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Classification of Generic 3-dimensional Lagrangian Singularities with $(Z_2)^l$-symmetries

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Abstract. The paper provides the complete list of local models for $\mathbf{Z}_2^l$-invariant generic germs of Lagrangian submanifolds of dimension ≤ 3. Classification is done directly for genrating functions of Lagrangian submanifolds and contains both elementary singularities and non-elementary ones with continuous moduli. The results demonstrate, in particular, that in contrast to the non-equivariant case the classification of equivariant Lagrangian singularities is not subordinated to the classification of symmetric functions up to the right equivariant equivalences.

1. Introduction

One of the most important steps in the initial development of singularity theory of Lagrangian submanifolds was finding that the singularities of (*non-equivariant*) canonical Lagrangian projections are completely determined (at least locally) by singularities of smooth generating functions (or generating families of functions). A crucial contribution to the problem was made by Arnold [2] who found the complete classification of stable singularities of Lagrangian submanifolds of dimension ≤ 5, inspiring further investigations in that direction (cf. [4, 3, 8, 25]). The standard (non-symmetric) theory of Lagrangian singularities has various important applications. In many of them non-trivial symmetries appear as an additional constraint and thus the problem of classification of $\mathcal{G}$-invariant Lagrangian submanifolds (with $\mathcal{G}$ being a compact Lie group of symmetry) emerges naturally. This problem was introduced and initially investigated in [13], then the formal stability theory was continued in the papers [15, 16].

In the present paper, we investigate the discrete groups of symmetries $\mathbf{Z}_2^l$. Such symmetries appear for instance in the problem of determination of symmetric caustics in geometrical optics of lenses [5, 17], in thermodynamical phase transitions in ordered systems [15] and in equivariant bifurcation theory [10]. The considerations are more complex and technical than in the non-equivariant case. On top of the usual complications caused by the symmetry we encounter additional obstacles. Firstly, in contrast to the non-equivariant case, the classifiction of equivariant Lagrangian singularities is not subordinated to the classification of symmetric functions up to the right equivariant equivalences. Secondly, at this stage the implication "infinitesimal stability $\Rightarrow$ stability" in our case has to be proved on a case by case basis. In the paper, this problem is implicit in the process of derivation of normal forms and is crucial, in particular, in providing the proper and rigorous identification of moduli.

One has the following two alternative approaches to the classification of equivariant Lagrangian singularities [16].

(i) An expansion of the original, partially heuristic approach of Arnold [2] which was based on derivation of the infinitesimal condition with the explicit use of hamiltonian vector fields on the ambient symplectic space.

(ii) A method based on direct use of versal unfoldings (following, in particular, Zakalyukin [25] and Arnold et al. [3]) modified by a respective addition of symmetry in the unfolding parameters (cf. [17]).

Unlike the non-symmetric case, the infinitesimal conditions yielded by (i) and (ii) have different forms. Roughly, (i) provides a more complicated, non-linear condition but involving fewer variables than the linear condition given by (ii). Those two infinitesimal conditions are equivalent at least at some particular simple cases of symmetry [16]. Leaving the question of equivalence in its full scope aside, we choose (i) for considerations in this paper. The reason for this is pragmatic: we find fewer variables as a considerable advantage in practical computations.

In the paper we give a complete list of local forms for $\mathbf{Z}_2^l$-invariant generic germs of Lagrangian submanifolds of dimension ≤ 3. We find that continuous invariants of normal forms appear in dimension 3 (cf. singularity Ξ_3 in Table 2 below) in contrast to the standard (non-symmetric) theory where the modal parameters appear from dimension 6 onwards [25]. Although the family Ξ_3 of all equivariant Lagrangian germs forms only two equivalence classes up to *non-equivariant* Lagrangian equivalences, it splits into uncountably many orbits under the action of *equivariant* Lagrangian equivalences (cf. Remark 3.8 below). Furthermore, examination of this family provides a direct elementary proof for independence of two classifications: (i) the classification of the equivariant Lagrangian singularities up to equivariant Lagrangian equivalences and (ii) the classification of the symmetric functions up to equivariant right equivalences (cf. [20, 24]).

The paper is organised as follows. Section 2 contains some introductory results concerning $\mathbf{Z}_2^l$ actions on $\mathbf{R}^2$ and $\mathbf{R}^3$, equivariant Lagrangian germs, pull-backs of Lagrangian manifolds and infinitesimal stability conditions. The main results of the paper are presented in Sect. 3. In particular the classification of generic Lagrangian singularities of dimension ≤ 3 with $\mathbf{Z}_2^l$ symmetry and their normal forms (up to an equivariant Lagrangian equivalence) are described by Theorems 3.3 and 3.6 and listed in Table 2; proofs are deferred to Sect. 4.

2. Preliminaries

We do not give proofs of most results in this section as they are fairly straightforward extensions of the results of Arnold [2] to the equivariant case. All functions, mappings, germs, etc., considered in this paper are C^∞. Symbol $\mathscr{G}$ will be used to denote a compact Lie group. All actions of $\mathscr{G}$ on the Euclidean spaces $\mathbf{R}^n$ are assumed to be orthogonal (with respect to the Euclidean inner product). Symbols $C^\infty_{\mathscr{G}}(n)$, $\mathscr{E}_{\mathscr{G}}(n)$, $C^\infty_{\mathscr{G}}(n,k)$ and $\mathscr{E}_{\mathscr{G}}(n,k)$ are used to denote the spaces of $\mathscr{G}$-invariant functions on $\mathbf{R}^n$ and their germs at 0, of $\mathscr{G}$-invariant mappings $\mathbf{R}^n \mapsto \mathbf{R}^k$ and their germs with the source and target at the origins, respectively. By $\mathbf{m}^k_{\mathscr{G}}(n)$ we denote the ideal in $\mathscr{E}_{\mathscr{G}}(n)$ of all germs with all derivatives of order up to $k-1$ vanishing at 0. If on $\mathbf{R}^n$ and $\mathbf{R}^k$ we have actions of different groups, say $\mathscr{G}$ and $\mathscr{G}'$, then the set of $\mathscr{G}\oplus\mathscr{G}'$-invariant functions, $\mathscr{G}\oplus\mathscr{G}'$-invariant germs, etc., on $\mathbf{R}^n\times\mathbf{R}^k$ will be denoted by $C^\infty_{\mathscr{G}\oplus\mathscr{G}'}(n+k)$, $\mathscr{E}_{\mathscr{G}\oplus\mathscr{G}'}(n+k)$, etc. It is known [18, 19, 6] that there exists a *Hilbert map* $\varrho:(\mathbf{R}^n,0)\to(\mathbf{R}^a,0)$ such that $C^\infty_{\mathscr{G}}(n)=\varrho^*C^\infty(a)$, $\mathscr{E}_{\mathscr{G}}(n)=\varrho^*\mathscr{E}_{\mathscr{G}}(a)$ and that the $\mathscr{E}_{\mathscr{G}}(n)$-module $\mathscr{E}_{\mathscr{G}}(n,m)$ is finitely generated over $\mathscr{E}_{\mathscr{G}}(n)$. In most of this paper the group $\mathscr{G}$ under consideration will be $\mathbf{Z}_2^l=\mathbf{Z}_2\oplus\ldots\oplus\mathbf{Z}_2$ (l terms), where $\mathbf{Z}_2$ denotes the multiplicative group $\{-1,+1\}$. For integers $s_1,\ldots,s_l>0$ and $s_{l+1}\ge 0$ we have the canonical representation ν of $\mathbf{Z}_2^l$ on $\mathbf{R}^n=\mathbf{R}^{s_1}\times\ldots\times\mathbf{R}^{s_l}\times\mathbf{R}^{s_{l+1}}$ defined as follows:

$$\nu_g(x_1,\ldots,x_l,x_{l+1})\stackrel{\text{def}}{=}(\varepsilon_1x_1,\ldots,\varepsilon_lx_l,x_{l+1})$$

for every $(x_1,\ldots,x_l,x_{l+1})\in\mathbf{R}^{s_1}\times\ldots\times\mathbf{R}^{s_l}\times\mathbf{R}^{s_{l+1}}$ and every $g=(\varepsilon_1,\ldots,\varepsilon_l)\in\mathbf{Z}_2^l$. The multi-index $(s_1,\ldots,s_l\,|\,s_{l+1})$ is called the *rank of the representation* ν on $\mathbf{R}^n$ [24]. We will refer to ν as the representation of $(l+1)$-terms sum $\mathbf{Z}_2\oplus\ldots\oplus\mathbf{Z}_2\oplus\mathbf{1}\approx\mathbf{Z}_2^l$ on the $(l+1)$-factors Cartesian product $\mathbf{R}^{s_1}\times\ldots\times\mathbf{R}^{s_l}\times\mathbf{R}^{s_{l+1}}$ if we want to avoid the explicit writing down of the rank of ν. By straightforward calculations and elimination of cases with zero-dimensional space of fixed points we obtain the following result (cf. [23, Sect. 6]).

