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General Integrable Problems of Classical Mechanics

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Abstract. Several classical problems of mechanics are shown to be integrable for the special systems of coupled rigid bodies, introduced in this work and called C^k-central configurations. It is proven that dynamics of an arbitrary C^k-central configuration in a Newtonian gravitational field with an arbitrary quadratic potential is integrable in the Liouville sense and in the theta-functions of Riemann surfaces. Hidden symmetry of the inertial dynamics of these configurations is disclosed and reductions of the Lagrange equations to the Euler equations on Lie coalgebras are obtained. Reductions and integrable cases of a heavy C^k-central configuration rotation around a fixed point are indicated. Separation of rotations of a space station type orbiting system, being a C^k-central configuration of rigid bodies, is proven. This result leads to the possibility of the independent stabilization of rotations of the rigid bodies in such orbiting configurations.

Contents

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1. Introduction and Summary

Recently Sreenath, Oh, Krishnaprasad and Marsden [25] investigated equilibria and stability of coupled rigid bodies by the energy-Casimir method and applied the Poincaré–Melnikov method to prove the stochastization of dynamics under small perturbations of homoclinic orbits. In the work [25] also the integrability of the inertial dynamics of two coupled planar (two-dimensional) rigid bodies is proven after indicating a reduction to a Hamiltonian system with one degree of freedom. Reductions, symmetry and phases in general problems of coupled rigid bodies dynamics were studied by Marsden, Montgomery and Ratiu [16]. The energy-momentum method and block diagonalization method were developed by Marsden, Simo, Lewis and Posbergh [15] and by Lewis, Marsden, Ratiu and Simo [14]. Applications of these methods to the coupled rigid bodies dynamics were considered by Krishnaprasad and Marsden [11], Krishnaprasad [12] and Patrick [21, 22]. Equilibrium rotations of planar kinematic chains of (two-dimensional) rigid bodies were studied by Baillieul [3]. Integrable cases of rotations of two multi-dimensional interacting (but not coupled) rigid bodies were constructed by Bobenko, Reyman and Semenov–Tian–Shansky [4]. Dynamics of orbiting multi-body configurations, including rigid and flexible bodies was studied by Modi and Suleman [17] and by Modi, Ng, Suleman and Morita [18]. General problems of coupled rigid bodies dynamics were considered in monographs by Routh [23], Leimanis [13] and Wittenburg [26].

The present work is devoted to the construction and investigation of integrable cases of three-dimensional coupled rigid bodies dynamics. We introduce the C^k-central configurations of rigid bodies, coupled by the ideal spherical ball-in-socket or hinge joints or by the Cardan suspensions without torque and friction. These configurations have tree structures with the main rigid body T_{0k} chosen, which is coupled with many branchy chains of rigid bodies, having lengths not greater than k, and without closed chains.

The rigid bodies in the C^k-central configurations are arbitrary, with arbitrary distributions of masses and inertia tensors. The spherical joints are placed in such a way, that the following mechanical property is fulfilled: If we fix the main rigid body T_{0k} and rotate all other rigid bodies independently around their spherical joints, then the mass center of the whole configuration does not move. The mechanical property is equivalent to the constructive definition of C^k-central configuration, presented in Sects. 2 and 3.

If one chooses an arbitrary system S of coupled rigid bodies, having tree structure without closed chains, then after appropriate shifts of spherical joints one obtains the C^k-central configuration. Therefore the dynamics of an arbitrary system S of coupled rigid bodies can be studied as perturbation of dynamics of an appropriate C^k-central configuration.

Our main theorem, proven in Sects. 2 and 3 states that dynamics of an arbitrary C^k-central configuration in the Newtonian gravitational field with an arbitrary quadratic potential is integrable in the Liouville sense and in the theta-functions of Riemann surfaces. This theorem holds also for dynamics of coupled n-dimensional rigid bodies ($n = 2, 3, 4, \dots$) in n-dimensional Euclidean space and has the same proof of all $n \geqq 2$. Here we have infinitely many arbitrary parameters, connected with the inertia tensors of the rigid bodies and concrete choice of graph of the tree structure and arbitrary form of the Newtonian quadratic potential. Therefore our main theorem describes the general integrable problems of classical mechanics.

The integrability of the problem considered is based on the fundamental physical law of the identity of the inertial and gravitational masses. Gravitational force, acting on a material point in the Newtonian gravitational field, is proportional to its mass. Therefore the kinetic energy and the potential energy of the C^k-central configuration depend on the same parameters, which are the entries of the inertia tensors of the coupled rigid bodies. Equations of rotation of even a single electrically charged rigid body in the electrostatic field with an arbitrary quadratic potential are not integrable.

In Sect. 4 a hidden symmetry of the inertial dynamics of an arbitrary C^k-central configuration is disclosed and reductions of the Lagrange equations to the Euler equations on the direct sums of Lie coalgebras are pointed out. In Sect. 5 reductions and integrable cases of rotation of a C^k-central configuration around a fixed point in Newtonian gravitational fields with quadratic and linear potentials are found. Multibody integrable generalization of the classical Neumann problem is presented in Sect. 6.

The problem of rotation around the mass center of a space station type orbiting system, being a C^k-central configuration of rigid bodies, is studied in Sect. 7. Separation of rotations of the rigid bodies in such a configuration is proven. The corresponding Lagrange equations are embedded into the Euler equations on the direct sums of the Lie coalgebras, which are autonomous in the case of the circular orbit and nonautonomous for the general elliptic orbit of the mass center.

Dynamics of more general CR^n-central configurations of coupled gyrostats, having all possible joints of rigid bodies: spherical or ball-in-socket joints, pin joints and universal joints, is studied in Sect. 8. Separation of rotations of an orbiting space station type CR^n-central configuration is obtained and integrability of inertial dynamics is proven.

2. Complete Integrability of Dynamics of a C^1-Central Configuration

I. Let us consider dynamics of a system S of $n + 1$ rigid bodies, consisting of the main rigid body T_0 and n rigid bodies T_α ($\alpha = 1, \dots, n$). Rigid body T_0 is coupled with each rigid body T_α in its (T_α) mass center $\mathbf{P}_\alpha$ by the ideal spherical ball-in-socket or hinge joint or by the Cardan suspension, so they are free to rotate relative to each other. Particles of rigid body T_0 have Lagrange coordinates r_0^i. Particles of rigid body T_α have Lagrange coordinates r_α^i; its mass center $\mathbf{P}_\alpha$ is in the origin (0, 0, 0) of coordinate system r_α^i. The center of the spherical joint $\mathbf{P}_\alpha$ in the rigid body T_0 has Lagrange coordinates $r_{0\alpha}^i$, which are arbitrary. Euler coordinates of the points $\mathbf{P}_\alpha$ are $q_\alpha^1(t)$, $q_\alpha^2(t)$, $q_\alpha^3(t)$.

Let T denote an effective rigid body, which is obtained from the rigid body T_0 by putting into the points $\mathbf{P}_\alpha$ masses m_α, equal to the masses of the rigid bodies T_α. Lagrange coordinates r_0^i are also used for the effective rigid body T; we choose them in such a way that the mass center $\mathbf{P}$ of the effective rigid body T has Lagrange coordinates $r_0^i = 0$. Euler coordinates of the mass center $\mathbf{P}$ are $q^i(t)$. Let $\rho_0(r_0^1, r_0^2, r_0^3)$ and $\rho_\alpha(r_\alpha^1, r_\alpha^2, r_\alpha^3)$ be the mass densities of the rigid bodies T_0 and T_α.

We shall call such system S, composed of the main rigid body T_0, coupled with n rigid bodies $T_1, \dots, T_n$ in their mass centers $\mathbf{P}_\alpha$ as the C^1-central configuration. The main mechanical property of the C^1-central configuration is the following: if we fix the main rigid body T_0 and rotate independently and arbitrarily all rigid bodies $T_1, \dots, T_n$ around their spherical joints (which coincide with their mass centers $\mathbf{P}_\alpha$) then the mass center $\mathbf{P}$ of the whole configuration does not move. The proof obviously consists in verifying that the mass center of the system S coincides with the mass center of the effective rigid body T, which is fixed.

The configuration space M of the system S is the direct product of Lie groups

$$M = R^3 \times \prod_{\alpha=0}^{n} SO(3)_\alpha . \tag{2.1}$$

Position of the mass center $\mathbf{P}$ is determined by the vector $\mathbf{q}(t) \in R^3$, rotation of the effective rigid body T around its mass center $\mathbf{P}$ is determined by the orthogonal matrix $Q^i_j(t) \in SO(3)_0$, rotation of the rigid body T_α around its mass center $\mathbf{P}_\alpha$ is determined by the orthogonal matrix $Q^i_{\alpha j}(t) \in SO(3)_\alpha$.

Euler coordinates of particles of the rigid body T_0 are determined by the formulae

$$x^i = \sum_{j=1}^{3} Q^i_j(t) r^j_0 + q^i(t) . \tag{2.2}$$

Euler coordinates of particles of the rigid body T_α are determined by the expressions

$$x^i_\alpha = \sum_{j=1}^{3} Q^i_{\alpha j}(t) r^j_\alpha + q^i_\alpha(t) , \tag{2.3}$$

where $q^1_\alpha(t)$, $q^2_\alpha(t)$, $q^3_\alpha(t)$ are Euler coordinates of the center of the spherical joint $\mathbf{P}_\alpha$, which belongs also to the rigid body T_0. In view of (2.2) we have

$$q^i_\alpha(t) = \sum_{j=1}^{3} Q^i_j(t) r^j_{0\alpha} + q^i(t) .$$

II.

Theorem 1. *Dynamics of an arbitrary C^1-central configuration S^1 in the Newtonian gravitational field with an arbitrary quadratic potential*

$$\varphi(x^1, x^2, x^3) = \frac{1}{2} \sum_{i,j=1}^{3} (a_{ij} x^i x^j + b_i x^i) \tag{2.4}$$

is described by a Hamiltonian system which is completely integrable in the Liouville sense. Dynamics of the mass center $\mathbf{P}$ is integrable in terms of elementary functions. Rotations of the rigid bodies $T, T_1, \ldots, T_n$ around their mass centers $\mathbf{P}, \mathbf{P}_1, \ldots, \mathbf{P}_n$ take place independently and are integrable in terms of the theta-functions of Riemann surfaces.

