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Generalized Drinfeld-Sokolov Reductions and KdV Type Hierarchies

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Abstract. Generalized Drinfeld-Sokolov (DS) hierarchies are constructed through local reductions of Hamiltonian flows generated by monodromy invariants on the dual of a loop algebra. Following earlier work of De Groot et al., reductions based upon graded regular elements of arbitrary Heisenberg subalgebras are considered. We show that, in the case of the nontwisted loop algebra $\ell(gl_n)$, graded regular elements exist only in those Heisenberg subalgebras which correspond either to the partitions of n into the sum of equal numbers $n = pr$ or to equal numbers plus one $n = pr + 1$. We prove that the reduction belonging to the grade 1 regular elements in the case $n = pr$ yields the $p \times p$ matrix version of the Gelfand-Dickey r-KdV hierarchy, generalizing the scalar case $p = 1$ considered by DS. The methods of DS are utilized throughout the analysis, but formulating the reduction entirely within the Hamiltonian framework provided by the classical r-matrix approach leads to some simplifications even for $p = 1$.

0. Introduction

The generalized KdV type hierarchies of Drinfeld and Sokolov (DS) are among the most important examples in the field of integrable evolution equations [1]. They also play an important rôle in current studies of two-dimensional gravity [2] and in conformal field theory [3]. The "second Gelfand-Dickey" Poisson bracket of these bihamiltonian systems is a reduction of the affine current algebra Lie-Poisson bracket, and it gives an extension of the Virasoro algebra by conformal tensors. Such extended conformal algebras are called $\mathscr{W}$-algebras and have received a lot of attention recently [4–6].

The motivation for the present work was to gain, from a purely Hamiltonian viewpoint, a better understanding of the reduction procedure used in [1] and the generalizations proposed in a recent series of papers [7–9] aimed at the construction of new integrable hierarchies and $\mathscr{W}$-algebras.

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In the DS construction [1] one starts by considering a first order matrix differential operator of the form

$$\mathscr{L} = \partial + \mu\,, \quad \text{with} \quad \mu = q + \Lambda\,, \tag{0.1}$$

where Λ is a matrix representing a grade 1 regular element of the principal Heisenberg subalgebra of a loop algebra, and q is a smooth mapping form S^1 into an appropriate subspace of the loop algebra represented by "lower triangular matrices." The crucial step of the construction is to transform μ into the Heisenberg subalgebra by a conjugation of $\mathscr{L}$. That this can be achieved by an algebraic, recursive procedure is due to the fact that the grades in q are lower than the grade of Λ and Λ is regular. The fact that Λ is regular also implies that the stabilizer of the transformed operator is given by the Heisenberg subalgebra. The compatible zero curvature equations are obtained from the positively graded generators of this Heisenberg subalgebra by transforming them into the stabilizer of $\mathscr{L}$ and applying a splitting procedure. The system exhibits a gauge invariance under a nilpotent, "strictly lower triangular," gauge group.

The authors of [7] realized (see also [10, 11]) that the DS construction can be applied in more general circumstances. They proposed to derive new hierarchies by replacing Λ in the above by *any* positive, graded regular element of *any* graded Heisenberg subalgebra, and correspondingly modifying the DS definition of the variable q and the gauge group. However, apart from some very simple examples, they did not investigate which systems can be obtained on the basis of this rather general proposal. We shall see here that the number of new hierarchies arising from this approach, which in [7] were termed type I hierarchies, is in fact rather limited, since graded regular elements do not exist in most Heisenberg subalgebras. For simplicity, we shall investigate here the case of the nontwisted loop algebra based on the general linear Lie algebra gl_n, in which case the inequivalent Heisenberg subalgebras are classified by the partitions on n [12]. By using the explicit description of the inequivalent Heisenberg subalgebras given in [13], we shall prove that graded regular elements exist only for the special partitions

$$n = pr \quad \text{and} \quad n = pr + 1\,. \tag{0.2}$$

After explaining this observation, we shall give a detailed analysis of the first series of "nice cases" under (0.2). The graded 1 regular elements of the corresponding Heisenberg subalgebra are the $n \times n$ matrices of the form $\Lambda_{r,p} = \Lambda_r \otimes D$, where D is a $p \times p$ diagonal matrix such that D^r has distinct, non-zero eigenvalues, and Λ_r is the usual $r \times r$ "Drinfeld-Sokolov matrix" containing 1's above the diagonal and the spectral parameter λ in the lower-left corner. The "constrained manifold" of the generalized DS reduction will be taken to be the space of $\mathscr{L}$'s of the form (0.1), where now $\Lambda = \Lambda_{r,p}$ and q is a mapping from S^1 into the block lower triangular subalgebra of gl_n, with $p \times p$ blocks. As in the $p = 1$ case considered by DS, the "reduced space" will be obtained by factorizing this constrained manifold by the group of nonabelian gauge transformations,

$$\mathscr{L} \to g\mathscr{L}g^{-1}\,, \tag{0.3}$$

where g is now block lower triangular, having $p \times p$ unit matrices in the diagonal blocks. We shall place the construction in a Hamiltonian setting from the very beginning, from which it will be clear that the reduced space is a bihamiltonian manifold that carries the commuting hierarchy of Hamiltonians provided by the monodromy invariants of $\mathscr{L}$. It will also be clear that the locality of the reduced system is guaranteed by construction.

In order to describe the reduced system (i.e., the generalized KdV hierarchy) in terms of gauge invariant variables, we shall prove the following facts, which extend the $p = 1$ results of [1]. First, the reduced space is the space of "matrix Lax operators" of the form

$$L = (-D)^{-r}\partial^r + u_1\partial^{r-1} + \ldots + u_{r-1}\partial + u_r\,, \tag{0.4}$$

where the u_i are now smooth mappings from S^1 into the space of $p \times p$ matrices. Second, the reduced bihamiltonian structure is that given by the two compatible (matrix) Gelfand-Dickey Poisson brackets; the second Poisson bracket algebra qualifies as a classical $\mathscr{W}$-algebra. Third, the hierarchy of commuting flows is generated by the Hamiltonians given by integrating the componentwise residues of the fractional (including integral) powers of the pseudo-differential operators obtained by diagonalizing the matrix Lax operators. In short, the DS reduction extends in this case to yield the $p \times p$ matrix version of the well-known (e.g. [14]) Gelfand-Dickey r-KdV hierarchy.

The main additional step required in computing the Hamiltonians in the $p \times p$ matrix case consists in the diagonalization of L. Those Hamiltonians which are obtained from the *integral* powers of the diagonalized Lax operators generate bihamiltonian ladders whose first elements are the Hamiltonians given by integrating the diagonal components of u_1, which are Casimirs of the first Poisson structure. The number of these bihamiltonian ladders (which are missing in the scalar case) is $p-1$ since the integral of $\mathrm{tr}(D^r u_1)$ is a Casimir of both Poisson structures. The other Hamiltonians can also be described as integrals of trace-residues of independent fractional powers of L, without diagonalization.

The KdV type hierarchies based on matrix Lax operators of the type (0.4) have been investigated before in [15–17] and more recently in [18]. In [15–17] the matrix $(-D)^{-r}$ (i.e., the coefficient of the leading term of L) was required to be regular because this implies the existence of the maximal number of independent r^{th} roots of L and corresponding commuting flows. It is interesting to see this condition re-emerge here from requiring that the matrix Λ used in the reduction procedure be a regular element of the Heisenberg subalgebra In these papers, the additional assumption was made that the diagonal part of u_1 vanishes. Setting $[u_1]_{\text{diag}} = 0$ is consistent with the equations of the hierarchy resulting from the DS reduction and in fact corresponds to an additional Hamiltonian symmetry reduction (see Remarks 2.5–2.7).

The recent preprint [18] deals with the hierarchy defined by the fractional powers of "covariant Lax operators," which are equivalent to operators of the form (0.4). More precisely, the case of the Lie algebra sl_n was considered, which in our approach corresponds to imposing the constraints $\mathrm{tr}(D^r u_1) = 0$. However, instead of taking a regular matrix for $(-D)^{-r}$, the unit matrix was used. As a result, the hierarchy obtained is much smaller than the one following from the DS reduction using a regular element.

To emphasize the link between the approach used in [1, 7] and that based upon the Adler-Kostant-Symes (AKS) construction as presented in the present work, we give the following elementary lemma.

Lemma 0.1. *Let $\mathscr{A}$ be a Lie algebra. Let $\mu_0 \in \mathscr{A}^*$ be given and $X \in \mathrm{cent}(\mathrm{stab}(\mu_0)) \subset \mathscr{A}$ be an element in the center of its stabilizer. Then there exists an element $\varphi \in I(\mathscr{A}^*)$ of the ring of ad^*-invariant functions on $\mathscr{A}^*$ such that $\nabla\varphi|_{\mu_0} = X$, where, as usual, an identification has been made between the cotangent space to $\mathscr{A}^*$ at μ_0 and $(\mathscr{A}^*)^* \sim \mathscr{A}$. Conversely, if $\varphi \in I(\mathscr{A}^*)$, then $\nabla\varphi|_{\mu_0} := X \in \mathrm{cent}(\mathrm{stab}(\mu_0))$.*

The point of this lemma is that, when applied to the loop algebra $\mathscr{A} = \ell(\mathscr{G})$ of a Lie algebra $\mathscr{G}$, with the splitting $\mathscr{A} = \mathscr{A}_+ + \mathscr{A}_-$ into the sum of subalgebras consisting of positive and negative powers in the loop parameter λ, it implies that, starting from any $X \in \mathrm{cent}(\mathrm{stab}(\mu_0))$, with $\mu_0 \in (\mathscr{A}_-)^* \sim (\mathscr{A}^*)_+$, the flow induced in $(\mathscr{A}_-)^*$ by exponentiation and factorization in the group is given by integration of a Lax type equation

$$\frac{d\mu}{dt} = \pm(\mathrm{ad}^*(\nabla\varphi|_\mu)_\pm)(\mu) \tag{0.5}$$

with $\varphi \in I(\mathscr{A}^*)$. Taking the Lie algebra $\mathscr{G}$ itself as a centrally extended loop algebra in the space variable x, Eq. (0.5) becomes a zero-curvature (Zakharov-Shabat) equation. The commutativity of such flows is part of the AKS theorem.

The above lemma underlies the equivalence between the DS approach [1, 7], which is based essentially upon cent(stab(μ)), and the AKS approach, based on $I(\mathscr{A}^*)$. This equivalence is certainly known to specialists, but in this paper it will be taken as the starting point and *all* results will be derived from the AKS Hamiltonian point of view.

This paper is organized as follows. In Sect. 1 we collect results that are relevant for understanding the DS type construction of compatible zero curvature equations in a Hamiltonian setting. In particular, in Sect. 1.1, we recall the relevant aspects of the AKS construction of commuting flows. In Sect. 2.2 we discuss a sufficient condition that can be used to obtain local ad*-invariant Hamiltonians from the asymptotic expansion of the monodromy matrix of an appropriate first order matrix differential operator. These two sections naturally lead us to look for the graded regular elements of the Heisenberg subalgebras of a loop algebra as the starting point for obtaining generalized DS hierarchies. The solution to this problem is given in the case of the nontwisted loop algebra $\ell(gl_n)$ by Theorem 1.7 in Sect. 1.3. Section 2 is devoted to the generalized DS reduction yielding the matrix Gelfand-Dickey hierarchy. The description of the reduced space is established in Sect. 2.1, the Hamiltonian structures are described in Sect. 2.2 and the Hamiltonians themselves are given in Sect. 2.3. The Poisson brackets and the first few Hamiltonians are computed explicitly for an example in Sect. 2.4. The main results of Sect. 2 are Theorem 2.4 which identifies the reduced Poisson structures as the first and second Gelfand-Dickey Poisson structures, and Corollary 2.11 of Theorem 2.10 which gives the generating set of commuting Hamiltonians. The paper concludes with remarks relating the matrix Gelfand-Dickey hierarchies to nonabelian "conformal" and "affine" Toda systems, comments on $\mathscr{W}$-algebras and on the case $n = pr + 1$, and further remarks concerning the literature and some open problems.

1. The AKS Construction and Local Reductions

In the first two sections we review some well-known results about the AKS approach in loop algebras. We shall naturally be led to considering the problem of finding all the graded regular elements of the Heisenberg subalgebras of $gl_n \otimes C[\lambda, \lambda^{-1}]$, which are given in Sect. 2.3.

1.1. The AKS Construction

Here we summarize those points of the AKS (or r-matrix) aproach which we shall need. Readers unfamiliar with the construction could consult, for example, [19–21] for further details.

Let $\mathscr{A}$ be a Lie algebra with Lie bracket $[\,,\,]$. Suppose that R is a classical r-matrix; that is, $R \in \operatorname{End}\mathscr{A}$ and the bracket $[X,Y]_R : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ given by

$$[X,Y]_R = \tfrac{1}{2}[RX,Y] + \tfrac{1}{2}[X,RY] \tag{1.1}$$

is also a Lie bracket. For a function φ in $C^\infty(\mathscr{A}^*)$ we define its differential $\nabla_\alpha \varphi \in \mathscr{A}$ at a point $\alpha \in \mathscr{A}^*$ by

$$\frac{d}{dt}\varphi(\alpha + t\beta)|_{t=0} = \langle \beta, \nabla_\alpha \varphi \rangle \quad \forall \beta \in \mathscr{A}^*\,, \tag{1.2}$$

where $\langle\,,\,\rangle$ is the dual pairing. Let

$$I(\mathscr{A}^*) = \{\varphi \in C^\infty(\mathscr{A}^*) \mid 0 = \langle (\mathrm{ad}^* X)(\alpha), \nabla_\alpha \varphi\rangle = \langle \alpha, [\nabla_\alpha \varphi, X]\rangle \ \ \forall X \in \mathscr{A}\}\,, \tag{1.3}$$

be the set of ad^*-invariant functions on $\mathscr{A}^*$. Note that in many cases we can think in terms of the Ad^*-action on $\mathscr{A}^*$ of a Lie group G corresponding to $\mathscr{A}$, in which case

$$I(\mathscr{A}^*) = \{\varphi \in C^\infty(\mathscr{A}^*) \mid \varphi(\mathrm{Ad}_g^*\,\alpha) = \varphi(\alpha) \ \ \forall g \in G\}\,. \tag{1.4}$$

Here ad^* means the action dual to the original Lie bracket, $[\,,\,]$ on $\mathscr{A}$; to refer to the action of $\mathscr{A}$ on $\mathscr{A}^*$ dual to that given by the bracket $[\,,\,]_R$, we write ad_R^*.

