(\frac{1 - \frac{u_1^2}{s^2}}{s} - \frac{(1 + 2z_1^2)A_1}{s} \right) - \left(\frac{1 - \frac{u_2^2}{s^2}}{s} - \frac{(1 + 2z_2^2)A_2}{s} \right) \right] ds ,$$

so

$$\|T_3(\theta_1) - T_3(\theta_2)\|_{2+\alpha} = \frac{1}{2} \left\| \frac{u_2^2 - u_1^2}{r^3} + \frac{(A_2 - A_1)}{r} + \frac{2}{r}(z_2^2 A_2 - z_1^2 A_1) \right\|_{1+\alpha} . \quad (5.12)$$

Thus as w_1 and $w_2 \in D_1$, it follows from Lemma 5.2e that $\|(u_1^2 - u_2^2)/r^3\|_{1+\alpha}$ can be made small if ε is small. Similarly, $\|(A_2 - A_1)/r\|_{1+\alpha}$ is small if ε is small. Finally, we have

$$\left\| \frac{2}{r}(z_2^2 A_2 - z_1^2 A_1) \right\|_{1+\alpha} \le \frac{2}{\bar{r}} [\|z_2^2(A_2 - A_1)\|_{1+\alpha} + \|(z_2^2 - z_1^2)A_1\|_{1+\alpha}] . \quad (5.13)$$

Now from Lemma 5.2d,

$$\|z_2^2(A_2 - A_1)\|_{1+\alpha} \le \|z_2^2\|_\infty \|A_2 - A_1\|_{1+\alpha} + \|z_2^2\|_{1+\alpha} \|A_2 - A_1\|_\infty . \quad (5.14)$$

Since z_2^2 is bounded and $\|A_2 - A_1\|_{1+\alpha} \to 0$ as $\varepsilon \to 0$, the first term on the rhs of (5.14) is small if ε is small. Since $\|z_2^2\|_{1+\alpha}$ is bounded, and $(A_2 - A_1) \in C^\alpha$, Lemma 5.2c shows that the second term on the rhs of (5.14) is small if ε is small. Similarly, we have

$$\|(z_2^2 - z_1^2)A_1\|_{1+\alpha} \le \|z_2^2 - z_1^2\|_{1+\alpha} \|A_1\|_\infty + \|z_2^2 - z_1^2\|_\infty \|A_1\|_{1+\alpha} , \quad (5.15)$$

and as $\|z_2^2 - z_1^2\|_\infty$ is bounded, and $\|A_1\|_{1+\alpha} \to 0$ as $\varepsilon \to 0$, the first term on the rhs of (5.14) tends to 0 as $\varepsilon \to 0$. Also, $\|A_1\|_{1+a}$ is bounded, and as $(z_2^2 - z_1^2) \in C^\alpha$, $\|z_2^2 - z_1^2\|_\infty \to 0$ as $\varepsilon \to 0$. Thus from (5.12) we see that T_3 is norm-decreasing if ε is small.

We now consider T_2. If $\theta = (w, z, A)$, we have

$$T_2(\theta) = \beta - \int_{\bar{r}}^{r} \frac{uw + \Phi z}{s^2 A} ds ,$$

so that

$$T_2(\theta_1) - T_2(\theta_2) = \int_{\bar{r}}^{r} \left[\frac{u_2 w_2}{s^2 A_2} - \frac{u_1 w_1}{s^2 A_1} + \frac{\phi_2 z_2}{s^2 A_2} - \frac{\phi_1 z_1}{s^2 A_1} \right] ds\,.$$

Now since (5.1) implies $\bar{\phi}\bar{z} + \bar{u}\bar{w} = 0$, we have

$$\begin{aligned} T_2(\theta_1) - T_2(\theta_2) &= \int_{\bar{r}}^{r} \left[\frac{u_2 w_2 - \bar{u}\bar{w}}{s^2 A_2} - \frac{u_1 w_1 - \bar{u}\bar{w}}{s^2 A_1} + \frac{\phi_2 z_2 - \bar{\phi}\bar{z}}{s^2 A_2} - \frac{\phi_1 z_1 - \bar{\phi}\bar{z}}{s^2 A_1} \right] ds \\ &\equiv \int_{\bar{r}}^{r} \left[\Delta\left(\frac{uw - \bar{u}\bar{w}}{s^2 A} \right) + \Delta\left(\frac{\phi z - \bar{\phi}\bar{z}}{s^2 A} \right) \right] ds\,. \end{aligned}$$