Proof. The Lagrangian of the C^1-central configuration S^1 in the fixed system of reference F is equal to the integral with respect to the volumes of all rigid bodies $T_0, T_1, \ldots, T_n$ of Lagrange functions of their particles:

$$L_S = L_0 + \sum_{\alpha=1}^{n} L_\alpha , \tag{2.5}$$

$$L_0 = \frac{1}{2} \int_{T_0} \rho_0(\mathbf{r}_0)(\dot{x}_0, \dot{x}_0) d^3\mathbf{r}_0 - \int_{T_0} \rho_0(\mathbf{r}_0)\varphi(x_0) d^3\mathbf{r}_0 , \tag{2.6}$$

$$L_\alpha = \frac{1}{2}\int_{T_\alpha} \rho_\alpha(\mathbf{r}_\alpha)\left(\sum_{i=1}^{3}\left(\sum_{j=1}^{3} \dot{Q}^i_{\alpha j} r^j_\alpha + \dot{q}^i_\alpha\right)^2\right) d^3\mathbf{r}_\alpha$$

$$- \int_{T_\alpha} \rho_\alpha(\mathbf{r}_\alpha)\varphi\left(\sum_{j=1}^{3} Q^1_{\alpha j} r^j_\alpha + q^1_\alpha, \sum_{j=1}^{3} Q^2_{\alpha j} r^j_\alpha + q^2_\alpha, \sum_{j=1}^{3} Q^3_{\alpha j} r^j_\alpha + q^3_\alpha\right) d^3\mathbf{r}_\alpha . \quad (2.7)$$

Lagrange coordinates r^i_α are chosen in such a way that mass center $\mathbf{P}_\alpha$ has zero coordinates, so the equalities hold

$$\int_{T_\alpha} \rho_\alpha(\mathbf{r}_\alpha) r^j_\alpha d^3\mathbf{r}_\alpha = 0 . \quad (2.8)$$

In view of the definition of the inertia tensor $I_{\alpha ik}$ we have

$$\int_{T_\alpha} \rho_\alpha(\mathbf{r}_\alpha) r^i_\alpha r^j_\alpha d^3\mathbf{r}_\alpha = \frac{1}{2}\mathrm{Tr}(I_\alpha)\delta_{ij} - I_{\alpha ij}$$
$$= J_{\alpha ij} . \quad (2.9)$$

Expression (2.7) after substituting formulae (2.4) and integrating with respect to the volume of the rigid body T_α takes the form

$$L_\alpha = L_{1\alpha} + L_{2\alpha} , \quad (2.10)$$

$$L_{1\alpha} = \frac{1}{2} m_\alpha(\dot{\mathbf{q}}_\alpha, \dot{\mathbf{q}}_\alpha) - m_\alpha \varphi(\mathbf{q}_\alpha) , \quad (2.11)$$

$$L_{2\alpha} = \frac{1}{2}\sum_{i,j,k=1}^{3} \dot{Q}^i_{\alpha j} J_{\alpha jk} \dot{Q}^i_{\alpha k} - \frac{1}{2}\sum_{i,j,k,m=1}^{3} a_{ij} Q^i_{\alpha k} Q^j_{\alpha m} J_{\alpha km}$$
$$= -\frac{1}{2}\mathrm{Tr}(J_\alpha Q^{-1}_\alpha \dot{Q}_\alpha Q^{-1}_\alpha \dot{Q}_\alpha) + \frac{1}{2}\mathrm{Tr}(Q^t_\alpha a Q_\alpha I_\alpha) , \quad (2.12)$$

where additive constants are omitted. In this calculation we used essentially the quadratic forms of the kinetic energy and of the Newtonian potential (2.4). In Lagrangian (2.7) all linear with respect to r^i_α summands are equal to zero after the integration in view of the equalities (2.8).

From the formulae (2.5) and (2.10) we receive

$$L_S = L_T + \sum_{\alpha=1}^{n} L_{2\alpha} , \quad (2.13)$$

where Lagrangian L_T in view of (2.6) and (2.11) has the form

$$L_T = L_0 + \sum_{\alpha=1}^{n} L_{\alpha 1}$$
$$= \frac{1}{2}\int_{T_0} \rho_0(\mathbf{r}_0)(\dot{\mathbf{x}}_0, \dot{\mathbf{x}}_0) d^3\mathbf{r}_0 - \int_{T_0} \rho_0(\mathbf{r}_0)\varphi(\mathbf{x}_0) d^3\mathbf{r}_0$$
$$+ \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha(\dot{\mathbf{q}}_\alpha, \dot{\mathbf{q}}_\alpha) - \sum_{\alpha=1}^{n} m_\alpha\varphi(\mathbf{q}_\alpha) . \quad (2.14)$$

Vector $\mathbf{q}_\alpha(t)$ describes the dynamics of the center of the spherical joint $\mathbf{P}_\alpha$, which by definition of the C^1-central configuration S^1 belongs to the rigid body T_0 (and

T_α also). Therefore the function L_T is the Lagrangian of the effective rigid body T, having masses m_α in the points $\mathbf{P}_\alpha$. The mass density $\rho(\mathbf{r}_0)$ of the effective rigid body T is determined by the formula

$$\rho(\mathbf{r}_0) = \rho_0(\mathbf{r}_0) + \sum_{\alpha=1}^{n} m_\alpha \delta(\mathbf{r}_0 - \mathbf{P}_\alpha)\,. \tag{2.15}$$

So Lagrangian (2.14) takes the form

$$L_T = \frac{1}{2}\int_T \rho(\mathbf{r}_0)(\dot{\mathbf{x}}_0, \dot{\mathbf{x}}_0) d_3\mathbf{r}_0 - \int_T \rho(\mathbf{r}_0)\varphi(\mathbf{x}_0) d_3\mathbf{r}_0\,. \tag{2.16}$$

Substituting here the formulae (2.2), we get

$$L_T = \frac{1}{2}\int_T \rho(\mathbf{r}_0) \sum_{i=1}^{3}\left(\sum_{j=1}^{3} \dot{Q}_j^i r_0^j + \dot{q}^i\right)^2 d^3\mathbf{r}_0$$
$$- \int_T \rho(\mathbf{r}_0)\varphi\left(\sum_{j=1}^{3} Q_j^1 r_0^j + q^1, \sum_{j=1}^{3} Q_j^2 r_0^j + q^2, \sum_{j=1}^{3} Q_j^3 r_0^j + q^3\right) d^3\mathbf{r}_0\,. \tag{2.17}$$

In view of the definitions of the inertia tensor I_{ik} and mass center of the effective rigid body T the formulae are valid

$$\int_T \rho(\mathbf{r}_0) r_0^i r_0^j d^3\mathbf{r}_0 = \frac{1}{2}\mathrm{Tr}(I)\delta_{ij} - I_{ij} = J_{ij}\,,$$
$$\int_T \rho(\mathbf{r}_0) r_0^i d^3\mathbf{r}_0 = 0\,. \tag{2.18}$$

Expression (2.17) after substituting the formula (2.4) and integrating with respect to the volume of the rigid body T in view of (2.18) takes the form

$$L_T = L_1 + L_2\,, \tag{2.19}$$

$$L_1 = \frac{1}{2} m(\dot{\mathbf{q}}, \dot{\mathbf{q}}) - m\varphi(\mathbf{q})\,, \tag{2.20}$$

$$L_2 = -\frac{1}{2}\mathrm{Tr}(JQ^{-1}\dot{Q}Q^{-1}\dot{Q}) + \frac{1}{2}\mathrm{Tr}(Q^t a Q I)\,. \tag{2.21}$$

Here m is the mass of the effective rigid body T, or total mass of all rigid bodies $T_0, T_1, \ldots, T_n$.

Thus, in view of (2.13) we receive the resulting formula for the Lagrangian L_{S^1} of the C^1-central configuration S^1:

$$L_{S^1} = L_1(\mathbf{q}, \dot{\mathbf{q}}) + L_2(Q, \dot{Q}) + \sum_{\alpha=1}^{n} L_{2\alpha}(Q_\alpha, \dot{Q}_\alpha)\,. \tag{2.22}$$

Time evolution of vector $\mathbf{q}$ and $n+1$ orthogonal matrices Q, Q_α ($\alpha = 1, \ldots, n$) describes completely the dynamics on the configuration manifold M (2.1), that is dynamics of the system of coupled rigid bodies S^1. The basic formula (2.22) shows that variables $\mathbf{q}$, Q, Q_α are separated and there is no interaction between dynamics of the mass center $\mathbf{P}$ and rotation of the rigid bodies T, T_α around their mass centers $\mathbf{P}$ and $\mathbf{P}_\alpha$.

The Lagrangian L_1 (2.20) in view of (2.4) is the Lagrangian of a harmonic oscillator. So the dynamics of the mass center $\mathbf{P}$ of the system S^1 is integrable in terms of elementary functions.

The Lagrangians L_2 (2.21) and $L_{2\alpha}$ (2.12) describe rotation of a rigid body around a fixed point in the gravitational field with the homogeneous quadratic potential (without linear terms). This problem is studied in detail by Bogoyavlenskij [6, 7], where it is proven that the corresponding Hamiltonian system is integrable in the Liouville sense and dynamics is integrable in terms of the theta-functions of the Riemann surfaces. (History of this problem, its generalizations and bibliography are presented in survey [10]). Therefore dynamics of the C^1-central configuration S^1 also is integrable in the Liouville sense and in the Riemann theta-functions. Theorem 1 is proven.

Remark 1. In Bogoyavlenskij [8, 9, 10] the integrability of dynamics of one arbitrary rigid body (without a fixed point) in a Newtonian gravitational field with an arbitrary quadratic potential is proven. This is the special case of Theorem 1.

3. General Integrable Problems of Classical Mechanics

The main consequence of the Theorem 1 is that the Lagrangian of an arbitrary C^1-central configuration S^1 in the Newtonian gravitational field with an arbitrary quadratic potential (2.4) is split into the sum (2.22), where the vector $\mathbf{q}(t)$ stays for the mass center of all system S^1.

We consider m C^1-central configurations denoted as $S_1^1, \ldots, S_m^1$ and let $\mathbf{q}_1^1(t), \ldots, \mathbf{q}_m^1(t)$ be vectors of their mass centers. We call the C^2-central configuration a system of rigid bodies, obtained by the spherical coupling of the main rigid body T_{02} with m C^1-central configurations $S_1^1, \ldots, S_m^1$ in their mass centers $\mathbf{q}_1^1, \ldots, \mathbf{q}_m^1$. The effective rigid body T_2 is obtained from the rigid body T_{02} by putting masses m_i^1 (the mass of the system S_i^1) into the points $\mathbf{q}_i^1$.