Lemma 2.1. *A linear action of $\mathbf{Z}_2^l$ on $\mathbf{R}^2$ or $\mathbf{R}^3$ is via a group homomorphism onto one of the following eight effective groups operating as described.*
For $\mathbf{R}^2$:
(i) *the trivial group* $\mathbf{1}$ *with the trivial action,*
(ii) $\mathbf{Z}_2$ *with the canonical action of* rank(2),
(iii) $\mathbf{Z}_2\approx\mathbf{Z}_2\oplus\mathbf{1}$ *with the canonical action of* rank(1 | 1),
(iv) $\mathbf{Z}_2\oplus\mathbf{Z}_2$ *with the canonical action of* rank(1, 1).
For $\mathbf{R}^3$:
(v) *the trivial group* $\mathbf{1}$ *with the trivial action,*
(vi) $\mathbf{Z}_2$ *with the canonical action of* rank(3),
(vii) $\mathbf{Z}_2\approx\mathbf{Z}_2\oplus\mathbf{1}$ *with the canonical action of* rank(2 | 1),
(viii) $\mathbf{Z}_2\approx\mathbf{Z}_2\oplus\mathbf{1}$ *with the canonical action of* rank(1 | 2),
(ix) $\mathbf{Z}_2\oplus\mathbf{Z}_2$ *with the canonical action of* rank(1, 2),
(x) $\mathbf{Z}_2\oplus\mathbf{Z}_2\approx\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{1}$ *with the canonical action of* rank(1, 1 | 1),
(xi) $\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2$ *with the canonical action of* rank(1, 1, 1),
(xii) $\mathbf{Z}_2\oplus\mathbf{Z}_2\approx\{(\alpha,\beta,\gamma)\in\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2;\alpha\beta\gamma=1\}$ *with the action induced by the canonical action in* (xi).

Lemma 2.2. *The Hilbert maps and generators of the modules of equivariant mappings for non-trivial effective group actions of $\mathbf{Z}_2^l$ on $\mathbf{R}^2$ and $\mathbf{R}^3$ with* dim{*space of fixed points*} ≥ 1 *are given in Table* 1.

Let us denote by (x,ξ) the canonical coordinates on $T^*\mathbf{R}^n\approx\mathbf{R}^{2n}$ induced by the natural coordinates x on $\mathbf{R}^n$, by π the canonical projection $T^*\mathbf{R}^n\to\mathbf{R}^n$, $(x,\xi)\mapsto x$ and by ω the canonical symplectic form $\sum_i d\xi_i\wedge dx_i$ on $T^*\mathbf{R}^n$. An orthogonal $\mathscr{G}$-action ν on $\mathbf{R}^n$ has the natural lifting $\nu^*\approx\nu\oplus\nu$ to $T^*\mathbf{R}^n$ of the form $\nu_g^*(x,\xi)\mapsto(\nu_g(x),\nu_g(\xi))$. An n-dimensional immersed submanifold $L=\iota(\mathbf{R}^n)$, $\iota:\mathbf{R}^n\to T^*\mathbf{R}^n$, is called a *Lagrangian $\mathscr{G}$-invariant submanifold (L$\mathscr{G}$-manifold)* if $\iota^*\omega=0$ and $\nu_g^*(L)=L$ for every $g\in\mathscr{G}$; such ι is called a *Lagrangian immersion.*

Table 1. Selected Hilbert maps and generators of $\mathscr{E}_{\mathscr{G}}(n,n)$

	$\mathbf{R}^n$/Effective group/ Repres. rank	Hilbert map $\varrho:\mathbf{R}^n \to \mathbf{R}^a$	Generators of $\mathscr{E}_{\mathscr{G}}(n,n)$ $x \mapsto V_i(x) \in \mathbf{R}^n$	$\varrho(\mathbf{R}^n) \subset \mathbf{R}^a$
1	$\mathbf{R}^2/\mathbf{Z}_2 \oplus 1/(1\,\vert\,1)$	(x_1^2, x_2)	$(x_1,0),(0,1)$	$z_1 \geq 0$
2	$\mathbf{R}^3/\mathbf{Z}_2 \oplus 1/(1\,\vert\,2)$	(x_1^2, x_2, x_3)	$(x_1,0,0),(0,1,0),(0,0,1)$	$z_1 \geq 0$
3	$\mathbf{R}^3/\mathbf{Z}_2 \oplus 1/(2\,\vert\,1)$	$(x_1^2, x_1x_2, x_2^2, x_3)$	$(x_1,0,0),(x_2,0,0),(0,x_1,0)$ $(0,x_2,0),(0,0,1)$	$z_1z_3 = z_2^2$
4	$\mathbf{R}^3/\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus 1/(1,1\,\vert\,1)$	(x_1^2, x_2^2, x_3)	$(x_1,0,0),(0,x_2,0),(0,0,1)$	$z_1, z_2 \geq 0$

Let I and J be two multi-indices such that $I \cap J = \emptyset$, $I \cup J = \{1,\ldots,n\}$ and $\mathbf{R}^I \stackrel{\text{def}}{=} \{y \in \mathbf{R}^n; y_J = 0\}$, $\mathbf{R}^J \stackrel{\text{def}}{=} \{y \in \mathbf{R}^n; y_I = 0\}$ are $\mathscr{G}$-invariant subspaces (for convenience we identify $y \in \mathbf{R}^n$ with the pair (y_I, y_J)). A transformation $\iota:\mathbf{R}^n \to T^*\mathbf{R}^n$ of the form

$$\iota(y_I, y_J) = \left(\frac{\partial S}{\partial y_I}(y_I,y_J), y_J, y_I, -\frac{\partial S}{\partial y_J}(y_I,y_J)\right), \tag{1}$$

where $S \in C^\infty_{\mathscr{G}}(\mathbf{R}^n)$, is a $\mathscr{G}$-invariant Lagrangian immersion. Function S will be called an *IJ-function* of the Lagrangian submanifold $L \stackrel{\text{def}}{=} \iota(\mathbf{R}^n) \subset T^*\mathbf{R}^n$. A germ $(L,0)$ of an $L\mathscr{G}$-manifold $L \subset T^*\mathbf{R}^n$ at $0 (\in L)$ will be called an *$L\mathscr{G}$-germ*. We say that $(S,0) = (S(\xi_I, x_J),0) \in \mathscr{E}_{\mathscr{G}}(n)$ is an *IJ-germ* of $(L,0)$ if the germ (L_S, q_0) $\Big($of the $L\mathscr{G}$-manifold L_S generated by the IJ-function S at $q_0 = \left(\left.\frac{\partial S}{\partial \xi_I}\right|_0, 0, 0, -\left.\frac{\partial S}{\partial x_J}\right|_0\right)\Big)$ equals $(L,0)$ up to translation $(x,\xi) \mapsto (x,\xi) - q_0$ of $T^*\mathbf{R}^n$. The IJ-germ $(S(\xi_I,x_J),0)$ is called *minimal* if $\partial^2 S/\partial\xi_i\partial\xi_{i'}|_0 = 0$ for every $i,i' \in I$. In this case $\#I$ (= cardinality of I) equals $\dim\ker((T\pi)|_{T_0L})$ and for any $I'J'$-germ of $(L,0)$ the inequality $\#I' \geq \#I$ holds. A minimal IJ-germ with $I \neq \emptyset$ will be called *non-trivial*. As we shall see soon, any $L\mathscr{G}$-germ can be transformed into one having a minimal IJ-germ by a smooth equivariant transformation $\Phi: T^*\mathbf{R}^n \to T^*\mathbf{R}^n$ preserving both the fibration π and the symplectic form ω (i.e. such that $\Phi^*\omega = \omega$). Such a Φ is called *an equivariant Lagrangian equivalence ($L\mathscr{G}$-equivalence)*. An IJ-germ and an $I'J'$-germ corresponding to $L\mathscr{G}$-equivalent $L\mathscr{G}$-germs will also be called *$L\mathscr{G}$-equivalent*. A germ of $L\mathscr{G}$-equivalence is of the form

$$\Lambda_{\psi,\alpha}(\eta) \stackrel{\text{def}}{=} (\psi^{-1})^*\eta + d\alpha, \qquad \eta \in T^*\mathbf{R}^n, \tag{2}$$

where $\psi:(\mathbf{R}^n,0) \to (\mathbf{R}^n,0)$ is an equivariant diffeomorphism and $\alpha \in \mathscr{E}_{\mathscr{G}}(n)$ [25]. With the help of $L\mathscr{G}$-equivalences the IJ-germs (or, equivalently, the immersion (1)) can be simplified as follows.

Lemma 2.3. *Any IJ-germ $(S(\xi_I,x_J),0) \in \mathscr{E}_{\mathscr{G}}(n)$ is $L\mathscr{G}$-equivalent to:*
(i) *the IJ-germ $(S - j_0^1 S, 0) \in \mathbf{m}^2_{\mathscr{G}}(n)$, where*

$$j_0^1 S \stackrel{\text{def}}{=} S(0) + \sum_{i\in I} \xi_i \partial S/\partial\xi_i(0) + \sum_{j\in J} x_j \partial S/\partial x_j(0),$$

(ii) *an* $\{1, \ldots, n\}\emptyset$*-germ* $(S'(\xi_1, \ldots, \xi_n), 0) \in \mathbf{m}^2_{\mathscr{G}}(n)$,
(iii) *an* $I''J''$*-germ* $(S(\xi_{I''}, x_{J''}), 0) \in \mathbf{m}^3_{\mathscr{G}}(n)$.

We omit the straightforward proof of this result.

