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q-Deformed Poincaré Algebra

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Abstract. The q-differential calculus for the q-Minkowski space is developed. The algebra of the q-derivatives with the q-Lorentz generators is found giving the q-deformation of the Poincaré algebra. The reality structure of the q-Poincaré algebra is given. The reality structure of the q-differentials is also found. The real Laplacian is constructed. Finally the comultiplication, counit and antipode for the q-Poincaré algebra are obtained making it a Hopf algebra.

1. Introduction

Quantum groups have already established themselves in such diverse branches of mathematics and theoretical physics as conformal field theory, integrable models, statistical mechanics, knot theory and topology of low-dimensional manifolds. Like many other notions (quantum mechanics, special relativity) quantum groups appear as some deformation of old "classical" objects, in this case groups. Although this type of deformation can be understood in terms of usual quantum mechanics, the idea of quantizing the symmetry itself is apparently new. The fruitfulness of this idea is supported by the number of geometric and algebraic notions which can be "q-deformed." First of all quantum groups can be viewed as symmetries of "quantum" spaces [1, 2]. Next the frame of differential calculus can be extended to include quantum groups and quantum spaces [3, 4].

The role of symmetry in physics is hard to overestimate. This explains the wide interest which quantum groups found among theoretical physicists. Particularly one is tempted to deform a real physical system in this spirit. This requires first of

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all a deep understanding of the q-deformation of Minkowski geometry. The quantum Minkowski space itself is more or less understood [5–8]. The quantum Lorentz group serves as the q-symmetry group of this space. One is naturally interested in the action of the q-Lorentz algebra on the q-Minkowski space. This question is nontrivial since the relation between Lie algebras and Lie groups becomes more involved on the quantum level, in particular as of now the exponential map is unknown. The q-Lorentz algebra was obtained in [8, 9] where the Lorentz generators were defined by their commutation relations with the q-spinors.

The next step is to define the quantum Poincaré algebra, or to add the infinitesimal translations to the q-Lorentz algebra. This is the aim of the present paper. Following the classical example we treat the q-derivatives as generators of translations. (Another approach was followed in [10] where the translations stayed undeformed.) The general theory [11, 12] giving the q-deformation of the universal enveloping algebra of any simple Lie group is not sufficient for the Poincaré algebra, since it is not simple. To find the algebra we use the action of the Lorentz generators and derivatives on the q-Minkowski space.

We discover a new effect absent in the classical Poincaré algebra. Namely, the operators conjugate to the derivatives cannot be expressed linearly in terms of the derivatives themselves (in contrast to the q-Minkowski coordinates for which the conjugation is linear and just given by the classical formulas). A similar phenomenon also occurs for the conjugated differentials.

We also construct the coproduct for the derivatives. We prove that this comultiplication is natural, or in other words is compatible with the action. Finally we find the counit and antipode to complete the Hopf algebra structure of the q-deformed Poincaré algebra.

The paper is organized as follows. In Sects. 2 and 3 we give preliminaries on the q-spinors, the q-Minkowski vectors and the R-matrices for them. Section 4 contains the necessary information about the q-Lorentz algebra and its action on the q-Minkowski space. In Sect. 5 we discuss the q-differential calculus on the q-Minkowski space. Section 6 is devoted to the reality structure for derivatives and differentials. There we also construct the real Laplacian. Finally, in Sect. 7 the comultiplication for the translation sector of the q-Poincaré algebra is given and its naturality is proved. Appendices contain technical formulas for the projector decomposition of the $\hat{R}$-matrices, commutation relations between coordinates and differentials, action of the conjugate derivatives, and relations for some q-differential operators needed for defining the reality structure. Many of the relations are given in a component form which is useful in checking some of the nonlinear relations in the text.

2. $\hat{R}$-Matrices for the q-Lorentz Group

The q-deformed Lorentz group has been studied in [5–7]. These analyses made use of the classical isomorphism $SO(3,1)\cong SL(2,\mathbf{C})/\mathbf{Z}_2$. Since the quantum group $SL_q(2,\mathbf{C})$ is well understood, it is natural to use it for the q-Lorentz group. The fundamental representation of $SL_q(2,\mathbf{C})$ consists of two-dimensional complex quantum spinors x^α and their complex conjugates $\bar{x}^{\dot\alpha}$. Minkowski vectors are constructed as bilinears of a spinor and a conjugate spinor. A vector is written as

$$X^{\dot\alpha\beta} = \bar{x}^{\dot\alpha} x^\beta \, . \tag{2.1}$$

The $\hat{R}$-matrix for the q-Lorentz group is determined by moving such a vector through another bi q-spinor $\bar{u}^{\dot{\gamma}} v^{\delta}$, where u and v are independent copies of q-spinors. However there is an ambiguity in choosing the q-relations between $\bar{x}$, x and $\bar{u}$, v. This results in two different $\hat{R}$-matrices for the q-Lorentz group. Both $\hat{R}$-matrices satisfy the Yang–Baxter equation. This construction of the $\hat{R}$-matrices was followed in [7], and we shall refer to them as $\hat{R}_{\mathrm{I}}$ and $\hat{R}_{\mathrm{II}}$.[1]

These two $\hat{R}$-matrices satisfy the characteristic equations

$$
\begin{aligned}
&(\hat{R}_{\mathrm{I}} - 1)(\hat{R}_{\mathrm{I}} + q^2)(\hat{R}_{\mathrm{I}} + q^{-2}) = 0 \,, \\
&(\hat{R}_{\mathrm{II}} + 1)(\hat{R}_{\mathrm{II}} - q^2)(\hat{R}_{\mathrm{II}} - q^{-2}) = 0 \,.
\end{aligned} \tag{2.2}
$$

Solution of the eigenvalue problem gives the decomposition of each matrix into three projectors. Taken together one finds four projectors: P_T which is the q-deformed trace projector, P_S which is the traceless part of the q-deformed symmetrizer, and P_+ and P_- which are the selfdual and antiselfdual parts of the q-deformed antisymmetrizer. These are the q-deformed versions of the classical projectors. Their explicit form is given in Appendix A. The four projectors sum to the identity matrix:

$$
\mathbb{1} = P_S + P_T + P_+ + P_- \,, \tag{2.3}
$$

and the $\hat{R}$ matrices are written as the sums

$$
\begin{aligned}
&\hat{R}_{\mathrm{I}} = P_S + P_T - q^2 P_+ - q^{-2} P_- \,, \\
&\hat{R}_{\mathrm{II}} = q^{-2} P_S + q^2 P_T - P_+ - P_- \,.
\end{aligned} \tag{2.4}
$$

These are the only linear combinations of the four projectors which are compatible with the relations between the components of a q-vector. A more precise statement will be given in [7].

The higher dimensional orthogonal q-groups are described by only one $\hat{R}$-matrix. In four dimensions this is the $\hat{R}_{\mathrm{II}}$-matrix. However, in four dimensions the situation is special in that the antisymmetric square of the vector representation is reducible. It decomposes into the selfdual and antiselfdual parts. The $\hat{R}_{\mathrm{II}}$-matrix takes the same eigenvalue on both. The selfdual and antiselfdual parts are distinguished by the $\hat{R}_{\mathrm{I}}$-matrix. Note that we need both $\hat{R}$-matrices since $\hat{R}_{\mathrm{I}}$ in turn does not split the q-symmetrizer into the trace and traceless parts.