By induction we call C^k-central configuration a system of rigid bodies, obtained by coupling the main rigid body T_{0k} with n C^{k-1}-central configurations $S_1^{k-1}, \ldots, S_n^{k-1}$ in their mass centers $\mathbf{q}_1^{k-1}, \ldots, \mathbf{q}_n^{k-1}$. The effective rigid body T_k is obtained from the rigid body T_{0k} by putting masses m_i^{k-1} (mass of the system S_i^{k-1}) into the points $\mathbf{q}_i^{k-1}$. The mass center $\mathbf{P}_k$ of C^k-central configuration S^k coincides with the mass center of the effective rigid body T_k. Therefore the mass center $\mathbf{P}_k$ does not move if all rigid bodies in this configuration (except T_{0k}) arbitrarily rotate around their spherical joints.

Theorem 2. *Dynamics of an arbitrary C^k-central configurations S^k of n coupled rigid bodies in the Newtonian gravitational field with an arbitrary quadratic potential* (2.4) *is described by the Hamiltonian system which is completely integrable in the Liouville sense. The Lagrangian L_{S^k} of the C^k-central configuration S^k is split into the sum*

$$L_{S^k} = L_1(\mathbf{q}, \dot{\mathbf{q}}) + \sum_{\alpha=1}^{N} L_{2\alpha}(Q_\alpha, \dot{Q}_\alpha) , \tag{3.1}$$

$$L_1(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m(\dot{\mathbf{q}}, \dot{\mathbf{q}}) - m\varphi(\mathbf{q}) , \tag{3.2}$$

$$L_{2\alpha}(Q_\alpha, \dot{Q}_\alpha) = -\frac{1}{2}\mathrm{Tr}(J_\alpha Q_\alpha^{-1} \dot{Q}_\alpha Q_\alpha^{-1} \dot{Q}_\alpha) + \frac{1}{2}\mathrm{Tr}(Q_\alpha^t a Q_\alpha I_\alpha) , \tag{3.3}$$

where vector $\mathbf{q}(t)$ *stays for the mass center of the configuration* S^k *(with the whole mass* m*) and the orthogonal matrix* Q_α *describes rotation of the* α*-th effective rigid body around its mass center. Dynamics of mass center* $\mathbf{q}(t)$ *is integrable in terms of elementary functions. Rotation of each rigid body* T_α *is integrable in terms of the Riemann theta-functions.*

Proof. Theorem 2 for the C^1-central configurations is proven in Sect. 2. Therefore let us suppose by induction the theorem is proven for C^{k-1}-central configurations $S_1^{k-1}, \ldots, S_n^{k-1}$. That means their Lagrangians $L_{S_j^{k-1}}$ have the form:

$$L_{S_j^{k-1}} = L_{1j}(\mathbf{q}_j, \dot{\mathbf{q}}_j) + \sum_{\alpha=1}^{n_j} L_{2\alpha}(Q_\alpha, \dot{Q}_\alpha)\,, \tag{3.4}$$

$$L_{1j} = \frac{1}{2} m_j(\dot{\mathbf{q}}_j, \dot{\mathbf{q}}_j) - m_j \varphi(\mathbf{q}_j)\,,$$

where vector $\mathbf{q}_j(t)$ stays for the mass center of the configuration S_j^{k-1} (with the whole mass m_j). The C^k-central configuration S^k is obtained by the spherical coupling of the main rigid body T_{0k} in the points $\mathbf{P}_j$ with the C^{k-1}-central configurations S_j^{k-1} ($j = 1, \ldots, n$) in their mass centers $\mathbf{q}_j$. The Lagrangian L_{S^k} of the configuration S^k has the form

$$L_{S^k} = L_{T_{0k}} + \sum_{j=1}^{n} L_{S_j^{k-1}}\,. \tag{3.5}$$

The Lagrangian $L_{T_{0k}}$ is given by the formula

$$L_{T_{0k}} = \frac{1}{2} \int_{T_{0k}} \rho_0(\mathbf{r}_0) \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \dot{Q}_j^i r_0^j + \dot{q}^i \right)^2 d^3\mathbf{r}_0$$

$$- \int_{T_{0k}} \rho_0(\mathbf{r}_0) \varphi\left(\sum_{j=1}^{3} Q_j^1 r_0^j + q^1, \sum_{j=1}^{3} Q_j^2 r_0^j + q^2, \sum_{j=1}^{3} Q_j^3 r_0^j + q^3 \right) d^3\mathbf{r}_0\,. \tag{3.6}$$

Here (r_0^1, r_0^2, r_0^3) are Lagrange coordinates of particles of the rigid body T_{0k}, $\rho_0(\mathbf{r}_0)$ is its mass density. Vector (q^1, q^2, q^3) stays for the mass center of the effective rigid body T_k and coincides with the mass center of the whole configuration S^k. The effective rigid body T_k is obtained from the rigid body T_{0k} by putting masses m_j into the coupling points $\mathbf{P}_j(\mathbf{q}_j)$. Its mass density has the form

$$\rho(\mathbf{r}_0) = \rho_0(\mathbf{r}_0) + \sum_{j=1}^{n} m_j \delta(\mathbf{r}_0 - \mathbf{P}_j)\,. \tag{3.7}$$

Orthogonal matrix Q_0 describes rotation of the effective rigid body T_k around its mass center $\mathbf{q}$.

Formula (3.5) after substituting the expressions (3.4), (3.6) and using the formula (3.7) is transformed into

$$L_{S^k} = L_{T_k} + \sum_{j=1}^{n} \sum_{\alpha=1}^{n_j} L_{2\alpha}(Q_\alpha, \dot{Q}_\alpha)\,. \tag{3.8}$$

Here the Lagrangian L_{T_k} of the effective rigid body T_k has the form

$$L_{T_k} = \frac{1}{2} \int_{T_k} \rho(\mathbf{r}_0) \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \dot{Q}^i_j r^j_0 + \dot{q}^i \right)^2 d^3\mathbf{r}_0$$

$$- \int_{T_k} \rho(\mathbf{r}_0) \varphi \left(\sum_{j=1}^{3} Q^1_j r^j_0 + q^1, \sum_{j=1}^{3} Q^2_j r^j_0 + q^2, \sum_{j=1}^{3} Q^3_j r^j_0 + q^3 \right) d^3\mathbf{r}_0 . \quad (3.9)$$

This Lagrangian coincides with (2.17), so it is transformed also into the form (2.19)–(2.21). Therefore Theorem 2 is proven by the induction.

The integrability of the dynamics of the mass center $\mathbf{q}(t)$ in terms of elementary functions and the integrability of rotations of all rigid bodies T_α in terms of the Riemann theta-functions obviously follow from the separation of variables in the Lagrangian L_{S^k} (3.1) and from the results of works [6, 7], see also the next section.

Remark 2. Let some C^k-central configuration S^k with the main rigid body T_{0k} be coupled with another system of rigid bodies S by one spherical joint, belonging to the rigid body T_{0k}. Then the Lagrangian of the whole system $S \cup S^k$ has the form

$$L_{S \cup S^k} = L_S + L_{S, T_k} + \sum_{\alpha=1}^{N-1} L_{2\alpha}(Q_\alpha, \dot{Q}_\alpha) . \quad (3.10)$$

Here L_S is the Lagrangian of the system S, function L_{S, T_k} describes interaction between system S and the effective rigid body T_k. The Lagrangians $L_{2\alpha}(Q_\alpha, \dot{Q}_\alpha)$ of all other $N-1$ rigid bodies of the C^k-central configuration S^k are separated (in the Newtonian gravitational field with quadratic potential). Rotations of these $N-1$ rigid bodies around their spherical joints are described by the independent Lagrange system with the Lagrangians of the form (3.3).

Remark 3. The construction used for the effective rigid bodies T_i does not coincide with that of the augmented rigid bodies, introduced by Wittenburg [26]. For example, all the augmented rigid bodies have equal masses, but effective rigid bodies T_i have different masses.

4. Hidden Symmetry of the Inertial Dynamics

I. The inertial dynamics of a C^k-central configuration S^k is described in view of Theorem 2 by the Lagrange system with the Lagrangian

$$L_{S^k} = \frac{1}{2} m(\dot{\mathbf{q}}, \dot{\mathbf{q}}) - \frac{1}{2} \sum_{\alpha=1}^{N} \mathrm{Tr}(J_\alpha Q_\alpha^{-1} \dot{Q}_\alpha Q_\alpha^{-1} \dot{Q}_\alpha) , \quad (4.1)$$

coinciding with the kinetic energy. The rotational part of the Lagrangian (4.1) has a large group of hidden symmetries $G = \prod_{\alpha=1}^{N} SO(3)_\alpha$, acting by the left multiplications

$$Q_\alpha \to R_\alpha Q_\alpha , \quad R_\alpha \in SO(3)_\alpha . \quad (4.2)$$

In view of the explicit form of the Lagrangian (4.1) all rigid bodies of the C^k-central configurations S^k rotate around their spherical joints independently. For example there are stable regimes of dynamics in which certain K effective rigid bodies inertially rotate and other $N - K$ rigid bodies do not rotate and move in space remaining parallel to themselves.