2.1. Morse families of Pullbacks

We recall that a germ $(F, 0) \in \mathbf{m}^2_{\mathscr{G}}(m+n)$ of a family of functions $\mathbf{R}^n \ni x \mapsto F(\lambda, x)$, $\lambda \in \mathbf{R}^m$, is called a *Morse family germ (*Mf*-germ)* if

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial\lambda\partial\lambda}, \frac{\partial^2 F}{\partial\lambda\partial x}\right)\Bigg|_0 = m\,. \tag{3}$$

As it is known from Arnold [2] (see [13] for the equivariant case) such Mf-germ $(F, 0)$ generates an $L\mathscr{G}$-germ in $T^*\mathbf{R}^n$, say $(L(F), 0)$, by means of the equations

$$\frac{\partial F}{\partial \lambda}(\lambda, x) = 0\,, \qquad \xi = \frac{\partial F}{\partial x}(\lambda, x)\,. \tag{4}$$

Any $L\mathscr{G}$-germ is generated by an Mf-germ (e.g. by $F(\lambda, x_I, x_J) \stackrel{\text{def}}{=} S(\lambda, x_J) - \sum_{i\in I} \lambda_i x_i$, where $S(\xi_I, x_J)$ is an IJ-germ).

Let $(\phi, \alpha) \in \mathscr{E}_{\mathscr{G}}(n, \tilde{n}) \times \mathscr{E}_{\mathscr{G}}(n)$ and $(\tilde{L}, 0)$, $\tilde{L} \subset T^*\mathbf{R}^{\tilde{n}}$ be an $L\mathscr{G}$-germ. Following [14] we define *the pullback* $(\phi, \alpha)^*\,(\tilde{L}, 0)$ of $(\tilde{L}, 0)$ *by means of a relation* (ϕ, α) as the germ at $0 \in T^*\mathbf{R}^n$ of the subset

$$\{\phi^*_x \eta + d\alpha|_x \in T^*\mathbf{R}^n; x \in \mathbf{R}^n, \phi(x) = \pi(\eta) \text{ and } \eta \in \tilde{L} \cap U\}\,,$$

where U is a sufficiently small neighbourhood of $0 \in T^*\mathbf{R}^{\tilde{n}}$.

Lemma 2.4. *Let* $(\tilde{F}, 0) \in \mathbf{m}^2_{\mathscr{G}}(\mathbf{R}^m \times \mathbf{R}^{\tilde{n}})$ *be an* Mf*-germ generating an* $L\mathscr{G}$*-germ* $(\tilde{L}, 0)$ *and*

$$F(\lambda, x) \stackrel{\text{def}}{=} \tilde{F}(\lambda, \phi(x)) + \alpha(x)\,,$$

where $(\phi, \alpha) \in \mathscr{E}_{\mathscr{G}}(n, \tilde{n}) \times \mathbf{m}^2_{\mathscr{G}}(n)$. *Then:*
(i) $(\phi, \alpha)^*\,(\tilde{L}, 0)$ *is given by Eqs.* (4).
(ii) *If* ϕ *satisfies the mapping condition*

$$\operatorname{Im}(T_0\phi) + \operatorname{Im}(T_0(\pi \mid \tilde{L})) = T_0\mathbf{R}^{\tilde{n}}\,, \tag{5}$$

then $(L, 0) \stackrel{\text{def}}{=} (\phi, \alpha)^*\,(\tilde{L}, 0)$ *is an* $L\mathscr{G}$*-germ.*
(iii) F *is an* Mf*-germ if and only if* (5) *holds.*

Proof. (i) can be checked directly. (ii) is obtained by a slight reformulation of the transversality condition in [11, 12, Proposition 4.1]. (iii) The equivalence of (5) and (3) can be verified directly. Q.E.D.

The $L\mathscr{G}$-germ $(L, 0)$ as in Lemma 2.4.ii will be called an $L\mathscr{G}$*-pullback* of $(\tilde{L}, 0)$ by means of the $L\mathscr{G}$*-relation* (ϕ, α). It is easily seen that if $\Lambda_{\psi\beta} : (T^*\mathbf{R}^{\tilde{n}}, 0) \to (T^*\mathbf{R}^{\tilde{n}}, 0)$ is an $L\mathscr{G}$-equivalence of the form (2), then $(\psi\circ\phi, \alpha+\beta\circ\phi)$ is an $L\mathscr{G}$-relation inducing $(L, 0)$ from $(\Lambda_{\psi\beta}\tilde{L}, 0)$.

Lemma 2.5. *Let* $(\tilde{L}, 0)$, $\tilde{L} \subset T^*(\mathbf{R}^m \times \mathbf{R}^k)$ *be an* $L\mathcal{G}$*-germ generated by an* Mf*-germ* $\tilde{F} \in \mathbf{m}^2_{\mathcal{G}}(m+m+k)$ *of the form*

$$\tilde{F}(\lambda, x, u) = f(\lambda, u) - \sum_{i=1}^{m} \lambda_i x_i\,, \quad \textit{for} \quad (\lambda, x, u) \in \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}^k\,, \tag{6}$$

where $f \in \mathbf{m}^3_{\mathcal{G}}(m+m)$. *Let* $(L, 0)$, $L \subset T^*\mathbf{R}^n$, *be an* $L\mathcal{G}$*-pullback of* $(\tilde{L}, 0)$ *by means of* $(\phi, \alpha) \in \mathcal{E}_{\mathcal{G}}(n, m+k) \times \mathcal{E}_{\mathcal{G}}(n)$ *such that* $\operatorname{rank}(\partial\phi/\partial x)|_0 = l$. *Then there exists a* $\mathcal{G}$*-space* $\mathbf{R}^s$, $s \overset{\text{def}}{=} n - m \geq 0$, *such that* $\mathcal{G}$*-spaces* $\mathbf{R}^m \times \mathbf{R}^s$ *and* $\mathbf{R}^n$ *are isomorphic and the* $L\mathcal{G}$*-germ* $(L, 0)$ *has, up to an* $L\mathcal{G}$*-equivalence, an* Mf*-germ* $(F, 0)$ *of the form*

$$F(\lambda, x, t) = f(\lambda, U(x, t)) - \sum_{i=1}^{m} \lambda_i x_i \qquad ((\lambda, x, t) \in \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}^s \approx \mathbf{R}^m \times \mathbf{R}^n)\,,$$

where $U : (\mathbf{R}^m \times \mathbf{R}^s, 0) \to (\mathbf{R}^m, 0)$ *is an equivariant transformation such that* $\operatorname{rank}(\partial U/\partial(x,t))|_0 \geq l - m$.

This follows directly from the above properties of pullbacks. Note that the $\mathcal{G}$-space $\mathbf{R}^s$ is isomorphic to the $\ker(T_0\phi) \subset T_0\mathbf{R}^n \approx \mathbf{R}^n$.

2.2. *Infinitesimal Stability Conditions*

A family of transformations $\Phi_t : T^*\mathbf{R}^n \to T^*\mathbf{R}^n$, $|t| < \varepsilon$, is a family of $L\mathcal{G}$-equivalences if and only if it is a flow of a time dependent Hamiltonian vector field on $T^*\mathbf{R}^n$:

$$X_{H_t} \overset{\text{def}}{=} \sum_{i=1}^{n} \frac{\partial H_t}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial H_t}{\partial x_i}\frac{\partial}{\partial \xi_i}\,,$$

where each H_t belongs to the space $\mathcal{H}_{\mathcal{G}}(T^*\mathbf{R}^n)$ of symmetric hamiltonians ($\subset C^\infty_{\mathcal{G}}(T^*\mathbf{R}^n)$) of the form

$$H(x, \xi) = \langle a(x) \,|\, \xi \rangle + b(x) \tag{7}$$

(see Arnold [2], or [15] for the equivariant case). Here $\langle\,|\,\rangle$ denotes the inner product on $\mathbf{R}^n$, $a \in C^\infty_{\mathcal{G}}(n, n)$ and $b \in C^\infty_{\mathcal{G}}(\mathbf{R}^n)$.

Proposition 2.6. *For a smooth family of* $L\mathcal{G}$*-germs* $(L_t, 0)$, $L_t \subset T^*\mathbf{R}^n$, $|t| < \varepsilon$, *corresponding to a family of* IJ*-germs* $(S_t, 0) \in \mathbf{m}^2_{\mathcal{G}}(\mathbf{R}^n)$, $S_t = S_t(\xi_I, x_J)$, *the following conditions are equivalent.*

(i) *There exists a family of* $L\mathcal{G}$*-equivalences* $\Phi_t : T^*\mathbf{R}^n \to T^*\mathbf{R}^n$ *and an open neighbourhood* U *of* $0 \in T^*\mathbf{R}^n$ *such that* $\Phi_t(L_0 \cap U) \subset L_t$ *and* $\Phi_t(0) = 0$ *for* $|t| < \varepsilon$.

(ii) *There exists a smooth family of Hamiltonians* $H_t \in \mathcal{H}_{\mathcal{G}}(T^*\mathbf{R}^n)$ *such that for each* t, $|t| < \varepsilon$, *we have*

$$-\frac{dS_t}{dt} = H_t\left(\frac{\partial S_t}{\partial \xi_I}, x_J, \xi_I, -\frac{\partial S_t}{\partial x_J}\right) \quad \textit{near} \quad (0, t) \in T^*\mathbf{R}^n \times \mathbf{R}\,, \tag{8}$$

$$(H_t, 0) \in \mathbf{m}^2_{\mathcal{G}}(T^*\mathbf{R}^n)\,. \tag{9}$$

Proof. This theorem is a sort of equivariant version of the Hamilton-Jacobi Theorem [1] (for the symplectic structure $\sum_{i\in I} d\xi_i \wedge dx_i - \sum_{j\in J} d\xi_j \wedge dx_j$ on $T^*\mathbf{R}^n$ rather than the canonical one). The proof is similar to the proof of Proposition 3.3 and Lemma 3.2 in [15]. Q.E.D.