A surprisingly large number of integrable systems arise as consequences of the following result [20], which is the r-matrix version of the AKS theorem, and whose proof is a direct application of (1.1) and (1.3).

Proposition 1.1. *The elements of $I(\mathscr{A}^*)$ are an involutive family in $C^\infty(\mathscr{A}^*)$ with respect to the R Lie-Poisson bracket on $\mathscr{A}^*$. That is, if $\varphi, \psi \in I(\mathscr{A}^*)$ then*

$$\{\varphi,\psi\}_R(\alpha) \equiv \langle \alpha, [\nabla_\alpha\varphi, \nabla_\alpha\psi]_R\rangle = 0\,. \tag{1.5}$$

The dynamical equation generated by the Hamiltonian $\varphi \in I(\mathscr{A}^*)$ through the R Lie-Poisson bracket has the generalized Lax form

$$\dot\alpha = (\mathrm{ad}^*\, \tfrac{1}{2} R\nabla_\alpha\varphi)(\alpha)\,. \tag{1.6}$$

Consider now the special case that will be of interest in what follows (see [21]). Let $\mathscr{G}$ be a Lie algebra and set $\mathscr{A} = \ell(\mathscr{G}) = \mathscr{G} \otimes C[\lambda, \lambda^{-1}] = \left\{ \sum_{i=r}^{s} X_i \lambda^i \,\middle|\, X_i \in \mathscr{G} \right\}$. Let $\mathscr{A}_+ = \mathscr{G} \otimes C[\lambda]$, $\mathscr{A}_- = \mathscr{G} \otimes \lambda^{-1} C[\lambda^{-1}]$, so that $\mathscr{A} = \mathscr{A}_+ + \mathscr{A}_-$. We let $P_\pm$ be the projection operators defined by this splitting and set $R = P_+ - P_-$. For any $\eta \in C[\lambda, \lambda^{-1}]$ we define $\hat\eta : \mathscr{A} \to \mathscr{A}$ by $(\hat\eta X)(\lambda) = \eta(\lambda) X(\lambda)$. Then

$$R_\eta := R \circ \hat\eta \tag{1.7}$$

defines a classical r-matrix for any η, and the corresponding Lie-Poisson brackets for different η are compatible.

We can identify $\ell(\mathscr{G})^*$ with $\ell(\mathscr{G}^*) = \mathscr{G}^* \otimes C[\lambda, \lambda^{-1}]$ by means of the pairing $\langle\,,\,\rangle$ given by

$$\langle \alpha, X\rangle := (\alpha(X))|_{-1}\,, \quad \text{i.e.} \quad \left\langle \sum_i \alpha_i \lambda^i, \sum_j X_j \lambda^j \right\rangle := \sum_{i+j=-1} \alpha_i(X_j)\,. \tag{1.8}$$

Remark 1.1. An equivalent procedure would be to keep the r-matrix fixed and define a dual pairing for each $\eta \in C[\lambda, \lambda^{-1}]$ by inserting a factor η into the definition (1.8). In view of this, we see that in this case there is no essential difference between the r-matrix formulation and the original AKS splitting procedure. In what follows, the collection of commuting Hamiltonian systems determined by the elements of $I(\ell(\mathcal{G}^*))$, together with the compatible Poisson brackets $\{\,,\}_{R_\eta}$ will be referred to as *the AKS system.*

The ad*-action of $\ell(\mathcal{G})$ on $\ell(\mathcal{G}^*)$ is given by pointwise evaluation in λ. This implies that one can in general obtain elements of $I(\ell(\mathcal{G}^*))$ in the following way. Let $\varphi \in I(\mathcal{G}^*)$ be an invariant polynomial; choose any element $\rho \in C[\lambda, \lambda^{-1}]$. Then $\varphi_\rho \in C^\infty(\ell(\mathcal{G}^*))$, given by

$$\varphi_\rho(\alpha) := (\rho(\lambda)\varphi(\alpha(\lambda)))|_{-1}\,, \qquad \alpha = \sum_i \alpha_i \lambda^i \in \ell(\mathcal{G}^*)\,, \tag{1.9}$$

is in $I(\ell(\mathcal{G}^*))$. If $I(\mathcal{G}^*)$ is generated by polynomials φ then $I(\ell(\mathcal{G}^*))$ is generated by the corresponding functions φ_ρ. The Hamiltonian vector fields determined by these invariant functions on $\ell(\mathcal{G}^*)$ obey the following relations,

$$\{f, \varphi_{\rho\chi}\}_{R_\eta} = \{f, \varphi_\rho\}_{R_{\eta\chi}}\,, \quad \forall f \in C^\infty(\ell(\mathcal{G}^*))\,, \quad \forall \eta, \rho, \chi \in C[\lambda, \lambda^{-1}]\,. \tag{1.10}$$

Notice [21] that the space

$$\hat{\mathcal{M}}_{-m,n} := \left\{ \sum_{i=-m}^{n} u_i \lambda^i \,\middle|\, u_i \in G^* \right\} \subset \ell(\mathcal{G}^*)\,, \quad \text{where} \quad m \geq 0, n \geq -1\,, \tag{1.11}$$

is a Poisson subspace for any of the r-matrices $R_k := R \circ \hat{\eta}_k$, where $\eta_k(\lambda) := \lambda^k$, if $-m \leq k \leq n+1$. Furthermore, if $k \leq n$ then u_n is a Casimir element; i.e., constant under any Hamiltonian flow of the R_k Lie-Poisson bracket. Hence the affine subspace $\mathcal{M}_{-m,n} \subset \hat{\mathcal{M}}_{-m,n}$ having u_n fixed has $n+m+1$ compatible Poisson brackets, and the restriction of the elements of $I(\ell(\mathcal{G}^*))$ provides a commuting family of Hamiltonians with respect to any of them. Note that the same computation which shows $\mathcal{M}_{-m,n}$ is a Poisson subspace of $\ell(G^*)$ justifies restricting ourselves to $\ell(\mathcal{G}^*)$, which is strictly speaking only a subspace of $\ell(\mathcal{G})^*$. In both cases one simply has to check that an arbitrary Hamiltonian flow determined by

$$\dot{\alpha} = (\mathrm{ad}^*_{R_\eta} \nabla_\alpha H)(\alpha) \tag{1.12}$$

does not leave the space.

From now on we take $\mathcal{G}$ to be $\widetilde{gl_n^\wedge}$, the central extension of the algebra of smooth loops in gl_n; i.e., $\mathcal{G} = \{(X,a) \mid X : S^1 \to gl_n, a \in \mathbf{C}\}$ with [1]

$$[(X,a),(Y,b)] = \left(XY - YX, \int_0^{2\pi} dx \,\mathrm{tr}\, X'(x)Y(x) \right). \tag{1.13}$$

[1] The periodic space variable parametrizing S^1 is usually denoted by $x \in [0, 2\pi]$, and tilde signifies here "loops in x"

As usual, we represent (a dense subspace of) the dual space $\mathscr{G}^*$ as the space of first order matrix differential operators

$$\mathscr{L} = (e\partial + \mu) \leftrightarrow (\mu, e) \in \mathscr{G}^*, \tag{1.14}$$

with $\mu \in \widetilde{gl}_n \sim \widetilde{gl}_n^*$, $e \in \mathbf{C}$ and dual pairing

$$\langle(\mu, e), (X, a)\rangle = ea + \int_0^{2\pi} dx \operatorname{tr} \mu(x) X(x). \tag{1.15}$$

Upon introducing the spectral parameter λ, the elements of $\ell(\mathscr{G})$ and $\ell(\mathscr{G}^*)$ are given by pairs (X, a) and (μ, e), where $X = \sum X_j \lambda^j$, $\mu = \sum \mu_i \lambda^i$ are now mappings from S^1 into $\ell(gl_n)$, and $a = \sum a_j \lambda^j$, $e = \sum e_i \lambda^i$ are elements of $C[\lambda, \lambda^{-1}]$. The ad*-action on $\ell(\mathscr{G}^*)$ is given by

$$\operatorname{ad}^*(X, a) : (\mu, e) \mapsto (X\mu - \mu X - eX', 0). \tag{1.16}$$

The Ad*-action is given by "nonabelian gauge transformations"; that is, by

$$\mathscr{L} = e\partial + \mu \mapsto g\mathscr{L}g^{-1} = e\partial + (g\mu g^{-1} - eg'g^{-1}), \tag{1.17}$$

where g is an element of the loop group $\ell(\widetilde{Gl}_n)$ associated to $\ell(\widetilde{gl}_n)$.

The ring of ad*-invariant functions $I(\ell(\mathscr{G}^*))$ can be specified as follows. Consider the linear problem

$$(e\partial_x + \mu(x, \lambda))\Phi(x, \lambda) = 0, \tag{1.18}$$

where $\Phi(x, \lambda) \in Gl_n$. Then the eigenvalues of the monodromy matrix

$$T(\lambda) = \Phi(2\pi, \lambda)(\Phi(0, \lambda))^{-1}, \tag{1.19}$$

viewed as functions on $\ell(\mathscr{G}^*)$, generate the ring $I(\ell(\mathscr{G}^*))$.

The number of invariants is infinite due to the parametric dependence of T on λ. In general these are not local functionals of μ; i.e., they cannot be expressed as functions of a finite number of integrals of local densities in the components $\mu_i(x)$ of $\mu(x, \lambda) = \sum_i \mu_i(x)\lambda^i$ and their derivatives. However, under certain conditions (see Sect. 1.2), it is possible, using the asymptotic expansion of $T(\lambda)$, to determine an infinite set of local, commuting Hamiltonians on appropriate Poisson subspaces of $\ell(\mathscr{G}^*)$, or reductions thereof.

Following the above, let us take η in (1.7) to be of the form

$$\eta(\lambda) = \eta_0 + \eta_1 \lambda, \tag{1.20}$$

and consider the subspace, $\mathscr{M}_{0,1} \subset \bar{\mathscr{M}}_{0,1}$ consisting of elements of the form

$$(\mu, e)(\lambda) = (J + C_1\lambda, e_0 + e_1\lambda), \tag{1.21}$$

where $C_1 \in gl_n$, $e_0, e_1 \in \mathbf{C}$ are constants and $J : S^1 \to gl_n$ is arbitrary. Since this is a Poisson subspace of $\ell(\mathscr{G}^*)$ with respect to the R_η Lie-Poisson bracket, the Poisson bracket can be restricted to functions that depend only on J, giving

$$\begin{aligned}\{\varphi, \psi\}_{R_\eta}(J) = &-\eta_0\left(\int_{S^1} \operatorname{tr} C_1\left[\frac{\delta\varphi}{\delta J}, \frac{\delta\psi}{\delta J}\right] + e_1 \int_{S^1} \operatorname{tr}\left(\frac{\delta\varphi}{\delta J}\right)' \frac{\delta\psi}{\delta J}\right) \\ &+ \eta_1\left(\int_{S^1} \operatorname{tr} J\left[\frac{\delta\varphi}{\delta J}, \frac{\delta\psi}{\delta J}\right] + e_0 \int_{S^1} \operatorname{tr}\left(\frac{\delta\varphi}{\delta J}\right)' \frac{\delta\psi}{\delta J}\right).\end{aligned} \tag{1.22}$$

The functional differential $\frac{\delta\varphi}{\delta J}: S^1 \to gl_n$ is defined here by the usual formula,

$$\frac{d}{dt}\,\varphi(J+tK)|_{t=0} = \int_{S^1} \operatorname{tr} K \,\frac{\delta\varphi}{\delta J} \quad \forall K: S^1 \to gl_n\,. \tag{1.23}$$

From now on we choose the values of the Casimirs e_0 and e_1 to be $e_0 = 1$ and $e_1 = 0$. The space of Poisson brackets given by (1.22) has a basis given by the choices $\eta(\lambda) = 1$ and by $\eta(\lambda) = \lambda$. A common way of describing the above is to say that the space of functions of J has two compatible Poisson brackets:

$$\{\varphi, \psi\}_1(J) = -\int_{S^1} \operatorname{tr} C_1 \left[\frac{\delta\varphi}{\delta J}, \frac{\delta\psi}{\delta J}\right] \tag{1.24a}$$

and

$$\{\varphi, \psi\}_2(J) = \int_{S^1} \operatorname{tr}\left(J\left[\frac{\delta\varphi}{\delta J}, \frac{\delta\psi}{\delta J}\right] + \left(\frac{\delta\varphi}{\delta J}\right)' \frac{\delta\psi}{\delta J}\right). \tag{1.24b}$$

We have denoted the Poisson bracket belonging to $\eta = \lambda^i$ as $\{\,,\}_{i+1}$ in order to be consistent with the traditional terminology of KdV systems later. Note that the first Poisson bracket is just the Lie derivative of the second Poisson bracket with respect to the vector field that generates translations of J in the direction $-C_1$.

