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A Relation Between a.c. Spectrum of Ergodic Jacobi Matrices and the Spectra of Periodic Approximants*

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Abstract. We study ergodic Jacobi matrices on $l^2(Z)$, and prove a general theorem relating their a.c. spectrum to the spectra of periodic Jacobi matrices, that are obtained by cutting finite pieces from the ergodic potential and then repeating them. We apply this theorem to the almost Mathieu operator: $(H_{\alpha,\lambda,\theta}u)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha n + \theta)u(n)$, and prove the existence of a.c. spectrum for sufficiently small λ, all irrational α's, and a.e. θ. Moreover, for $0 \le \lambda < 2$ and (Lebesgue) a.e. pair α, θ, we prove the explicit equality of measures: $|\sigma_{\rm ac}| = |\sigma| = 4 - 2\lambda$.

1. Introduction

In this paper, we study one dimensional ergodic Jacobi matrices. These are families of (bounded, self adjoint) operators H_ω on $l^2(Z)$, defined by:

$$H_\omega = H_0 + V_\omega\,, \qquad (H_0 u)(n) = u(n+1) + u(n-1)\,, \qquad (V_\omega u)(n) = V_\omega(n)u(n)\,, \tag{1.1}$$

where V_ω is a (real) stationary bounded ergodic potential, that is: we consider a probability measure space (Ω, dp), a measure preserving invertible ergodic transformation T, and a bounded measurable real-valued function f, and define: $V_\omega(n) = f(T^n\omega)$.

For such a family $\{H_\omega\}_{\omega\in\Omega}$, it is known [9] that the spectrum of H_ω, and its decomposition into a.c., s.c., and p.p. parts are a.e. constant with respect to ω. Namely, there are subsets: $\sigma, \sigma_{\rm ac}, \sigma_{\rm sc}, \sigma_{\rm pp}$ of R, such that for a.e. $\omega: \sigma_\omega \equiv \mathrm{Spec}(H_\omega) = \sigma$ and $\sigma_{\rm ac}, \sigma_{\rm sc}, \sigma_{\rm pp}$ are (respectively) the absolutely continuous, singular continuous and pure point spectra of H_ω ($\sigma_{\rm pp}$ being the closure of the set of eigenvalues).

The Lyapunov exponent $\gamma(E)$, characterizing solutions of the equation:

$$u(n+1) + u(n-1) + V_\omega(n)u(n) = Eu(n) \tag{1.2}$$

is defined for the family $\{H_\omega\}_{\omega\in\Omega}$, as follows:

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For each $n > 0$ define:

$$T_n^\omega(E) = \begin{pmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Phi_n^\omega(E) = T_n^\omega(E) T_{n-1}^\omega(E) \dots T_1^\omega(E), \tag{1.3}$$

such that: $\mathbf{u}(n) = \Phi_n^\omega(E)\mathbf{u}(0)$, where:

$$\mathbf{u}(n) \equiv \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}, \tag{1.4}$$

and denote:

$$\gamma_\omega(E) \equiv \lim_{n\to\infty} \frac{1}{n} \ln \|\Phi_n^\omega(E)\| \tag{1.5}$$

($\|\cdot\|$ being the operator norm of the 2×2 matrices). It is a result of Furstenberg and Kesten [9, 13], that for every E, for a.e. ω: the limit $\gamma_\omega(E)$ exists, and is independent of ω. This (a.e. ω) common limit is called the Lyapunov exponent and denoted by $\gamma(E)$.

For the family $\{H_\omega\}_{\omega\in\Omega}$, we define a subset A of R by:

$$A \equiv \{E \mid \gamma(E) = 0\}. \tag{1.6}$$

A is related to σ_{ac} by the Ishii-Pastur-Kotani Theorem [9], which states: $\sigma_{\mathrm{ac}} = \bar{A}^{\mathrm{ess}}$, where: $\bar{A}^{\mathrm{ess}} \equiv \{E \in R \mid |A \cap (E-\varepsilon, E+\varepsilon)| > 0 \,\forall \varepsilon > 0\}$, and $|\cdot|$ denotes Lebesgue measure. In particular this theorem implies that A is contained in σ_{ac} up to a set of zero measure, and thus: $|\sigma_{\mathrm{ac}}| \geq |A|$.