We say that an $L\mathscr{G}$-germ $(L,0)$, $L \subset T^*\mathbf{R}^n$, is *infinitesimally stable* [16] if

$$\mathscr{E}_{\mathscr{G}}(T^*\mathbf{R}^n)|_L = \mathscr{H}_{\mathscr{G}}(T^*\mathbf{R}^n)|_L\,. \tag{10}$$

This condition has a form independent of the immersion $\iota\colon \mathbf{R}^n \to L \subset T^*\mathbf{R}^n$ but it is not very convenient for explicit calculations. We introduce below (after [16]) a few more useful versions of (10).

Lemma 2.7. *Let $(S(\xi_I, x_J), 0) \in \mathbf{m}^2_{\mathscr{G}}(\mathbf{R}^n)$ be an IJ-germ of $(L,0)$, let $\varrho : \mathbf{R}^n \to \mathbf{R}^a$ be a Hilbert map and $\tilde{V}_1, \ldots, \tilde{V}_b$ be generators of $\mathscr{E}_{\mathscr{G}}(n,n)$ over $\mathscr{E}_{\mathscr{G}}(n)$. Let $V_i \in \mathbf{m}(a)$ and $U = (U_1, \ldots, U_a) \in \mathscr{E}(a,a)$ be such that*

$$V_i \circ \varrho(\xi_I, x_J) = \left\langle \tilde{V}_i\left(\frac{\partial S}{\partial \xi_i}, x_J\right) \,\middle|\, \left(\xi_I, -\frac{\partial S}{\partial x_J}\right)\right\rangle \quad (i = 1, \ldots, b), \tag{11}$$

$$U \circ \varrho(\xi_I, x_J) = \varrho\left(\frac{\partial S}{\partial \xi_I}, x_J\right). \tag{12}$$

The following conditions are equivalent.

(i) *The $L\mathscr{G}$-germ $(L,0)$ is infinitesimally stable.*

(ii)

$$\mathscr{E}(a) = \langle U_1, \ldots, U_a\rangle_{\mathscr{E}(a)} + \langle V_1, \ldots, V_b, 1\rangle_{\mathbf{R}} + \mathscr{M}_{\varrho}(a)\,, \tag{13}$$

where $\mathscr{M}_\varrho(a)$ denotes the ideal in $\mathscr{E}(a)$ of all germs vanishing on $\varrho(\mathbf{R}^n)$.

(iii) *Let a compact Lie group $\mathscr{G}'$ operate orthogonally on $\mathbf{R}^k$, and let $(S'(\xi_I, x_J, t), 0) \in \mathbf{m}^2_{\mathscr{G}\oplus\mathscr{G}'}(n+k)$ be such that $S'|_{t=0} = S$. Every $\alpha \in \mathscr{E}_{\mathscr{G}\oplus\mathscr{G}'}(n+k)$ has the expansion*

$$\alpha(\xi_I, x_J, t) = \sum_{i=1}^{b} \left\langle \tilde{V}_i\left(\frac{\partial S'}{\partial \xi_I}, x_J\right) \,\middle|\, \left(\xi_I, -\frac{\partial S'}{\partial x_J}\right)\right\rangle \times \tilde{H}_i\left(\frac{\partial S'}{\partial \xi_I}, x_J, t\right) + \tilde{B}\left(\frac{\partial S'}{\partial \xi_I}, x_J, t\right), \tag{14}$$

where $\tilde{H}_i, \tilde{B} \in \mathscr{E}_{\mathscr{G}\oplus\mathscr{G}'}(n+k)$.

Proof of the above result is straightforward. Every IJ-germ $(S,0)$ satisfying one of the conditions (ii)–(iii) will be called *infinitesimally stable*. Equation (13) depends actually on $(F,0) \in \mathscr{E}(a)$ such that $S = F \circ \varrho$ and on the choice of I and J. We will call it an inf-*IJ-stability condition* for $(F,0) \in \mathscr{E}(a)$. From (i) the following two corollaries follow immediately.

Corollary 2.8. *If the IJ-germ $(S,0)$ is infinitesimally stable, then the IJ'-germ $(S',0)$ as in* (v), *where $J' = J \cup \{n+1, \ldots, n+k\}$, is also infinitesimally stable.*

Corollary 2.9. *Condition* (ii) *of Proposition* 2.6 *holds for every smooth family of IJ-germs $(S_t, 0) \in \mathbf{m}^2_{\mathscr{G}}(\mathbf{R}^n)$, $|t| < \varepsilon$, such that $(S_0,0) = (S,0)$ if and only of the IJ-germ $(S,0)$ is infinitesimally stable.*

Example 2.10. Let us consider the group $\mathscr{G} \stackrel{\text{def}}{=} \mathbf{Z}_2 \oplus \mathbf{Z}_2 = \{\pm 1, \pm 1\}$ acting on $\mathbf{R}^4$ as $\nu_{(\varepsilon_1,\varepsilon_2)}:(x_1, \ldots, x_4) \mapsto (\varepsilon_1 x_1, \varepsilon_2 x_2 x_2, x_4)$ for each $(\varepsilon_1, \varepsilon_2) \in \mathbf{Z}_2 \oplus \mathbf{Z}_2$, a Hilbert map $\varrho: \mathbf{R}^4 \to \mathbf{R}^4$, $\varrho(x) \stackrel{\text{def}}{=} (x_1^2, x_2^2, x_3, x_4)$ and the following four generators of $\mathscr{E}_{\mathbf{Z}_2 \oplus \mathbf{Z}_2}(4,4): \tilde{V}_1 \stackrel{\text{def}}{=} (x_1, 0, 0, 0)$, $\tilde{V}_2 \stackrel{\text{def}}{=} (0, x_2, 0, 0)$, $\tilde{V}_3 \stackrel{\text{def}}{=} (0, 0, 1, 0)$ and $\tilde{V}_4 \stackrel{\text{def}}{=} (0, 0, 0, 1)$. Obviously $\varrho(\mathbf{R}^4) = \{z \in \mathbf{R}^4; z_1 \geq 0 \text{ and } z_2 \geq 0\}$ and so $\mathscr{M}_\varrho(4) \subset \mathbf{m}^\infty(4)$. Easy calculations show that for an IJ-germ

$$(S(\xi_I, x_J), 0) = (S(\xi_1, x_2, \xi_3, x_4), 0) = (F(z)|_{z=(\xi_1^2, x_2^2, \xi_3, x_4)}, 0) \in \mathbf{m}^2_{\mathbf{Z}_2 \oplus \mathbf{Z}_2}(4)$$

the condition of inf-IJ-stability (Eq. (13)) takes the form

$$\begin{aligned}\mathscr{E}(z_1, \ldots, z_4) = &\left\langle z_1 \left(\frac{\partial F}{\partial z_1}\right)^2, z_2, \frac{\partial F}{\partial z_3}, z_4 \right\rangle_{\mathscr{E}(z_1,z_2,z_3,z_4)} \\ &+ \left\langle z_1 \frac{\partial F}{\partial z_1}, z_2 \frac{\partial F}{\partial z_2}, z_3, \frac{\partial F}{\partial z_4}, 1 \right\rangle_{\mathbf{R}} + \mathscr{M}_\varrho(4).\end{aligned}$$

By Nakayama's Lemms [3, 7] this is equivalent to the equation:

$$\mathscr{E}(z_1, z_3) = \langle z_1 \bar{F}_{,1}^2, \bar{F}_{,3} \rangle_{\mathscr{E}(z_1,z_3)} + \langle z_1 \bar{F}_{,1}, z_3, 1 \rangle_{\mathbf{R}}, \tag{15}$$

where $\bar{F}_{,1} \stackrel{\text{def}}{=} \partial F / \partial z_i|_{z_2=z_4=0}$.

3. Classification of Generic IJ-Germs with $\mathbf{Z}_2^l$ Symmetry

In this paper we consider a restricted notion of genericity of Lagrangian submanifolds expressed in terms of their IJ-functions. It is based on an observation that an IJ-function, if exists, is determined up to an additive constant by its $L\mathscr{G}$-manifold. We shall say that a class of germs $\mathscr{C} \subset \mathscr{E}_\mathscr{G}(n)$ is *generic (non-generic)* if there exists a residual subset $\mathscr{F} \subset \mathscr{C}_\mathscr{G}^\infty(n)$ such that for every $F \in \mathscr{F}$ and every $\mathscr{G}$*-invariant point* $x_0 \in \mathbf{R}^n$ the germ $(F(x + x_0), 0)$ belongs (does not belong) to $\mathscr{C}$.

Now we state an equivariant version of Thom's Transversality Lemma useful as a technical tool for finding generic classes. It is a simple consequence of continuity and openness of the transformation $\varrho^*: C^\infty(a) \to C_\mathscr{G}^\infty(n)$. Below the s-jet extension of $f \in C^\infty(\mathbf{R}^n)$ is denoted by $j^s f$, the (trivial) fibre bundle of j^s-jets over $\mathbf{R}^n$ by $J^s(\mathbf{R}^n)$ and its fibre over $x \in \mathbf{R}^n$ by $J_x^s(\mathbf{R}^n)$.