In general, "interesting" examples result from appropriate further reductions of a space $\mathscr{M}_{0,1}$. A key observation in this respect is that the group consisting of those λ-independent nonabelian gauge transformations $g: S^1 \to Gl_n$ for which

$$gC_1g^{-1} = C_1 \tag{1.25}$$

is a *symmetry group* of the AKS system restricted to $\mathscr{M}_{0,1}$. Indeed the transformations of $\mathscr{M}_{0,1}$ obtained from (1.17) by using this group preserve both the compatible Poisson structures and the monodromy invariants. Thus one can use this group or any of its subgroups when searching for "nice symmetry reductions" of the AKS system.

Remark 1.2. The apparent generalization obtained by considering $\bar{\bar{\mathscr{M}}}_{-m,n}$ with $n+m > 1$ instead of $\bar{\mathscr{M}}_{0,1}$ does not add any new interesting structure, since the reduction only affects u_{n-1} and u_n; the other terms remain generic.

1.2. How to Obtain Local Invariants

Let us consider the operator $\mathscr{L} = (\partial + \mu)$, where $\mu: S^1 \to \ell(gl_n)$. There is a fairly well-known sufficient condition on the form of the function μ that one can impose in order to guarantee the locality of the monodromy invariants of $\mathscr{L}$. This condition involves the *graded regular* elements of the affine algebra $\ell(gl_n)$, and we shall explain a variant of it below.

Consider a fixed element $\Lambda \in \ell(gl_n)$ and denote its kernel and image in the adjoint representation by

$$\mathscr{K} = \operatorname{Ker}(\operatorname{ad}\Lambda) \quad \text{and} \quad \mathscr{I} = \operatorname{Im}(\operatorname{ad}\Lambda)\,. \tag{1.26}$$

Of course, $\mathscr{K}$ is a subalgebra of $\ell(gl_n)$. For Λ a *regular* element one has

$$\ell(gl_n) = \mathscr{K} + \mathscr{I}\,, \qquad \mathscr{K} \cap \mathscr{I} = \{0\}\,, \qquad \mathscr{K}: \text{abelian}\,. \tag{1.27}$$

We are interested in regular elements that are *homogeneous with positive degree* with respect to some integral grading. An integral grading of $\ell(gl_n)$ can be defined by the eigenspaces of a linear operator $d_{N,H}: \ell(gl_n) \to \ell(gl_n)$ of the form

$$d_{N,H} = N\lambda \frac{d}{d\lambda} + \mathrm{ad}\, H\,, \tag{1.28}$$

where N is a non-zero integer and H is a diagonalizable element of gl_n with integral spectrum in the adjoint representation. This formula in fact defines a derivation on $\ell(gl_n)$ with integral eigenvalues and finite dimensional eigenspaces.

The following proposition, which generalizes the corresponding well-known result for the homogeneous grading case [1, 19, 22], states the existence of a solution of the linear problem given by a series that can be computed by an algorithm involving only linear algebraic operations and integrations.

Proposition 1.2. *Let a grading* (1.28) *of* $\ell(gl_n)$ *be given. Let* Λ *be a regular homogeneous element of grade* $l > 0$. *Consider a function* $\mu: S^1 \to \ell(gl_n)$ *of the form*

$$\mu(x) = (q(x) + \Lambda)\,, \tag{1.29a}$$

where

$$q(x) = \sum_{k<l} q^k(x) \quad \text{with} \quad d_{N,H}(q^k) = kq^k\,. \tag{1.29b}$$

Then the linear problem

$$(\partial_x + q(x) + \Lambda)\Phi(x) = 0 \tag{1.30}$$

has a unique solution of the form

$$\Phi(x) = (I + W(x))e^{F(x)}(I + W(0))^{-1}\Phi(0)\,, \tag{1.31}$$

where

$$F(x) \in \mathscr{K}, W(x) \in \mathscr{I}, \text{ and } W(x) = \sum_{k<0} W^k(x) \text{ with } d_{N,H}(W^k) = kW^k\,. \tag{1.32}$$

Here the W^k*'s are uniquely determined differential polynomials in the components of* q *and* F *is given by*

$$F(x) = -\int_0^x dy[q_{\mathscr{K}}(y) + (q(y)W(y))_{\mathscr{K}} + \Lambda]\,, \tag{1.33}$$

where the subscript $\mathscr{K}$ *refers to the* $\mathscr{K}$ *component in the decomposition* $\ell(gl_n) = \mathscr{K} + \mathscr{I}$.

Corollary 1.3. *The monodromy matrix of* $\mathscr{L} = (\partial + q + \Lambda)$ *is conjugate to*

$$\exp(F(2\pi)) = \exp\left(-\int_0^{2\pi} dx[q_{\mathscr{K}}(x) + (q(x)W(x))_{\mathscr{K}} + \Lambda]\right)\,, \tag{1.34}$$

and thus its invariants are functions of integrals of local densities in $q(x)$.

Proof. The procedure is essentially the same as given in a special case in [22]. By substituting the ansatz

$$\Phi = (I + W)e^F\Psi\,, \tag{1.35}$$

where I is the unit matrix and Ψ is a constant, into (1.30) we obtain

$$W' + (I + W)F' + (q + \Lambda)(I + W) = 0\,. \tag{1.36}$$

If we decompose this equation according to $\ell(gl_n) = \mathscr{I} + \mathscr{K}$ by using (1.32) together with $\mathscr{I}\mathscr{K} \subset \mathscr{I}$ and $\mathscr{K}\mathscr{K} \subset \mathscr{K}$, then the $\mathscr{K}$-component gives

$$F' + [q(I + W)]_{\mathscr{K}} + \Lambda = 0\,, \tag{1.37}$$

which (up to a constant) can be integrated to give (1.33), since $\mathscr{K}$ is abelian. By substituting (1.37) into (1.36) the $\mathscr{I}$-component gives

$$[\Lambda, W] + W' - W(q + qW)_{\mathscr{K}} + (q + qW)_{\mathscr{I}} = 0\,. \tag{1.38}$$

One can solve this equation recursively for the W^k's by using the grading assumptions of (1.29) and (1.32) together with the fact that $\mathrm{ad}\,\Lambda$ maps $\mathscr{I}$ to $\mathscr{I}$ in a one-to-one manner since Λ is regular. This procedure obviously yields the W^k's as differential polynomials in q. Finally, the integration constant Ψ in (1.35) is fixed by the initial condition, giving (1.31). $\square$

Note that the recursive procedure appearing in the proof, combined with the diagonalization of the generators of $\mathscr{K}$, is also useful for computing the monodromy invariants in practice. It should also be noted that in the above we have not considered the convergence of the series solution at all. It is well known that such series do not convergence in general, and are to be considered as asymptotic expansions in λ (or alternatively as formal series).

1.3. The List of Graded Regular Elements

We have seen that the graded regular elements of $\ell(gl_n)$ can be used to impose constraints on the form of $\mathscr{L} = (\partial + \mu)$ leading to local monodromy invariants. The suggestion of De Groot et al. [7] was to use the graded regular elements of the inequivalent graded Heisenberg subalgebras of the loop algebras [12] to construct generalizations of the Drinfeld-Sokolov hierarchies. The graded Heisenberg subalgebras of the loop algebra $\ell(gl_n)$ (maximal abelian subalgebras that acquire a central extension in $\ell(gl_n)^\wedge$ have been given an explicit description recently in [13], where the authors were interested in the related vertex operator constructions. By using this description, we shall show that graded regular elements exist only in some exceptional Heisenberg subalgebras. The complete list is given by Theorem 1.7 at the end of this section.

The graded Heisenberg subalgebras of $\ell(gl_n)$ can be associated to the *partitions* of n in the following way [13]. First, for m any natural number, we define the following $m \times m$ matrices:

$$\begin{aligned} \Lambda_m &= \lambda e_{m,1} + \sum_{k=1}^{m-1} e_{k,k+1}\,, \\ H_m &= \mathrm{diag}[j, (j-1), \ldots, -(j-1), -j]\,, \qquad j = \frac{m-1}{2}\,, \end{aligned} \tag{1.39}$$

and

$$\sigma_m = \exp[2\pi i H_m/m]\,,$$

where $e_{i,j}$ is the standard elementary matrix with entry 1 in the ij^{th} place and 0 elsewhere. Let a partition of n be given by

$$n = n_1 + n_2 + \ldots + n_k\,, \quad \text{where} \quad n_1 \geq n_2 \geq \ldots \geq n_k \geq 1\,. \tag{1.40}$$

We associate to this partition the $n \times n$ matrix

$$\sigma = \mathrm{diag}[\sigma_{n_1}, \sigma_{n_2}, \ldots, \sigma_{n_k}]\,, \tag{1.41a}$$

and denote by N the order of the inner automorphism of gl_n acting through conjugation by σ. If we let

$$N' = \mathrm{lcm}(n_1, n_2, \ldots, n_k)\,, \tag{1.41b}$$

we have

$$N = \begin{cases} N'\,, & \text{if } N'\left(\dfrac{1}{n_i} + \dfrac{1}{n_j}\right) \text{ is even for all } i,j\,; \\[2ex] 2N'\,, & \text{if } N'\left(\dfrac{1}{n_i} + \dfrac{1}{n_j}\right) \text{ is odd for some } i,j\,. \end{cases} \tag{1.41c}$$

We then introduce the $n \times n$ diagonal matrix H via the equation

$$\sigma = \exp[2\pi i H/N]\,, \tag{1.41d}$$

and consider the grading of $\ell(gl_n)$ given by the eigenvalues of

$$d = N\lambda \frac{d}{d\lambda} + \mathrm{ad}\, H\,. \tag{1.42}$$

The Heisenberg subalgebra corresponding to the partition (1.40) is spanned by the $n \times n$ "block-diagonal" matrices Λ of the following form:

$$\Lambda = \begin{bmatrix} y_1 \Lambda_{n_1}^{l_1} & & & \\ & y_2 \Lambda_{n_2}^{l_2} & & \\ & & \ddots & \\ & & & y_k \Lambda_{n_k}^{l_k} \end{bmatrix}, \tag{1.43}$$

where the $l_i (i = 1, 2, \ldots, k)$ are arbitrary integers and the y_i are arbitrary numbers. This maximal abelian subalgebra of $\ell(gl_n)$ is invariant under the grading operator (1.42). An element Λ of the form given by (1.43) is *regular* if $\mathcal{K} = \mathrm{Ker}(\mathrm{ad}\,\Lambda)$ is exactly the Heisenberg subalgebra (and not a larger space). We next investigate the existence of the graded regular elements for some simple partitions, from which we shall then be able to read off the answer for the general case.

The simplest case is that of the *homogeneous* Heisenberg subalgebra, which belongs to the partition $n = 1 + 1 + \ldots + 1$, when $d = \lambda \dfrac{d}{d\lambda}$ and the graded regular elements are of the form

$$\Lambda = \lambda^k \,\mathrm{diag}[y_1, y_2, \ldots, y_n]\,, \qquad y_i \neq y_j\,, \ \forall k\,. \tag{1.44}$$

The *principal* Heisenberg subalgebra belongs to the other extreme case when n is "not partitioned at all". In this case $d = n\lambda \dfrac{d}{d\lambda} + \mathrm{ad}\, H_n$ and the graded generators are the powers of the "Drinfeld-Sokolov matrix" Λ_n. We have

$$\Lambda_n^{l+mn} = \lambda^m \Lambda_n^l\,, \quad d(\Lambda_n^{l+mn}) = (l+mn)\Lambda_n^{l+mn}\,, \quad 0 \leq l \leq (n-1)\,, \ \forall m\,. \tag{1.45}$$

It is obvious that Λ_n^{l+mn} is regular if and only if Λ_n^l is regular. The Drinfeld-Sokolov matrix Λ_n itself is *regular* since its eigenvalues are the n *distinct* n^{th}-roots of λ, and from this one also easily verifies the following by looking at the eigenvalues of Λ_n^l.

Lemma 1.4. *The element* Λ_n^l $(1 \leq l \leq (n-1))$ *is regular if and only if* n *and* l *are relatively prime.*

Consider now a partition of the type

$$n = n_1 + n_2\,, \quad \text{with} \quad n_1 > n_2 > 1\,. \tag{1.46}$$

Lemma 1.5. *In the case* (1.46) *there is no graded regular element in the Heisenberg subalgebra.*

Proof. As candidates for graded regular elements, it is enough to consider the matrices of the form

$$\Lambda = \begin{bmatrix} y_1\Lambda_{n_1}^{l_1} & \\ & y_2\Lambda_{n_2}^{l_2} \end{bmatrix} \quad \text{with} \quad y_1y_2 \neq 0\,, \qquad l_i \neq 0 \bmod n_i\,. \tag{1.47}$$

Case (i): Assume that n_1 and n_2 are relatively prime. We can check from the definition of the grading that there is no graded element of the form (1.47), since

$$\begin{bmatrix} \Lambda_{n_1}^{l_1} & \\ & 0_{n_2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0_{n_1} & \\ & \Lambda_{n_2}^{l_2} \end{bmatrix} \tag{1.48}$$

have different grades.

Case (ii): If n_1 and n_2 are not relatively prime and $m > 1$ is their greatest common divisor then the graded elements of the form (1.47) are those for which

$$l_1 = k\,\frac{n_1}{m} \quad \text{and} \quad l_2 = k\,\frac{n_2}{m}\,, \qquad k: \text{ any integer}\,. \tag{1.49}$$

This implies by Lemma 1.4 that neither of the $\Lambda_{n_i}^{l_i}$ $(i = 1, 2)$ is regular, and therefore there is no graded regular element of the type (1.47) either. $\square$

Let us also note the following rather obvious fact.

Lemma 1.6. *There is no graded regular element in the Heisenberg subalgebra if the partition is of the type*

$$n = m + 1 + \ldots + 1\,, \qquad 1 < m < (n-1)\,; \tag{1.50}$$

i.e., if it consists of a "non-singlet", m*, and more than one "singlets"* (1*'s*).