The $\hat{R}$-matrix used to define q-relations between elements of the q-space depends on the projector decomposition needed. For the coordinates X^i the antisymmetrizers acting on the tensor product of two coordinates must give zero. Suppressing indices we write this as

$$
P_+ XX = 0, \quad P_- XX = 0 \,. \tag{2.5}
$$

Using this fact we then have

$$
\mathbb{1} XX = (P_S + P_T) XX = \hat{R}_{\mathrm{I}} XX \,, \tag{2.6}
$$

[1] In [5] the matrix $\hat{R}_{\mathrm{II}}$ was derived in a different way. There a matrix of the q-Lorentz group was constructed as a tensor product of an $SL_q(2, \mathbf{C})$ matrix and its conjugate. The q-relations between elements of the Lorentz group matrix were determined by the R-matrix for $SL_q(2, \mathbf{C})$. These q-relations then give rise to the $\hat{R}_{\mathrm{II}}$-matrix for the q-Lorentz group

where in the first step we used (2.3) for $\mathbb{1}$ and in the second step we used (2.4) for $\hat{R}_{\rm I}$. Including the indices we then have

$$X^i X^j = \hat{R}_{\rm I}{}^{ij}{}_{kl} X^k X^l \tag{2.7}$$

for the q-relations between coordinates.

A differential calculus is established on this algebra by introducing an exterior derivative d with the usual properties of nilpotency and Leibniz rule:

$$d^2 = 0, \quad d(fg) = (df)g + f(dg), \tag{2.8}$$

where f and g are functions of the coordinates. The differential of X^i is called ξ^i. The action of d on the coordinates and differentials is

$$dX^i = \xi^i + X^i d, \quad d\xi^i = -\xi^i d. \tag{2.9}$$

We will need q-relations between the differentials themselves and between the differentials and coordinates.

Classically the ξ^i are anticommuting objects, so in the q-deformed case we require that a tensor product of two differentials is annihilated by the symmetrizers:

$$P_S \xi\xi = 0, \quad P_T \xi\xi = 0. \tag{2.10}$$

Then we write

$$\mathbb{1}\xi\xi = (P_+ + P_-)\xi\xi = -\hat{R}_{\rm II}\xi\xi \tag{2.11}$$

using (2.3) and (2.4) for the projector decompositions of $\mathbb{1}$ and $\hat{R}_{\rm II}$, respectively. Then the equation

$$\xi^i \xi^j = -\hat{R}_{\rm II}{}^{ij}{}_{kl} \xi^k \xi^l \tag{2.12}$$

gives q-relations between the differentials.

For q-relations between coordinates and differentials assume that $X\xi = C\xi X$ for some matrix C. Applying d to this equation gives $\xi\xi = -C\xi\xi$. Comparing with (2.12) we see that $C = \hat{R}_{\rm II}$ and

$$X^i \xi^j = \hat{R}_{\rm II}{}^{ij}{}_{kl} \xi^k X^l \tag{2.13}$$

gives the desired q-relations.

Derivatives are introduced by the usual expansion of the exterior derivative:

$$d = \xi^i \partial_i. \tag{2.14}$$

Then applying d to a coordinate X^i, using the Leibniz rule (2.8) for d, and (2.13) to move the ξ's to the left, we find

$$\partial_i X^j = \delta_i^j + \hat{R}_{\rm II}{}^{jk}{}_{il} X^l \partial_k \tag{2.15}$$

for the action of derivatives on coordinates.

We also need q-relations among the derivatives themselves. Assume a relation of the form $\partial\partial = F\partial\partial$ for some matrix F. Applying both sides of this equation to a coordinate X^i and using (2.15) gives the consistency condition $(1 - F)(1 + \hat{R}_{\rm II}) = 0$. A check of (2.4) shows that this is satisfied if $F = \hat{R}_{\rm I}$, and the q-relations between derivatives are given by

$$\partial_i \partial_j = \hat{R}_{\rm I}{}^{lk}{}_{ji} \partial_k \partial_l. \tag{2.16}$$

Note the reversed order of index summation compared to the other $\hat{R}$-matrix equations.

Although we will not need them, one can also find q-relations between the differentials and derivatives. Assume a relation of the form $\partial\xi = D\xi\partial$ for some matrix D. Applying both sides to a coordinate X^i and using already established relations one finds $D = \hat{R}_{\mathrm{II}}^{-1}$ and we have

$$\partial_i \xi^j = \hat{R}_{\mathrm{II}}^{-1\,jk}{}_{il}\xi^l \partial_k \,. \tag{2.17}$$

This completes the algebra of coordinates, differentials and derivatives.

Throughout this discussion we could have used the inverse of the $\hat{R}$-matrices instead. This would leave the XX, $\xi\xi$ and $\partial\partial$ q-relations unchanged. However the $X\xi$, ∂X and $\partial\xi$ relations would be different. This would give a second possible choice for the differential calculus. However this second choice coincides with the complex conjugates of the derivatives. With the definition of $\hat{\partial}$ in Appendix C this can be seen by conjugating the above relations and using

$$\hat{R}_{\mathrm{II}}^{-1\,ij}{}_{kl} = q^2 g^{im}\hat{R}_{\mathrm{II}}{}^{jn}{}_{mk}g_{nl} = q^2 g_{km}\hat{R}_{\mathrm{II}}{}^{mi}{}_{ln}g^{nj} \tag{2.18}$$

(this is a property shared by all orthogonal and symplectic quantum groups [1]) and

$$\hat{R}_{\mathrm{II}}{}^{ij}{}_{kl} = f^i_a\, f^j_b \hat{R}_{\mathrm{II}}{}^{ba}{}_{dc}\, f^c_k\, f^d_l \,, \tag{2.19}$$

where f is the matrix appearing in the complex conjugation of coordinates: $\overline{X^i} = f^i_j X^j$.

3. Minkowski Coordinates

In this section we introduce the quantum Minkowski coordinates. Explicit formulas for their q-relations are given. The invariant Minkowski length is also presented.

In [8] the q-Minkowski coordinates were constructed as bilinears of q-spinors (x, y) and their conjugates $(\bar{x}, \bar{y})$. We take as coordinates the four quantities

$$\begin{aligned} A &= \bar{x}y, \quad C = \bar{x}x \,, \\ B &= \bar{y}x, \quad D = \bar{y}y \,. \end{aligned} \tag{3.1}$$

Their reality properties are

$$\begin{aligned} \bar{A} &= B, \quad \bar{C} = C \,, \\ \bar{B} &= A, \quad \bar{D} = D \,. \end{aligned} \tag{3.2}$$

Real Minkowski coordinates are defined by the linear combinations

$$\begin{aligned} X^0 &= \frac{1}{\sqrt{2}}(C + D), \quad X^1 = \frac{1}{\sqrt{2}}(A + B) \,, \\ X^3 &= \frac{1}{\sqrt{2}}(C - D), \quad X^2 = \frac{i}{\sqrt{2}}(A - B) \,. \end{aligned} \tag{3.3}$$

In the following we will refer to general Minkowski coordinates as X^i, but for explicit calculations the basis (A, B, C, D) is convenient.