Lemma 3.1. *Let $\mathscr{G}$ operate trivially on $\mathbf{R}^k$ and orthogonally on $\mathbf{R}^n$ with* 0 *being the only fixed point. Let $\varrho: \mathbf{R}^n \to \mathbf{R}^a$ be a Hilbert map for $(\mathbf{R}^n, \mathscr{G})$ and X be a stratified submanifold of $J_0^s(\mathbf{R}^a \times \mathbf{R}^k)$ of codimension c_X. Then the subset of $\mathscr{E}_\mathscr{G}(n+k)$ of all germs $(S, 0) \in \mathscr{E}_\mathscr{G}(n+k)$ such that $S(x,t) = F(\varrho(x), t)$, where $F \in C^\infty(a+k)$ and $j_0^s F \in X$, is generic if $c_X \leq k$ and non-generic if $k < c_X$.*

Up to an $L\mathscr{G}$-equivalence (a translation in $T^*\mathbf{R}^n$) any IJ-germ belongs to $\mathbf{m}_\mathscr{G}^2(n)$ (Lemma 2.3). To accommodate this simplification in our task of classification of typical $L\mathscr{G}$-germs up to $L\mathscr{G}$-equivalences we modify slightly the above definitions as follows. We shall say that a *class $C \subset \mathbf{m}_\mathscr{G}^2(n)$ of IJ-germs is generic (non-generic)* if the class $\tilde{C}$ of all germs in $\mathscr{E}_\mathscr{G}(n)$ of the form

$$(\xi_I, x_J) \mapsto S(\xi_I, x_J) + c_0 + \sum_{i \in I} c_i \xi_i + \sum_{j \in J} c_j x_j,$$

where $S \in C$ and $c_\alpha \in \mathbf{R}$, is generic (non-generic) in $\mathscr{E}_{\mathscr{G}}(n)$ (in the previously defined sense).

According to Lemma 2.3, an $L\mathscr{G}$-germ $(L, 0)$ is either trivial $(\dim \operatorname{Ker}(\pi \mid L) = 0)$ or, up to an $L\mathscr{G}$-equivalence, is generated by a non-trivial minimal IJ-germ from $\mathbf{m}^3_{\mathscr{G}}(n)$. Thus we observe the following.

Remark 3.2. If non-trivial generic IJ-germs exist in $\mathbf{m}^2_{\mathscr{G}}(n)$, then the subspace of fixed points of $\mathbf{R}^n$ has a positive dimension.

Now we formulate the main results of the paper.

Theorem 3.3. *We are given an orthogonal action of $\mathscr{G} = \mathbf{Z}_2^l$ on $\mathbf{R}^n$ $(n = 2, 3)$ and multi-indices I, J, $(I \cap J = \{1, \ldots, n\}, I \cap J = \emptyset)$. Any IJ-germ $(S, 0) \in \mathbf{m}^2_{\mathscr{G}}((n)$ either*

(i) *belongs, up to an equivariant orthogonal change of coordinates on $\mathbf{R}^n$, to one of the non-trivial, minimal and generic classes of IJ-germs (Cases) in Table 2, or*

(ii) *belongs to a generic class of trivial IJ-germs, or*

(iii) *belongs to the non-generic class of all remaining IJ-germs.*

Proof. In virtue of Remark 3.2 and Lemma 2.2 it is enough to consider all possible pairs of multi-indices IJ for all cases in Table 2. The proof is similar in all these cases. We consider here in detail the Case 2 of Table 1 ($\mathscr{G} \overset{\text{def}}{=} \mathbf{Z}_2 \oplus \mathbf{1}$ operates of $\mathbf{R}^3$, $\varrho(x) = (x_1^2, x_2, x_3)$) and $I \overset{\text{def}}{=} \{1\}$, $J \overset{\text{def}}{=} \{2, 3\}$, only.

Let $(z_i, f, f_{\alpha\beta\gamma})$ be coordinates in 5-jet space $J^5(\mathbf{R}^3)$ corresponding to

$$\left(z_i, F(z_i), \frac{\partial^{\alpha+\beta+\gamma} F}{\partial z_1^\alpha \partial z_2^\beta \partial z_3^\gamma}(z_i)\right).$$

Let us consider the following submanifolds of $J_0^5(\mathbf{R}^3)$: $M_0 \overset{\text{def}}{=} \{f_{100} \neq 0\}$, $M_1 \overset{\text{def}}{=} \{f_{100} = 0 \neq f_{200}(f_{110}^2 + f_{101}^2)\}$ and M_2 defined by the conditions $f_{100} = f_{200} = 0 \neq f_{300}$ and

$$\operatorname{rank} \begin{bmatrix} 0 & 0 & 3f_{300} & 4f_{400} \\ f_{110} & f_{210} & f_{310} & f_{410} \\ f_{101} & f_{201} & f_{301} & f_{401} \end{bmatrix} = 3\,.$$

Obviously M_i, $i = 0, 1, 2$, is a submanifold of $J_0^5(\mathbf{R}^3)$ of codimension i and $N \overset{\text{def}}{=} J_0^5(\mathbf{R}^3) - (M_0 \cup M_1 \cup M_2)$ is a stratified submanifold of codimension 3 in $J_0^5(\mathbf{R}^3)$ [3]. In virtue of Lemma 3.1 IJ-germs in $\mathbf{m}^2_{\mathscr{G}}(3)$ are divided into the following four classes. (We recall $(S, 0) \in \mathbf{m}^2_{\mathscr{G}}(3)$ is of the form $S \overset{\text{def}}{=} F \circ \varrho$, $F \in \mathbf{m}(3)$.)

(a) The generic class of germs $F \circ \varrho$ such that $j_0^5 F \in M_0$, containing no minimal IJ-germs.

(b) The generic class of germs $F \circ \varrho$ such that $j_0^5 F \in M_1$, containing only minimal IJ-germs.

(c) The generic class of germs $F \circ \varrho$ such that $j_0^5 F \in M_2$, containing only minimal IJ-germs.

(d) The non-generic class of germs $F \circ \varrho$ such that $j_0^5 F \in N_1$ containing only minimal IJ-germs.

Obviously classes (b) and (c) correspond to case (i) (Cases 4 and 5 of Table 2) while (a) and (d) correspond to the cases (ii) and (iii) of the Theorem. Q.E.D.

Table 2. Non-trivial minimal generic IJ-germs for the orthogonal actions of $\mathbf{Z}_2^l$ on $\mathbf{R}^2$ and $\mathbf{R}^3$. In Case 5 we denote $\mathscr{M}(F) \overset{\text{def}}{=} \begin{vmatrix} f_{110} & f_{210} & f_{310} & f_{410} \\ f_{101} & f_{201} & f_{301} & f_{401} \\ 0 & 0 & 3f_{300} & 4f_{400} \end{vmatrix}$

Case	$\mathbf{R}^n$ Effective group Repr. rank	IJ-germs $S = F \circ \varrho(\xi_I, x_J) \in \mathscr{E}_{\mathscr{G}}(n)$ $F = \sum f_{i_1 \dots i_a} z^{i_1} \dots z_a^{i_a} + o^{\infty}(z) \in \mathscr{E}(z)$ (conditions for $F \circ \varrho \in \mathbf{m}^2_{\mathscr{G}}(n)$)	Conditions for classes	Normal forms
1	2	3	4	5
1	$\mathbf{R}^2/\mathbf{Z}_2 \oplus 1/(1 \mid 1)$	$S(x_1, \xi_2) = F(x_1^2, \xi_2)$ $(f_{00} = f_{01} = 0)$	$f_{02} = 0 \neq f_{02}$	A_2: $S = \xi_2^3$
2		$S(\xi_1, x_2) = F(\xi_1^2, x_2)$ $(f_{00} = f_{01} = 0)$	$f_{10} = 0 \neq f_{20} f_{11}$	A_3: $S = \pm\xi_1^4 + x_2\xi_1^2$
3	$\mathbf{R}^3/\mathbf{Z}_2 \oplus 1/(1 \mid 2)$	$S(\xi_1, \xi_2, x_3) = F(\xi_1^2, \xi_2, x_3)$ $(f_{000} = f_{010} = f_{001} = 0)$	$f_{100} = f_{020} = 0 \neq f_{030} f_{110}$ $f_{110} f_{021} - 3 f_{030} f_{101} \neq 0$	$D_4^{\pm}$: $S = \pm\xi_1^2\xi_2 + \xi_2^3 + x_3\xi_2^2$
4		$S(\xi_1, x_2, x_3) = F(\xi_1^2, x_2, x_3)$	$f_{100} = 0 \neq f_{200} \neq (f_{110}^2 + f_{101}^2)$	A_3: $S = \pm\xi_1^4 + x_2\xi_1^2$
5		$(f_{000} = f_{010} = f_{001} = 0)$	$f_{100} = f_{200} = 0 \neq f_{300}$ rank $\mathscr{M}(F) = 3$	Ξ_3: $S = \mathscr{F}(\lambda, \mu, x_2, x_3) + \xi_1 \mu \big\vert_{\frac{\partial \mathscr{F}}{\partial \lambda}=0, \frac{\partial \mathscr{F}}{\partial \mu}=0}$, $\mathscr{F} = \pm\lambda^6 + a\lambda^8 + u_1\lambda^8 + u_2\lambda^4 + u_3\lambda^2 - \lambda\mu$, $(u_{i_1}, u_{i_2}, u_{i_3}) = (x_2, x_3, \phi(\mu^2, x_2, x_3))$, $\phi \in \mathbf{m}(3)$ and $a \overset{\text{def}}{=} f_{400}\, \lvert f_{300} \rvert^{-4/3} \in \mathbf{R}$
6		$S(x_1, \xi_2, x_3) = F(x_1^2, \xi_2, x_3)$	$f_{020} = 0 \neq f_{030}$	A_2: $S = \xi_2^3$
7		$(f_{000} = f_{010} = f_{001} = 0)$	$f_{020} = f_{030} = 0 \neq f_{040} f_{021}$	A_3: $S = \pm\xi_2^4 + x_3\xi_2^2$
8	$\mathbf{R}^3/\mathbf{Z}_2 \oplus 1/(2 \mid 1)$	$S(\xi_1, x_2, x_3) = F(\xi_1^2, \xi_1 x_2, x_2^2, x_3)$ $(f_{0000} = f_{0001} = 0)$	$f_{1000} = 0 \neq f_{2000} f_{1001}$	A_3: $S = \pm\xi_1^4 + x_3\xi_1^2$
9		$S(x_1, x_2, \xi_3) = F(x_1^2, x_1 x_2, x_2^2, \xi_3)$ $(f_{0000} = f_{0001} = 0)$	$f_{0002} = 0 \neq f_{0003}$	A_2: $S = \xi_3^3$
10	$\mathbf{R}^3/\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{1}/(1, 1 \mid 1)$	$S(\xi_1, x_2, x_3) = F(\xi_1^2, x_2^2, x_3)$ $(f_{000} = f_{001} = 0)$	$f_{100} = 0 \neq f_{200} f_{101}$	A_3: $S = \pm\xi_1^4 + x_3\xi_1^2$
11		$S(x_1, x_2, \xi_3) = F(x_1^2, x_2^2, \xi_3)$ $(f_{000} = f_{001} = 0)$	$f_{002} = 0 \neq f_{003}$	A_3: $S = \xi_3^3$