The Lagrangian (4.1) determines the left-invariant metric on the configuration space M, which is the direct product of Lie groups

$$M = R^3 \times \prod_{\alpha=1}^{N} SO(3)_\alpha \tag{4.3}$$

and has symmetry group $E_3 \times G$, where E_3 is the group of all Euclidean motions of the space R^3. The corresponding momentum map, see Abraham and Marsden [1], has the form

$$(q, \dot{q}, Q_\alpha, \dot{Q}_\alpha) \to (\mathbf{p} = m\dot{\mathbf{q}}, \quad \bar{M}_\alpha = Q_\alpha M_\alpha Q_\alpha^{-1}) . \tag{4.4}$$

The vector $\mathbf{p}$ and N skew-symmetric matrices $\bar{M}_\alpha$ are first integrals of the Lagrange system. Skew-symmetric matrices of angular velocity ω_α and angular momentum m_α in the rotating frames of reference, connected with the α^{th} effective rigid body, are determined by the formulae

$$\begin{aligned} \omega_\alpha &= Q_\alpha^{-1}\dot{Q}_\alpha , \\ M_\alpha &= J_\alpha\omega_\alpha + \omega_\alpha J_\alpha , \\ J_{\alpha ij} &= \frac{1}{2}\mathrm{Tr}(I_\alpha)\delta_{ij} - I_{\alpha ij} . \end{aligned} \tag{4.5}$$

Here $I_{\alpha ij}$ are entries of the inertia tensor of the α^{th} effective rigid body. Lagrangian equations with the Lagrangian L_{S^k} (4.1) admit the reduction to the Euler equations on the Lie coalgebra $\mathscr{A}_{3N} = \bigoplus_{\alpha=1}^{N} SO(3)_\alpha$:

$$\dot{M}_\alpha = [M_\alpha, \omega_\alpha] , \quad \alpha = 1, \ldots, N . \tag{4.6}$$

These equations have $2N$ independent and involutive first integrals

$$I_{1\alpha} = \frac{1}{2}\mathrm{Tr}(M_\alpha^2) , \quad I_{2\alpha} = \frac{1}{2}\mathrm{Tr}(M_\alpha\omega_\alpha) . \tag{4.7}$$

The total kinetic energy of the inertial dynamics of the C^k-central configuration has the form

$$T = \frac{1}{2}m(\dot{\mathbf{q}}, \dot{\mathbf{q}}) - \frac{1}{2}\sum_{\alpha=1}^{N} \mathrm{Tr}(M_\alpha\omega_\alpha) .$$

Inertial dynamics of all rigid bodies in the C^k-central configuration is integrable in terms of the elliptic functions.

Remark 4. The momentum map (4.4) determines $3N + 3$ adiabatic invariants of the inertial dynamics of system S of N coupled rigid bodies which is a small perturbation of some C^k-central configuration. Inertial dynamics of such systems, which is described by dynamics of geodesics of certain metrics on the configuration space M (4.3) can be studied by the methods of the Kolmogorov–Arnold–Moser theory.

II. Dynamics of a C^k-central configuration in a Newtonian gravitational field with an arbitrary quadratic potential $\varphi(\mathbf{q})$ (2.4) is described by the Lagrange system with the Lagrangian (3.1). This system is equivalent to the split system of matrix equations

$$\dot{M}_\alpha = [M_\alpha, \omega_\alpha] + [u_\alpha, J_\alpha] , \quad \dot{u}_\alpha = [u_\alpha, \omega_\alpha] , \tag{4.8}$$

where $u_\alpha = Q_\alpha^t a Q_\alpha$ and the Lagrange equations for the dynamics of the mass center $\mathbf{q}(t)$ with the Lagrangian $L_1 = \frac{1}{2}m(\dot{\mathbf{q}}, \dot{\mathbf{q}}) - m\varphi(\mathbf{q})$.

Equations (4.8), as shown by Bogoyavlenskij [6, 7] are the Euler equations on the dual space to the Lie algebra A_9 and have the Hamiltonian

$$H_\alpha = \frac{1}{2}\mathrm{Tr}(M_\alpha \omega_\alpha) + \mathrm{Tr}(J_\alpha u_\alpha) . \tag{4.9}$$

Vectors a of the Lie algebra A_9 have the form $a = M + u$, where M is a skew-symmetric 3×3 matrix and u is a symmetric 3×3 matrix. Commutators of these matrices are determined by the formulae

$$\begin{aligned} [M_1, M_2] &= M_1 M_2 - M_2 M_1 , \\ [M, u] &= Mu - uM , \\ [u_1, u_2] &= 0 . \end{aligned} \tag{4.10}$$

Thus the Lie algebra A_9 is the semidirect sum $SO(3) \dot{\oplus} R^6$.

Therefore the discussed Lagrange system has the reduction to the Euler equations on the Lie coalgebra $\mathscr{A}_{9N}^*$, where $\mathscr{A}_{9N} = \bigoplus_{\alpha=1}^N A_9$.

Systems (4.8) are equivalent to the Lax equations with spectral parameter E:

$$\begin{aligned} \dot{L}_\alpha &= [L_\alpha, A_\alpha] , \\ L_\alpha &= J_\alpha^2 E^2 + M_\alpha E + u_\alpha , \\ A_\alpha &= J_\alpha E + \omega_\alpha . \end{aligned} \tag{4.11}$$

Therefore N Riemann surfaces Γ_α, determined by equations

$$R_\alpha(w, E) = \det(J_\alpha^2 E^2 + M_\alpha E + u_\alpha - w \cdot 1) = 0 \tag{4.12}$$

are associated with trajectories of the systems (4.8). All coefficients of Eq. (4.12) are involutive first integrals of (4.8), for example, $3N$ independent first integrals

$$I_{1\alpha} = H_\alpha , \quad I_{2\alpha} = \mathrm{Tr}(M_\alpha^2 + 2J_\alpha^2 u_\alpha) , \quad I_{3\alpha} = \mathrm{Tr}(M_\alpha^2 u_\alpha + J_\alpha^2 u_\alpha^2) . \tag{4.13}$$

Trajectories of systems (4.8) are integrable in terms of the theta-functions of Riemann surface $\mathbf{\Gamma}_\alpha$, see works [6, 7].

Three-dimensional dynamics of each rigid body, belonging to the C^k-central configuration S^k, is integrable in terms of the theta-functions of several Riemann surfaces $\mathbf{\Gamma}_\alpha$, the number of which is equal to the number of rigid bodies, coupling the chosen rigid body with the main rigid body T_{0k}.

Remark 5. Three-dimensional dynamics of a system S of N coupled rigid bodies, which is a small perturbation of a C^k-central configuration S^k, in the Newtonian gravitational field with quadratic potential $\varphi(\mathbf{q})$ (2.4), possesses $3N$ adiabatic invariants $I_{1\alpha}$, $I_{2\alpha}$, $I_{3\alpha}$ (4.13). Dynamics of such a system S can be studied by the methods of the Kolmogorov–Arnold–Moser theory.

5. Reductions and Integrable Cases of C^k-Central Configuration Rotation Around a Fixed Point in the Newtonian Gravitational Fields with Quadratic and Linear Potentials

In this section we consider rotation of some C^k-central configuration S^k with the main rigid body T_{0k} around a fixed point P_0, belonging to T_{0k}. Let the origin of the Lagrange coordinates r_0^1, r_0^2, r_0^3 (for particles of the rigid body T_{0k}) and the origin of the Euler coordinates x^1, x^2, x^3 coincide with the fixed point P_0, so the corresponding vector $\mathbf{q}(t)$ has coordinates (0, 0, 0).

The Lagrangian of the C^k-central configuration S^k in the Newtonian gravitational field with quadratic potential (2.4) has the form (3.8), where Lagrangian L_{T_k} is calculated by the formula (3.9) with $q^i(t) \equiv 0$. The mass center of the effective rigid body T_k has Lagrange coordinates

$$R^i = \int_{T_k} \rho(\mathbf{r}_0) r_0^i d^3\mathbf{r}_0 . \tag{5.1}$$

The inertia tensor I_{kij} is calculated with respect to the fixed point P_0:

$$\begin{aligned} J_{kij} &= \frac{1}{2}\mathrm{Tr}(I_k)\delta_{ij} - I_{kij} \\ &= \int_{T_k} \rho(\mathbf{r}_0) r_0^i r_0^j d^3\mathbf{r}_0 . \end{aligned} \tag{5.2}$$

In these notations the Lagrangian L_{T_k} (3.9) has the form

$$L_{T_k} = -\frac{1}{2}\mathrm{Tr}(J_k Q^{-1}\dot{Q} Q^{-1}\dot{Q}) + \frac{1}{2}\mathrm{Tr}(Q^t a Q I_k) - \frac{1}{2}(\mathbf{b}, Q\mathbf{R}) . \tag{5.3}$$

Corollary 1. *In the considered problem all N effective rigid bodies of the C^k-central configuration S^k rotate around their spherical joints independently. Rotation of each of the $N - 1$ rigid bodies, except the effective rigid body T_k, is described by the integrable Lagrange systems with Lagrangians $L_{2\alpha}(Q_\alpha, \dot{Q}_\alpha)$ of the form* (3.3). *Dynamics of the configuration S^k possesses $3(N - 1)$ involutive first integrals* (4.13), *where $\alpha = 1, \ldots, N - 1$.*

Corollary 2. *Rotation of the effective rigid body T_k is integrable in the following cases*:

1) *Fixed point P_0 coincides with the mass center of the effective rigid body T_k, that means $R^i = 0$.*
2) *Fixed point P_0 is arbitrary, but potential $\rho(x^1, x^2, x^3)$ is pure quadratic, that means $b_i = 0$.*

In both these cases the Lagrangian (5.3) coincides with the integrable Lagrangian (3.5) and dynamics of all configuration S^k possesses $3N$ involutive first integrals (4.13), where $\alpha = 1, \ldots, N$.

Corollary 3. *Dynamics of an arbitrary C^k-central configuration S^k around a fixed point in the Newtonian gravitational field with linear potential $\varphi(x^1, x^2, x^3)$, $a_{ij} \equiv 0$,* *possesses a large group of hidden symmetries,* $G_1 = \prod_{\alpha=1}^{N-1} SO(3)_\alpha$, *acting as in* (3.10). *The momentum map has the form* (4.4), *where* $\mathbf{p} = 0$, $\alpha = 1, \ldots, N-1$. *Rotation of each of the* $N-1$ *effective rigid bodies is pure inertial and is described by geodesics of the left-invariant metrics on the Lie group* $SO(3)_\alpha$ *with the Lagrangian*

$$L = -\frac{1}{2}\mathrm{Tr}(J_\alpha Q_\alpha^{-1}\dot{Q}_\alpha Q_\alpha^{-1}\dot{Q}_\alpha) . \tag{5.4}$$

Rotation of the effective rigid body T_k *is described by the Lagrangian*

$$L = -\frac{1}{2}\mathrm{Tr}(JQ^{-1}\dot{Q}Q^{-1}\dot{Q}) - \frac{1}{2}(\mathbf{b}, Q\mathbf{R}) . \tag{5.5}$$

The corresponding Lagrange equations are reduced to the Euler–Poisson equations which are integrable only in three cases, discovered by Euler ($R^i = 0$), *Lagrange* ($I_1 = I_2$, $R^1 = R^2 = 0$) *and Kovalevskaya* ($I_1 = I_2 = 2I_3$, $R^3 = 0$).

The previous considerations lead to the consequence.