Quantum relations between the coordinates are provided by the $\hat{R}$-matrix for the Lorentz group through the $\hat{R}$-matrix relation (2.7). Explicitly the q-relations for (A, B, C, D) are

$$\begin{aligned} AB &= BA - q^{-1}\lambda CD + q\lambda D^2, & BC &= CB - q^{-1}\lambda BD\,, \\ AC &= CA + q\lambda AD, & BD &= q^2 DB\,, \\ AD &= q^{-2}DA, & CD &= DC\,, \end{aligned} \tag{3.4}$$

where q is real and $\lambda = q - q^{-1}$. These relations are invariant under complex conjugation accompanied by reversal of the variable order. Also these relations allow ordering of any monomial in the coordinates.

A quantum Minkowski metric may be obtained from the trace projector which is one of the projectors comprising the $\hat{R}$-matrix. In the basis (A, B, C, D) the metric g_{ij} and inverse g^{ij} are

$$g_{ij} = \begin{pmatrix} 0 & q^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & q\lambda \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q^{-2} & 0 & 0 & 0 \\ 0 & 0 & -q\lambda & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.5}$$

Written out using the q-relations for the coordinates the Minkowski length of a four vector is

$$L = (q^2 + 1)^{-1} g_{ij} X^i X^j = AB - q^{-2} CD\,. \tag{3.6}$$

This length is real and commutes with the coordinates: $LX^i = X^i L$.

4. Lorentz Algebra

In this section we review the previous results on the q-deformed Lorentz algebra [8, 9]. Lorentz generators are defined by their action on the four vectors. From this the algebra and coproduct of the generators is determined. The counit, antipode, and reality conditions complete the description of the Hopf algebra.

In [8, 9] the q-deformed Lorentz algebra was presented. There the generators were defined by their action on two dimensional complex quantum spinors. Here we will confine the discussion to the quantum Minkowski coordinates which can be constructed as bi q-spinors. The three generators of $SU_q(2)$, T^+, T^-, and T^3, form the rotation subalgebra. The Lorentz algebra is completed by adding non-compact generators T^1, T^2, S^1, and S^2. Although this is an algebra with seven generators, we had shown [9] that they are not independent and one generator may be eliminated. All these generators annihilate the constant monomial: $T^i 1 = 0$. For the diagonal generators T^3, T^1, and S^2 it is convenient to define $\tau^3 = 1 - \lambda T^3$, $\tau^1 = 1 + \lambda T^1$, and $\sigma^2 = 1 + \lambda S^2$ which obey $\tau^i 1 = 1$.

The $SU_q(2)$ generators have the following action on the Minkowski coordinates:

$$\begin{aligned}
&T^+A = q^{-2}AT^+, && T^+C = CT^+ + q^{-1}A\,,\\
&T^+B = q^2BT^+ + qD - q^{-1}C, && T^+D = DT^+ - q^{-1}A\,,\\
&T^-A = q^{-2}AT^- + q^{-1}C - qD, && T^-C = CT^- - qB\,,\\
&T^-B = q^2BT^-, && T^-D = DT^- + qB\,,\\
&\tau^3A = q^{-4}A\tau^3, && \tau^3C = C\tau^3\,,\\
&\tau^3B = q^4B\tau^3, && \tau^3D = D\tau^3\,.
\end{aligned} \tag{4.1}$$

Note that all of these generators commute with the time coordinate $X^0 \propto C + D$. Thus they do not generate boosts. The additional generators have action

$$\begin{aligned}
&T^2A = qAT^2, && T^2C = qCT^2 + qA\tau^1\,,\\
&T^2B = q^{-1}BT^2 + q^{-1}D\tau^1, && T^2D = q^{-1}DT^2\,,\\
&S^1A = q^{-1}AS^1 + q^{-1}D\sigma^2, && S^1C = (q^{-1}C + q\lambda^2D)S^1 + qB\sigma^2\,,\\
&S^1B = qBS^1, && S^1D = qDS^1\,,\\
&\tau^1A = qA\tau^1 + q\lambda^2DT^2, && \tau^1C = (q^{-1}C + q\lambda^2D)\tau^1 + q\lambda^2BT^2\,,\\
&\tau^1B = q^{-1}B\tau^1, && \tau^1D = qD\tau^1\,,\\
&\sigma^2A = q^{-1}A\sigma^2, && \sigma^2C = qC\sigma^2 + q\lambda^2AS^1\,,\\
&\sigma^2B = qB\sigma^2 + q\lambda^2DS^1, && \sigma^2D = q^{-1}D\sigma^2\,.
\end{aligned} \tag{4.2}$$

As shown in [8], these generators produce a linear combination of rotations and boosts in the limit $q \to 1$, and they complete the Lorentz algebra. Also, it should be noted that all generators of the q-Lorentz algebra commute with the q-Minkowski length L.

The algebra of the generators follows from this action. To this end one finds bilinear combinations of the generators having an action proportional to the action of some linear combinations. The full algebra of the seven generators is

$$\begin{aligned}
&\tau^1T^+ = T^+\tau^1 + \lambda T^2, && T^+T^2 = q^{-2}T^2T^+\,,\\
&\tau^1T^- = q^{-2}T^-\tau^1 - \lambda S^1, && T^-T^2 = T^2T^- + \lambda^{-1}(\sigma^2 - \tau^1)\,,\\
&\tau^1T^2 = q^2T^2\tau^1, && T^+S^1 = q^2S^1T^+ + \lambda^{-1}(\tau^3\tau^1 - \sigma^2)\,,\\
&\tau^1S^1 = S^1\tau^1, && T^-S^1 = S^1T^-\,,\\
& && T^+T^- = q^2T^-T^+ + q\lambda^{-1}(1 - \tau^3)\,,\\
&\sigma^2T^+ = T^+\sigma^2 - q^2\lambda\tau^3T^2, && T^2S^1 = S^1T^2\,,\\
&\sigma^2T^- = q^2T^-\sigma^2 + q^2\lambda S^1, && \\
&\sigma^2T^2 = q^{-2}T^2\sigma^2, && \tau^1\sigma^2 = \sigma^2\tau^1 + q\lambda^3T^2S^1\,,\\
&\sigma^2S^1 = S^1\sigma^2, && \tau^3\tau^1 = \tau^1\tau^3\,,\\
& && \tau^3\sigma^2 = \sigma^2\tau^3\,.
\end{aligned}$$

$$\tau^3 T^+ = q^{-4} T^+ \tau^3 \,,$$
$$\tau^3 T^- = q^4 T^- \tau^3 \,,$$
$$\tau^3 T^2 = q^{-4} T^2 \tau^3 \,,$$
$$\tau^3 S^1 = q^4 S^1 \tau^3 \,, \tag{4.3}$$

The algebra may be written in a more conventional form by the substitutions $\tau^1 = 1 + \lambda T^1$, $\sigma^2 = 1 + \lambda S^2$, and $\tau^3 = 1 - \lambda T^3$.

This algebra appears to have seven generators. However there is an extra relation in the algebra which allows elimination of one of the diagonal generators. Consider the quantity

$$Z = \tau^1 \sigma^2 - q^2 \lambda^2 T^2 S^1 \,. \tag{4.4}$$

One finds that Z is central in the algebra and commutes with all of the coordinates: $ZX^i = X^i Z$. Therefore Z is 1. Then one could eliminate τ^1 or σ^2 from the algebra, for example by the substitution $\sigma^2 = (\tau^1)^{-1}(1 + q^2\lambda^2 T^2 S^1)$. However this would leave the algebra with inverse powers of the remaining diagonal generator. In the following it will be convenient to keep all seven generators, having in mind that they are not independent.