By straightforward check we obtain:

Proposition 3.4. *Generic IJ-germs of all classes in Table 2 except Case 5 are* inf-*IJ-stable.*

An I_1J_1-germ and an I_2J_2-germ from $\mathbf{m}^2_{\mathscr{G}}(n)$ are called *$L\mathscr{G}$-equivalent* if the $L\mathscr{G}$-germs generated by them are $L\mathscr{G}$-equivalent.

Lemma 3.5. *Let $(S_t, 0) = (F_t \circ \varrho, 0) \in \mathbf{m}^3_{\mathscr{G}}(\mathbf{R}^n)$, $-a^2 \leq t \leq b^2$, be a smooth family of IJ-germs belonging to a single Case* 1–4 *or* 6–11 *in Table* 2 *(with the appropriate action of $\mathscr{G}$ on $\mathbf{R}^2$ or $\mathbf{R}^3$). Then any two germs $(S_{t_1}, 0)$ and $(S_{t_2}, 0)$ in this family are $L\mathscr{G}$-equivalent.*

Outline of a proof. In virtue of Proposition 2.6 it is enough to show that there exists a smooth family of Hamiltonians $H_t \in \mathscr{H}_{\mathscr{G}}(\mathbf{R}^n)$ (of the form (7)) satisfying Eqs. (8)–(9) in some open neighbourhood of $(0,0) \in T^*\mathbf{R}^n \times \mathbf{R}$. From Proposition 3.4 and Lemma 2.7.iii (for $\mathscr{G}' = \mathbf{1}$, $k = 1$) we obtain the expansion

$$-\frac{dS_t}{dt}(\xi_I, x_J) = H_t\left(\frac{\partial S_t}{\partial \xi_I}, x_J, \xi_I, -\frac{\partial S_t}{\partial x_J}\right)$$
$$\text{(for } (\xi_I, x_J, t) \text{ near } (0,0,0) \in \mathbf{R}^n \times \mathbf{R}), \tag{16}$$

where the symmetric hamiltonians $(H_t, 0) \in \mathscr{H}_{\mathscr{G}}(T^*\mathbf{R}^n)$ have the form

$$H_t(x, \xi) \stackrel{\text{def}}{=} \sum_{i=1}^{b} \langle \tilde{V}_i(x) \,|\, \xi \rangle \tilde{A}_i(x, t) + \tilde{B}(x, t), \tag{17}$$

with $(\tilde{A}_i, 0)$, $(\tilde{B}, 0) \in \mathscr{E}_{\mathscr{G}}(n)$ and $(\tilde{V}_i, 0)$'s being generators of $\mathscr{E}_{\mathscr{G}}(n, n)$. The idea is to show that in all cases of interest (16), (17) and additionally (9) can be satisfied for all t sufficiently close to 0. This can be shown case by case with some mundane but straightforward algebra, with the use of the following equivalent from of (16):

$$-\frac{dF_t}{dt}(z) = \sum_{i=1}^{b} V_i(z, t) A_i \circ U(z, t) + B \circ U(z, t)\Big|_{z=\varrho\left(\frac{\partial S_t}{\partial \xi_I}, x_J\right)}, \tag{18}$$

where $(A_i, 0)$, $(B, 0) \in \mathscr{E}(a)$, $V_i, \ldots, V_b \in \mathbf{m}_{\mathscr{G}\oplus 1}(n+1)$ and the $\mathscr{G}$-invariant transformation $U : \mathbf{R}^n \to \mathbf{R}^a$ is defined by (11)–(12) (with S_t instead of S).

Now we shall discuss the derivation of normal forms.

Theorem 3.6. *Column* 5 *of Table* 2 *contains normal forms for all $L\mathscr{G}$-equivalence classes of all non-trivial, minimal and generic IJ-germs for all non-trivial orthogonal actions of $\mathbf{Z}_2^l$ on $\mathbf{R}^2$ and $\mathbf{R}^3$.*

Proof. At first we make a simplification. Let

$$F(z) = \sum f_{i_1 \ldots i_a} z_1^{i_1} \ldots z_a^{i_a} + o^\infty(z) \in \mathscr{E}(a)$$

satisfy the set of conditions in Columns 3 and 4 of Table 2 for one of the Cases 1–4 or 6–11. Then there exists a *family* F_t satisfying the same set of conditions and joining F to a polynomial $\tilde{F} \in \mathscr{E}_{\mathscr{G}}(n)$ having all coefficients zero except those explicitly appearing in Table 2. By Corollary 3.5 IJ-germs $(F, 0)$ and $(\tilde{F}, 0)$ are $L\mathscr{G}$-equivalent, so further considerations can be restricted to finding such families and the normal forms for such polynomial germs $(\tilde{F}, 0)$.

Case 1. The family $F_t(z) \overset{\text{def}}{=} \operatorname{sgn}(f_{03})((1-t)|f_{03}| - t)z_2^3 = f_{03}^t z_2^3$ satisfies the conditions of Case 1 and joins $F_0(z) = f_{03}z_2^3$ with $F_1(z) = \varepsilon z_2^3$, where $\varepsilon \overset{\text{def}}{=} \operatorname{sgn} f_{03}$. So $\{1\}\{2\}$-germs $(S(x_1,\xi_2),0) \overset{\text{def}}{=} (f_{03}\xi_2^3, 0)$ and $(S_1(x_1,\xi_2),0) \overset{\text{def}}{=} (\varepsilon\xi_2^3,0)$ are $L\mathscr{G}$-equivalent. Now note that the change of sign $(x_1,x_2) \mapsto (x_1,-x_2)$ on $\mathbf{R}^2$ induces a $L\mathscr{G}$-equivalence of $T^*\mathbf{R}^n$ making $\{1\}\{2\}$-germs $(-\xi_2^3,0)$ and $(\xi_2^3,0)$ $L\mathscr{G}$-equivalent.

Cases 2, 3 and 6–11. These cases can be proved analogously as in Case 1 (we omit this part of the proof). Note that in cases 3 and 4 we can utilise an observation that the subspace of matrices of rank n is open in the space of all $n \times m$ matrices, and each connected component of it contains a matrix $[d_{ij}]$ with entries $d_{ii} = \pm 1$ and $d_{ij} = 0$ for $i \neq j$.

Case 5. We have separated this case from the others since it requires a slightly different approach and is the only non-elementary "catastrophy" [3, 7, 22] we encounter here. We divide this part of the proof into several steps.

We shall consider $\mathbf{R}^4$ with the canonical action of $\mathscr{G} \overset{\text{def}}{=} \mathbf{Z}_2 \oplus \mathbf{1}$ of rank$(1\,|\,3)$ and the Hilbert map $\varrho: \mathbf{R}^4 \to \mathbf{R}^4$, $(x_1, \ldots, x_4) \mapsto (x_1^2, x_2, x_3, x_4)$. Assume $I \overset{\text{def}}{=} (1)$ and $J \overset{\text{def}}{=} (2,3,4)$.