Corollary 4. *The problem of rotation of an arbitrary* C^k*-central configuration* S^k *of* N *coupled rigid bodies around a fixed point in the Newtonian gravitational field with quadratic or linear potential* $\varphi(x^1, x^2, x^3)$ *is reduced to the classical problem of rotation of only one effective rigid body* T_k *around a fixed point. Dynamics of the whole* C^k*-central configuration* S^k *is integrable if and only if rotation of the effective rigid body* T_k *is integrable.*

6. Multibody Integrable Generalization of the Neumann Problem

The classical Neumann problem is concerned with the dynamics of a material point on the surface of a sphere

$$(q^1)^2 + (q^2)^2 + (q^3)^2 = R_0^2 \tag{6.1}$$

in a field with homogeneous quadratic potential

$$\varphi(x^1, x^2, x^3) = \frac{1}{2}\sum_{i,j=1}^{3} a_{ij}x^i x^j . \tag{6.2}$$

The Neumann problem is completely integrable by the Hamilton–Jacobi method of separation of variables, see Neumann [20], Moser [19].

The following theorem describes generalization of the Neumann problem.

Theorem 3. *Dynamics of an arbitrary* C^k*-central configuration* S^k *in the Newtonian gravitational field with the homogeneous quadratic potential* (6.2) *under constraint that the mass center* $\mathbf{P}$ *of the whole configuration* S^k *moves on the sphere* (6.1) *is completely integrable. Rotations of all rigid bodies around their spherical joints and dynamics of the mass center* $\mathbf{P}$ *are described by the separated integrable Lagrange systems.*

Proof. In view of Theorem 2 dynamics of the C^k-central configuration S^k is described by the Lagrangian L_{S^k} (3.1), which is split into the sum of the separated Lagrangians $L_1(\mathbf{q}, \dot{\mathbf{q}})$ and $L_{2\alpha}(Q_\alpha, \dot{Q}_\alpha)$. The Lagrangians $L_{2\alpha}(Q_\alpha, \dot{Q}_\alpha)$ (3.3) describe integrable rotation of the α^{th} effective rigid body around the corresponding spherical joint. The Lagrangian L_1 (3.2) describes dynamics of the mass center $\mathbf{P}$, which is supposed to move on the sphere (6.1). The Lagrangian L_1 coincides with the Lagrangian of the integrable Neumann problem. Therefore the dynamics of the mass center $\mathbf{P}$ is integrable by the Hamilton–Jacobi method, and rotation of α^{th} rigid body is integrable in terms of the theta-functions of Riemann surface $\mathbf{\Gamma}_\alpha$, determined by Eq. (4.11).

7. Separation of Rotations of an Orbiting Space Station Type C^k-Central Configuration

I. Let us consider the problem of rotation of an artificial satellite of the Earth around its mass center. We suppose the gravitational field of the Earth is spherically-symmetric with the potential $\varphi = -Gm_0|\mathbf{r}|^{-1}$, where G is the gravity constant and m_0 is the mass of the Earth. It is known that the orbit of the satellite mass center is an ellipse with high accuracy and the Earth is in one of its focuses. We suppose that the origin of the fixed frame of references x_0^1, x_0^2, x_0^3 coincides with the center of the Earth, the satellite orbit lies in the plane $x^3 = 0$, and $\mathbf{q}(t) = (q^1(t), q^2(t), q^3(t) = 0)$ is the vector of the satellite mass center $\mathbf{P}$ and $\gamma(t) = \dfrac{\mathbf{q}(t)}{|\mathbf{q}(t)|}$.

Let x^1, x^2, x^3 be local Cartesian coordinates with the origin in the mass center $\mathbf{P}$: $x^i = x_0^i - q^i(t)$. The potential energy of the satellite in the gravitational field of the Earth, having potential $\varphi = -Gm_0|\mathbf{r}|^{-1}$, is equal

$$U = -\int_T \frac{Gm_0\,\rho(\mathbf{r}_0)}{|\mathbf{r}(\mathbf{r}_0)|} d^3\mathbf{r}_0\,, \tag{7.1}$$

where $\mathbf{r}_0$ are Lagrange coordinates of particles of the satellite T. We assume that the scales l of the satellite are much smaller than the distance to the Earth: $l \ll |\mathbf{q}|$. For the function $|\mathbf{r}|$ we have

$$\begin{aligned} |\mathbf{r}| &= (\mathbf{q} + \mathbf{x}, \mathbf{q} + \mathbf{x})^{\frac{1}{2}} \\ &= |\mathbf{q}|\left(1 + \frac{2}{|\mathbf{q}|}(\gamma, \mathbf{x}) + \frac{1}{|\mathbf{q}|^2}(\mathbf{x}, \mathbf{x})\right)^{\frac{1}{2}}, \end{aligned} \tag{7.2}$$

where $(\mathbf{x}, \mathbf{y})$ is the scalar product. The Taylor expansion of $\dfrac{1}{|\mathbf{r}|}$ in the neighbourhood of the point $q^1(t), q^2(t), q^3(t)$ has the form

$$\frac{1}{|\mathbf{r}|} = \frac{1}{|\mathbf{q}|} - \frac{1}{|\mathbf{q}|^2}(\gamma, \mathbf{x}) + \frac{1}{|\mathbf{q}|^3}\left(3(\gamma, \mathbf{x})^2 - \frac{1}{2}(\mathbf{x}, \mathbf{x})\right) + \mathcal{O}\frac{1}{|\mathbf{q}|^4}\,. \tag{7.3}$$

Substituting this expression into (7.1) and integrating with respect to the Lagrange coordinates $\mathbf{r}_0$ we get

$$U = -\frac{Gmm_0}{|\mathbf{q}|} - \frac{Gm_0}{|\mathbf{q}|^3}\int_T \rho(\mathbf{r}_0)\left(3(\gamma, \mathbf{x})^2 - \frac{1}{2}(\mathbf{x}, \mathbf{x})\right)d^3\mathbf{r}_0 + \mathcal{O}\frac{1}{|\mathbf{q}|^4}\,. \tag{7.4}$$

Here we use that in view of the definition of the mass center the equality holds

$$\int_T \rho(\mathbf{r}_0)(\gamma, \mathbf{x}) d^3\mathbf{r}_0 = 0 . \tag{7.5}$$

Expression (7.4) for the potential energy is used often, see for example Beletskij [5], Modi and Suleman [17], Saruchev [24].

In view of the formula (7.4) the Lagrangian L of the satellite in the gravitational field of the Earth in the main approximation has the form

$$L = L_1 + L_2 . \tag{7.6}$$

Here L_1 is the Lagrangian of the Kepler problem for dynamics of the satellite mass center $\mathbf{q}(t)$ around the Earth

$$L_1 = \frac{1}{2} m(\dot{\mathbf{q}}, \dot{\mathbf{q}}) + \frac{Gmm_0}{|\mathbf{q}|} . \tag{7.7}$$

Lagrangian L_2 describes rotation of the satellite around its mass center in the nonstationary gravitational field with homogeneous quadratic potential

$$\varphi(x^1, x^2, x^3) = -\frac{Gm_0}{|\mathbf{q}(t)|^3} \sum_{i,j=1}^{3} \left(3\gamma^i(t)\gamma^j(t) - \frac{1}{2}\delta_{ij} \right) x^i x^j . \tag{7.8}$$

II. Assume now the artificial satellite is a space station type C^k-central configuration S^k of N coupled rigid bodies. The mass center of such configuration does not move if all $N-1$ rigid bodies rotate arbitrarily around their spherical joints and the main effective rigid body T_k rotates around its mass center. The method of investigation of the Lagrangian L_{S^k} of the C^k-central configuration S^k developed for the proof of Theorems 1 and 2 in Sects. 2 and 3 is applicable also when the quadratic gravitational potential (2.4) is a nonstationary one and has the form (7.8), for example. Therefore we prove by the same method the following result.

Theorem 4. *The Lagrangian L_{S^k}, describing rotation of a C^k-central configuration S^k around its mass center, moving around the Earth on the orbit $\mathbf{q}(t)$, is split into the sum*

$$L_{S^k} = \sum_{\alpha=1}^{N} L_\alpha . \tag{7.9}$$

Lagrangians L_α have the form

$$L_\alpha = -\frac{1}{2}\mathrm{Tr}(J_\alpha Q_\alpha^{-1} \dot{Q}_\alpha Q_\alpha^{-1} \dot{Q}_\alpha) + \frac{1}{2}\mathrm{Tr}(Q_\alpha^t a Q_\alpha I_\alpha) , \tag{7.10}$$

where matrix a is nonstationary with the entries

$$a_{ij}(t) = -\frac{3Gm_0}{|\mathbf{q}(t)|^3}\gamma^i(t)\gamma^j(t) , \quad \gamma(t) = \frac{\mathbf{q}(t)}{|\mathbf{q}(t)|} . \tag{7.11}$$

In view of the separation of rotations in Lagrangian L_{S^k} (7.9) each effective rigid body T_α from the C^k-central configuration S^k rotate around its spherical joint independently, as well as the main effective rigid body T_k rotates around its mass center. Therefore we obtain the following consequence.

Corollary 5. *The problem of stabilization of rotations may be solved independently for each rigid body T_α from the orbiting space station type C^k-central configuration S^k.*

III. Let $\Omega = \dot{\varphi} = \mu|\mathbf{q}(t)|^{-2}$ be the angular velocity of the mass center $\mathbf{q}(t)$, μ is its orbital angular momentum. By definition we have

$$\begin{aligned} \gamma^1(t) &= \cos\varphi(t) , \\ \gamma^2(t) &= \sin\varphi(t) , \\ \gamma^3(t) &= 0 , \\ \frac{d\gamma}{dt} &= -\Omega\gamma\times\mathbf{n} , \\ \Omega &= \dot{\varphi} , \end{aligned} \tag{7.12}$$

where $\mathbf{n}$ is the constant unit vector, orthogonal to the orbit $\mathbf{q}(t)$.