For each ω, we define a sequence $\{V_m^\omega\}_{m=1}^\infty$ of periodic potentials, by:

$$V_m^\omega(n) = V_\omega(n), \qquad n = 1, 2, \dots, m, \qquad V_m^\omega(n+m) = V_m^\omega(n), \tag{1.7}$$

such that V_m^ω is obtained from V_ω by "cutting" a finite piece of length m, and then repeating it. For each pair ω, m we denote by σ_m^ω the spectrum of the periodic Jacobi matrix: $H_m^\omega \equiv H_0 + V_m^\omega$. As is known for periodic Jacobi matrices, this spectrum consists of m bands (closed intervals). We call $\{V_m^\omega\}$ (respectively $\{H_m^\omega\}$) canonical periodic approximants of V_ω (respectively H_ω).

The main result of this paper, is the following theorem, which relates the set A (and thus σ_{ac}) to the spectra of the periodic approximants:

Theorem 1. *For* a.e. ω*:*

$$\left| \limsup_{m\to\infty} \sigma_m^\omega \backslash A \right| = 0,$$

where:

$$\limsup_{m\to\infty} \sigma_m^\omega \equiv \bigcap_{l=1}^{\infty} \bigcup_{m=l}^{\infty} \sigma_m^\omega .$$

Since $\left| \bigcup_{m=1}^{\infty} \sigma_m^\omega \right| < \infty$, we have (see e.g. [21]): $|\limsup \sigma_m^\omega| \geq \limsup |\sigma_m^\omega|$, and thus Theorem 1 implies:

Corollary 1.1. *For* a.e. ω*:*

$$\limsup_{m\to\infty} |\sigma_m^\omega| \leq |A| \leq |\sigma_{\mathrm{ac}}| .$$

If the potential is "highly random," it is known [9], that there is no absolutely continuous spectrum, and thus Corollary (1.1) simply implies that for such potentials: $\limsup |\sigma_m^\omega| = 0$ for a.e. ω. For certain almost periodic potentials, however, a.c. spectrum does occur, and Theorem 1 (or Corollary 1.1) can serve as a tool for establishing its existence and estimating its Lebesgue measure.

In this paper we will apply Theorem 1 to the study of the almost Mathieu operator $H_{\alpha,\lambda,\theta}$, defined by:

$$H_{\alpha,\lambda,\theta} = H_0 + V_{\alpha,\lambda,\theta}, \qquad V_{\alpha,\lambda,\theta}(n) = \lambda\cos(2\pi\alpha n + \theta). \tag{1.8}$$

It is known [9] that if α is irrational, then the family: $\{H_{\alpha,\lambda,\theta}\}_{\theta\in[0,2\pi]}$ is ergodic in the sense discussed above, if we take Lebesgue measure for the measure dp on $\Omega = [0, 2\pi]$ (i.e. $dp(\theta) = d\theta$). The spectrum of $H_{\alpha,\lambda,\theta}$, denoted by $\sigma(\alpha, \lambda)$, is known in this case (α irrational) to be completely independent of θ, and the sets $\sigma_{ac}(\alpha, \lambda)$, $\sigma_{sc}(\alpha, \lambda)$, and $\sigma_{pp}(\alpha, \lambda)$ are defined for the family $\{H_{\alpha,\lambda,\theta}\}_{\theta\in[0,2\pi]}$, being the (Lebesgue) a.e. θ independent a.c., s.c., and p.p. spectra of $H_{\alpha,\lambda,\theta}$. The set $A(\alpha, \lambda)$ is also defined for the family, as the set where the Lyapunov exponent vanishes.

The almost Mathieu operator was extensively studied by many authors [1–5, 7, 8, 11, 12, 18, 19], and many of its spectral characteristics are known. Bellissard, Lima, and Testard [4] have shown that for (Lebesgue) a.e. α, all θ and sufficiently small coupling λ, $H_{\alpha,\lambda,\theta}$ has some a.c. spectrum, and moreover for a.e. irrational $\alpha: |\sigma_{ac}(\alpha, \lambda)| \to 4$ as $\lambda \to 0$. Chulaevsky and Delyon [8] have recently shown that for a.e. pair α, θ and sufficiently small λ, $H_{\alpha,\lambda,\theta}$ has purely a.c. spectrum. In this paper we combine Theorem 1 with some results of Avron, van Mouche, and Simon [3] to show:

Theorem 2. (i) *For all irrational α's:*

$$|\sigma_{ac}(\alpha, \lambda)| \ge |A(\alpha, \lambda)| \ge 4 - \left(2 + \frac{\pi}{\sqrt{5}}\right)\lambda.$$

(ii) *If $0 \le \lambda < 2$ and α is an irrational obeying.*

$$\lim_{n\to\infty} q_n^2 \left|\alpha - \frac{p_n}{q_n}\right| = 0$$

for some sequence $\{p_n/q_n\}_{n=1}^{\infty}$ of rationals, then:

$$|A(\alpha, \lambda)| = |\sigma_{ac}(\alpha, \lambda)| = |\sigma(\alpha, \lambda)| = 4 - 2\lambda.$$

Remarks. 1) The set of irrationals characterized in (ii), has full Lebesgue measure [14].
2) The equality $|\sigma(\alpha, \lambda)| = 4 - 2\lambda$ was conjectured by Aubry and Andre [1] to hold for every irrational α and $0 \le \lambda \le 2$, and was studied by Thouless [19] and by Avron, van Mouche, and Simon [3]. While they did not succeed to prove it for the general case, and only obtained the lower bound: $|\sigma(\alpha, \lambda)| \ge 4 - 2\lambda$, we will show that for $\lambda \ne 2$, and a.e. α (i.e. α as in (ii)), it does follow immediately from the results of Avron et al.

Theorem 2 extends the previously known results, giving explicit measures for "most" α's, and establishing the existence of a.c. spectrum (at sufficiently small coupling) for all α's. The somewhat striking part of this extension, is the existence of a.c. spectrum for the case where α is a Liouville number. In this case the spectrum was previously conjectured to be purely singular continuous [10, 17], and the existence of the a.c. spectrum disproves a general conjecture of Deift and Simon about the existence of certain types of eigenfunctions associated to a.c. spectra of ergodic Jacobi matrices (see Sect. 7 of [10] for details).

In Sect. 2 we describe some preliminaries from the theory of periodic Jacobi matrices. In Sect. 3 we prove Theorem 1, and in Sect. 4 we prove Theorem 2.

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2. Preliminaries

In this section, we describe some basic results from the theory of periodic Jacobi matrices. The description is brief, and mostly without proofs. For further details the reader is referred to [20] and [16].

Let $V = \{V(n)\}$ be a periodic potential with period p (i.e. $V(n+p) = V(n)$ for all n). We denote: $H \equiv H_0 + V$, $\sigma \equiv \mathrm{Spec}(H)$, and define the 2×2 transfer matrices $\Phi_n(E)$, by (1.3), with $V_\omega = V$. From the periodicity of V, we have:

$$\lim_{n\to\infty} \|\Phi_n(E)\|^{1/n} = [\mathrm{Spr}(\Phi_p(E))]^{1/p}, \tag{2.1}$$

where $\mathrm{Spr}(\cdot)$ is the spectral radius. Thus, the Lyapunov exponent $\gamma(E)$ exists (for the individual operator H) for every $E \in R$, and is determined by the one period transfer matrix $\Phi_p(E)$ as:

$$\gamma(E) \equiv \lim_{n\to\infty} \frac{1}{n} \ln\|\Phi_n(E)\| = \frac{1}{p} \ln[\mathrm{Spr}(\Phi_p(E))]. \tag{2.2}$$

Since $\det \Phi_n(E) = 1$ (for all n), we see that for $E \in A \equiv \{E \mid \gamma(E) = 0\}$ $\Phi_p(E)$ has eigenvalues presentable as: $e^{\pm ikp}$ (with $k \in R$), and for $E \notin A$ it has eigenvalues presentable as: $e^{\pm\gamma p}$. The corresponding eigenvectors give rise (as initial conditions) to solutions $\{u^\pm(n)\}_{n=-\infty}^{\infty}$ of Eq. (1.2) (with $V_\omega = V$), obeying: $u^\pm(n+p) = e^{\pm ikp}u^\pm(n)$ for $E \in A$, and $u^\pm(n+p) = e^{\pm\gamma p}u^\pm(n)$ for $E \notin A$. From this, it is not hard to verify the existence of $(E-H)^{-1}$ (as a bounded operator) for $E \notin A$, and its inexistence for $E \in A$. Thus, we have: $\sigma = A$.