In [8, 9] the coproduct for the generators was found by considering their action on functions of the spinors. The same results can be obtained using functions of the Minkowski coordinates. The counit and antipode are determined by the coproduct, and conjugation of the action on the coordinates (including reversal of variable order) yields the real structure of the generators. For the $SU_q(2)$ generators the coproduct Δ is

$$\Delta(T^\pm) = T^\pm \otimes 1 + (\tau^3)^{\frac{1}{2}} \otimes T^\pm \,, \quad \Delta(\tau^3) = \tau^3 \otimes \tau^3 \,. \tag{4.5}$$

The counit ε and antipode S are

$$\varepsilon(T^\pm) = 0, \quad S(T^\pm) = -(\tau^3)^{-\frac{1}{2}} T^\pm \,,$$
$$\varepsilon(\tau^3) = 1, \quad S(\tau^3) = (\tau^3)^{-1} \,. \tag{4.6}$$

Under conjugation the $SU_q(2)$ generators obey

$$\overline{T^\pm} = q^{\mp 2} T^\mp, \quad \overline{\tau^3} = \tau^3 \,. \tag{4.7}$$

The results for the remaining generators are slightly more complicated. The coproduct is

$$\Delta(\tau^1) = \tau^1 \otimes \tau^1 + \lambda^2 S^1 (\tau^3)^{-\frac{1}{2}} \otimes T^2 \,,$$
$$\Delta(\sigma^2) = \sigma^2 \otimes \sigma^2 + \lambda^2 T^2 (\tau^3)^{\frac{1}{2}} \otimes S^1 \,,$$
$$\Delta(T^2) = T^2 \otimes \tau^1 + (\tau^3)^{-\frac{1}{2}} \sigma^2 \otimes T^2 \,,$$
$$\Delta(S^1) = S^1 \otimes \sigma^2 + (\tau^3)^{\frac{1}{2}} \tau^1 \otimes S^1 \,. \tag{4.8}$$

The counit and antipode are

$$\begin{aligned}
&\varepsilon(\tau^1) = 1, \quad S(\tau^1) = \sigma^2 , \\
&\varepsilon(\sigma^2) = 1, \quad S(\sigma^2) = \tau^1 , \\
&\varepsilon(T^2) = 0, \quad S(T^2) = -q^2(\tau^3)^{\frac{1}{2}} T^2 , \\
&\varepsilon(S^1) = 0, \quad S(S^1) = -(\tau^3)^{-\frac{1}{2}} S^1 .
\end{aligned} \tag{4.9}$$

In checking the antipode property one needs the fact that $Z = 1$. Finally the reality conditions for the new generators are

$$\begin{aligned}
&\overline{\tau^1} = (\tau^3)^{-\frac{1}{2}}\sigma^2, \quad \overline{T^2} = -(\tau^3)^{-\frac{1}{2}} S^1 , \\
&\overline{\sigma^2} = (\tau^3)^{\frac{1}{2}}\tau^1, \quad \overline{S^1} = -q^2(\tau^3)^{\frac{1}{2}} T^2 .
\end{aligned} \tag{4.10}$$

This completes the construction of the Hopf algebra of the Lorentz generators.

5. Poincaré Algebra

In this section we add translation generators to the Lorentz algebra. As translation generators we take the q-deformed four-vector derivatives. The action of the derivatives on the Minkowski coordinates and the algebra of the derivatives is defined by the $\hat{R}$-matrix. The action also allows one to find the commutation relations of the derivatives with the Lorentz generators.

The action of the derivatives on the q-Minkowski space reads

$$\begin{aligned}
&\partial_A A = 1 + q^{-2} A\partial_A + \lambda^2 B\partial_B + \lambda(qD - q^{-1}C)\partial_C - q^{-1}\lambda D\partial_D , \\
&\partial_A B = B\partial_A , \\
&\partial_A C = q^{-2} C\partial_A + \lambda^2 D\partial_A + q\lambda(1 - q\lambda)B\partial_C - q\lambda B\partial_D , \\
&\partial_A D = D\partial_A - q\lambda B\partial_C , \\
&\partial_B A = A\partial_B , \\
&\partial_B B = 1 + q^{-2} B\partial_B - q^{-1}\lambda D\partial_C , \\
&\partial_B C = C\partial_B - q\lambda A\partial_C , \\
&\partial_B D = q^{-2} D\partial_B , \\
&\partial_C A = A\partial_C - q^{-1}\lambda D\partial_B , \\
&\partial_C B = q^{-2} B\partial_C , \\
&\partial_C C = 1 + q^{-2} C\partial_C + \lambda^2 D\partial_C - q^{-1}\lambda B\partial_B , \\
&\partial_C D = D\partial_C ,
\end{aligned}$$

$$\begin{aligned}
\partial_D A &= q^{-2} A \partial_D + \lambda (qD - q^{-1} C) \partial_B \,, \\
\partial_D B &= B \partial_D - q^{-1} \lambda D \partial_A + \lambda^2 B \partial_C \,, \\
\partial_D C &= C \partial_D - q^{-1} \lambda A \partial_A + q \lambda B \partial_B - q \lambda^2 (qD - q^{-1} C) \partial_C \,, \\
\partial_D D &= 1 + q^{-2} D \partial_D - q^{-1} \lambda B \partial_B \,.
\end{aligned} \tag{5.1}$$

The algebra of derivatives is

$$\begin{aligned}
&\partial_A \partial_B = \partial_B \partial_A - q\lambda \partial_C \partial_C + q\lambda \partial_D \partial_C, && \partial_B \partial_C = q^{-2} \partial_C \partial_B \,, \\
&\partial_A \partial_C = q^2 \partial_C \partial_A, && \partial_B \partial_D = \partial_D \partial_B + q\lambda \partial_C \partial_B \,, \\
&\partial_A \partial_D = \partial_D \partial_A - q^3 \lambda \partial_C \partial_A, && \partial_C \partial_D = \partial_D \partial_C \,.
\end{aligned} \tag{5.2}$$

This algebra is consistent with the action on coordinates.

To find the commutation relations between the derivatives and the Lorentz generators we follow the same procedure used for the Lorentz algebra alone. The procedure is straightforward but somewhat lengthy. For the rotation subalgebra one obtains

$$\begin{aligned}
&T^+ \partial_A = q^2 \partial_A T^+ - q\partial_C + q\partial_D, && T^+ \partial_C = \partial_C T^+ + q^{-1} \partial_B \,, \\
&T^+ \partial_B = q^{-2} \partial_B T^+, && T^+ \partial_D = \partial_D T^+ - q\partial_B \,, \\
&T^- \partial_A = q^2 \partial_A T^-, && T^- \partial_C = \partial_C T^- - q^{-1} \partial_A \,, \\
&T^- \partial_B = q^{-2} \partial_B T^- + q^{-1} \partial_C - q^{-1} \partial_D, && T^- \partial_D = \partial_D T^- + q\partial_A \,, \\
&\tau^3 \partial_A = q^4 \partial_A \tau^3, && \tau^3 \partial_C = \partial_C \tau^3 \,, \\
&\tau^3 \partial_B = q^{-4} \partial_B \tau^3, && \tau^3 \partial_D = \partial_D \tau^3 \,,
\end{aligned} \tag{5.3}$$