It follows from the "block structure" of the Heisenberg subalgebras given by (1.39)–(1.43) above that graded regular elements do not exist for any of those partitions which contain a subset of the type appearing in Lemmas 1.5 and 1.6. Hence the only cases not excluded are the "partitions into equal blocks",

$$n = pr = \overbrace{r + \ldots + r}^{p\text{ times}}\,, \tag{1.51}$$

and the cases "equal blocks plus a singlet",

$$n = pr + 1 = \overbrace{r + \ldots + r}^{p\text{ times}} + 1\,. \tag{1.52}$$

On the other hand, by inspecting the eigenvalues of the generators of the Heisenberg subalgebras, we can establish that graded regular elements do indeed exist in the cases (1.51), (1.52), and the following theorem gives the complete list.

Theorem 1.7. *Graded regular elements exist only in those Heisenberg subalgebras of* $\ell(gl_n)$ *which belong to the special partitions* (1.51) *or* (1.52). *In the equal block case* (1.51) *with* $r > 1$ *the graded regular elements are of the form*

$$\Lambda = \lambda^m \begin{bmatrix} y_1 \Lambda_r^l & & & \\ & y_2 \Lambda_r^l & & \\ & & \ddots & \\ & & & y_p \Lambda_r^l \end{bmatrix}, \tag{1.53}$$

where

$$1 \leq l \leq (r-1), \qquad y_i \neq 0, \qquad y_i^r \neq y_j^r, \qquad i,j = 1, \ldots, p, \qquad i \neq j,$$

with l *relatively prime to* r *and* m *any integer. The element* Λ *is of grade* $(l + mr)$ *with respect to the grading operator given by* (1.42), *where* $N = r$ *and*

$$H = \mathrm{diag}[\overbrace{H_r, H_r, \ldots, H_r}^{p \text{ times}}]. \tag{1.54}$$

In the equal-blocks-plus-singlet case (1.52), *the graded regular elements are those* $n \times n$ *matrices which contain an* $(n-1) \times (n-1)$ *block of the form given by* (1.53) *in the "top-left corner" and an arbitrary entry in the "lower-right corner." The relevant grading operator is given by* (1.42) *with* $N = r$,

$$H = \mathrm{diag}[\overbrace{H_r, H_r, \ldots, H_r}^{p \text{ times}}, 0] \quad \textit{if } r \textit{ is odd;} \tag{1.55a}$$

and with $N = 2r$,

$$H = \mathrm{diag}[\overbrace{2H_r, 2H_r, \ldots, 2H_r}^{p \text{ times}}, 0] \quad \textit{if } r \textit{ is even.} \tag{1.55b}$$

Remark 1.3. Let us designate the ordered eigenvector basis of the matrix H in (1.54) as

$$X_{j,1}, X_{j-1,1}, \ldots, X_{-j,1}; X_{j,2}, X_{j-1,2}, \ldots, X_{-j,2}; \ldots; X_{j,p}, X_{j-1,p}, \ldots, X_{-j,p}. \tag{1.56}$$

Here $j = (r-1)/2$, the first index is the eigenvalue and the second one orders the $r \times r$ blocks. It is often convenient to use the re-ordered basis

$$X_{j,1}, X_{j,2}, \ldots, X_{j,p}; X_{j-1,1}, X_{j-1,2}, \ldots, X_{j-1,p}; \ldots; X_{-j,1}, X_{-j,2}, \ldots, X_{-j,p}. \tag{1.57}$$

When expressed in the new basis the matrix H of (1.54) may be written as $H_r \otimes 1_p$ and the graded regular element Λ given by (1.53) takes the form

$$\lambda^m \Lambda_r^l \otimes D, \quad \text{where} \quad D = \mathrm{diag}(y_1, y_2, \ldots, y_p). \tag{1.58}$$

In the following section we shall consider reductions based on the grade 1 regular elements belonging to the equal block case.

2. Equal Block Reduction of the AKS System to the Matrix Gelfand-Dickey Hierarchy

We have seen that graded regular elements exist only in those Heisenberg subalgebras of $\ell(gl_n)$ which correspond to partitions into equal blocks or equal blocks plus a singlet. The purpose of this section is to study in some detail symmetry reductions of the general AKS system that are based upon the grade 1 regular elements of the Heisenberg subalgebras belonging to the partitions into equal blocks, $n = pr$, generalizing the $p = 1$ case considered in [1]. The final result of the analysis below is that the reduction of the bihamiltonian manifold $\mathcal{M} := \mathcal{M}_{0,1}$ carrying the commuting family of ad^*-invariant Hamiltonians yields the $p \times p$ matrix version of the well-known Gelfand-Dickey r-KdV hierarchy. More exactly, we shall establish the following:

1. The reduced phase space is the space of r^{th} order, $p \times p$ matrix differential operators carrying the first and second *Gelfand-Dickey Poisson brackets*. The second Poisson bracket algebra is an example of a classical $\mathscr{W}$-algebra.
2. The commuting hierarchy of Hamiltonians resulting from the monodromy invariants is given by the componentwise residues of the fractional (including integral) powers of the *pseudo-differential operators* obtained by *diagonalizing* the matrix differential operators.

The exact statements are given by Theorem 2.4 and Corollary 2.11 in Sects. 2.2 and 2.3. These results generalize the analogous results proven by Drinfeld and Sokolov for the scalar case $p = 1$. We shall in fact use many of their methods, but at the same time introduce some simplifications (at least to our taste) in the proofs.

Remark 2.1. For $p > 1$ the subhierarchy provided by the trace-residues of the *fractional* powers of the matrix differential operators is not exhaustive, since it does not include the Hamiltonians obtained form the *integral* powers of the corresponding diagonal pseudo-differential operators, which also appear in Corollary 2.11.

2.1. A Local Symmetry Reduction of the AKS System

After reordering the basis as explained previously, the generators of our Heisenberg subalgebra are the $n \times n$ matrices of the form $\Lambda_r^k \otimes D$, where Λ_r^k is the k^{th} power of the $r \times r$ DS matrix, k is an arbitrary integer, and D is an arbitrary $p \times p$ diagonal matrix. The generator $\Lambda_r^k \otimes D$ is of grade k under the grading defined by $d = r\lambda \dfrac{d}{d\lambda} + \mathrm{ad}\, H$ with

$$H = H_r \otimes 1_p = \mathrm{diag}[j 1_p, (j-1) 1_p, \ldots, -(j-1) 1_p, -j 1_p]\,, \qquad j = \frac{(r-1)}{2}\,. \tag{2.1}$$

Choose a *grade* 1 *regular element* of the Heisenberg algebra, [cf. (1.53), (1.58)]; i.e., an element

$$\Lambda := \Lambda_r \otimes D \tag{2.2}$$

such that D^r has distinct, non-zero eigenvalues. Define the λ-independent constant matrices C_0 and C_1 by the equality

$$\Lambda = \Lambda_r \otimes D := C_0 + \lambda C_1\,. \tag{2.3a}$$

More explicitly, these matrices are given in block form by

$$\Lambda = \begin{bmatrix} 0 & D & 0 & \dots & 0 \\ \vdots & 0 & D & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & D \\ \lambda D & 0 & \dots & \dots & 0 \end{bmatrix} := C_0 + \lambda C_1 \,. \tag{2.3b}$$

We start reducing the general AKS system carried by $\ell(\widetilde{gl_n^\wedge})^*$ by confining ourselves to the space $\mathscr{M}$ consisting of elements of the form

$$\mathscr{L} = \partial + J + \lambda C_1 \,, \qquad J : S^1 \to gl_n \,. \tag{2.4}$$

This is an example of a Poisson subspace of the type $\mathscr{M}_{0,1}$ considered in Sect. 1.1, and therefore it carries the two compatible Poisson brackets given by (1.24a, b).

Consider the decomposition

$$gl_n = gl_n^- + gl_n^0 + gl_n^+ \tag{2.5}$$

induced by the eigenvalues of $\operatorname{ad} H$, Eq. (2.1), where the summands are the subalgebras of block lower triangular, block diagonal, and block upper triangular matrices (with $p \times p$ blocks), respectively. On account of the relation

$$[gl_n^-, C_1] = 0 \,, \tag{2.6}$$

the group $\mathscr{N}$ of transformations

$$e^f : \mathscr{L} \mapsto e^f \mathscr{L} e^{-f} \,, \quad \text{with} \quad f : S^1 \to gl_n^- \,, \tag{2.7}$$

is a symmetry group of the AKS system carried by $\mathscr{M}$; i.e., these transformations preserve the two Poisson structures and the monodromy invariants. Next we define a *symmetry reduction of the AKS system* using $\mathscr{N}$ in such a way as to ensure the *locality* of the reduced system. That is, consider the following two step reduction process, which is an obvious generalization of the one used in [1]. First, we restrict our system to the "constrained manifold" $\mathscr{M}_c \subset \mathscr{M}$, defined as the set of $\mathscr{L}$'s of the following special form:

$$\mathscr{L} = \partial + (q + C_0) + \lambda C_1 = \partial + q + \Lambda \,, \qquad q : S^1 \to (gl_n^- + gl_n^0) \,. \tag{2.8}$$

Here C_0 is the constant matrix given by (2.3), and q is required to be block lower triangular. (Note that with respect to the second Poisson structure (1.24b), this is just fixing a level set of the moment map generating the Hamiltonian group action (2.7), and the reduction procedure is essentially that of Marsden-Weinstein at the level of a Poisson manifold rather than a symplectic one. With respect to the first Poisson structure (1.24a), the constrained quantities determining the form of $\mathscr{L}$ in (2.8) are all Casimirs and hence this just determines a Poisson submanifold.) Second, we factorize this constrained manifold by the symmetry group $\mathscr{N}$, defining the reduced phase space

$$\mathscr{M}_{\text{red}} = \mathscr{M}_c / \mathscr{N} \,. \tag{2.9}$$

To put it another way, we factorize out the "gauge transformations" generated by $\mathscr{N}$ by declaring that only the $\mathscr{N}$-invariant functions of $\mathscr{L}$ are physical. The nice features of this reduction are that

i) The monodromy invariants of $\mathscr{L} \in \mathscr{M}_c$ can be computed algebraically as asymptotic series in λ which depend on q through integrals of local densities formed form its components and their derivatives.
ii) The compatible Poisson structures on $\mathscr{M}$ induce compatible Poisson structures on $\mathscr{M}_{\text{red}}$.
iii) The gauge orbits in $\mathscr{M}_c$ allow for global, differential polynomial gauge sections, which give rise to complete sets of gauge invariant differential polynomials.

Statement i) follows immediately since we have chosen $\mathscr{M}_c$ so that the conditions of Proposition 1.2 are satisfied. Statement ii) means that the compatible Poisson brackets carried by $\mathscr{M}$ can be consistently restricted to the *gauge invariant* functions in $C^\infty(\mathscr{M}_c)$, whose space can be naturally identified with $C^\infty(\mathscr{M}_{\text{red}})$. This can be seen as a consequence of the Dirac theory or reduction by constraints as follows. We first note that, by choosing some basis $\{\gamma_i\}$ of gl_n^-, the constraints defining $\mathscr{M}_c \subset \mathscr{M}$ can be written as

$$\chi_i(x) = 0 \quad \text{where} \quad \chi_i(x) = \operatorname{tr} \gamma_i (J(x) - C_0)\,. \tag{2.10a}$$

It is easy to verify that these constraints are *first class*; i.e.,

$$\{\chi_i(x), \chi_k(y)\}|_{\mathscr{M}_c} = 0\,, \tag{2.10b}$$

for any of the compatible Poisson brackets on $\mathscr{M}$. We next notice that the functions $\chi_i(x)$ are the generating densities (i.e., components of the moment map) of the $\mathscr{N}$ symmetry transformations with respect to the second Poisson bracket (1.24b). Thus the theory of reduction by constraints (which in this case is just the Poisson version of Marsden-Weinstein reduction) tells us to factorize the constrained manifold by these transformations; the second Poisson bracket algebra closes on the gauge invariant functions on $\mathscr{M}_c$, inducing a Poisson structure on the factor space. On the other hand, the χ_i do not generate any transformations on $\mathscr{M}$ under the first Poisson bracket (1.24a); i.e., they are Casimir functions. Therefore the first Poisson bracket can in principle already be restricted to $C^\infty(\mathscr{M}_c)$ without any factorization by $\mathscr{N}$. Then $\mathscr{N}$ becomes a group of Poisson maps with respect to the restricted bracket, which can further be reduced to a Poisson bracket on the invariant functions. In this way, we naturally obtain two induced Poisson brackets on $\mathscr{M}_{\text{red}}$ from those on $\mathscr{M}$, and the induced Poisson brackets are compatible because the original brackets (1.24a, b) were compatible.

One is always interested in gauge invariant objects and convenient gauge fixings when describing systems with gauge symmetries. In the present example, as in the $p = 1$ case of [1], a convenient gauge section is defined by the subspace

$$V := \left\{ \partial + \sum_{i=1}^{r} e_{r,i} \otimes v_{r-i+1} + \Lambda \,\middle|\, v_k : S^1 \to gl_p \right\}, \tag{2.11}$$

i.e., the space of $\mathscr{L}$'s in which the matrix function q is allowed to have non-zero ($p \times p$ block) entries only in the last row. This global section of the gauge orbits can be reached from an arbitrary point $\mathscr{L} = (\partial + q + \Lambda) \in \mathscr{M}_c$ by a unique gauge transformation that depends on q in a differential polynomial way. It follows that the components of the v_i provide a basis (free generating set) for the gauge invariant differential polynomials which can be formed from the components of q.