Thus

$$\begin{aligned} \|T_2(\theta_1) - T_2(\theta_2)\|_{1+\alpha} &= \left\| \Delta\left(\frac{uw - \bar{u}\bar{w}}{r^2 A} \right) + \Delta\left(\frac{\phi z - \bar{\phi}\bar{z}}{r^2 A} \right) \right\|_\alpha \\ &\leq \left\| \Delta\left(\frac{uw - \bar{u}\bar{w}}{r^2 A} \right) \right\|_\alpha + \left\| \Delta\left(\frac{\phi z - \bar{\phi}\bar{z}}{r^2 A} \right) \right\|_\alpha . \end{aligned} \tag{5.16}$$

Writing

$$\frac{uw - \bar{u}\bar{w}}{r_2 A} = \frac{1}{cBr^2} \left(\frac{uw - \bar{u}\bar{w}}{\varrho} \right),$$

we may apply Lemma 5.3 to conclude that

$$\left\| \Delta\left(\frac{uw - \bar{u}\bar{w}}{r^2 A} \right) \right\|_\alpha \to 0 \quad \text{as } \varepsilon \to 0\,. \tag{5.17}$$

Now write

$$\frac{\phi z - \bar{\phi}\bar{z}}{r^2 A} = \frac{\phi z - \bar{\phi} z}{r^2 A} + \frac{\bar{\phi} z - \bar{\phi}\bar{z}}{r^2 A}\,, \tag{5.18}$$

and as

$$\frac{\phi z - \bar{\phi} z}{r^2 A} = \frac{z}{r^2 cB} \left(\frac{\phi - \bar{\phi}}{\varrho} \right),$$

we may apply Lemma 5.3 to conclude that

$$\left\| \Delta\left(\frac{\phi z - \bar{\phi} z}{r^2 A} \right) \right\|_\alpha \to 0 \quad \text{as } \varepsilon \to 0\,. \tag{5.19}$$

Next, we have

$$\begin{aligned} \left\| \Delta\left(\frac{\bar{\phi} z - \bar{\phi}\bar{z}}{r^2 A} \right) \right\|_\alpha &= \left\| \Delta\left(\frac{\bar{\phi} z - d\varrho\bar{\phi} - \bar{\phi}\bar{z}}{r^2 A} \right) \right\|_\alpha \\ &= \left\| \Delta\left(\frac{\bar{\phi} z - d\varrho\bar{\phi} - \bar{\phi}\bar{z}}{cr^2 \varrho B} \right) \right\|_\alpha . \end{aligned} \tag{5.20}$$

Now define n by

$$n(r) = \frac{\bar{\phi} z - d\varrho\bar{\phi} - \bar{\phi}\bar{z}}{cr^2 B}\,;$$

then from (5.20),

$$\left\|\Delta\left(\frac{\bar{\phi}z-\bar{\phi}\bar{z}}{r^2A}\right)\right\|_\alpha=\left\|\frac{n_2}{\varrho B_2}-\frac{n_1}{\varrho B_1}\right\|_\alpha\leq\left\|\frac{n_2}{\varrho B_2}-\frac{n_2}{\varrho B_1}\right\|_\alpha+\left\|\frac{n_2}{\varrho B_1}-\frac{n_1}{\varrho B_1}\right\|_\alpha. \quad (5.21)$$

But

$$\frac{n_2}{\varrho B_2}-\frac{n_2}{\varrho B_1}=\frac{n_2}{B_1B_2}\left(\frac{B_1-B_2}{\varrho}\right),$$

so that from Lemma 5.2d,

$$\left\|\frac{n_2}{\varrho B_2}-\frac{n_2}{\varrho B_1}\right\|_\alpha\leq\left\|\frac{n_2}{B_1B_2}\right\|_\infty\left\|\frac{B_1-B_2}{\varrho}\right\|_\alpha+\left\|\frac{n_2}{B_1B_2}\right\|_\alpha\left\|\frac{B_1-B_2}{\varrho}\right\|_\infty.$$

Now as n_2/B_1B_2 is in $C^{1+\alpha}$ and

$$n=\frac{\bar{\Phi}}{cr^2}(z-d\varrho-\bar{z}),$$

we see that $n_2(\bar{r})=n_2'(\bar{r})=0$. Thus since $\|(B_1-B_2)/\varrho\|_\alpha$ is bounded, we may apply Lemma 5.2c to conclude that

$$\left\|\frac{n_2}{B_1B_2}\right\|_\infty\left\|\frac{B_1-B_2}{\varrho}\right\|_\alpha\to 0 \quad \text{as } \varepsilon\to 0. \quad (5.23)$$

We also have from Lemma 5.2b that $\left\|\frac{n_2}{B_1B_2}\right\|_\alpha\to 0$ as $\varepsilon\to 0$, and since $\|(B_1-B_2)/\varrho\|_\infty\leq\varepsilon^\alpha\|(B_2-B_2)/\varrho\|_\alpha\leq\frac{\varepsilon^\alpha}{1+\alpha}(B_1-B_2)\|_{1+\alpha}$, (by Lemma 5.2c, a), we have

$$\left\|\frac{n_2}{B_1B_2}\right\|_\alpha\left\|\frac{B_1-B_2}{\varrho}\right\|_\infty\to 0 \quad \text{as } \varepsilon\to 0.$$