and for the noncompact generators the result is

$$\begin{aligned}
&T^2 \partial_A = q^{-1} \partial_A T^2 - q\partial_C \tau^1, && T^2 \partial_C = q^{-1} \partial_C T^2 \,, \\
&T^2 \partial_B = q\partial_B T^2, && T^2 \partial_D = q\partial_D T^2 + q\lambda^2 \partial_C T^2 - q\partial_B \tau^1 \,, \\
&S^1 \partial_A = q\partial_A S^1, && S^1 \partial_C = q\partial_C S^1 \,, \\
&S^1 \partial_B = q^{-1} \partial_B S^1 - q^{-1} \partial_C \sigma^2, && S^1 \partial_D = q^{-1} \partial_D S^1 - q^{-1} \partial_A \sigma^2 \,, \\
&\tau^1 \partial_A = q^{-1} \partial_A \tau^1, && \tau^1 \partial_C = q\partial_C \tau^1 \,, \\
&\tau^1 \partial_B = q\partial_B \tau^1 - q\lambda^2 \partial_C T^2, && \tau^1 \partial_D = q^{-1} \partial_D \tau^1 - q^{-1} \lambda^2 \partial_A T^2 \,, \\
&\sigma^2 \partial_A = q\partial_A \sigma^2 - q^3 \lambda^2 \partial_C S^1, && \sigma^2 \partial_C = q^{-1} \partial_C \sigma^2 \,, \\
&\sigma^2 \partial_B = q^{-1} \partial_B \sigma^2, && \sigma^2 \partial_D = q\partial_D \sigma^2 + q\lambda^2 \partial_C \sigma^2 - q\lambda^2 \partial_C S^1 \,.
\end{aligned} \tag{5.4}$$

This completes the q-deformed Poincaré algebra.

6. Real Structure

In the classical case it is straightforward to find conjugation rules for derivatives. However in the q-generalization one encounters difficulties. The action of the conjugated derivatives is given in Appendix C. Comparing with (5.1) one observes that these operators cannot be expressed linearly in terms of the q-derivatives themselves. This exhibits a new effect which does not appear on the classical level. The conjugation operation becomes nonlinear. In this section we give the explicit nonlinear relations between the derivatives and their conjugates.

To this end we will need to define several operators. First is the Laplacian of the q-derivatives

$$\Delta = (q^{-2}+1)^{-1} g^{ij} \partial_j \partial_i = \partial_A \partial_B - q^2 \partial_C \partial_D \, , \tag{6.1}$$

which is the only quadratic central element in the algebra of derivatives. Also we have the conjugated Laplacian:

$$\hat{\Delta} = q^{-2} \hat{\partial}_A \hat{\partial}_B - \hat{\partial}_C \hat{\partial}_D \, . \tag{6.2}$$

The two are related by

$$\bar{\Delta} = q^{10} \hat{\Delta} \, . \tag{6.3}$$

Note that $\hat{\Delta}$ commutes with the hatted derivatives. With the unhatted derivatives it obeys

$$\partial_i \hat{\Delta} = q^2 \hat{\Delta} \partial_i \, . \tag{6.4}$$

Next we define the operators E and $\hat{E}$:

$$E = X^i \partial_i, \quad \hat{E} = X^i \hat{\partial}_i \, , \tag{6.5}$$

which are related by

$$\bar{E} = -q^2 (q^2+1)^2 - q^8 \hat{E} \, . \tag{6.6}$$

The action of these operators on the coordinates and derivatives is given in Appendix D.

These operators together with the Minkowski length L serve as building blocks for two more operators Λ and $\hat{\Lambda}$:

$$\begin{aligned} \Lambda &= 1 - q^{-1} \lambda E + q^{-2} \lambda^2 L \Delta \, , \\ \hat{\Lambda} &= 1 + q \lambda \hat{E} + q^4 \lambda^2 L \hat{\Delta} \, . \end{aligned} \tag{6.7}$$

Using the formulas from Appendix D one finds

$$\Lambda \hat{\Lambda} = 1, \quad \bar{\Lambda} = q^8 \hat{\Lambda} \, . \tag{6.8}$$

The operators Λ and $\hat{\Lambda}$ act on both coordinates and derivatives multiplicatively. For Λ we have

$$\begin{aligned} \Lambda X^i &= q^{-2} X^i \Lambda \, , \\ \Lambda \partial_i &= q^2 \partial_i \Lambda \, , \\ \Lambda \hat{\partial}_i &= q^2 \hat{\partial}_i \Lambda \end{aligned} \tag{6.9}$$

with the corresponding relations for $\hat{\Lambda}$ given by relation (6.8).

It is clear from the construction of these operators (Δ, E, Λ and their conjugates) that they are Lorentz scalars. A check verifies that they all commute with the Lorentz generators.

The hatted derivatives can now be expressed in terms of the unhatted derivatives as

$$\hat{\partial}_i = q^{-2}\Lambda^{-1}[\Delta, g_{ij}X^j] . \tag{6.10}$$

Using the formulas in Appendix D we can write this in the form

$$\hat{\partial}_i = \Lambda^{-1}(\partial_i - q^{-3}\lambda g_{ij}X^j\Delta) \tag{6.11}$$

or explicitly

$$\begin{aligned}
\hat{\partial}_A &= \Lambda^{-1}(\partial_A - q^{-1}\lambda B\Delta) , \\
\hat{\partial}_B &= \Lambda^{-1}(\partial_B - q^{-3}\lambda A\Delta) , \\
\hat{\partial}_C &= \Lambda^{-1}(\partial_C + q^{-3}\lambda D\Delta) , \\
\hat{\partial}_D &= \Lambda^{-1}(\partial_D + q^{-3}\lambda(C - q\lambda D)\Delta) .
\end{aligned} \tag{6.12}$$

The inverse map is

$$\begin{aligned}
\partial_A &= \hat{\Lambda}^{-1}(\hat{\partial}_A + q^5\lambda B\hat{\Delta}) , \\
\partial_B &= \hat{\Lambda}^{-1}(\hat{\partial}_B + q^3\lambda A\hat{\Delta}) , \\
\partial_C &= \hat{\Lambda}^{-1}(\hat{\partial}_C - q^3\lambda D\hat{\Delta}) , \\
\partial_D &= \hat{\Lambda}^{-1}(\hat{\partial}_D - q^3\lambda(C - q\lambda D)\hat{\Delta}) .
\end{aligned} \tag{6.13}$$

One verifies these relations by checking that the left- and right-hand sides (rhs) have the same action on the coordinates. Note that the terms proportional to the Laplacians in the rhs of these equations have the same transformation properties as the derivatives. Thus these mappings are covariant under the global q-Lorentz group. Along with (4.7) and (4.10) this describes the real structure of the q-Poincaré algebra.