Step 5.1. An IJ-germ $(S,0) \in \mathbf{m}^2_{\mathscr{G}}(n)$, $S(\xi_1,x_2,x_3,x_4) = F(z)|_{z=(\xi_1^2,x_2,x_3,x_4)}$ is inf-IJ-stable if

$$f_{1000} = f_{2000} = 0 \neq f_{3000}\,, \tag{19}$$

$$\operatorname{rank}\begin{bmatrix} 4f_{4000} & f_{4100} & f_{4010} & f_{4001} \\ 3f_{3000} & f_{3100} & f_{3010} & f_{3001} \\ 0 & f_{2100} & f_{2010} & f_{2001} \\ 0 & f_{1100} & f_{1010} & f_{1001} \end{bmatrix} = 4\,. \tag{20}$$

Proof. The inf-IJ-stability condition has the following form:

$$\mathscr{E}(z_1) = \langle z_1 \bar{F}_{,1}^2 \rangle_{\mathscr{E}(z_1)} + \langle z_1\bar{F}_{,1}, \bar{F}_{,2}, \bar{F}_{,3}, \bar{F}_{,4}, 1\rangle_{\mathbf{R}}\,, \tag{21}$$

where '$^-$' indicates the restriction to $z_2 = z_3 = z_4 = 0$. From the assumptions of the lemma it follows immediately that $\mathbf{m}^5(z_1) = \langle z_1\bar{F}_{,1}^2\rangle_{\mathscr{E}(z_1)}$. Thus to show (21) it is enough to prove that every polynomial in z_1 of degree ≤ 4 belongs to $\langle z_1\bar{F}_{,1}, \bar{F}_{,2}, \bar{F}_{,3}, \bar{F}_{,4}, 1\rangle_{\mathbf{R}} \bmod \mathbf{m}^5(z_1)$. This easily translates into a 4×4 system of linear equations with non-vanishing determinant of the form (20) (up to a multiplicative constant).

Step 5.2. Let $(S_t,0) = (F_t \circ \varrho, 0) \in \mathbf{m}_{\mathscr{G}}(4)$, $-\varepsilon < t < 1+\varepsilon$, $0 < \varepsilon$, be a smooth family of IJ-germs such that conditions (19) and (20) are satisfied for each t and

$$\dot{f}^t_{3000} = 3\beta(t) f^t_{3000}\,, \qquad \dot{f}^t_{4000} = 4\beta(t) f^t_{4000}\,, \tag{22}$$

where β is a smooth function and the expansion $dF_t/dt = \sum \dot{f}^t_{ijkl} z_1^i z_2^j z_3^k z_4^l + o^\infty(z)$ is assumed. Then there exists a smooth family of Hamiltonians $H_t \in \mathscr{H}_{\mathscr{G}}(T^*\mathbf{R}^4)$ of the form (7) satisfying Eqs. (8) and (9) in some neighbourhood of $\{0\}\times[0,1] \subset T^*\mathbf{R}^4 \times \mathbf{R}$.

Let $(S_t(\xi_1,x_2,x_3,x_4),0) = (F_t(\xi_1^2,x_2,x_3,x_4),0) \in \mathbf{m}^3_{\mathbf{Z}_2\oplus\mathbf{1}}(1+3)$ be a family satisfying assumptions (19)–(22) of Step 5.2. Equations (16) and (18) take the following form in this case:

$$
\begin{aligned}
&-\frac{dS_t}{dt}(\xi_1, x_2, x_3, x_4) = H_t\left(\frac{\partial S_t}{\partial \xi_1}, x_2, x_3, x_4, \xi_1, -\frac{\partial S_t}{\partial x_2}, -\frac{\partial S_t}{\partial x_3}, -\frac{\partial S_t}{\partial x_4}\right), \\
&-\frac{dF_t}{dt}(z) = z_1 F_{t,1} A_{t1} \circ U + \sum_{i=2}^{4} F_{t,1} A_{ti} \circ U + B_t \circ U|_{z=(\xi_1^2, x_2, x_3, x_4)},
\end{aligned}
\tag{23}
$$

where

$$
H_t(\xi, x) \stackrel{\text{def}}{=} x_1 \xi_1 A_{t1}(z) + \sum_{i=2}^{4} \xi_i A_{ti}(z) + B_t(z)|_{z=(x_1^2, x_2, x_3, x_4)}, \tag{24}
$$

$U \stackrel{\text{def}}{=} (\xi_1^2 F_{t,1}^2(z), x_2, x_3, x_4)$ and A_{ti}, $B \in C^\infty(4)$. To show that (9) holds let us consider

$$
\alpha^t(\xi_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} -\frac{dF_t}{dt}(z) - z_1 F_{t,1}(z)\beta(t)|_{z=(\xi_1^2, x_2, x_3, x_4)}.
$$

We find by virtue of (22) that

$$
\alpha^t = 0 \bmod \langle \xi_1^{10}, x_2, x_3, x_4 \rangle .
$$

According to Step 5.1 there exists the following smooth family of expansions (for sufficiently small t):

$$
\alpha^t(z) = \xi_1^2 F_{t,1}(z) \tilde{A}_{t1} \circ U(z) + \sum_{i=2}^{4} F_{t,1}(z) A_{ti} \circ U(z) + B_t \circ U(z)|_{z=(\xi_1^2, x_2, x_3, x_4)},
$$

where $\tilde{A}_{t1}, A_{ti}, B_t \in \mathscr{E}(4)$ and $U(z) \stackrel{\text{def}}{=} (z_1 F_{t,1}^2(z), z_2, z_3, z_4)$. From this equation taken $\bmod \langle \xi_1^{10}, x_2, x_3, x_4 \rangle_{\mathscr{E}(\xi_1^2, x_2, x_3, x_4)}$ we find immediately that $A_{tj}(0) = B_t(0) = \partial B_t / \partial x_i(0) = 0$. Now setting $A_{t1}(z) \stackrel{\text{def}}{=} \tilde{A}_{t1}(z) + z_1 F_{t,1}(z)\beta(t)$ for $z \in \mathbf{R}^4$ we obtain Hamiltonian (24) belonging to $\mathbf{m}^2_{\mathbf{Z}_2}(T^*\mathbf{R}^3)$ and satisfying (23).

Step 5.3. We show that an IJ-germ $(S, 0) = (F \circ \varrho, 0)$ as in Step 5.1 is $L\mathscr{G}$-equivalent to the IJ-germ $(\tilde{F} \circ \varrho, 0)$, where $\tilde{F}$ has the form

$$
\tilde{F} = \varepsilon z_1^3 + a z_1^4 + x_2 z_1 + x_3 z_1^2 + x_4 z_1^4, \tag{25}
$$

with $\varepsilon = \pm 1$ and $a \stackrel{\text{def}}{=} f_{4000} |f_{3000}|^{-4/3}$.

By virtue of the previous two steps and an argument analogous to that in the beginning of the proof of this theorem the IJ-germ $(S, 0)$ is $L\mathscr{G}$-equivalent to an IJ-germ $(S', 0) = (F' \circ \varrho, 0)$ with $F' = \sum f'_{\alpha_1 \dots \alpha_4} z_1^{\alpha_1} \dots z_4^{\alpha_4}$ being a polynomial with all coefficients vanishing except, perhaps, those appearing explicitly in (19), (20) and f'_{4000}. Now utilizing Step 5.2, an observation on homotopic connected components in the space of $n \times m$ matrices (cf. Cases 2, 3 and 6–11, above) and solving Eq. (22) for suitable β we find an $L\mathscr{G}$-equivalence of $(S', 0)$ with $(S'', 0) = (F'' \circ \varrho, 0)$, where

$$
F''(z_1, \dots, z_4) \stackrel{\text{def}}{=} \operatorname{sgn}(f_{3000}) z_1^3 + a z_1^4 \pm z_{i_2} z_1 \pm z_{i_3} z_1^2 \pm z_{i_4} z_1^4,
$$

where $\{i_2, i_3, i_4\} = \{2, 3, 4\}$ and a is as above. Finally we get the form (25) by applying to $(S'', 0)$ an $L\mathscr{G}$-equivalence of $T^*\mathbf{R}^4$ induced by a suitable diffeomorphism of the form $(x_1, \dots, x_4) \mapsto (x_1, \pm x_{i_2}, \pm x_{i_3}, \pm x_{i_4})$.

Step 5.4. Let $(L',0)$, $L' \subset T^*\mathbf{R}^3$, be an $L\mathscr{G}$-germ generated by an $\{1\}\{2,3\}$-germ $(S',0) = (F' \circ \varrho, 0) \in \mathbf{m}^2_{\mathscr{G}}(\mathbf{R}^3)$ as in Case 5 of Table 2 (Columns 1–4). Then $(L',0)$ has an Mf-germ

$$\mathscr{F}(\lambda, x) \stackrel{\text{def}}{=} \pm\lambda^6 + a\lambda^8 + u_1(x)\lambda^2 + u_2(x)\lambda^4 + u_3(x)\lambda^8 - \lambda x_i \in \mathbf{m}_{\mathbf{Z}_2\oplus\mathscr{G}}(1+3)\,, \tag{26}$$

$$(u_{i_1}, u_{i_2}, u_{i_3}) = (x_2, x_3, f(x_1^2, x_2, x_3))\,, \quad \text{where } f \in \mathbf{m}_{\mathscr{G}}(3) \text{ and } \{i_1, i_2, i_3\} = \{1,2,3\}\,. \tag{27}$$