For the matrix $\bar{a}_{ij} = \gamma_i\gamma_j$ we have

$$\begin{aligned} \bar{a} &= Q_1 e Q_1^t , \\ \dot{Q}_1 &= \Omega n_0 Q_1 , \end{aligned}$$

where

$$Q_1 = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad n_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the matrices

$$\begin{aligned} u_\alpha &= -Q_\alpha^t \bar{a} Q_\alpha \\ &= -Q_\alpha^t Q_1 e Q_1^t Q_\alpha , \\ n_\alpha &= Q_\alpha^t n_0 Q_\alpha \end{aligned} \tag{7.13}$$

satisfy the equations ($\dot{Q}_\alpha = Q_\alpha\omega_\alpha$)

$$\dot{u}_\alpha = [u_\alpha, \omega_\alpha - \Omega n_\alpha] , \quad \dot{n}_\alpha = [n_\alpha, \omega_\alpha] . \tag{7.14}$$

Therefore the Lagrange equations with the Lagrangian L_α (7.10) are embedded into the matrix equations, generalizing (4.8):

$$\begin{aligned} \dot{M}_\alpha &= [M_\alpha, \omega_\alpha] - a_0[u_\alpha, I_\alpha] , \\ \dot{u}_\alpha &= [u_\alpha, \omega_\alpha - \Omega n_\alpha] , \\ \dot{n}_\alpha &= [n_\alpha, \omega_\alpha] , \\ a_0 &= \frac{3Gm_0}{|\mathbf{q}(t)|^3} , \\ \Omega &= \frac{\mu}{|\mathbf{q}(t)|^2} . \end{aligned} \tag{7.15}$$

The Lagrange equations on the Lie group $SO(3)$, determined by the Lagrangian L_α (7.10) are embedded also into the system of vector equations

$$\begin{aligned} \dot{\mathbf{M}}_\alpha &= \mathbf{M}_\alpha \times \omega_\alpha + a_0 \gamma_\alpha \times I_\alpha \gamma_\alpha\,, \\ \dot{\gamma}_\alpha &= \gamma_\alpha \times (\omega_\alpha - \Omega \mathbf{n}_\alpha)\,, \\ \dot{\mathbf{n}}_\alpha &= \mathbf{n}_\alpha \times \omega_\alpha\,. \end{aligned} \tag{7.16}$$

Here all vectors $\mathbf{M}_\alpha$, ω_α, γ_α, $\mathbf{n}_\alpha$ are considered in the rotating frame of reference, connected with the effective rigid body T_α, its angular velocity is ω_α, its angular momentum is $\mathbf{M}_\alpha = I_\alpha \omega_\alpha$.

Equations (7.15) and (7.16) for the circular orbits of the mass center are autonomous, in this case $a_0 = 3\Omega^2$.

IV. Let us consider the Lie algebra L_9, which is the semi-direct sum of Lie algebras $SO(3) + R^3 + R^3$ with basis X_i, Y_j^β, $(i, j = 1, 2, 3,\ \beta = 1, 2)$, and commutator relations

$$[X_i, X_j] = \varepsilon_{ijk} X_k\,, \quad [X_i, Y_j^\beta] = \varepsilon_{ijk} Y_k^\beta\,, \quad [Y_i^\alpha, Y_j^\beta] = 0\,.$$

Vectors of the dual space L_9^* are represented in the form $\mathbf{M} + \gamma + \mathbf{n}$, according to the decomposition indicated. The Euler equations with the Hamiltonian $H(\mathbf{M}, \gamma, \mathbf{n})$ in the space L_9^* have the form

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M} \times \frac{\partial H}{\partial \mathbf{M}} + \gamma \times \frac{\partial H}{\partial \gamma} + \mathbf{n} \times \frac{\partial H}{\partial \mathbf{n}}\,, \\ \dot{\gamma} &= \gamma \times \frac{\partial H}{\partial \mathbf{M}}\,, \\ \dot{\mathbf{n}} &= \mathbf{n} \times \frac{\partial H}{\partial \mathbf{M}}\,. \end{aligned} \tag{7.17}$$

It is simple to check that Eqs. (7.16) have form (7.17) with the Hamiltonian

$$H(\mathbf{M}, \gamma, \mathbf{n}) = \frac{1}{2}(\mathbf{M}, I^{-1}\mathbf{M}) + \frac{a_0}{2}(\gamma, I\gamma) - \Omega(\mathbf{M}, \mathbf{n})\,. \tag{7.18}$$

Therefore Eqs. (7.17) are nonautonomous Euler equations on the Lie coalgebra L_9^*.

The Euler equations (7.16), (7.17) in view of the general theory by Arnold [2] are Hamiltonian with the Hamiltonian (7.18) on six-dimensional simplectic submanifolds $M^6 = SO(3) \times R^3$, determined in the space L_9^* by three geometric constraints:

$$(\gamma, \gamma) = 1\,, \quad (\mathbf{n}, \mathbf{n}) = 1\,, \quad (\gamma, \mathbf{n}) = 0\,. \tag{7.19}$$

Thus we obtain the following consequence from Theorem 4.

Corollary 6. *The equations of rotation of an arbitrary C^k-central configuration S^k around its mass center $\mathbf{q}(t)$ moving around the Earth are split into the direct sum of the Euler equations on N Lie coalgebras L_9^*. These Hamiltonian equations are autonomous in the case of circular orbit $\mathbf{q}(t)$ and nonautonomous for the general elliptic orbit of the mass center.*

8. Separation of Rotations of CR^n-Central Configuration of Coupled Gyrostats

I. In this section we study dynamics of more general configurations of coupled rigid bodies, including gyrostats. As known, gyrostat T consists of a rigid body carrier T_0 and N rigid rotors T_α, rotating around axes ℓ_α, fixed in the carrier T_0. It is supposed that the mass center $\mathbf{q}_\alpha(t)$ of the rotor T_α lies on the axis ℓ_α and its inertia tensor I_α is invariant under rotations around axis ℓ_α. Thus position of the mass center $\mathbf{q}(t)$ of the gyrostat T and its inertia tensor I do not change when all rotors T_α rotate and the carrier T_0 is fixed.

We call the CR^1-central configuration C^1 a system of rigid bodies obtained by spherical ball-in-socket coupling of some rigid body $\tilde{T}_{10}$, carrying n_1 rotors $\tilde{T}_{1\alpha}$ ($\alpha = 1, \ldots, n$), with s gyrostats $T_1, \ldots, T_S$ (having masses $m_1, \ldots, m_S$). It is supposed that centers of joints coincide with mass centers of gyrostats $T_1, \ldots, T_S$, having coordinates $\mathbf{q}_1(t), \ldots, \mathbf{q}_S(t)$. Let us denote $\mathbf{P}_1, \ldots, \mathbf{P}_S$ the corresponding coupling points of the rigid body $\tilde{T}_{10}$ or rotors $\tilde{T}_{1\alpha}$. Effective rigid body T_{10} and effective rotors $T_{1\alpha}$ are obtained by putting masses $m_1, \ldots, m_S$ into points $\mathbf{P}_1, \ldots, \mathbf{P}_S$. We suppose that the effective rigid body T_{10} with effective rotors $T_{1\alpha}$ is a gyrostat, so the mass center of each effective rotor $T_{1\alpha}$ lies on its axis of rotation ℓ_α and its inertia tensor $I_{1\alpha}$ is invariant under rotations around axis ℓ_α.

By induction CR^k-central configuration C^k is obtained by ideal spherical coupling of some rigid body $\tilde{T}_{k0}$ carrying n_k rotors $\tilde{T}_{k\alpha}$ ($\alpha = 1, \ldots, n$) with s_k CR^{k-1} central configurations $C_1^{k-1}, \ldots, C_{S_k}^{k-1}$ in their mass centers $\mathbf{q}_1(t), \ldots, \mathbf{q}_{S_k}(t)$. It is supposed that effective rigid body T_{k0} with effective rotors $T_{k\alpha}$ obtained by putting masses of the corresponding CR^{k-1} central configurations $m_1, \ldots, m_{S_k}$ into coupling points $\mathbf{P}_1, \ldots, \mathbf{P}_{S_k}$ is a gyrostat, as in the previous construction.

Remark 6. Dynamics of coupled gyrostats where all rotors are symmetric and all spherical joints belong to carriers, but not to rotors, was studied by Wittenburg and Lilov [27]. Equilibria of a spacecraft with rotors were studied by Krishnaprasad and Berenstein [28]. The introduced CR^n-central configurations include more complicated systems of rigid bodies, where some rotors themselves play role of carriers and are coupled through spherical joints with CR^k-central configurations, $k < n$.

II. **Theorem 5.** *Lagrangian describing dynamics of CR^n-central configuration in the Newtonian gravitational field with an arbitrary quadratic potential* (2.4) *is split into sum of noninteracting Lagrangians, describing rotations of effective gyrostats.*

Proof. We shall prove Theorem 5 by induction, analogously to the proof of Theorem 2 in Sect. 3. On each step of the induction we study the Lagrangian of an effective gyrostat T_k carrying n_k effective rotors $T_{k\alpha}$ ($\alpha = 1, \ldots, n$). The term effective means that the gyrostat has point masses, so the mass density $\rho_k(\mathbf{r}_k)$ of the carrier T_{k0} and mass densities $\rho_{k\alpha}(\mathbf{r}_{k\alpha})$ of rotors $T_{k\alpha}$ contain δ-functions and have form (2.15).

Dynamics of particles of the carrier T_{k0} in the Euler coordinates x^1, x^2, x^3 is described by the formulae

$$x^i = \sum_{j=1}^{3} Q^i_{kj}(t) r^j_k + q^i_k(t)\,, \tag{8.1}$$

where matrix $Q_k(t)$ with entries $Q^i_{kj}(t)$ is orthogonal. The mass center of the effective gyrostat T_k is in the origin (0, 0, 0) of the Lagrange coordinates r_k, its Euler coordinates are $q^i_k(t)$.

Dynamics of the rotor $T_{k\alpha}$ is determined by the expression

$$x^i = \sum_{j=1}^{n_k} (Q_k S_{k\alpha} R_{k\alpha})^i_j r^j_\alpha + q^i_\alpha(t) \,. \tag{8.2}$$

Here $R_{k\alpha}(t)$ is the matrix of rotation

$$R_{k\alpha}(t) = \begin{pmatrix} \cos\varphi_{k\alpha} & -\sin\varphi_{k\alpha} & 0 \\ \sin\varphi_{k\alpha} & \cos\varphi_{k\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.3}$$

Constant orthogonal matrix $S_{k\alpha}$ defines the orientation of the rotor $T_{k\alpha}$ in the carrier T_{k0}, so $S_{k\alpha}\mathbf{e}_3 = \ell_{k\alpha}$, where vector $\mathbf{e}_3 = (0, 0, 1)$.