The properties of the differentials under conjugation are also nontrivial. Again we introduce several relevant scalar quantities. Define

$$W = g_{ij}\xi^i X^j = q^2\xi^A B + \xi^B A - \xi^C D - \xi^D(C - q\lambda D) . \tag{6.14}$$

Note that $g_{ij}X^i\xi^j = q^2 W$. The quantity

$$U = W - q\lambda L d \tag{6.15}$$

commutes with all coordinates: $UX^i = X^i U$. The relations between the coordinates and the differentials are given by (2.13). One can check that the quantities

$$\phi^i = \xi^i - q\lambda X^i d \tag{6.16}$$

satisfy the following relations with the coordinates:

$$q^2 X^i\phi^j = \hat{R}_{\mathrm{II}}^{-1}{}^{ij}{}_{kl}\phi^k X^l + q\lambda g^{ij}U . \tag{6.17}$$

Up to the factor q^2 in the lhs and the last term in the rhs these are the relations (B.4) for X's and ξ's. There is another set of quantities which satisfy relations with X's similar to (6.17). Namely one can rewrite the relations (2.15) in the form

$$q^2 X^i\hat{\partial}^j = \hat{R}_{\mathrm{II}}^{-1}{}^{ij}{}_{kl}\hat{\partial}^k X^l - q^{-2}g^{ij} , \tag{6.18}$$

where $\hat{\partial}^i = g^{ij}\hat{\partial}_j$. Since U commutes with coordinates we can compensate the extra term in (6.17) by adding $q^3\lambda U\hat{\partial}^k$ to ϕ^k. To get rid of the factor q^2 in the lhs of (6.17) we use the same scaling operator Λ. Thus the quantities $\Lambda(\phi^i + q^3\lambda U\hat{\partial}^i)$ have the same commutation relations with $\hat{X}$'s as ξ's. Since these commutation relations are homogeneous we can conclude only that the $\hat{\xi}^i$ are proportional to $\Lambda(\phi^i + q^3\lambda U\hat{\partial}^i)$. The proportionality factor can be found from the requirement that the square of the conjugation operation is unity on the ξ's. Finally we obtain

$$\hat{\xi}^i = q^{-4}\Lambda(\phi^i + q^3\lambda U\hat{\partial}^i) = q^{-4}\Lambda(\xi^i - q\lambda X^i d) + q^{-3}\lambda U(\partial^i - q^{-3}\lambda X^i \Delta) \tag{6.19}$$

with $\partial^i = g^{ij}\partial_j$. This implies the following reality property for the exterior derivative:

$$\hat{d} \equiv \hat{\xi}^i\hat{\partial}_i = q^{-4}\Lambda d + q^{-5}\lambda U\Delta \,. \tag{6.20}$$

This can be verified by a direct check. We note also that

$$\hat{d} = -\bar{d}\,. \tag{6.21}$$

To write the inverse map define $\hat{W} = g_{ij}\hat{\xi}^i X^j$. Using (6.19) for the $\hat{\xi}$'s one finds

$$U = \hat{W} + q\lambda L\hat{d}\,. \tag{6.22}$$

By conjugating (6.14) one finds that $\hat{W} = q^2\bar{W}$. This implies that U is real, $\bar{U} = U$. Then the inverse map reads

$$\xi^i = \hat{\Lambda}(\hat{\xi}^i + q^3\lambda X^i\hat{d} - q\lambda U\partial^i)\,, \tag{6.23}$$

where we use (6.22) to write U in terms of $\hat{\xi}$'s. For the exterior derivative we have

$$d = q^4\hat{\Lambda}\hat{d} - q^5\lambda U\hat{\Delta}\,. \tag{6.24}$$

Again we note that the mappings (6.19) and (6.23) do not spoil the global q-Lorentz covariance.

The Laplacians Δ and $\hat{\Delta}$ defined above are not real as seen in (6.3). However relations (6.12) allow the construction of the real Laplacian. Substituting them into (6.2) one finds

$$\hat{\Delta} = q^{-4}\Lambda^{-1}\Delta\,. \tag{6.25}$$

Therefore using (6.3) we obtain

$$\bar{\Delta} = q^6\Lambda^{-1}\Delta\,. \tag{6.26}$$

Now define the real Laplacian to be

$$\Delta_R = q^{-2}\Lambda^{-\frac{1}{2}}\Delta = q^2\Lambda^{\frac{1}{2}}\hat{\Delta}\,. \tag{6.27}$$

Using relations (6.8), (6.9) one checks that this Laplacian is indeed real, $\overline{\Delta_R} = \Delta_R$. It q-commutes with the derivatives:

$$\Delta_R\partial_i = q^{-1}\partial_i\Delta_R\,, \quad \Delta_R\hat{\partial}_i = q\hat{\partial}_i\Delta_R\,. \tag{6.28}$$

The properties of this Laplacian will be discussed elsewhere.

7. Hopf Structure of q-Derivatives

In this section we complete the Hopf structure of the q-derivatives. The coproduct is found by a heuristic method. Arguments are made for the validity of this coproduct. Then this coproduct is used to determine the counit and antipode.

In [8, 9] the coproduct for the Lorentz generators was found by considering the action of the generators on monomials in the spinors. The same could be done for the derivatives on monomials in the Minkowski coordinates. However the complexity of the action on coordinates (5.1) makes this a difficult task. Here we use a more heuristic approach. We make an ansatz for a coproduct of the form

$$\Delta(\partial_i) = \partial_i \otimes 1 + \mathcal{O}_i^{\,j} \otimes \partial_j \,, \tag{7.1}$$

where the operators $\mathcal{O}_i^{\,j}$ are made up of the Lorentz generators and the scaling operator Λ. An inspection of the derivative action shows that ∂_B and ∂_C have a simple form. For these derivatives it is not difficult to find combinations of the Lorentz generators for $\mathcal{O}_i^{\,j}$ which produce the correct action on coordinates. But there remains an undetermined power of Λ in the $\mathcal{O}_i^{\,j}$. It is also easy to evaluate ∂_B on the monomial B^n. This fixes the power of Λ. The coproduct for the remaining derivatives is then found using the algebra with the $SU_q(2)$ generators $T^{\pm}$. By this procedure the coproduct is found to be

$$\begin{aligned}
\Delta(\partial_A) &= \partial_A \otimes 1 + \Lambda^{\frac{1}{2}}(\tau^3)^{\frac{1}{2}}\tau^1 \otimes \partial_A + q^3\lambda^2\Lambda^{\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}T^- S^1 \otimes \partial_B \\
&\quad - \lambda\Lambda^{\frac{1}{2}}T^-\tau^1 \otimes \partial_C - q\lambda\Lambda^{\frac{1}{2}}S^1 \otimes \partial_D \,, \\
\Delta(\partial_B) &= \partial_B \otimes 1 + \Lambda^{\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}\sigma^2 \otimes \partial_B - q\lambda\Lambda^{\frac{1}{2}}T^2 \otimes \partial_C \,, \\
\Delta(\partial_C) &= \partial_C \otimes 1 + \Lambda^{\frac{1}{2}}\tau^1 \otimes \partial_C - q\lambda\Lambda^{\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}S^1 \otimes \partial_B \,, \\
\Delta(\partial_D) &= \partial_D \otimes 1 + \Lambda^{\frac{1}{2}}\sigma^2 \otimes \partial_D - q\lambda\Lambda^{\frac{1}{2}}(\tau^3)^{\frac{1}{2}}T^2 \otimes \partial_A \,, \\
&\quad - q^2\lambda\Lambda^{\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}T^-\sigma^2 \otimes \partial_B + q\lambda^2\Lambda^{\frac{1}{2}}T^- T^2 \otimes \partial_C \,.
\end{aligned} \tag{7.2}$$

This coproduct is a homomorphism of the entire Poincaré algebra and is coassociative.

The comultiplication was found by direct inspection. We know only that it is a homomorphism of the algebra. We now discuss the naturality of this comultiplication. In other words we wish to prove that the comultiplication is compatible with the action.