Remark 2.2. A distinguished gauge invariant differential polynomial is obtained by restricting

$$\mathcal{V}_H = \tfrac{1}{2}\,\mathrm{tr}(J^2) + \mathrm{tr}(HJ') \tag{2.12}$$

from $\mathcal{M}$ to $\mathcal{M}_c$. The density $\mathcal{V}_H$ satisfies the Virasoro algebra under the second Poisson bracket, and therefore it generates an action of Diff(S^1) on $\mathcal{M}$ that survives the present reduction. Since it contains this Virasoro density, the second Poisson bracket algebra of the gauge invariant differential polynomials can be regarded as an extended conformal algebra; that is, a classical $\mathcal{W}$-algebra. This $\mathcal{W}$-algebra is a member of a natural family of extended conformal algebras that recently has been studied in [4, 5]. The reader is referred to [5] for a detailed description of differential polynomial gauge fixings, like the gauge (2.11), and for the related construction of a generating set for the $\mathcal{W}$-algebra consisting of $\mathcal{V}_H$ and conformal tensors.

In the next section we shall also need the "block diagonal gauge" given by

$$\Theta := \left\{ \partial + \sum_{i=1}^{r} e_{i,i} \otimes \theta_i + \Lambda \,\middle|\, \theta_i : S^1 \to gl_p \right\}. \tag{2.13}$$

It should be noted that Θ defines only a partial gauge fixing, which has a finite dimensional residual gauge freedom. Of course it is nevertheless true that any gauge invariant element of $C^\infty(\mathcal{M}_c)$ can be recovered from its restriction to Θ. In particular, the Poisson bracket of any two gauge invariant functions in $C^\infty(\mathcal{M}_c)$ can be recovered from its restriction to Θ, and the nice feature is that for the second Poisson bracket this restriction can be computed in terms of the "free current algebra" of the θ_i's. For later reference we summarize this fact as a lemma. The proof is similar to that for the $p = 1$ case [1].

Lemma 2.1. *Let $\varphi, \psi \in C^\infty(\mathcal{M}_c)$ be gauge invariant functions and consider $\xi = \{\varphi, \psi\}_2 \in C^\infty(\mathcal{M}_c)$, which is well defined and is also gauge invariant. Let us denote the restriction of these functions to Θ by $\bar\varphi, \bar\psi$ and $\bar\xi$, respectively. Then we have*

$$\bar\xi(\theta_i, \ldots, \theta_r) = \sum_{i=1}^{r} \int_{S^1} \mathrm{tr}\left(\theta_i \left[\frac{\delta\bar\varphi}{\delta\theta_i}, \frac{\delta\bar\psi}{\delta\theta_i} \right] + \left(\frac{\delta\bar\varphi}{\delta\theta_i} \right)' \frac{\delta\bar\psi}{\delta\theta_i} \right). \tag{2.14}$$

Proof. We consider the decomposition $J = J_- + J_0 + J_+$ defined by means of (2.5) and write out formula (1.24b) in terms of the partial functional derivatives corresponding to these variables. We then compute the value of $\{\varphi, \psi\}_2$ at an arbitrary point on the "gauge slice" Θ of the block diagonal gauge, where $J_- = 0$, $J_0 = \mathrm{diag}[\theta_1, \ldots, \theta_r]$, $J_+ = C_0$. (For this computation we can extend φ and ψ from $\mathcal{M}_c$ to $\mathcal{M}$ in an arbitrary way since they are invariant under the gauge transformations generated by the first class constraints specifying this constrained manifold.) Using the grading structure, we find that

$$\begin{aligned}\bar\xi(J_0) := \{\varphi, \psi\}_2(J_0 + C_0) &= \int_{S^1} \mathrm{tr}\left(J_0 \left[\frac{\delta\varphi}{\delta J_0}, \frac{\delta\psi}{\delta J_0} \right] + \left(\frac{\delta\varphi}{\delta J_0} \right)' \frac{\delta\psi}{\delta J_0} \right) \\ &= \sum_{i=1}^{r} \int_{S^1} \mathrm{tr}\left(\theta_i \left[\frac{\delta\varphi}{\delta\theta_i}, \frac{\delta\psi}{\delta\theta_i} \right] + \left(\frac{\delta\varphi}{\delta\theta_i} \right)' \frac{\delta\psi}{\delta\theta_i} \right),\end{aligned} \tag{2.15}$$

where $\frac{\delta\varphi}{\delta\theta_i}, \frac{\delta\psi}{\delta\theta_i}$ are understood as the $p\times p$ blocks constituting the block diagonal $\frac{\delta\varphi}{\delta J_0}, \frac{\delta\psi}{\delta J_0}$. This immediately gives (2.14) since at the point $J = J_0 + C_0$ we have $\frac{\delta\varphi}{\delta\theta_i} = \frac{\delta\bar{\varphi}}{\delta\theta_i}$, and similarly for ψ. $\square$

Now let M be the space of $p\times p$ "matrix Lax operators" of the following form:

$$L = (-D^{-1})^r \partial^r + \sum_{i=1}^{r} u_i \partial^{r-i}, \qquad u_i : S^1 \to gl_p . \tag{2.16}$$

As in the $p = 1$ case [1], M can serve as a natural model of the reduced space $\mathscr{M}_c/\mathscr{N}$. To see this let us consider the linear problem

$$\mathscr{L}\Psi = \mathscr{L}\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_r\end{pmatrix} = 0, \tag{2.17}$$

for $\mathscr{L}$ given by (2.8), where the entries ψ_i of Ψ are p-component column vectors. This system of equations is covariant under (2.7) if we let Ψ transform as

$$\Psi \to e^f \Psi . \tag{2.18}$$

Denoting the ij^{th} $p\times p$ block of q by q_{ij}, the system (2.17) can be recast in the following form:

$$\begin{aligned} L\psi_1 &= \lambda\psi_1 ,\\ \psi_2 &= (-D)^{-1}(\partial + q_{11})\psi_1 ,\\ \psi_3 &= (-D)^{-1}[q_{21} + (\partial + q_{22})(-D)^{-1}(\partial + q_{11})]\psi_1 ,\\ &\ \vdots \end{aligned} \tag{2.19}$$

where L is an operator of the form (2.16) whose coefficients u_i are uniquely determined differential polynomials in q. Notice now that the component ψ_1 is invariant under (2.18), because f is a strictly block lower triangular matrix. This implies that the potentials $u_i = u_i[q]$ entering in $L\psi_1 = \lambda\psi_1$ must also be invariant functions of q under (2.18). In other words, the operator L attached to $\mathscr{L}$ by the equivalence of (2.17) and (2.19) actually depends only on the $\mathscr{N}$-orbit of $\mathscr{L}$ in $\mathscr{M}_c$. Thus it gives rise to a map $m : \mathscr{M}_{\text{red}} \to M$. It is easy to see that this is a one-to-one map. The inverse m^{-1} can be given by attaching to an arbitrary operator L in (2.16) the unique orbit in $\mathscr{M}_c$ which intersects the gauge section V (2.11) in the point

$$\mathscr{L} = \partial + \sum_{i=1}^{r} e_{r,i} \otimes v_{r-i+1} + \Lambda, \quad \text{where} \quad v_i = (-1)^{r+1-i} D u_i D^{r-i} . \tag{2.20}$$

This equation provides an identification of the space M with the gauge section V and hence with the reduced phase space $\mathscr{M}_{\text{red}}$,

$$M \cong V \cong \mathscr{M}_{\text{red}} . \tag{2.21}$$

It follows from the above that the natural mapping $\pi : \Theta \to M$ is given by the factorization formula:

$$L = (-D^{-1})^r \partial^r + u_1 \partial^{r-1} + \ldots + u_r = (-D^{-1}(\partial + \theta_r)) \ldots (-D^{-1}(\partial + \theta_1)) \,, \quad (2.22)$$

generalizing the scalar case [23, 1]. Lemma 2.1 can be reformulated as saying that π, which is the usual Miura map when $p = 1$, is a *Poisson mapping* if the space Θ carries the Poisson bracket appearing on the right-hand side of (2.14) and M carries the Poisson bracket induced from the second Poisson bracket on $\mathscr{M}$ via the reduction and the identification $M \cong \mathscr{M}_{\rm red}$.

2.2. *The Gelfand-Dickey Form of the Reduced Poisson Structures*

In this section we prove a theorem that establishes the equivalence of the Poisson structures naturally carried by the spaces $\mathscr{M}_{\rm red}$ and M. This theorem will rely on the preliminary Lemma 2.2 which follows [2]. This concerns the Poisson Lie group property of the so-called Sklyanin bracket as pointed out by Semenov-Tian-Shansky [20].

Let $\mathscr{A}$ be an associative algebra. Suppose that $\mathrm{Tr} : \mathscr{A} \to \mathbf{C}$ is a non-degenerate trace-form on $\mathscr{A}$; i.e., a linear mapping with the property that the formula $\langle a, b \rangle = \mathrm{Tr}\, ab$ defines a non-degenerate, symmetric bilinear form on $\mathscr{A}$. As usual, identify the space $\mathscr{A}$ with its dual $\mathscr{A}^*$ (or a subspace thereof) by means of the pairing defined by Tr. Suppose that A and B are disjoint subalgebras of $\mathscr{A}$ such that $\mathscr{A} = A + B$ and that with respect to the pairing $\langle\, ,\rangle$ we have $A^\perp = A$ and $B^\perp = B$. Let P_A and P_B be the projection maps on $\mathscr{A}$ defined by the splitting $\mathscr{A} = A + B$. Since A and B turn into Lie subalgebras of $\mathscr{A}$ with respect to the natural Lie algebra structure on $\mathscr{A}$ given by $[a, b] = ab - ba$, then $R = P_A - P_B$ is a skew-symmetric r-matrix on $\mathscr{A}$, and the AKS construction applies here too.

For any function $\varphi \in C^\infty(\mathscr{A})$, we define $\nabla_a \varphi$, its gradient at the point $a \in \mathscr{A}$, by

$$\frac{d}{dt}\, \varphi(a + tb)|_{t=0} = \langle b, \nabla_a \varphi \rangle = \mathrm{Tr}\, b \nabla_a \varphi \,, \quad \forall b \in \mathscr{A} \,. \qquad (2.23)$$

In addition to the R Lie-Poisson bracket given by

$$\{\varphi, \psi\}^{(1)}(a) := \mathrm{Tr}(a[X, Y]_R) \,, \quad \text{where} \quad X = \nabla_a \varphi, Y = \nabla_a \psi \,, \qquad (2.24)$$

we can also define a second Poisson bracket on $\mathscr{A}$:

$$\{\varphi, \psi\}^{(2)}(a) := \tfrac{1}{2}\, \mathrm{Tr}(Y a R(Xa) - a Y R(aX)) \,. \qquad (2.25)$$

This Poisson bracket is known as the quadratic r-bracket, the Skylanin bracket, or the second Gelfand-Dickey bracket. The proof of the Jacobi identity for the Poisson bracket (2.25) is easy if we remember that because of the derivation property it is sufficient to check it for linear functions; i.e., functions of the form $f_X(a) = \mathrm{Tr}(aX)$, $X \in \mathscr{A}$. In the proof we also use the fact that R satisfies the modified Yang-Baxter equation,

$$[RX, RY] = R([RX, Y] + [X, RY]) - [X, Y] \quad \forall X, Y \in \mathscr{A} \,. \qquad (2.26)$$

Note that the R Lie-Poisson bracket and the quadratic r-bracket are *compatible*.

---

[2] This lemma was explained to us by A. G. Reyman

Lemma 2.2. *With respect to the quadratic r-bracket on $\mathscr{A}$, the mapping $m: \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ given by $m(a, b) = ab$ is a Poisson mapping when $\mathscr{A} \times \mathscr{A}$ has the product Poisson structure corresponding to the quadratic r-bracket on each of its components.*

Proof. For a pair of elements $X, Y \in \mathscr{A}$, consider the linear functions on $\mathscr{A}$ given by

$$f_X(l) = \operatorname{Tr} lX\,, \qquad f_Y(l) = \operatorname{Tr} lY\,.$$

Then

$$m^* f_X(a, b) = \operatorname{Tr} abX \quad \text{and} \quad m^* f_Y(a, b) = \operatorname{Tr} abY\,.$$

Variation of $m^* f_X$ with respect to a for fixed b gives

$$\nabla_a m^* f_X|_{(a,b)} = bX\,,$$

similarly

$$\nabla_b m^* f_X|_{(a,b)} = Xa\,.$$

Thus

$$\begin{aligned}
\{m^* f_X, m^* f_Y\}^{(2)}_{\mathscr{A} \times \mathscr{A}}(a, b) &= \tfrac{1}{2} \operatorname{Tr}(bYaR(bXa) - abYR(abX)) \\
&\quad + \tfrac{1}{2} \operatorname{Tr}(YabR(Xab) - bYaR(bXa)) \\
&= \tfrac{1}{2} \operatorname{Tr}(YabR(Xab) - abYR(abX)) \\
&= (m^* \{f_X, f_Y\}^{(2)}_{\mathscr{A}})\,(a, b)\,.
\end{aligned}$$
□

We shall apply this lemma to the following example:

$$\mathscr{A} = \left\{ X = \sum_{s=-\infty}^{N} X_s \partial^s \,\middle|\, \forall X_s : S^1 \to p \times p \text{ matrices}, \forall N \geq 0 \right\} := \mathrm{GD_M}\,, \tag{2.27}$$

the space of pseudo-differential operators with $p \times p$ matrix coefficients. We call this space $\mathrm{GD_M}$ for "matrix Gelfand-Dickey." Multiplication of matrix pseudo-differential operators is defined in the usual way [14, 16]. The trace form, $\operatorname{Tr} : \mathrm{GD_M} \to \mathbf{C}$, is given by

$$\operatorname{Tr} X := \int_{S^1} \operatorname{tr} \operatorname{res}(X) = \int_{S^1} \operatorname{tr} X_{-1}\,, \tag{2.28}$$

where tr is the ordinary matrix trace. The two subalgebras A and B in this case are given by

$$A := \mathrm{GD_{M_+}} = \left\{ X = \sum_{s=0}^{N} X_s \partial^s \right\}, \quad B := \mathrm{GD_{M_-}} = \left\{ X = \sum_{s=-\infty}^{-1} X_s \partial^s \right\}. \tag{2.29}$$

It is usual to write $P_A(X) = X_+$ and $P_B(X) = X_-$. One sees by inspection that for any $k > 0$, the space

$$M_{0,k} := \{ X = X_0 \partial^k + X_1 \partial^{k-1} + \ldots + X_k \mid X_0 \text{ fixed} \} \subset \mathrm{GD_M} \tag{2.30}$$

is a Poisson subspace with respect to both Poisson brackets (2.24) and (2.25).