The mass center of the rotor $T_{k\alpha}$ is in the origin (0, 0, 0) of the Lagrange coordinates $r^j_{k\alpha}$, its Euler coordinates are $q^i_\alpha(t)$. So the equalities hold

$$\int_{T_{k\alpha}} \rho_{k\alpha}(\mathbf{r}_{k\alpha}) r^i_{k\alpha} d^3\mathbf{r}_{k\alpha} = 0 \,, \tag{8.4}$$

where $\rho_{k\alpha}(\mathbf{r}_{k\alpha})$ is the mass density of the rotor $T_{k\alpha}$.

Lagrangian L_k describing rotation of an effective gyrostat T_k in the Newtonian gravitational field with an arbitrary quadratic potential (2.4) has the form

$$L_k = L_{k0} + \sum_{\alpha=1}^{n_k} L_{k\alpha} \,. \tag{8.5}$$

Here L_{k0} is Lagrangian of the carrier T_{k0}. Lagrangian $L_{k\alpha}$ describes dynamics of the effective rotor $T_{k\alpha}$ and has the form

$$\begin{aligned} L_{k\alpha} = {} & \frac{1}{2}\int_{T_{k\alpha}} \rho_{k\alpha}(\mathbf{r}_{k\alpha'}) \sum_{i=1}^{3} \left(\sum_{j=1}^{3} (\dot{U}_{k\alpha})^i_j r^j_{k\alpha} + \dot{q}^i_{k\alpha} \right)^2 d^3\mathbf{r}_{k\alpha} \\ & - \int_{T_{k\alpha}} \rho_{k\alpha}(\mathbf{r}_{k\alpha}) \varphi\left(\sum_{j=1}^{3} (U_{k\alpha})^1_j r^j_{k\alpha} + q^1_{k\alpha}, \sum_{j=1}^{3} (U_{k\alpha})^2_j r^j_{k\alpha} + q^2_{k\alpha}, \right. \\ & \left. \sum_{j=1}^{3} (U_{k\alpha})^3_j r^j_{k\alpha} + q^3_{k\alpha} \right) d^3\mathbf{r}_{k\alpha} \,, \end{aligned} \tag{8.6}$$

where $U_{k\alpha} = Q_k S_{k\alpha} R_{k\alpha}$. Expression (8.6) in view of the equalities (8.4) obtains the form

$$L_{k\alpha} = L_{k1\alpha} + L_{k2\alpha} \,, \tag{8.7}$$

$$L_{k1\alpha} = \frac{1}{2} m_{k\alpha}(\dot{\mathbf{q}}_{k\alpha}, \dot{\mathbf{q}}_{k\alpha}) - m_{k\alpha}\varphi(\mathbf{q}_{k\alpha}) \,, \tag{8.8}$$

$$L_{k2\alpha} = \frac{1}{2}\mathrm{Tr}((Q_k S_{k\alpha} R_{k\alpha})^{\boldsymbol{\cdot}} \tilde{J}_{k\alpha} (Q_k S_{k\alpha} R_{k\alpha})^{t\boldsymbol{\cdot}}) - \frac{1}{2}\mathrm{Tr}(Q_k S_{k\alpha} R_{k\alpha} \tilde{J}_{k\alpha} (Q_k S_{k\alpha} R_{k\alpha})^t a) \,, \tag{8.9}$$

where a is symmetric matrix with entries a_{ij} (2.4), $m_{k\alpha}$ is the mass of the effective rotor $T_{k\alpha}$ and $\tilde{J}_{k\alpha}$ is the symmetric matrix with entries

$$(\tilde{J}_{k\alpha})_{ij} = \int_{T_{k\alpha}} \rho_{k\alpha}(\mathbf{r}_{k\alpha}) r^i_{k\alpha} r^j_{k\alpha} d^3\mathbf{r}_{k\alpha} \,. \tag{8.10}$$

Inertia tensor $I_{k\alpha}$ is connected with $\tilde{J}_{k\alpha}$ (8.10) by the relation

$$(I_{k\alpha})_{ij} = \mathrm{Tr}(\tilde{J}_{k\alpha})\delta_{ij} - (\tilde{J}_{k\alpha})_{ij} \,. \tag{8.11}$$

Both tensors $I_{k\alpha}$, $\tilde{J}_{k\alpha}$ are supposed to be invariant under rotations $R_{k\alpha}(t)$ of rotor $T_{k\alpha}$. Hence we have

$$R_{k\alpha}\tilde{J}_{k\alpha}R^t_{k\alpha} = \tilde{J}_{k\alpha} \,, \quad R_{k\alpha}\tilde{J}_{k\alpha} = \tilde{J}_{k\alpha}R_{k\alpha} \,. \tag{8.12}$$

This relation obviously implies that matrix $\tilde{J}_{k\alpha}$ is diagonal with entries $(\tilde{J}_{k\alpha})_{ij} = \mathrm{diag}(B_{k\alpha'} B_{k\alpha'} C_{k\alpha})$. In view of relations (8.12) Lagrangian $L_{k2\alpha}$ (8.9) takes the form

$$L_{k2\alpha} = -\frac{1}{2}\mathrm{Tr}(J_{k\alpha}Q_k^{-1}\dot{Q}_k Q_k^{-1}\dot{Q}_k) - \frac{1}{2}\mathrm{Tr}(J_{k\alpha}R_{k\alpha}^{-1}\dot{R}_{k\alpha}R_{k\alpha}^{-1}\dot{R}_{k\alpha})$$
$$- \mathrm{Tr}(S_{k\alpha}\tilde{J}_{k\alpha}R_{k\alpha}^{-1}\dot{R}_{k\alpha}S^t_{k\alpha}Q_k^{-1}\dot{Q}_k) - \frac{1}{2}\mathrm{Tr}(Q_k J_{k\alpha} Q_k^t a) \,. \tag{8.13}$$

Here we denote $J_{k\alpha} = S_{k\alpha}\tilde{J}_{k\alpha}S^t_{k\alpha}$.

For the rotation matrix $R_{k\alpha}(t)$ (8.3) the equality holds

$$R_{k\alpha}^{-1}\dot{R}_{k\alpha} = \dot{\varphi}_{k\alpha}\sigma \,, \quad \sigma = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.14}$$

Therefore Lagrangian (8.13) is transformed into Lagrangian

$$L_{k2\alpha} = -\frac{1}{2}\mathrm{Tr}(J_{k\alpha}Q_k^{-1}\dot{Q}_k Q_k^{-1}\dot{Q}_k) - \frac{1}{2}\mathrm{Tr}(Q_k J_{k\alpha} Q_k^t a)$$
$$+ \frac{1}{2}A_{k\alpha}\dot{\varphi}^2_{k\alpha} + B_{k\alpha}\dot{\varphi}_{k\alpha}\mathrm{Tr}(z_{k\alpha}Q_k^{-1}\dot{Q}_k) \,, \tag{8.15}$$

where $A_{k\alpha} = -\mathrm{Tr}(S_{k\alpha}\tilde{J}_{k\alpha}S^t_{k\alpha}\sigma^2)$ is constant and $z_{k\alpha}$ is constant skew-symmetric matrix

$$z_{k\alpha} = -S_{k\alpha}\sigma S^t_{k\alpha} \,. \tag{8.16}$$

By definition the mass centers $\mathbf{q}_{k\alpha}(t)$ of the effective rotors $T_{k\alpha}$ are immovable relative to the carrier T_{k0}. Effective rigid body T_{k0} is obtained from $\tilde{T}_{k0}$ by putting masses $m_{k\alpha}$ of rotors $T_{k\alpha}$ into the points $\mathbf{q}_{k\alpha}(t)$. So the mass density of the effective rigid body T_{k0} is determined by the formula

$$\rho_k(\mathbf{r}_k) = \rho_{k0}(\mathbf{r}_k) + \sum_{\alpha=0}^{n_k} m_{k\alpha}\delta(\mathbf{r}_k - \mathbf{P}_{k\alpha}) \,, \tag{8.17}$$

where vectors $\mathbf{P}_{k\alpha}$ define positions of mass centers $\mathbf{q}_{k\alpha}$ in the Lagrange coordinates r^1_k, r^2_k, r^3_k.

In view of the formula (8.5)–(8.8) Lagrangian $\tilde{L}_{k0}$ of the effective rigid body T_{k0} has the form

$$\tilde{L}_{k0} = L_{k0} + \sum_{\alpha=1}^{n_k} L_{k1\alpha} = L_{k1} + \tilde{L}_{k2} , \tag{8.18}$$

$$L_{k1} = \frac{1}{2} m_k(\dot{\mathbf{q}}_k, \dot{\mathbf{q}}_k) - m_k \varphi(\mathbf{q}_k) , \tag{8.19}$$

$$\tilde{L}_{k2} = -\frac{1}{2}\mathrm{Tr}(\tilde{J}_k Q_k^{-1} \dot{Q}_k Q_k^{-1} \dot{Q}_k) - \frac{1}{2}\mathrm{Tr}(Q_k \tilde{J}_k Q_k^t a) . \tag{8.20}$$

Here m_k is the mass of the whole gyrostat T_k, vector $\mathbf{q}_k(t)$ determines position of its mass center, tensor $\tilde{J}_k$ is defined by the formula

$$(\tilde{J}_k)_{ij} = \int_{\tilde{T}_{k0}} \rho_k(r_k) r_k^i r_k^j d^3 r_k .$$

Substituting formulae (8.18)–(8.20), (8.15) and (8.7) into the initial sum (8.5) we obtain the following expression for the Lagrangian L_k of the gyrostat T_k:

$$L_k = L_{k1} + L_{k2} , \tag{8.21}$$

$$L_{k2} = -\frac{1}{2}\mathrm{Tr}(J_k Q_k^{-1} \dot{Q}_k Q_k^{-1} \dot{Q}_k) - \frac{1}{2}\mathrm{Tr}(Q_k J_k Q_k^t a) + \sum_{\alpha=1}^{n_k} \left(\frac{1}{2} A_{k\alpha} \dot{\varphi}_{k\alpha}^2 + B_{k\alpha} \dot{\varphi}_{k\alpha} \mathrm{Tr}(z_{k\alpha} Q_k^{-1} \dot{Q}_k) \right) , \tag{8.22}$$

where tensor J_k has the form

$$J_k = \tilde{J}_k + \sum_{\alpha=1}^{n_k} J_{k\alpha} . \tag{8.23}$$

Lagrangian L_{k2} (8.22) has n_k cyclic coordinates $\varphi_{k\alpha}$. Therefore applying the Routh transformation

$$L_{k3} = L_{k2} - \sum_{\alpha=1}^{n_k} p_{k\alpha} \dot{\varphi}_{k\alpha} , \tag{8.24}$$

$$p_{k\alpha} = \frac{\partial L_{k2}}{\partial \dot{\varphi}_{k\alpha}} = A_{k\alpha} \dot{\varphi}_{k\alpha} + B_{k\alpha} \mathrm{Tr}(z_{k\alpha} Q_{k\alpha}^{-1} \dot{Q}_{k\alpha}) , \tag{8.25}$$

we obtain that Lagrange equations with Lagrangian L_{k2} (8.22) are equivalent to the Routh equations

$$\frac{\partial L_{k3}}{\partial Q_{ki}^j} = \frac{d}{dt} \frac{\partial L_{k3}}{\partial \dot{Q}_{ki}^j} , \tag{8.26}$$

$$\dot{p}_{k\alpha} = \frac{\partial L_{k3}}{\partial \varphi_{k\alpha}} = 0 , \quad \dot{\varphi}_{k\alpha} = -\frac{\partial L_{k3}}{\partial p_{k\alpha}} . \tag{8.27}$$