The solutions obeying: $u^\pm(n+p) = e^{\pm ikp}u^\pm(n)$, that characterize σ, are called Bloch wave solutions, and the corresponding (real) k's are called Bloch wave numbers. The existence of these solutions for every $E \in \sigma$, allows to transform the spectral problem for H into a finite matrix eigenvalue problem, by defining:

$$\mathscr{A}(k) \equiv \begin{pmatrix} V(1) & 1 & & & e^{-ikp} \\ 1 & V(2) & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{ikp} & & & 1 & V(p) \end{pmatrix} \tag{2.3}$$

and noting that:

$$\sigma = \bigcup_k \mathrm{Spec}(\mathscr{A}(k)). \tag{2.4}$$

By direct expansion in minors, it can be shown that:

$$\det(E - \mathscr{A}(k)) = \Delta(E) - 2\cos kp, \tag{2.5}$$

where $\Delta(E)$ (called the discriminant, and easily shown to coincide with the trace of $\Phi_p(E)$) is a polynomial of degree p, independent of k, with real coefficients and p distinct real zeroes. Moreover, (for $E \in R$) we have: $|\Delta(E)| \geq 2$ in all the extremum points of $\Delta(E)$. This shows that σ consists of p bands (closed intervals), such that $\Delta(E)$ is strongly monotone on each band. By considering only $k \in [0, \pi/p]$ (which is always assumed throughout the rest of this paper), (2.5) defines p dispersion functions: $E_l(k)$, $l = 1, 2, \ldots, p$, such that each $E_l(k)$ is a strongly monotone C^∞ function of k, from the interval $[0, \pi/p]$ onto the l^{th} band of σ. In particular the width of the l^{th} band is given by: $\int_0^{\pi/p} \left| \frac{dE_l(k)}{dk} \right| dk$.

We denote by $\text{Intb}(\sigma)$ the union of the interiors of the bands of σ. For each $E \in \text{Intb}(\sigma)$ there is a unique pair l, k, such that $E = E_l(k)$, and moreover there is a unique corresponding derivative: $\frac{dE_l(k)}{dk}$, which we denote simply by: $\frac{dE}{dk}$. We have (Lemma 4.1 of [16]):

Proposition 2.1. *Let $E \in \text{Intb}(\sigma)$, and let $\{u(n)\}$ be a corresponding Bloch wave solution of Eq. (1.1). normalized such that:*

$$\sum_{n=1}^{p} |u(n)|^2 = 1 ,$$

then for each n:

$$\left| \frac{dE}{dk} \right| = 2p \left| \text{Im}[u(n)\overline{u(n+1)}] \right| ,$$

where $\bar{\cdot}$ denotes complex conjugation.

3. Proof of Theorem 1

Lemma 3.1. *For every ω and $m \geq 2$, let $E \in \text{Intb}(\sigma_m^\omega)$, and let $\Phi_m^\omega(E)$ be the 2×2 transfer matrix defined by (1.3), then:*

$$\|\Phi_m^\omega(E)\| \leq 2m \left| \frac{dE}{dk_m^\omega} \right|^{-1} ,$$

where k_m^ω is the appropriate Bloch wave number.

Proof. Since $E \in \text{Intb}(\sigma_m^\omega)$, and since $\Phi_m^\omega(E)$ is a one period transfer matrix for H_m^ω, it has two eigenvalues: $e^{\pm i k_m^\omega m}$, and corresponding normalized eigenvectors: $\mathbf{x}^\pm = (x_1^\pm, x_2^\pm)^T$, which are uniquely defined up to a phase. The eigenvectors $\mathbf{x}^\pm$ can be chosen to be complex conjugates of each other, and thus we can assume: $x_1^+ = x_1$, $x_2^+ = x_2$, $x_1^- = \overline{x_1}$, $x_2^- = \overline{x_2}$, where $|x_1|^2 + |x_2|^2 = 1$. Let $a, b \in C$ obey: $|a|^2 + |b|^2 = 1$, and denote: $\mathbf{y} = a\mathbf{x}^+ + b\mathbf{x}^-$, then we have:

$$\begin{aligned}\frac{\|\Phi_m^\omega(E)\mathbf{y}\|^2}{\|\mathbf{y}\|^2} &= \frac{\|ae^{ik_m^\omega m}\mathbf{x}^+ + be^{-ik_m^\omega m}\mathbf{x}^-\|^2}{\|a\mathbf{x}^+ + b\mathbf{x}^-\|^2}\\ &\le \frac{(|a|+|b|)^2}{|a|^2+|b|^2-2|a|\,|b|\,|\langle\mathbf{x}^+,\mathbf{x}^-\rangle|}\\ &\le \frac{2}{1-|\langle\mathbf{x}^+,\mathbf{x}^-\rangle|}\\ &= \frac{2(1+|\langle\mathbf{x}^+,\mathbf{x}^-\rangle|)}{1-|\langle\mathbf{x}^+,\mathbf{x}^-\rangle|^2}\\ &\le \frac{4}{1-|\langle\mathbf{x}^+,\mathbf{x}^-\rangle|^2}\,. \end{aligned} \tag{3.1}$$

Denote by φ the phase difference between x_1 and x_2, such that: $x_1\overline{x_2} = |x_1|\,|x_2|e^{i\varphi}$, then:

$$\begin{aligned}1-|\langle\mathbf{x}^+,\mathbf{x}^-\rangle|^2 &= (|x_1|^2+|x_2|^2)^2 - |x_1^2+x_2^2|^2\\ &= 2|x_1|^2|x_2|^2 - 2\,\mathrm{Re}(x_1^2\overline{x_2^2})\\ &= 2|x_1|^2|x_2|^2(1-\cos 2\varphi)\\ &= 4|x_1|^2|x_2|^2\sin^2\varphi\\ &= 4(\mathrm{Im}(x_1\overline{x_2}))^2\,, \end{aligned} \tag{3.2}$$

and thus (3.1) implies:

$$\|\Phi_m^\omega(E)\| \le \frac{1}{|\mathrm{Im}(x_1\overline{x_2})|}\,. \tag{3.3}$$

As described in Sect. 2, we have: $x_1 = u(1)$, $x_2 = u(0)$, for some Bloch wave solution $\{u(n)\}$ of Eq. (1.1) (with $V_\omega = V_m^\omega$). Since $|x_1|^2+|x_2|^2 = 1$, we have:

$$N^2 \equiv \sum_{n=1}^{m} |u(n)|^2 \ge 1\,, \tag{3.4}$$

and $\{u(n)/N\}$ is a normalized (over one period) Bloch wave solution. Thus, Proposition 2.1 implies:

$$\left|\frac{dE}{dk_m^\omega}\right| = \frac{2m}{N^2}\,|\mathrm{Im}(x_1\overline{x_2})|\,, \tag{3.5}$$

and from (3.3), we obtain:

$$\|\Phi_m^\omega(E)\| \le \frac{2m}{N^2}\left|\frac{dE}{dk_m^\omega}\right|^{-1} \le 2m\left|\frac{dE}{dk_m^\omega}\right|^{-1}.\quad\square \tag{3.6}$$

Proof of Theorem 1. For every ω, $m \ge 2$ and $\varepsilon > 0$, we define:

$$S_m^\omega(\varepsilon) \equiv \left\{E \mid E \in \mathrm{Intb}(\sigma_m^\omega),\ \left|\frac{dE}{dk_m^\omega}\right| \ge \varepsilon\right\}, \tag{3.7}$$

and for each ω, we define:

$$S_\omega \equiv \limsup_{m\to\infty} S_m^\omega(m^{-2})\,. \tag{3.8}$$

Since there are m bands in σ_m^ω, and $0 \leq k_m^\omega \leq \pi/m$ for each band, we clearly have:

$$|\sigma_m^\omega \backslash S_m^\omega(\varepsilon)| < \pi\varepsilon\,, \tag{3.9}$$

from which follows that the sum $\sum\limits_{m=2}^{\infty} |\sigma_m^\omega \backslash S_m^\omega(m^{-2})|$ converges. As can be easily seen from the definition of lim sup, this implies:

$$\left|\limsup_{m\to\infty} (\sigma_m^\omega \backslash S_m^\omega(m^{-2}))\right| = 0\,. \tag{3.10}$$

Thus, since: $\limsup \sigma_m^\omega \backslash S_\omega \subseteq \limsup(\sigma_m^\omega \backslash S_m^\omega(m^{-2}))$, and clearly: $S_\omega \subseteq \limsup \sigma_m^\omega$, we have:

$$\limsup_{m\to\infty} \sigma_m^\omega \approx S_\omega\,, \tag{3.11}$$

where $\approx$ denotes set equality up to sets of zero (Lebesgue) measure.