Proposition. *Let f and g be functions of A, B, C and D. Let $\Delta(\psi) = \sum_\alpha a_\alpha \otimes b_\alpha$, where ψ is any element of the q-Poincaré algebra. Then*

$$\psi(fg) = \sum_\alpha a_\alpha(f) b_\alpha(g) \,. \tag{7.3}$$

Sketch of Proof. 1. A straightforward calculation shows that Δ is coassociative. Explicitly, if

$$\Delta(a_\alpha) = \sum_\beta \lambda_{\alpha\beta} \otimes \phi_{\alpha\beta} \,, \quad \Delta(b_\alpha) = \sum_\gamma \mu_{\alpha\gamma} \otimes \nu_{\alpha\gamma} \,, \tag{7.4}$$

then

$$\sum_{\alpha\beta} \lambda_{\alpha\beta} \otimes \phi_{\alpha\beta} \otimes b_\alpha = \sum_{\alpha\gamma} a_\alpha \otimes \mu_{\alpha\gamma} \otimes \nu_{\alpha\gamma} \,. \tag{7.5}$$

2. Comparing with the action we conclude

$$\psi(X^i g) = \sum_\alpha a_\alpha(X^i) b_\alpha(g) \tag{7.6}$$

for all q-Minkowski coordinates X^i.

3. Induction in $\deg f$. For $\operatorname{def} f = 0$ the proposition obviously holds. By (7.6) we have

$$\psi(X^i fg) = \sum_\alpha a_\alpha(X^i) b_\alpha(fg) \,. \tag{7.7}$$

For $\psi = b_\alpha$ in (7.3) the statement holds by induction assumption for $\deg f = n$. Therefore

$$b_\alpha(fg) = \sum_\gamma \mu_{\alpha\gamma}(f) \nu_{\alpha\gamma}(g) \tag{7.8}$$

and

$$\psi(X^i fg) = \sum_{\alpha\gamma} a_\alpha(X^i) \mu_{\alpha\gamma}(f) \nu_{\alpha\gamma}(g) \,. \tag{7.9}$$

Using coassociativity (7.5) we can rewrite it in the form

$$\psi(X^i fg) = \sum_{\alpha\beta} \lambda_{\alpha\beta}(X^i) \phi_{\alpha\beta}(f) b_\alpha(g) \,. \tag{7.10}$$

Now using (7.6) with $\psi = a_\alpha$:

$$a_\alpha(X^i f) = \sum_\beta \lambda_{\alpha\beta}(X^i) \phi_{\alpha\beta}(f) \,, \tag{7.11}$$

we conclude that

$$\psi(X^i fg) = \sum_\alpha a_\alpha(X^i f) b_\alpha(g) \tag{7.12}$$

and (7.3) holds for $\deg f = n + 1$.

4. Induction in deg g. For $\deg f = 0$ and $\deg g = 1$ one easily sees that the statement holds. Above induction in $\deg f$ shows then that it holds for any f and $g = X^i$ (that is, $\deg g = 1$). Now since it holds for g we have

$$\psi(f X^i g) = \sum_\alpha a_\alpha(f X^i) b_\alpha(g) \,. \tag{7.13}$$

Using (7.3) with $\psi = a_\alpha$ we can write

$$a_\alpha(f X^i) = \sum_\beta \lambda_{\alpha\beta}(f) \phi_{\alpha\beta}(X^i) \,. \tag{7.14}$$

Now we use coassociativity again:

$$\psi(f X^i g) = \sum_{\alpha\beta} \lambda_{\alpha\beta}(f) \phi_{\alpha\beta}(X^i) b_\alpha(g) = \sum_{\alpha\gamma} a_\alpha(f) \mu_{\alpha\gamma}(X^i) \nu_{\alpha\gamma}(g) \,. \tag{7.15}$$

For $\psi = b_\alpha$ in (7.6) we already proved that

$$b_\alpha(X^i g) = \sum_\gamma \mu_{\alpha\gamma}(X^i)\nu_{\alpha\gamma}(g) \, . \tag{7.16}$$

Therefore

$$\psi(fX^i g) = \sum_\alpha a_\alpha(f) b_\alpha(X^i g) \, , \tag{7.17}$$

and the statement holds for $X^i g$. This finishes induction in deg g and the proof.

With this coproduct the counit and antipode are determined. The counit for all derivatives vanishes:

$$\varepsilon(\partial_i) = 0 \, . \tag{7.18}$$

The antipode is

$$\begin{aligned} S(\partial_A) &= -\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}(\sigma^2\partial_A + q\lambda^2 S^1 T^- \partial_B + \lambda\sigma^2 T^- \partial_C + q\lambda S^1 \partial_D) \, , \\ S(\partial_B) &= -\Lambda^{-\frac{1}{2}}(\tau^3)^{\frac{1}{2}}(\tau^1\partial_B + q^3\lambda T^2 \partial_C) \, , \\ S(\partial_C) &= -\Lambda^{-\frac{1}{2}}(\sigma^2\partial_C + q^{-1}\lambda S^1 \partial_B) \, , \\ S(\partial_D) &= -\Lambda^{-\frac{1}{2}}(\tau^1\partial_D + q\lambda T^2\partial_A + \lambda\tau^1 T^- \partial_B + q\lambda^2 T^2 T^- \partial_C) \, . \end{aligned} \tag{7.19}$$

In checking the antipode property ones uses the fact that $Z = 1$ in (4.4).

The coproduct for the derivatives includes the derivatives themselves, Lorentz generators and the scaling operator Λ. We note that Λ does not belong to the q-Lorentz algebra. This is seen by the fact that $\Lambda L = q^{-4} L\Lambda$, whereas the q-Lorentz generators commute with L. A similar effect already occurs for the q-derivatives in two dimensions: their coproduct includes the scaling operator which does not belong to $SL_q(2)$ [4]. The Hopf structure of the scaling operator is

$$\Delta(\Lambda) = \Lambda \otimes \Lambda \, , \quad \varepsilon(\Lambda) = 1 \, , \quad S(\Lambda) = \Lambda^{-1} \, . \tag{7.20}$$

This completes the Hopf structure of the Poincaré algebra.

A. Projector Decomposition of $\hat{R}$-Matrices

In this appendix we list the four projectors extracted from the two forms of the $\hat{R}$-matrix for the Lorentz group. They act on the tensor product of two coordinate spaces, so are 16×16 matrices. However, they are block diagonal and decompose into two 1-, two 4- and one 6-dimensional blocks. The bases for these blocks are labeled by pairs of coordinates, and are

$$\begin{aligned} &(1): (AA) \quad (4): (DA, CA, AD, AC) \\ &(1'): (BB) \quad (4'): (CB, DB, BC, BD) \\ &(6): (BA, DD, DC, CD, CC, AB). \end{aligned} \tag{A.1}$$

We write the projectors in blocks with these bases.