The last equation (8.27) has the form

$$\dot{\varphi}_{k\alpha} = \frac{1}{A_{k\alpha}} p_{k\alpha} - \frac{B_{k\alpha}}{A_{k\alpha}} \mathrm{Tr}(z_{k\alpha} Q_k^{-1} \dot{Q}_k) . \tag{8.28}$$

Substituting these expressions into (8.24) we obtain the formula

$$L_{k3} = -\frac{1}{2}\mathrm{Tr}(J_k Q_k^{-1}\dot{Q}_k Q_k^{-1}\dot{Q}_k) - \frac{1}{2}\mathrm{Tr}(Q_k J_k Q_k^t a)$$

$$- \sum_{\alpha=1}^{n_k}\left(\frac{B_{k\alpha}^2}{2A_{k\alpha}}(\mathrm{Tr}(z_{k\alpha}Q_k^{-1}\dot{Q}_k))^2 + p_{k\alpha}\frac{B_{k\alpha}}{A_{k\alpha}}\mathrm{Tr}(z_{k\alpha}Q_{k\alpha}^{-1}\dot{Q}_{k\alpha}) - \frac{p_{k\alpha}^2}{2A_{k\alpha}}\right). \quad (8.29)$$

Here $A_{k\alpha}$, $B_{k\alpha}$, $p_{k\alpha}$ are constants and $z_{k\alpha}$ is the constant skew-symmetric matrix (8.16). Therefore we get that dynamics of the effective gyrostat T_k is described by Lagrange equations with Lagrangian

$$\tilde{L}_k = L_{k1} + L_{k3} \quad (8.30)$$

and Eqs. (8.28), determining the relative rotations of the effective rotors $T_{k\alpha}$.

Formula (8.30) with Lagrangians L_{k1} and L_{k3} defined by (8.19) and (8.29) shows that Lagrangian $\tilde{L}_k$ is split into two noninteracting parts, describing dynamics of mass center of the effective gyrostat T_k and its rotation around mass center. Therefore the first and consequent statements of the induction are proven. Hence Lagrangian of the whole CR^n-central configuration has the form

$$L = L_1 + \sum_{k=1}^{N} L_{k3} . \quad (8.31)$$

Here Lagrangian $L_1 = m(\dot{\mathbf{q}}, \dot{\mathbf{q}}) - \varphi(\mathbf{q})$ describes dynamics of the mass center $\mathbf{q}(t)$ of the whole configuration. Lagrangians L_{k3} describe rotations of effective gyrostats and have the form (8.29). Theorem 5 is proven.

III. Lagrange equations with Lagrangian (8.29) are equivalent to the following matrix equations:

$$\begin{aligned} \dot{M}_k &= [M_k + N_k, \omega_k] + [u_k, J_k] , \\ \dot{u}_k &= [u_k, \omega_k] , \end{aligned} \quad (8.32)$$

where the notations are used

$$M_k = J_k\omega_k + \omega_k J_k + 2\sum_{\alpha=1}^{n_k}\frac{B_{k\alpha}^2}{A_{k\alpha}}\mathrm{Tr}(z_{k\alpha}\omega_k)z_{k\alpha} ,$$

$$N_k = -2\sum_{\alpha=1}^{n_k}\frac{B_{k\alpha}}{A_{k\alpha}}p_{k\alpha}z_{k\alpha} , \quad \omega_k = Q_k^{-1}\dot{Q}_k , \quad u_k = Q_k^t a Q_k . \quad (8.33)$$

Here only skew-symmetric matrix N_k depends on conserved momenta $p_{k\alpha}$ and is constant.

Equations (8.32) have first integral

$$H_k = \frac{1}{2}\mathrm{Tr}(M_k\omega_k) + \mathrm{Tr}(u_k J_k) , \quad (8.34)$$

which does not depend on the constant matrix N_k and momenta $p_{k\alpha}$.

IV. The proof of Theorem 5 is valid as well in the case of time-dependent gravitational potential (2.4) which has the form (7.8) for example. Therefore applying Theorem 5 to the problem of rotation of space station type orbiting multibody configuration we obtain the following result.

Corollary 7. *Equations of rotation of an arbitrary CR^n-central configuration around its mass center $\mathbf{q}(t)$ (moving around the Earth in an elliptic orbit) are split into system of noninteracting equations describing rotation of the effective gyrostats T_k around their mass centers.*

Equations of rotation of the effective gyrostats T_k have the form, generalizing (8.32) and (7.15),

$$\dot{M}_k = [M_k + N_k, \omega_k] - a_0[u_k, I_k] ,$$
$$\dot{u}_k = [u_k, \omega_k - \Omega n_k] , \quad \dot{n}_k = [n_k, \omega_k] , \tag{8.35}$$
$$a_0 = \frac{3Gm_0}{|\mathbf{q}(t)|^3} , \quad \Omega = \frac{\mu}{|\mathbf{q}(t)|^2} ,$$

where skew-symmetric matrices M_k, ω_k, N_k are determined by expressions (8.33).

For a circular orbit we have $a_0 = 3\Omega^2 = \text{const}$. Equations (8.35) in this case have first integral

$$H_k = \frac{1}{2}\mathrm{Tr}(M_k\omega_k) - a_0\,\mathrm{Tr}(u_k I_k) - \Omega\,\mathrm{Tr}((M_k + N_k)n_k) . \tag{8.36}$$

Equations of rotation of the effective gyrostat T_k have the following vector form

$$\dot{\mathbf{M}}_k = (\mathbf{M}_k + \mathbf{N}_k)\times\boldsymbol{\omega}_k + a_0\,\gamma_k\times I_k\gamma_k ,$$
$$\dot{\gamma}_k = \gamma_k\times(\boldsymbol{\omega}_k - \Omega\,\mathbf{n}_k) , \quad \dot{\mathbf{n}}_k = \mathbf{n}_k\times\boldsymbol{\omega}_k . \tag{8.37}$$

These equations in case of circular orbit ($a_0 = 3\Omega^2 = \text{const}$) have first integral

$$H_k = \frac{1}{2}(\mathbf{M}_k, \boldsymbol{\omega}_k) + \frac{a_0}{2}(\gamma_k, I_k\gamma_k) - \Omega(\mathbf{M}_k + \mathbf{N}_k, \mathbf{n}_k) . \tag{8.38}$$

V. Applying Theorem 5 to the problem of the inertial dynamics of a multibody configuration we obtain the consequence.

Corollary 8. *The inertial dynamics of an arbitrary CR^n-central configuration is integrable.*

Indeed, system (8.32) in the case of inertial dynamics ($u_k = 0$) is reduced to a one matrix equation

$$\dot{M}_k = [M_k + N_k, \omega_k] . \tag{8.39}$$

This equation has two first integrals

$$H_k = \frac{1}{2}\mathrm{Tr}(M_k\omega_k) , \quad H_{k2} = \mathrm{Tr}(M_k + N_k)^2 . \tag{8.40}$$

Therefore Eq. (8.39) in the space of 3×3 skew-symmetric matrices is integrable in elliptic functions. Rotation of the effective gyrostat T_k is determined from the linear equation $\dot{Q}_k = Q_k\omega_k$.

Remark 7. Lagrangians (8.22), (8.29) in the case of absence of gravitational field are left-invariant on the Lie group $SO(3)\times S^1\times\cdots\times S^1$. So inertial dynamics of an arbitrary CR^n-central configuration possesses large group G of hidden symmetries

$$G=\prod_{k=1}^{N} SO(3)_k\times(S^1)^M .$$

Here commutative subgroup $(S^1)^M$ acts as independent rotations of all effective rotors.

VI. By definition two rigid bodies are coupled by a universal joint if they are free to rotate around two intersecting axes ℓ_1,ℓ_2 which are fixed with respect to each other, see Wittenburg [26]. We consider system U_k obtained by a universal joint of arbitrary many rigid bodies $T_{k\alpha}$ ($\alpha=1,\ldots,n$) having inertia tensors $I_{k\alpha}$ symmetric under rotations around axes $\ell_{k\alpha}$ and with mass centers lying on the axes $\ell_{k\alpha}$. Such systems U_k of universally joint rigid bodies $T_{k\alpha}$ is obviously a particular case of a gyrostat T_k, for which carrier $\tilde{T}_{k0}$ consists of all connected axes $\ell_{k\alpha}$ (not intersecting in general) and is massless.

Therefore dynamics of system U_k in the Newtonian gravitational field with an arbitrary quadratic potential (2.4) is described by Lagrange system with the Lagrangian L_k (8.21), (8.22). Here the mass density of the effective rigid body T_{k0} is determined by the formula (8.17) with $\rho_{k0}(r_k)=0$.

Lagrange equations for inertial dynamics of system U_k are reduced to one matrix equation (8.39), which possesses two first integrals (8.40) and hence is integrable in elliptic functions.

We call the CU^n-central configuration of a particular case of CR^n-central configuration, where some of carriers $\tilde{T}_{k0}$ are massless, and so corresponding rotors $T_{k\alpha}$ are coupled by universal joints. Applying Theorem 5 we get the consequence.

Corollary 9. *Equations of inertial dynamics of an arbitrary CU^n-central configuration are integrable.*

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