Let $E \in S_\omega$, then $E \in S_{m_l}^\omega(m_l^{-2})$ for some subsequence $\{m_l\}_{l=1}^\infty$ of positive integers. From (3.7) and Lemma 3.1 we have for all l:

$$\|\Phi_{m_l}^\omega(E)\| \leq 2m_l^3\,, \tag{3.12}$$

which implies:

$$\liminf_{m\to\infty} \frac{1}{m} \ln\|\Phi_m^\omega(E)\| = 0\,. \tag{3.13}$$

From the Furstenberg-Kesten Theorem [9, 13], we know that for a.e. ω, for a.e. E, the limit: $\lim \frac{1}{m}\ln\|\Phi_m^\omega(E)\|$ exists, and is equal to $\gamma(E)$. Therefore, we obtain for a.e. ω, for a.e. $E \in S_\omega : \gamma(E) = 0$. From (3.11), this implies that for a.e. ω, for a.e. $E \in \limsup \sigma_m^\omega : \gamma(E) = 0$, and this is precisely the statement of Theorem 1. □

4. Proof of Theorem 2

If $\alpha = p/q$ is a rational number, then for each θ, the almost Mathieu operator defined by (1.8), is a periodic Jacobi matrix with period q, and θ dependent spectrum: $\sigma(p/q, \lambda, \theta)$. For every irrational α, $\theta \in [0, 2\pi]$ and $\lambda \geq 0$, denote by: $\{\sigma_m(\alpha, \lambda, \theta)\}_{m=1}^\infty$, the spectra of the canonical periodic approximants of $H_{\alpha,\lambda,\theta}$, as defined in Sect. 1, then:

Lemma 4.1. *For every sequence of rationals: $\alpha_j = p_j/q_j \to \alpha$, there is a sequence $\{\theta_j\}_{j=1}^\infty$, such that:*

$$\left||\sigma_{q_j}(\alpha, \lambda, \theta)| - |\sigma(\alpha_j, \lambda, \theta_j|\right| \leq \pi\lambda q_j^2 |\alpha - \alpha_j|\,.$$

Proof. Both $H_{\alpha_j,\lambda,\theta_j}$ and the $q_j{}^{\text{th}}$ canonical periodic approximant of $H_{\alpha,\lambda,\theta}$, are periodic Jacobi matrices with period q_j, and the difference of the measures of their spectra must be smaller or equal to the sum of the distances between their corresponding band edges. These band edges are eigenvalues of the corresponding $\mathscr{A}(k)$ matrices with $k = 0$ and $k = \pi/q_j$ (see Sect. 2). Since the difference of two corresponding (i.e. with the same k) $\mathscr{A}(k)$ matrices is diagonal, it follows from Lidskii's Theorem [6, 15], that the sum of the distances between their corresponding eigenvalues must be smaller or equal to the sum of the absolute values of the (diagonal) entries of the difference matrix. This implies:

$$\left||\sigma_{q_j}(\alpha,\lambda,\theta)| - |\sigma(\alpha_j,\lambda,\theta_j)|\right| \leq 2\sum_{n=1}^{q_j} |\lambda\cos(2\pi\alpha n+\theta) - \lambda\cos(2\pi\alpha_j n+\theta_j)|$$

$$\leq 2\lambda\sum_{n=1}^{q_j} |2\pi(\alpha-\alpha_j)n + (\theta-\theta_j)|\,. \tag{4.1}$$

By choosing θ_j such that: $(\theta-\theta_j) = -\pi(\alpha-\alpha_j)(q_j+1)$, the last sum (which is the sum of two arithmetic series) is seen to be smaller than $\pi\lambda|\alpha-\alpha_j|q_j^2$, and this proves the lemma. □

For every real α, λ, we denote:

$$S(\alpha,\lambda) \equiv \bigcup_\theta \mathrm{Spec}(H_{\alpha,\lambda,\theta})\,. \tag{4.2}$$

If α is irrational then we have: $S(\alpha,\lambda) = \sigma(\alpha,\lambda)$, and if $\alpha = p/q$ is rational then the set $S(\alpha,\lambda)$ is similar to $\sigma(\alpha,\lambda,\theta)$, consisting of q bands. We denote by $G(\alpha,\lambda)$ the union of the gaps in $S(\alpha,\lambda)$, such that:

$$|S(\alpha,\lambda)| = \max S(\alpha,\lambda) - \min S(\alpha,\lambda) - |G(\alpha,\lambda)|\,. \tag{4.3}$$