Thus

$$\left\|\frac{n_2}{\varrho B_2}-\frac{n_2}{\varrho B_1}\right\|_\alpha\to 0 \quad \text{as } \varepsilon\to 0. \quad (5.24)$$

Finally, if $h(r)=\bar{\phi}/B_1r^2c$, then

$$\left\|\frac{n_2}{\varrho B_1}-\frac{n_1}{\varrho B_1}\right\|_\alpha=\left\|\left(\frac{\bar{\phi}}{B_1r^2c}\right)\left(\frac{z_2-z_1}{\varrho}\right)\right\|_\alpha=\left\|h\,\frac{z_2-z_1}{\varrho}\right\|_\alpha.$$

Now $h\in C^{1+\alpha}$, $h(\bar{r})=1$ (because $c=\bar{\phi}/\bar{r}^2$), so $(h-1)\in C^{1+\alpha}$, and thus $\|h-1\|_\infty\to 0$ as $\varepsilon\to 0$ (Lemma 5.2c). Hence if δ is chosen to satisfy $0<\delta<\alpha$, we can make

$$|h|_\infty\leq 1+\delta \quad \text{if } \varepsilon \text{ is small}.$$

Then from Lemma 5.2d,

$$\begin{aligned}\left\|\frac{n_2}{\varrho B_1}-\frac{n_1}{\varrho B_1}\right\|_\alpha&\leq\|h\|_\infty\left\|\frac{z_2-z_1}{\varrho}\right\|_\alpha+\|h\|_\alpha\left\|\frac{z_2-z_1}{\varrho}\right\|_\infty\\&\leq(1+\delta)\frac{1}{1+\alpha}\|z_2-z_1\|_{1+\alpha}+\|h\|_\alpha\left\|\frac{z_2-z_1}{\varrho}\right\|_\infty,\end{aligned}$$

where we have used Lemma 5.2a. Now $\|h\|_\alpha$ is bounded, and $\left\|\frac{z_2 - z_1}{\varrho}\right\|_\infty \to 0$, as $\varepsilon \to 0$, by Lemma 5.2c, a, (as in the proof above that $\|(B_1 - B_2)/\varrho\|_\infty \to 0$). Thus if $k = (1+\delta)/(1+\alpha)$, then $k < 1$ and

$$\left\|\frac{n_2}{\varrho B_1} - \frac{n_1}{\varrho B_1}\right\|_\alpha \leq k\|z_2 - z_1\|_{1+\alpha} + O(\varepsilon),$$

as $\varepsilon \to 0$. This, together with (5.24), (5.23), and (5.21) shows that

$$\left\|\Delta\left(\frac{\bar{\phi}z - \bar{\phi}\bar{z}}{r^2 A}\right)\right\|_\alpha \leq k\|z_2 - z_1\|_{1+\alpha} + O(\varepsilon).$$

Thus (5.17) together with (5.16) shows that T_2 is norm-decreasing.

It follows that T has a unique fixed point $(A(r,\bar{w}), w(r,\bar{w}), w'(r,\bar{w}))$ in X which is defined on some interval $\bar{r} \leq r \leq \bar{r} + s(\bar{w})$. Moreover, $s(\bar{w})$ depends continuously on $\bar{w}$, as follows from our proof (see [7, p. 141]). Thus the parametrized family $(A(r,\bar{w}), w(r,\bar{w}), w'(r,\bar{w}))$ is continuous in the sense of Definition 2.1.

Next, fix $\bar{w}$ as before, and consider the set of analytic functions on the disk $|r - \bar{r}| < s(\bar{w})$, which are continuous in the closure. (Note that if w, z and A are analytic, then $\tilde{w}$ and $\tilde{A}$ are obviously analytic. To see that $\tilde{z}$ is analytic note that the numerator is analytic and vanishes at $\bar{r}$, by (5.1). Since $A'(\bar{r}) = c \neq 0$, it follows that $\tilde{z}$ is analytic. Thus T preserves this subspace of analytic functions.) Endow this space with the L^∞-norm. Since for any two functions ϕ_1 and ϕ_2 in this space we have $\|\phi_1 - \phi_2\|_\infty \leq \|\phi_1 - \phi_2\|_\alpha s(\bar{w})^\alpha$, it follows that T is a contraction on this space. Thus our solution is analytic in the disk $|r - \bar{r}| < s(\bar{w})$. The proof of Theorem 5.1 is complete. □

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