In particular, the space M of operators L defined by (2.16) is such a subspace, on which the two Poisson brackets take the form

$$\{\varphi,\psi\}^{(1)}(L) = \int_{S^1} \operatorname{tr}\operatorname{res}(L[Y_-, X_-]), \tag{2.31}$$

$$\{\varphi,\psi\}^{(2)}(L) = \int_{S^1} \operatorname{tr}\operatorname{res}(YL(XL)_+ - LY(LX)_+), \tag{2.32}$$

where $X := \nabla_L\varphi$, $Y := \nabla_L\psi$ for $\varphi, \psi \in C^\infty(M)$. It is now easy to prove the following corollary of Lemma 2.2.

Corollary 2.3. *The mapping $\pi:\Theta \to M$ given by Eq. (2.22) is a Poisson mapping where the Poisson bracket of $f, h \in C^\infty(\Theta)$ is given by*

$$\{f,h\}(\theta_1,\ldots,\theta_r) = \sum_{i=1}^{r} \int_{S^1} \operatorname{tr}\left(\theta_i\left[\frac{\delta f}{\delta\theta_i}, \frac{\delta h}{\delta\theta_i}\right] + \left(\frac{\delta f}{\delta\theta_i}\right)' \frac{\delta h}{\delta\theta_i}\right), \tag{2.33}$$

and the Poisson bracket of $\varphi,\psi \in C^\infty(M)$ is given by the second Gelfand-Dickey bracket (2.32) *on the manifold M.*

Proof. It is sufficient to check that on the Poisson subspace $\{-D^{-1}(\partial+\theta)\}$ of $\mathrm{GD_M}$ (a subspace of type $M_{0,1}$) the Poisson bracket (2.25) becomes just the $r = 1$ case of (2.33), that is, if $\varphi,\psi \in C^\infty(\{\theta\})$, then

$$\{\varphi,\psi\}^{(2)}(-D^{-1}(\partial+\theta)) = \int_{S^1} \operatorname{tr}\left(\theta\left[\frac{\delta\varphi}{\delta\theta}, \frac{\delta\psi}{\delta\theta}\right] + \left(\frac{\delta\varphi}{\delta\theta}\right)' \frac{\delta\psi}{\delta\theta}\right). \quad \square \tag{2.34}$$

We may now state the main result of this section.

Theorem 2.4. *Under the identification $\mathscr{M}_{\mathrm{red}} \cong M$ given in* (2.21), *the Poisson brackets on the space $\mathscr{M}_{\mathrm{red}} = \mathscr{M}_c/\mathscr{N}$ obtained by reduction from the Poisson brackets* (1.24a) *and* (1.24b) *on $\mathscr{M}$ are equal respectively to the first and second Gelfand-Dickey Poisson brackets given by* (2.31) *and* (2.32).

Proof. Consider the one parameter group of transformations on $\mathscr{M}$ defined by

$$g_\tau : \mathscr{L} \mapsto (\mathscr{L} - \tau C_1), \qquad \tau \in \mathbf{R}. \tag{2.35}$$

These transformations preserve $\mathscr{M}_c \subset \mathscr{M}$ and commute with the action of $\mathscr{N}$. Thus we have a corresponding one parameter group of transformations $\{\bar g_\tau\}$ on $M \cong \mathscr{M}_c/\mathscr{N}$, operating as

$$\bar g_\tau : (L \mapsto L + \tau 1_p) \quad (1_p\text{: the } p\times p \text{ unit matrix}). \tag{2.36}$$

We now recall that $\{\ ,\ \}_1$ given by (1.24a) is the Lie derivative of $\{\ ,\ \}_2$ given by (1.24b) with respect to the vector field generating the flow (2.35). It follows that a similar relation holds for the corresponding induced brackets on $M \cong \mathscr{M}_{\mathrm{red}}$ with respect to the generator of the projected flow (2.36). On the other hand, one also sees by inspection that the first Gelfand-Dickey bracket on M; i.e., the restriction of (2.24) given by (2.31), is the Lie derivative of the second Gelfand-Dickey bracket (2.32) with respect to the generator of (2.36). Therefore the equality of the "first brackets" claimed in the theorem follows if we prove the equality of the "second brackets."

This latter is obtained as an immediate consequence of Lemma 2.1 and of the above corollary to Lemma 2.2. □

Remark 2.3. It is clear from the proof that Theorem 2.4 remains valid if we let D be any invertible $p \times p$ matrix in the construction. The choice of a graded regular element for Λ in (2.2) is made in order to guarantee the existence of local monodromy invariants by Proposition 1.2. In the next section this assumption will be fully used in determining the Hamiltonians of the reduced AKS system.

Remark 2.4. Let $R_c : \mathrm{GD_M} \to \mathrm{GD_M}$ be the mapping defined by right-multiplication by the invertible (constant) matrix c; i.e., $R_c(a) = ac$ for any $a \in \mathrm{GD_M}$. This is a one-to-one mapping that preserves the second Gelfand-Dickey bracket and carries the first Gelfand-Dickey bracket into the Poisson bracket $\{\ ,\ \}_c^{(1)}$ on $\mathrm{GD_M}$ given by

$$\{\varphi, \psi\}_c^{(1)}(a) = \mathrm{Tr}(ac^{-1}[cX, cY]_R)\,, \quad \text{where} \quad X = \nabla_a \varphi, Y = \nabla_a \psi\,, \tag{2.37}$$

which is compatible with $\{\ ,\ \}^{(2)}$ for any c. By means of this mapping, with $c = (-D)^r$, we can map the space M of L's in Eq. (2.16) onto the space $\tilde{M}$ consisting of the r^{th} order differential operators of the form

$$\tilde{L} = \partial^r + \sum_{i=1}^{r} \tilde{u}_i \partial^{r-i}\,, \qquad \tilde{u}_i : S^1 \to gl_p\,. \tag{2.38}$$

This is the reduced space we would have obtained by replacing the matrix D by the unit matrix in the reduction procedure described in the previous section. (The results of Theorem 2.10 and Corollary 2.11 below, which characterize the reduced monodromy invariants, would not then be applicable.) The one-to-one mapping between M and $\tilde{M}$ provided by R_c can be made into an isomorphism of bihamiltonian manifolds if one lets $M \subset \mathrm{GD_M}$ carry the restrictions of the two Gelfand-Dickey brackets, while $\tilde{M} \subset \mathrm{GD_M}$ carries the restriction of the "modified first bracket" $\{\ ,\ \}_c^{(1)}$ with $c = (-D)^r$, plus the restriction of the second Gelfand-Dickey bracket.

2.3. *Computation of the Hamiltonians*

In the previous section, we identified the Poisson brackets carried by the reduced space $\mathscr{M}_{\mathrm{red}} \cong M$ as the first and second Gelfand-Dickey Poisson brackets. In this section we establish a description of the Hamiltonians of the reduced AKS system in terms of the reduced-space variables $u_1, \ldots, u_r$. This will result from a computation of the eigenvalues of the monodromy matrix of the linear problem $\mathscr{L}\Psi = 0$ for an arbitrary $\mathscr{L} \in V$; i.e., for

$$\mathscr{L} = \partial + C_0 + \lambda C_1 + \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ v_r & \ldots & v_1 \end{pmatrix}. \tag{2.39}$$

Define the matrix Δ by

$$\Delta := (-D)^{-1}\,. \tag{2.40a}$$

Note that Δ^r is a nondegenerate and invertible diagonal matrix, by Eq. (2.2). Let

$$L = \Delta^r \partial^r + u_1 \partial^{r-1} + \ldots + u_r \tag{2.40b}$$

be the element of M corresponding to $\mathscr{L} \in V$. If $\{\phi_\alpha\}_{\alpha=1,\ldots,rp}$ is a complete set of independent (p-component column vector) solutions to

$$L\phi = \lambda\phi\,, \tag{2.41}$$

define the $n \times n$ matrix Φ by

$$\Phi := \begin{pmatrix} \phi_1 & \ldots & \phi_{rp} \\ \Delta\phi_1' & \ldots & \Delta\phi_{rp}' \\ \vdots & & \vdots \\ \Delta^{r-1}\phi_1^{(r-1)} & \ldots & \Delta^{r-1}\phi_{rp}^{(r-1)} \end{pmatrix}. \tag{2.42}$$

Then the columns of Φ are a complete set of solutions to $\mathscr{L}\Psi = 0$ and $T = \Phi(2\pi)\Phi(0)^{-1}$ is the monodromy matrix, whose invariants will be computed.

Since Δ^r is nondegenerate and diagonal, we can, through the usual, recursive approach, find an operator g of the form

$$g = I + \sum_{i=1}^{\infty} g_i \partial^{-i}\,, \tag{2.43}$$

with $g_i(x + 2\pi) = g_i(x)$ for all i, such that

$$L = g\hat{L}g^{-1} \tag{2.44}$$

with $\hat{L}$ a *diagonal* matrix pseudo-differential operator; i.e., with $\hat{L}$ of the form

$$\hat{L} = \Delta^r\partial^r + \sum_{i=1}^{\infty} a_i\partial^{r-i}\,, \qquad a_i\colon \text{ all diagonal matrices}\,. \tag{2.45}$$

For example, if we require the g_i's to be off-diagonal then we can recursively uniquely determine both the g_i's and the a_i's as differential polynomials in the u_i's, by comparing the two sides of $Lg = g\hat{L}$ term-by-term according to powers of ∂. Later we shall need that

$$a_1 = [u_1]_{\text{diag}}\,. \tag{2.46}$$

Fix $-\zeta$ to be any r^{th} root of λ,

$$(-\zeta)^r = \lambda\,. \tag{2.47}$$

Consider the $p \times p$ diagonal matrix asymptotic series

$$\hat{\psi}(x,\zeta) = e^{\zeta Dx}(\hat{\phi}_0(x) + \zeta^{-1}\hat{\phi}_1(x) + \ldots) \quad \text{for} \quad \zeta \sim \infty \tag{2.48}$$

satisfying the equation

$$(\hat{L}\hat{\psi})(x,\zeta) = (-\zeta)^r\,\hat{\psi}(x,\zeta) \tag{2.49}$$

in the following sense (see e.g. [24]). Using the definition

$$(\partial^{-s}e^{\zeta Dx}) := (\zeta D)^{-s}\,e^{\zeta Dx} \tag{2.50}$$

for s any integer, and extending this in the obvious way to pseudo-differential operators, we can write

$$\hat{\psi}(x,\zeta) = (\mathscr{D}e^{\zeta Dx})\,, \tag{2.51a}$$

where

$$\mathscr{D} = d_0(x) + d_1(x)\partial^{-1} + \dots \tag{2.51b}$$

is a uniquely determined diagonal pseudo-differential operator. Then the action of any pseudo-differential operator $P \in \mathrm{GD_M}$ on $\hat{\psi}$ is defined by multiplication of pseudo-differential operators,

$$(P\hat{\psi})(x,\zeta) := ((P\mathscr{D})e^{\zeta Dx}). \tag{2.52}$$

The left-hand side of Eq. (2.49) is understood in this sense. If we assume that $\det \hat{\phi}_0(x) \neq 0$ in (2.48), then $\hat{\psi}$ is uniquely determined up to multiplication by an x-independent diagonal matrix of the form

$$c(\zeta) = c_0 + \zeta^{-1}c_1 + \dots, \quad \text{with} \quad \det c_0 \neq 0. \tag{2.53}$$

At this point we make use of a result given in [1, Theorem 2.9, pp. 1986–1987]. The proposition below states the result in terms of the diagonal pseudo-differential operator $\hat{L}$, and therefore we are dealing with p different copies of the scalar case.

Proposition 2.5. *If $\hat{L}$ is of the form* (2.45), *∂ may be expressed as:*

$$\partial = -D\hat{L}_1^{1/r} + \sum_{i=0}^{\infty} F_i(-D\hat{L}_1^{1/r})^{-i}, \qquad (-D = \Delta^{-1}), \tag{2.54}$$

with

$$F_0 = -\frac{1}{r}(-D)^r a_1 \quad \textit{and} \quad \int_0^{2\pi} (kF_k + (-D)^k \operatorname{res}(\hat{L}_1^{k/r})) = 0, \quad \textit{for} \quad k > 0, \tag{2.55}$$

where $\hat{L}_1^{1/r}$ is the unique diagonal pseudo-differential operator such that

$$\hat{L}_1^{1/r} = \Delta\partial + \sum_{i=0}^{\infty} b_i \partial^{-i} \quad \text{and} \quad (\hat{L}_1^{1/r})^r = \hat{L}. \tag{2.56}$$

A further argument from [1] can be applied directly to deduce from (2.49)

Lemma 2.6.

$$(\hat{L}_1^{1/r}\hat{\psi})(x,\zeta) = -\zeta\hat{\psi}(x,\zeta). \tag{2.57}$$

Applying ∂ to $\hat{\psi}$ and making use of Proposition 2.5 and Lemma 2.6, we get

$$\hat{\psi}' = \left(\zeta D + \sum_{i=0}^{\infty} F_i(\zeta D)^{-i}\right)\hat{\psi}. \tag{2.58}$$

This equation can be solved, giving

$$\hat{\psi}(x,\zeta) = \exp\left(\zeta Dx + \sum_{k=0}^{\infty}(\zeta D)^{-k}\int_0^x F_k\right)\hat{\psi}(0,\zeta). \tag{2.59}$$

It follows that

$$\hat{\psi}(x+2\pi,\zeta) = \hat{\psi}(x,\zeta)\gamma(\zeta), \tag{2.60}$$

where

$$\gamma(\zeta) = \hat{\psi}(0,\zeta)^{-1} \exp\left(2\pi\zeta D + \sum_{k=0}^{\infty}(\zeta D)^{-k} \int_0^{2\pi} F_k\right)\hat{\psi}(0,\zeta)\,. \tag{2.61}$$

Define the $p \times p$ matrix asymptotic series $\psi(x,\zeta)$ by

$$\psi(x,\zeta) := (g\hat{\psi})(x,\zeta)\,, \tag{2.62}$$

where g is the pseudo-differential operator appearing in (2.43). We then have the self-evident

Lemma 2.7. *The columns of* ψ *defined by* (2.62) *are solutions of* (2.41).