Avron, van Mouche, and Simon [3], proved the following:

Proposition 4.1. (i) *For every rational p/q, $\lambda \geq 0$ and $\theta \in [0,2\pi]$:*

$$|\sigma(p/q,\lambda,\theta)| \geq 4 - 2\lambda\,.$$

(ii) *For every $0 \leq \lambda < 2$, and a sequence of rationals $\alpha_j = p_j/q_j$, obeying: $q_j \to \infty$ with p_j, q_j relatively prime:*

$$\lim_{j\to\infty} |S(\alpha_j,\lambda)| = 4 - 2\lambda\,.$$

(iii) *For every $\lambda > 0$, there is a constant C, such that if $|\alpha-\alpha'| < C$, then for every gap in $S(\alpha,\lambda)$ with midpoint E_g, and measure $|g|$ larger than $12(\lambda|\alpha-\alpha'|)^{1/2}$, there is a corresponding (containing E_g) gap in $S(\alpha',\lambda)$ with measure larger than:*

$$|g| - 12(\lambda|\alpha-\alpha'|)^{1/2}\,.$$

(iv) *The same continuity as in* (iii) *also holds for the extreme edges of $S(\alpha,\lambda)$, namely for $|\alpha-\alpha'| < C$:*

$$\left| \begin{matrix}\max\\ \min\end{matrix} S(\alpha,\lambda) - \begin{matrix}\max\\ \min\end{matrix} S(\alpha',\lambda)\right| < 6(\lambda|\alpha-\alpha'|)^{1/2}\,.$$

Proof of Theorem 2. (i) Let α be any irrational, then it is known [14] that there is a sequence of rationals: $\alpha_j = p_j/q_j \to \alpha$, obeying: $q_j^2|\alpha-\alpha_j| \leq 1/\sqrt{5}$. Thus, from statement (i) of Proposition 4.1 and from Lemma 4.1 we obtain:

$$|\sigma_{q_j}(\alpha,\lambda,\theta)| \geq 4 - 2\lambda - \pi\lambda q_j^2|\alpha-\alpha_j|$$

$$\geq 4 - 2\lambda - \frac{\pi\lambda}{\sqrt{5}}$$

$$= 4 - \left(2 + \frac{\pi}{\sqrt{5}}\right)\lambda\,. \tag{4.4}$$

Since $q_j \to \infty$ as $j \to \infty$, this implies (for every θ):

$$\limsup_{m\to\infty} |\sigma_m(\alpha,\lambda,\theta)| \geq 4 - \left(2 + \frac{\pi}{\sqrt{5}}\right)\lambda\,, \tag{4.5}$$

and from Corollary 1.1 the statement follows.

(ii) By repeating the arguments of (i), but with the sequence of rationals obeying: $q_j^2|\alpha-\alpha_j| \to 0$, we clearly obtain: $|A(\alpha,\lambda)| \geq 4-2\lambda$. In order to complete the proof it is sufficient to show: $|\sigma(\alpha,\lambda)| \leq 4-2\lambda$. Since there are at most $q_j - 1$ gaps in $S(\alpha_j,\lambda)$, we obtain from statement (iii) of Proposition 4.1 (for $|\alpha-\alpha_j| < C$):

$$|G(\alpha,\lambda)| > |G(\alpha_j,\lambda)| - 12(q_j-1)(\lambda|\alpha-\alpha_j|)^{1/2}\,, \tag{4.6}$$

and from (4.3) and statement (iv) of Proposition 4.1, this implies:

$$|S(\alpha,\lambda)| < |S(\alpha_j,\lambda)| + 12q_j(\lambda|\alpha-\alpha_j|)^{1/2}\,. \tag{4.7}$$

As $j \to \infty$ we have from statement (ii) of Proposition 4.1: $|S(\alpha_j,\lambda)| \to 4-2\lambda$, and by our assumption on $\{\alpha_j\}: q_j|\alpha-\alpha_j|^{1/2} \to 0$. Thus, (4.7) implies:

$$|\sigma(\alpha,\lambda)| = |S(\alpha,\lambda)| \leq 4-2\lambda\,, \tag{4.8}$$

which completes the proof. □

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