To show this let us define an $\{1\}\{2,3,4\}$-germ

$$(S(\eta_1, y_2, y_3, y_4), 0) \stackrel{\text{def}}{=} (F(z)|_{z=(\eta_1^2, y_2, y_3, y_4)}, 0) \in \mathbf{m}^2_{\mathscr{G}}(4)\,,$$

where $F(z) \stackrel{\text{def}}{=} F'(z_1, z_2, z_3) + z_4 \sum_{i=1}^{4} c_i z_1^i$ and $c_1, \ldots, c_4 \in \mathbf{R}$ are such that (20) holds. Let $(L,0)$, $L \subset T^*\mathbf{R}^4$ be the $L\mathscr{G}$-germ generated by $(S,0)$. The $L\mathscr{G}$-germ $(L',0)$ is an $L\mathscr{G}$-pullback of $(L,0)$ via an $L\mathscr{G}$-relation $(\phi, 0) \in \mathscr{E}_{\mathscr{G}}(3,4) \times \mathscr{E}_{\mathscr{G}}(3)$, where transformation $\phi\colon (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 0)$ has rank 3. So, according to Step 5.3, $(L,0)$ is an $L\mathscr{G}$-pullback of an $L\mathscr{G}$-germ $(\tilde{L}, 0)$, $\tilde{L} \subset T^*\mathbf{R}^4$, generated by IJ-germ $(\tilde{F}(\xi_1^2, x_2, x_3, x_4), 0) \in \mathbf{m}^3_{\mathscr{G}}(4)$ with $\tilde{F}$ of the form (25), by means of an $L\mathscr{G}$-relation $(\tilde{\phi}, \alpha) \in \mathscr{E}_{\mathscr{G}}(3,4) \times \mathscr{E}_{\mathscr{G}}(3)$, with rank $\tilde{\phi}|_0 = 3$. On applying Lemma 2.5 we find that, up to an $L\mathscr{G}$-equivalence $T^*\mathbf{R}^3 \to T^*(\mathbf{R}^3)$, $(L',0)$ has a Mf-germ of the form (26), where $u_i(x_1^2, x_2, x_3) \in \mathbf{m}_{\mathscr{G}}(3)$, $i = 1,2,3$, are such that the $\mathscr{G}$-invariant transformation $u = (u_1, u_2, u_3)\colon (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ has rank ≥ 2 at 0. It follows that the $\det[\partial(u_{i_1}, u_{i_2})/\partial(x_2, x_3)]|_0 \neq 0$, for some indices i_1 and i_2, so choosing (x_1, u_{i_1}, u_{i_2}) as new equivariant coordinates on $\mathbf{R}^3$, we find that u has the form (27).

Step 5.6. To complete the proof it remains to notice that in Column 4, Case 5 of Table 1 we have the implicit definition of a minimal $\{1\}\{2,3\}$-germ of $(L',0)$ in terms of an Mf-germ (26)–(27). Q.E.D.

Remark 3.7. All IJ-germs described by Column 3 of Table 2, with the exception of Case 5, are *structurally stable* in the following sense. Let $(S(\xi_I, x_J), 0) \in \mathbf{m}^2_{\mathscr{G}}(n)$ be an IJ-germ belonging to one of these Cases. If $(S',0) \in C^\infty_{\mathscr{G}}(n)$ is sufficiently close to S in Whitney's topology and $L' \subset T^*\mathbf{R}^n$ is a Lagrangian submanifold given by equations:

$$x_I = \frac{\partial S'}{\partial \xi_I}\,, \qquad \xi_J = -\frac{\partial S'}{\partial x_J}\,,$$

then there exists a $\mathscr{G}$-invariant point $p_o \in L'$ such that $L\mathscr{G}$-germs (L', p_o) and $(L,0)$ are $L\mathscr{G}$-equivalent.

Remark 3.8. The natural question arises at this stage: are singularities of type $\boldsymbol{\Xi_3}$ $L\mathscr{G}$-equivalent among themselves or not. We will not attempt to resolve this question here in its full scope but only show that there exist uncountably many different $L\mathscr{G}$-equivalence orbits (*$L\mathscr{G}$-orbits*) in this family.

In a particular case, when $(u_1, u_2, u_3)(x) \stackrel{\text{def}}{=} (0, x_2, x_3)$ and $a \in \mathbf{R}$, the Lagrangian germs of type $\boldsymbol{\Xi_3}$ (Table 2, Case 5), denoted here by $(L_a^\pm, 0)$, are given by the equations

$$\frac{\partial M_a^{\pm}}{\partial \lambda} = 0\,, \qquad \frac{\partial M_a^{\pm}}{\partial x} = \xi\,,$$

where $M_a^{\pm}(\lambda, x) \stackrel{\mathrm{def}}{=} \pm\lambda^6 + a\lambda^8 + x_2\lambda^4 + x_3\lambda^2 - \lambda x_1$ is a Morse family belonging to $\mathscr{E}_{\mathbf{Z}_2}(\mathbf{R} \times \mathbf{R}^3)$ with $\mathbf{Z}_2$ having the canonical representation ν of rank$(2\,|\,2)$ on $\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^3$, i.e. such that $\nu_\varepsilon : (\lambda, x_1, x_2, x_3) \mapsto (\varepsilon\lambda, \varepsilon x_1, x_2, x_3)$. We claim that $L\mathscr{G}$-germs $(L_a^+, 0)$ and $(L_b^+, 0)$ are not $L\mathscr{G}$-equivalent if $a \neq b$.

Indeed, assume that $(L_a^+, 0)$ and $(L_b^+, 0)$ are $L\mathscr{G}$-equivalent. If so, their Morse families, M_a^+ and M_b^+, must be *equivariantly* $\mathbf{R}^+$-equivalent. This means that there exist an equivariant diffeomorphism germ $\Phi : \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R} \times \mathbf{R}^3$ of the form $(\lambda, x) \mapsto (\Lambda(\lambda, x), X(x))$ and a function $\alpha \in \mathscr{E}_{\mathbf{Z}_2}(\mathbf{R}^4)$ such that

$$M_a^+(\lambda, x) = M_b^+(\Lambda(\lambda, x), X(x)) + \alpha(x) + \mathrm{const}\,, \tag{28}$$

which can be proved by generalization of its non-equivariant version in [25] or in [3, pp. 303–309]. Now, a straightforward symmetry argument shows that Λ and X_1, the first coordinate of $X = (X_1, X_2, X_3)$, must be as follows:

$$\Lambda(\lambda, x) = \lambda A(\lambda^2) + x_1 B(\lambda^2) + C(\lambda, x)\,,$$
$$X_1(x) = x_1(d + D(x_1^2, x_2, x_3))\,,$$

where C and D, as well as X_2, X_3 and α, are certain C^∞ functions belonging to the ideal $\langle x_1^2, x_2, x_3\rangle_{C^\infty(\lambda,x)}$. Assuming the expansion $A(\lambda^2) = a_o + a_1\lambda^2 + o(\lambda^4)$ and taking (28) modulo $\langle x_1^2, x_2, x_3, \lambda^{10}, x_1\lambda^5\rangle_{C^\infty(\lambda,x)}$ we get the equation

$$\lambda^6 + a\lambda^8 - \lambda x_1 = a_o^6\lambda^6 + (6a_o^5 a_1 + a_o^8 b)\lambda^8 - a_o d\lambda x_1 - a_1 d\lambda^3 x_1 \quad (\text{for } (\lambda, x_1) \in \mathbf{R}^2)\,. \tag{29}$$

It follows immediately that we must have $1 = a_o d$ and $0 = a_1 d$, so $a_1 = 0$. Further, Eq. (29) implies that $1 = a_o^6$, thus comparison of coefficients of λ^8 yields equations $a = 6a_o^5 a_1 + a_o^8 b = a_o^8 b = b$ which finally proves our claim about splitting of the family Ξ_3 into infinitely many $L\mathscr{G}$-orbits.

The symmetric function $\lambda \mapsto \lambda^6 + a\lambda^8$ on $\mathbf{R}$ with the canonical representation of $\mathbf{Z}_2$ of the rank$(1\,|\,0)$, i.e. such that $\lambda \mapsto \varepsilon\lambda (\varepsilon \in \mathbf{Z}_2)$, is equivariantly right equivalent to the function $\lambda \mapsto \lambda^6$ (note that $\lambda \mapsto \lambda(1 + a\lambda^2)^{1/6}$ is the appropriate diffeomorphism). Given that, we conclude immediately that the classification of equivariant Lagrangian singularities up to $L\mathscr{G}$-equivalences is not subordinated, in general, to the classification of symmetric functions up to right equivariant equivalence. For this reason it is independent from classifications of versal unfoldings like those of Wasserman [24] or Siersma [20].

4. Final Remarks and Applications

Let L be a $\mathscr{G}$-invariant Lagrangian submanifold in T^*X. In usual applications to physics L represents the space of states attainable, in the phase space T^*X, by the concrete, mechanical, thermodynamical or optical system. In optics of symmetric optical instruments L is a system of rays orthogonal to the transformed wavefront. The set of critical values of the projection $\pi_X : L \to X$ $(\pi_X : T^*X \to X)$, which is also the envelope of the outgoing rays, is a caustic of the system (cf. [5]). The list of classified symmetric Lagrangian submanifolds provides the classification of symmetric caustics. An alternative approach to the classification of caustics is presented in [17]. In that approach classification is obtained through Morse generating families and so-called "caustic equivalence." However this equivalence, keeping the diffeomorphic type of the caustic invariant, is not preserving the physical sense of the Lagrangian submanifold itself. Symmetric Lagrangian submanifolds arise naturally in a number of contexts. The magnetic stable and metastable phases of a ferromagnetic crystal can be modelled by symmetric Lagrangian submanifolds with invariant generating functions, F, as a free energy function (cf. [15, 13]). The point symmetry group of the crystal acts on internal and external variables and F must be invariant under this action (32 point groups). The locus of hysteresis and phase transitions can be identified with the local bifurcation set of F. In other problems with the general branching of symmetry the classification of generic symmetric Lagrangian submanifolds provides the typical models for stationary processes with qualitative changements. It forms a basis for a thermoelastic analysis of phase transitions in elastic crystals as it was formulated by [9].

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