If we expand g in descending powers of $-\hat{L}_1^{1/r}$ instead of descending powers of ∂ we have

$$g = I + m_1(-\hat{L}_1^{1/r})^{-1} + m_2(-\hat{L}_1^{1/r})^{-2} + \dots \tag{2.63}$$

for some $m_1, m_2, \dots$, with $m_i(x+2\pi) = m_i(x)$ for all i. Then (2.57) gives

$$\psi(x,\zeta) = (g\hat{\psi})(x,\zeta) = m(x,\zeta)\hat{\psi}(x,\zeta)\,, \tag{2.64a}$$

where

$$m(x,\zeta) := (I + m_1(x)\zeta^{-1} + m_2(x)\zeta^{-2} + \dots)\,. \tag{2.64b}$$

From this we obtain

Lemma 2.8.

$$\psi(x+2\pi,\zeta) = \psi(x,\zeta)\gamma(\zeta)\,. \tag{2.65}$$

Now consider the set of r independent roots of λ, given by

$$\{-\zeta_k = -e^{2\pi i k/r}\zeta\}_{k=0,\dots,r-1}\,, \tag{2.66}$$

and define ψ_i by

$$\psi_i(x,\zeta) := \psi(x,\zeta_i) = m(x,\zeta_i)\hat{\psi}(x,\zeta_i)\,, \qquad i = 0,1,\dots,r-1\,. \tag{2.67}$$

The point of this construction is:

Lemma 2.9. *The columns of the* $n \times n$ *matrix*

$$\Phi = \begin{pmatrix} \psi_0 & \dots & \psi_{r-1} \\ \Delta\psi_0' & \dots & \Delta\psi_{r-1}' \\ \vdots & & \vdots \\ \Delta^{r-1}\psi_0^{(r-1)} & \dots & \Delta^{r-1}\psi_{r-1}^{(r-1)} \end{pmatrix} \tag{2.68}$$

are a complete set of solutions to $\mathscr{L}\Psi = 0$.

Proof. We only have to prove that $\det\Phi \neq 0$. This follows by checking the leading term for $\zeta \sim \infty$. Using (2.48), (2.64a, b) and (2.67) we have

$$\Phi(x,\zeta) \sim \begin{pmatrix} 1 & \dots & 1 \\ -\zeta_0 & \dots & -\zeta_{r-1} \\ \vdots & \dots & \vdots \\ (-\zeta_0)^{r-1} & \dots & (-\zeta_{r-1})^{r-1} \end{pmatrix} \begin{pmatrix} e^{\zeta_0 Dx}\hat{\phi}_0(x) & & \\ & \ddots & \\ & & e^{\zeta_{r-1}Dx}\hat{\phi}_0(x) \end{pmatrix} \tag{2.69}$$

for $\zeta \sim \infty$. The determinant of the matrix on the right-hand side of Eq. (2.69) is non-zero. □

From Lemma 2.8 we obtain

$$\Phi(x+2\pi,\zeta) = \Phi(x,\zeta)\Gamma(\zeta)\,, \tag{2.70}$$

where

$$\Gamma(\zeta) = \mathrm{diag}(\gamma(\zeta_0),\,\dots,\,\gamma(\zeta_{r-1}))\,. \tag{2.71}$$

It follows that $T = \Phi(2\pi)\Phi(0)^{-1}$ is conjugate to Γ. By combining this with (2.61), (2.55) and (2.46), we arrive at the main result of this section.

Theorem 2.10. *The monodromy matrix of $\mathscr{L}$ is conjugate to the diagonal matrix $\mathscr{T}$ given by*

$$\mathscr{T} = \exp\begin{pmatrix} 2\pi\zeta_0 D - \frac{1}{r}\mathscr{H}(\zeta_0) & & & \\ & 2\pi\zeta_1 D - \frac{1}{r}\mathscr{H}(\zeta_1) & & \\ & & \ddots & \\ & & & 2\pi\zeta_{r-1} D - \frac{1}{r}\mathscr{H}(\zeta_{r-1}) \end{pmatrix}, \tag{2.72}$$

where

$$\mathscr{H}(\zeta) = \int_0^{2\pi} (-D)^r\,[u_1]_{\mathrm{diag}} + \sum_{k=1}^{\infty}(-\zeta)^{-k}\frac{r}{k}\int_0^{2\pi}\mathrm{res}(\hat{L}_1^{k/r})\,, \tag{2.73}$$

and $\hat{L}$ is the diagonalized form of the operator $L \in M$ corresponding to $\mathscr{L} \in \mathscr{M}_c$.

Corollary 2.11. *All Hamiltonians of the reduced* AKS *system carried by M are generated by the ones in the following list:*

$$\begin{aligned} \mathscr{H}_{0,i} &= (-1)^r \int_0^{2\pi} (D^r u_1)_{ii}\,, \\ \mathscr{H}_{k,i} &= \frac{r}{k}\int_0^{2\pi} \mathrm{res}(\hat{L}_1^{k/r})_{ii}\,, \qquad i = 1,\,\dots,\,p;\ k = 1,2,\,\dots\,. \end{aligned} \tag{2.74}$$

The corresponding flows are subject to the relations

$$\{f, \mathscr{H}_{k,i}\}^{(2)} = \{f, \mathscr{H}_{k+r,i}\}^{(1)} \qquad \forall f \in C^\infty(M),\ \forall i,k\,. \tag{2.75}$$

The first Hamiltonians in each of the bihamiltonian ladders belonging to fixed $k \bmod r$ and fixed i are Casimirs of the first Gelfand-Dickey bracket,

$$\{f, \mathscr{H}_{k,i}\}^{(1)} = 0\,, \qquad \forall f \in C^\infty(M) \quad \textit{for} \quad k = 0,1,\,\dots,\,(r-1)\,. \tag{2.76}$$

The number of independent bihamiltonian ladders in (2.75) *is $pr - 1 = n - 1$ since*

$$\sum_{i=1}^{p} \mathscr{H}_{mr,i} = 0\,, \quad \textit{for any } m = 1,2,\,\dots\,, \tag{2.77}$$

and $\sum_{i=1}^{p} \mathscr{H}_{0,i}$ is a Casimir with respect to both Gelfand-Dickey Poisson brackets.

Proof of Corollary 2.11. It is clear from the form of the matrix $\mathscr{T}$ in (2.72) that the Hamiltonians of the reduced AKS system; i.e., those obtained by reduction from the ad*-invariant Hamiltonians on the dual of the loop algebra, are generated by those in the list (2.74). The relations given by (2.75) and (2.76) can be traced back to the "general recursion relation" given by (1.10), but can also be verified directly by using the expressions of the Gelfand-Dickey brackets, (2.31), (2.32), and the Hamiltonians (2.74). That $\sum_{i=1}^{p} \mathscr{H}_{0,i}$ is a Casimir of both Poisson brackets also follows by direct verification. As for (2.77), note that

$$\sum_{i=1}^{p} \mathscr{H}_{mr,i} = \int_0^{2\pi} \operatorname{tr}\operatorname{res} \hat{L}^m = \int_0^{2\pi} \operatorname{tr}\operatorname{res} L^m = 0\,, \tag{2.78}$$

where the second equality follows from (2.44) and the third equality holds because L is a differential operator. □

Remark 2.5. Let ρ be a $p \times p$ diagonal matrix whose entries are all r^{th} roots of 1,

$$\rho = \operatorname{diag}(\rho_1, \ldots, \rho_p) \quad \text{with} \quad (\rho_i)^r = 1\,. \tag{2.79}$$

There is a unique r^{th} root of L whose leading term is given by $\rho\Delta\partial$. We denote this pseudo-differential operator by $L_\rho^{1/r}$. By using (2.43), (2.44), and (2.56) we can write

$$L_\rho^{1/r} = g\rho\hat{L}_1^{1/r} g^{-1}\,, \tag{2.80}$$

which implies that

$$\frac{r}{k}\operatorname{Tr}(L_\rho^{k/r}) = \frac{r}{k}\operatorname{Tr}(\rho^k \hat{L}_1^{k/r}) = \sum_{i=1}^{p} \rho_i^k \mathscr{H}_{k,i}\,. \tag{2.81}$$

For the Hamiltonian $\mathscr{H}_k^\rho := \frac{r}{k}\operatorname{Tr} L_\rho^{k/r} (k \neq 0 \bmod r)$ we have

$$\nabla_L \mathscr{H}_k^\rho = L_\rho^{k/r-1}\,. \tag{2.82}$$

From this we obtain, as in the scalar case [1],

$$\dot{L} = \{L, \mathscr{H}_k^\rho\}^{(2)} = \{L, \mathscr{H}_{k+r}^\rho\}^{(1)} = [(L_\rho^{k/r})_+, L]\,. \tag{2.83}$$

There are r^p possible choices of the matrix ρ in (2.79) and it is possible to single out p of them in such a way that the corresponding $\mathscr{H}_k^\rho$'s form a basis for the linear space spanned by the $\mathscr{H}_{k,i}$'s, for any fixed $k \neq 0 \bmod r$. Thus for $k \neq 0 \bmod r$ the relations (2.75) and (2.76) follow from (2.83).

Remark 2.6. Note that the diagonal terms of the potential u_1 of L in (2.40b) are constant along the flows of the hierarchy. This is obvious for flows of the type (2.83), and follows for all the flows generated by the ad*-invariants by combining the following two facts, which are easy to verify. First, the function

$$\mathscr{H}_K(L) = (-1)^r \int_0^{2\pi} \operatorname{tr}(KDu_1D^{r-1})\,, \tag{2.84}$$

where K is any gl_p valued function on S^1, generates via the second Gelfand-Dickey bracket the following one parameter group of transformations:

$$f_t : L \mapsto e^{tK} L e^{-tK} . \tag{2.85}$$

Second, these transformations leave the Hamiltonians (2.74) invariant for arbitrary *diagonal* K. For example, if $K = \mathrm{diag}(K_1, \ldots, K_p)$ is a constant diagonal matrix then

$$\mathscr{H}_K = \sum_{i=1}^{p} K_i \mathscr{H}_{0,i} \tag{2.86}$$

is one of the Hamiltonians of the hierarchy. It generates the evolution equation

$$\dot{L} = \{L, \mathscr{H}_K\}^{(2)} = [K, L] , \tag{2.87}$$

whose flow (2.85) indeed leaves $[u_1]_{\mathrm{diag}}$ invariant (but not the full matrix u_1). Finally, we could also consider symmetry reductions of the hierarchy under the abelian group action given by (2.85) with diagonal K's. The simplest reduction would be defined by setting $\mathrm{tr}(D^r u_1) = 0$. In this case, the reduced system is the same as the one obtained by using the Lie algebra sl_n instead of gl_n throughout the construction. Another symmetry reduction, which in a sense is maximal, is obtained by setting $[u_1]_{\mathrm{diag}} = 0$. This leads to the system studied previously by Gelfand-Dickey [14], Manin [16] and Wilson [17].

2.4. *Some Explicit Formulae in the Case $r = 2$*

Here we consider the simplest example $r = 2$ in order to make our general results more concrete. We shall display the explicit form of the Poisson brackets and the first two Hamiltonians for each bihamiltonian ladder in our reduced AKS system.

Denote the general element of the phase space M as

$$L = \Delta^2 \partial^2 + u\partial + w . \tag{2.88}$$

We can easily write out the evolution equations generated by an arbitrary $\mathscr{H} \in C^\infty(M)$ via either of the two Gelfand-Dickey Poisson brackets by using

$$\nabla_L \mathscr{H} = \partial^{-2} \frac{\delta \mathscr{H}}{\delta u} + \partial^{-1} \frac{\delta \mathscr{H}}{\delta w} . \tag{2.89}$$

The Hamiltonian equation generated by means of the first Poisson bracket (2.31) is given by

$$\dot{u} = \{u, \mathscr{H}\}^{(1)} = \left[\Delta^2, \frac{\delta \mathscr{H}}{\delta w}\right] , \tag{2.90a}$$

and

$$\dot{w} = \{w, \mathscr{H}\}^{(1)} = \Delta^2 \left(\frac{\delta \mathscr{H}}{\delta w}\right)' + \left(\frac{\delta \mathscr{H}}{\delta w}\right)' \Delta^2 + \left[\Delta^2, \frac{\delta \mathscr{H}}{\delta u}\right] + \left[u, \frac{\delta \mathscr{H}}{\delta w}\right] . \tag{2.90b}$$

Observe that $[u]_{\mathrm{diag}}$ is in the centre of the first Poisson bracket since it does not change under any Hamiltonian flow. In the scalar case $p = 1$ all the commutator terms drop out of (2.90), and for $u = 0$ we would recover the first Poisson structure of the standard KdV hierarchy.

The Hamiltonian equation generated by $\mathscr{H}$ through the second Poisson bracket (2.32) is given by

$$\dot{u} = \{u, \mathscr{H}\}^{(2)} = \left(\Delta^2 \frac{\delta\mathscr{H}}{\delta u} u - u \frac{\delta\mathscr{H}}{\delta u} \Delta^2\right) + \left(\Delta^2 \frac{\delta\mathscr{H}}{\delta w} w - w \frac{\delta\mathscr{H}}{\delta w} \Delta^2\right) - \Delta^2\left(\frac{\delta\mathscr{H}}{\delta w} u\right)' - 2\Delta^2\left(\frac{\delta\mathscr{H}}{\delta u}\right)'\Delta^2 + \Delta^2\left(\frac{\delta\mathscr{H}}{\delta w}\right)''\Delta^2 , \tag{2.91a}$$

and

$$\dot{w} = \{w, \mathscr{H}\}^{(2)} = \Delta^2\left(\frac{\delta\mathscr{H}}{\delta w} w\right)' + w\left(\frac{\delta\mathscr{H}}{\delta w}\right)'\Delta^2 + \Delta^2\left(\frac{\delta\mathscr{H}}{\delta w}\right)'''\Delta^2 - \Delta^2\left(\frac{\delta\mathscr{H}}{\delta u}\right)''\Delta^2 + \left(\Delta^2 \frac{\delta\mathscr{H}}{\delta u} w - w \frac{\delta\mathscr{H}}{\delta u} \Delta^2\right) + \left(u \frac{\delta\mathscr{H}}{\delta w} w - w \frac{\delta\mathscr{H}}{\delta w} u\right) - u\left(\frac{\delta\mathscr{H}}{\delta w} u\right)' + u\left(\frac{\delta\mathscr{H}}{\delta w}\right)''\Delta^2 - \Delta^2\left(\frac{\delta\mathscr{H}}{\delta w} u\right)'' - u\left(\frac{\delta\mathscr{H}}{\delta u}\right)'\Delta^2 . \tag{2.91b}$$

By straightforward computation from (2.43–2.45) we obtain

$$\mathscr{H}_{2,i}(u, w) = \int_0^{2\pi} \operatorname{res}(\hat{L})_{ii} = \int_0^{2\pi} h_{2,i} , \tag{2.92a}$$

with

$$h_{2,i} = \sum_{k\neq i} \frac{\Delta_i^2 + \Delta_k^2}{(\Delta_i^2 - \Delta_k^2)^2} u_{ik} u'_{ki} - \sum_{k\neq i} \frac{u_{ik} u_{ki} u_{ii}}{(\Delta_i^2 - \Delta_k^2)^2} + \sum_{k\neq i}\sum_{l\neq i} \frac{u_{ik} u_{kl} u_{li}}{(\Delta_i^2 - \Delta_k^2)(\Delta_i^2 - \Delta_l^2)} + \sum_{k\neq i} \frac{u_{ik} w_{ki} + w_{ik} u_{ki}}{(\Delta_i^2 - \Delta_k^2)} , \tag{2.92b}$$

where the Δ_i's are the components the $p \times p$ diagonal matrix Δ. The above formulae may be checked by verifying that the first evolution equation belonging to the bihamiltonian ladder containing $\mathscr{H}_{2,i}$ can be written as

$$\dot{L} = \{L, \mathscr{H}_{0,i}\}^{(2)} = \{L, \mathscr{H}_{2,i}\}^{(1)} = [e_{ii}, L] , \qquad \left(\mathscr{H}_{0,i}(u, w) = \int_0^{2\pi} \frac{u_{ii}}{\Delta_i^2}\right) , \tag{2.93}$$

consistently with (2.75), (2.86) and (2.87). The equation generated by $\mathscr{H}_{2,i}$ via the second Poisson bracket can now be obtained by substituting the expressions for the functional derivatives $\frac{\delta\mathscr{H}_{2,i}}{\delta u}$ and $\frac{\delta\mathscr{H}_{2,i}}{\delta w}$ into (2.91a, b). The result is somewhat lengthy, so we do not display it here.

We next give the first two Hamiltonians in the bihamiltonian ladder of Eq. (2.83); i.e.,

$$\mathscr{H}_k^\rho = \sum_{i=1}^{p} (\rho_i)^k \mathscr{H}_{k,i} = \frac{2}{k} \int_0^{2\pi} \operatorname{tr} \operatorname{res} L_\rho^{k/2} = \int_0^{2\pi} h_k^\rho \quad \text{for} \quad k = 1, 3 , \tag{2.94}$$

where the entries of $\rho = \mathrm{diag}(\rho_1, \ldots, \rho_p)$ are ± 1's [see (2.79)]. To write down the result, it is useful to define the diagonal matrix σ by

$$\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_p) := \rho\Delta\,, \tag{2.95}$$

and to associate to any $p \times p$ matrix v the matrix $\bar{v}$ given by the formula

$$(\bar{v})_{ij} := \frac{v_{ij}}{(\sigma_i + \sigma_j)}\,. \tag{2.96}$$

Using this notation, we have

$$h_1^\rho = \mathrm{tr}\,\sigma^{-1} w - \mathrm{tr}\,\sigma^{-1}\bar{u}^2\,, \tag{2.97}$$

and

$$\begin{aligned} h_3^\rho &= \mathrm{tr}\,\bar{w}\left(w - \tfrac{4}{3}\,\bar{u}'\sigma - 2\sigma\bar{u}' - 2\bar{u}^2\right) + \mathrm{tr}\,\bar{u}'\left(\sigma\bar{\bar{u}}'\sigma + \tfrac{4}{3}\,\sigma^2\bar{\bar{u}}'\right) \\ &\quad + \mathrm{tr}\,\bar{u}^2\left(\overline{(\bar{u}^2)} + 2\sigma\bar{\bar{u}}' + \tfrac{4}{3}\,\bar{\bar{u}}'\sigma^2\right). \end{aligned} \tag{2.98}$$

The first case of Eq. (2.83) is given explicitly as follows:

$$\dot{u} = \{u, \mathscr{H}_1^\rho\}^{(2)} = \{u, \mathscr{H}_3^\rho\}^{(1)} = \sigma u' - 2\sigma^2\bar{u}' + [\bar{u}, u] + [\sigma, w]\,, \tag{2.99a}$$

$$\dot{w} = \{w, \mathscr{H}_1^\rho\}^{(2)} = \{w, \mathscr{H}_3^\rho\}^{(1)} = \sigma w' - \sigma^2\bar{u}'' - u\bar{u}' + [\bar{u}, w]\,. \tag{2.99b}$$

These equations simplify to the free chiral wave equation in the scalar case $p = 1$ after putting $u = 0$, as they should. The analogue of the KdV equation is generated by $\mathscr{H}_3^\rho$ through the second Poisson bracket (2.91a, b). It is straightforward to derive it from the above, but the final expression is quite long.

Remark 2.7. There is no singularity in the formulae (2.92b), (2.96–2.98) above since the diagonal matrix $\Delta^2 = (-D)^{-2}$ has distinct, non-zero eigenvalues. This goes back to choosing $\Lambda = \Lambda_r \otimes D$ as a *regular* element of the Heisenberg subalgebra at the beginning of the construction. In general, for any r, if we continuously deform D to the unit matrix, which represents a singular case (type II in the terminology of [7]), then the Hamiltonians $\mathscr{H}_{mr,i}$ in (2.74) $(m = 1, 2, \ldots)$ may be expected to become ∞ and thus disappear from the hierarchy. In addition, it follows from a result in [16] that in the $D = I$ case there exists only a single r^{th} root of L, up to a scalar multiple.

3. Discussion

This paper was aimed at further exploring the integrable systems that can be associated to graded regular elements of loop algebras by the method of Drinfeld and Sokolov. We have concentrated on the simplest case, given by the nontwisted loop algebra $\ell(gl_n)$, and have shown that graded regular elements exist only in those Heisenberg subalgebras that correspond to partitions into equal blocks $n = pr$ or equal blocks plus a singlet $n = pr + 1$. We further analyzed the first case by taking the grade 1 regular elements and proved that the generalized DS reduction results in the matrix version of the r-KdV hierarchy of Gelfand-Dickey. We wish to close by mentioning some other interesting models, which are related to this KdV type hierarchy in a way that is familiar in the scalar case $p = 1$.

First of all, one has the modified KdV type hierarchy which is obtained from the general AKS system by using the block diagonal gauge. This means that the

Hamiltonians of the modified hierarchy are still given by the functions of L listed in (2.74), but L (and consequently $\hat{L}$) is considered to depend on

$$\mathscr{L} = \partial + \theta + \Lambda\,, \quad \text{where} \quad \theta := \mathrm{diag}[\theta_1, \ldots, \theta_r]\,, \tag{3.1}$$

through the Miura map (2.22). The evolution equations of the modified hierarchy are generated by means of the free current algebra Poisson bracket carried by the θ_i's, which are the basic fields of the modified hierarchy.

The nonabelian affine (periodic) Toda field equation may be defined as follows. Take the basic Toda field to be a Gl_n-valued block diagonal matrix $g(x,t)$,

$$g = \mathrm{diag}[g_1, \ldots, g_r]\,, \tag{3.2}$$

where the g_i's ($i = 1, \ldots, r$) are Gl_p-valued functions, periodic in the space variable x. In addition to the grade 1 regular element Λ, choose also a grade -1 regular element $\bar{\Lambda}$ and define the Toda equation to be the zero curvature equation

$$[\mathscr{L}_+, \mathscr{L}_-] = 0\,, \tag{3.3a}$$

where

$$\mathscr{L}_+ := \partial_+ + g^{-1}\partial_+ g + \Lambda\,, \qquad \mathscr{L}_- := \partial_- + g^{-1}\bar{\Lambda} g\,, \tag{3.3b}$$

and $\partial_\pm = (\partial_x \pm \partial_t)$. Here, $\mathscr{L}_+$ is obtained from the modified KdV operator $\mathscr{L}$ in (3.1) by substituting ∂_+ for ∂ and $g^{-1}\partial_+ g$ for θ, and therefore we have the same relationship between the conservation laws of the present Toda model and the modified KdV hierarchy as is familiar in the abelian case [1, 10, 25]. In particular, by the same arguments as in [1], we can construct an infinite number of conserved local currents for the Toda equation by transforming $\mathscr{L}_+$ into the Heisenberg subalgebra. Note also that another infinite set of conserved currents can be constructed by utilizing the fact that the Toda equation can equivalently be written as

$$[\tilde{\mathscr{L}}_+, \tilde{\mathscr{L}}_-] = 0\,, \tag{3.4a}$$

with

$$\tilde{\mathscr{L}}_+ := \partial_+ + g\Lambda g^{-1}\,, \qquad \tilde{\mathscr{L}}_- := \partial_- - \partial_- g \cdot g^{-1} + \bar{\Lambda}\,, \tag{3.4b}$$

and transforming $\tilde{\mathscr{L}}_-$ into the Heisenberg subalgebra.

The nonabelian affine Toda model defined above was proposed originally by Mikhailov [26] and further studied, e.g., in [27]. More precisely, these authors took $\Lambda = \Lambda_r \otimes 1_p$ and $\bar{\Lambda} = \Lambda_r^{-1} \otimes 1_p$, where 1_p is the $p \times p$ unit matrix. These are not regular elements of the Heisenberg subalgebra and therefore the DS construction of local conservation laws corresponding to graded generators of the Heisenberg subalgebra would not be applicable with this choice without modifications.

A related nonabelian conformal (open) Toda model can be obtained by omitting the λ dependent terms from $\mathscr{L}_\pm(\tilde{\mathscr{L}}_\pm)$ in the above. This model is a member of a family of models described in [28] and can also be obtained by a Hamiltonian symmetry reduction of the Wess-Zumino-Novikov-Witten (WZNW) model. By using the WZNW picture it is clear that the $\mathscr{W}$-algebra given by the second matrix Gelfand-Dickey Poisson bracket can be realized as the algebra of Noether currents in the nonabelian conformal Toda model. Details can be found in [4, 5, 18]. Note that in these papers the group Sl_n was used instead of Gl_n, but one can impose the Sl_n constraints $\mathrm{tr}(D^r u_1) = 0$, $\mathrm{tr}\,\theta = 0$ and $\det g = 1$ without changing the essential features of the KdV, modified KdV or Toda systems, or their relationship.

It seems plausible that the nonabelian affine Toda model (as well as an appropriate "conformal affine" variant of it) can be viewed as a Hamiltonian symmetry reduction of the affine WZNW model, generalizing the abelian case [29]. This Toda model is also known to be a reduction of the multicomponent Toda lattice hierarchy [30]. Similarly, the matrix r-KdV hierarchy should be related to the multicomponent KP hierarchy [30, 31].

The other case in which a graded regular element exists in the Heisenberg subalgebra, corresponding to the partition $n = pr + 1$, has not been pursued in the present work. By taking an arbitrary regular element of minimal positive grade it may be verified that the generalized DS reduction proposed in [7] leads to a $\mathscr{W}$-algebra which is again equal to one of those studied in [4, 5] in the context of WZNW reductions. In these papers a family of $\mathscr{W}$-algebras was associated to the sl_2 subalgebras of gl_n (sl_n). The $\mathscr{W}$-algebra arising from a regular element of minimal grade corresponds to the sl_2 subalgebra under which the defining representation of gl_n decomposes into p copies of the r-dimensional $sl(2)$ irreducible representation plus a singlet.

Recall that both the sl_2 subalgebras of gl_n and the Heisenberg subalgebras of $\ell(gl_n)$ are classified by the partitions of n. It is unclear whether there is a general relationship between the $\mathscr{W}$-algebras associated to sl_2 embeddings and KdV type hierarchies or not, since there is a $\mathscr{W}$-algebra for any partition, but graded regular elements exist only in exceptional cases.

The present work was based entirely on the Hamiltonian AKS approach to integrable systems. The Grassmannian approach [32, 34] has also been generalized to other Heisenberg subalgebras than the principal one in [33, 31, 9]. It is clear that the integrable systems associated to graded regular elements in arbitrary loop algebras would deserve further study from both viewpoints. The starting point could be the explicit description of Heisenberg subalgebras recently worked out in [34].

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