The symmetrizing projectors are P_T and P_S. P_T is the trace projector, and may be written in terms of the metric as

$$P^{ij}_{T\,kl} = \frac{1}{(q+q^{-1})^2} g^{ij} g_{kl} \,. \tag{A.2}$$

Explicitly it is

$$P_{T(1)} = P_{T(1')} = 0 \,, \quad P_{T(4)} = P_{T(4')} = 0 \,,$$

$$P_{T(6)} = \frac{1}{(q+q^{-1})^2} \begin{pmatrix} q^{-2} & q^{-1}\lambda & -q^{-2} & -q^{-2} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -q\lambda & 1 & 1 & 0 & -q^2 \\ -1 & -q\lambda & 1 & 1 & 0 & -q^2 \\ -q\lambda & -q^2\lambda^2 & q\lambda & q\lambda & 0 & -q^3\lambda \\ 1 & q\lambda & -1 & -1 & 0 & q^2 \end{pmatrix} \tag{A.3}$$

where $\lambda = q - q^{-1}$. The traceless part of the symmetrizer P_S has the form

$$P_{S(1)} = P_{S(1')} = 1 \,,$$

$$P_{S(4)} = \frac{1}{q^2+1} \begin{pmatrix} q^2 & 0 & q^2 & 0 \\ -q\lambda & q^2 & q^2\lambda^2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & q^2 & q^3\lambda & 1 \end{pmatrix},$$

$$P_{S(4')} = \frac{1}{q^2+1} \begin{pmatrix} 1 & q^3\lambda & q^2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & q^2\lambda^2 & q^2 & -q\lambda \\ 0 & q^2 & 0 & q^2 \end{pmatrix},$$

$$P_{S(6)} = \frac{1}{(q+q^{-1})^2}$$

$$\times \begin{pmatrix} q^2 & -q\lambda(2+q^{-2}) & 1 & 1 & 0 & 1 \\ 0 & (q+q^{-1})^2 & 0 & 0 & 0 & 0 \\ q^2 & q^3\lambda & 1 & 1 & 0 & 1 \\ q^2 & q^3\lambda & 1 & 1 & 0 & 1 \\ -q\lambda & -q^2\lambda^2 & -q^{-1}\lambda & -q^{-1}\lambda & (q+q^{-1})^2 & q\lambda(2+q^2) \\ 1 & q\lambda & q^{-2} & q^{-2} & 0 & q^{-2} \end{pmatrix}. \tag{A.4}$$

The selfdual and antiselfdual parts of the antisymmetrizer are P_+ and P_-:

$$P_{+(1)} = P_{+(1')} = 0\,,$$

$$P_{+(4)} = \frac{1}{q^2+1}\begin{pmatrix} 1 & 0 & -q^2 & 0 \\ q\lambda & 0 & -q^3\lambda & 0 \\ -1 & 0 & q^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_{+(4')} = \frac{1}{q^2+1}\begin{pmatrix} q^2 & -q^3\lambda & -q^2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & q\lambda & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_{+(6)} = \frac{1}{(q+q^{-1})^2}\begin{pmatrix} 1 & q\lambda & q^{-2} & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & q\lambda & q^{-2} & -1 & 0 & -1 \\ -q^2 & -q^3\lambda & -1 & q^2 & 0 & q^2 \\ q\lambda & q^2\lambda^2 & q^{-1}\lambda & -q\lambda & 0 & -q\lambda \\ -1 & -q\lambda & -q^{-2} & 1 & 0 & 1 \end{pmatrix}, \tag{A.5}$$

and

$$P_{-(1)} = P_{-(1')} = 0\,,$$

$$P_{-(4)} = \frac{1}{q^2+1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & q\lambda & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -q^2 & -q^3\lambda & q^2 \end{pmatrix},$$

$$P_{-(4')} = \frac{1}{q^2+1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^2 & 0 & -1 \\ 0 & -q^3\lambda & 0 & q\lambda \\ 0 & -q^2 & 0 & 1 \end{pmatrix},$$

$$P_{-(6)} = \frac{1}{(q+q^{-1})^2}\begin{pmatrix} 1 & q\lambda & -1 & q^{-2} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -q^2 & -q^3\lambda & q^2 & -1 & 0 & q^2 \\ 1 & q\lambda & -1 & q^{-2} & 0 & -1 \\ q\lambda & q^2\lambda^2 & -q\lambda & q^{-1}\lambda & 0 & -q\lambda \\ -1 & -q\lambda & 1 & -q^{-2} & 0 & 1 \end{pmatrix}. \tag{A.6}$$

Classically these reduce to the four usual projectors. Plugging $q = 1$ into the above expressions it may be verified that

$$P_{T}{}^{ij}{}_{kl} = \frac{1}{4} g^{ij} g_{kl} ,$$

$$P_{S}{}^{ij}{}_{kl} = \frac{1}{2}(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k) - \frac{1}{4} g^{ij} g_{kl} ,$$

$$P_{+}{}^{ij}{}_{kl} = \frac{1}{4}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) - \frac{i}{4} \varepsilon^{ij}{}_{kl} ,$$

$$P_{-}{}^{ij}{}_{kl} = \frac{1}{4}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) + \frac{i}{4} \varepsilon^{ij}{}_{kl} . \tag{A.7}$$

Here the ε-tensor is defined so that in the real basis $\varepsilon^{0123} = 1$.

B. q-Differentials

As discussed in Sect. 2 the first step in finding the derivative action is to find the q-relations between coordinates and differentials. These are given by the $\hat{R}$-matrix equation (2.13). In the (A, B, C, D) basis we have explicitly

$$\begin{aligned}
A\xi^A &= q^{-2}\xi^A A ,\\
A\xi^B &= \xi^B A + q\lambda\xi^D D - q^{-1}\lambda\xi^D C - q^{-1}\lambda\xi^C D + \lambda^2\xi^A B ,\\
A\xi^C &= \xi^C A + q\lambda\xi^A D - q^{-1}\lambda\xi^A C ,\\
A\xi^D &= q^{-2}\xi^D A - q^{-1}\lambda\xi^A D ,\\
\\
B\xi^A &= \xi^A B - q^{-1}\lambda\xi^D D ,\\
B\xi^B &= q^{-2}\xi^B B ,\\
B\xi^C &= q^{-2}\xi^C B + \lambda^2\xi^D B - q^{-1}\lambda\xi^B D ,\\
B\xi^D &= \xi^D B ,\\
C\xi^A &= q^{-2}\xi^A C + \lambda^2\xi^A D - q^{-1}\lambda\xi^D A ,\\
C\xi^B &= \xi^B C - q^{-1}\lambda\xi^C B + q\lambda\xi^D B ,\\
C\xi^C &= q^{-2}\xi^C C - q\lambda\xi^B A - q^2\lambda^2\xi^D D\\
&\qquad + \lambda^2\xi^D C + \lambda^2\xi^C D + q\lambda(1 - q\lambda)\xi^A B ,\\
C\xi^D &= \xi^D C - q\lambda\xi^A B ,\\
\\
D\xi^A &= \xi^A D ,\\
D\xi^B &= q^{-2}\xi^B D - q^{-1}\lambda\xi^D B ,\\
D\xi^C &= \xi^C D - q\lambda\xi^A B ,\\
D\xi^D &= q^{-2}\xi^D D .
\end{aligned} \tag{B.1}$$

We define the hatted differentials by

$$\hat{\xi}^A = \overline{\xi^B}, \quad \hat{\xi}^B = \overline{\xi^A}, \quad \hat{\xi}^C = \overline{\xi^C}, \quad \hat{\xi}^D = \overline{\xi^D} . \tag{B.2}$$

With these definitions, conjugation of the above q-relations results in

$$\begin{aligned}
A\hat{\xi}^A &= q^2 \hat{\xi}^A A , \\
A\hat{\xi}^B &= \hat{\xi}^B A + q\lambda \hat{\xi}^D D , \\
A\hat{\xi}^C &= q^2 \hat{\xi}^C A + q^3 \lambda \hat{\xi}^A D , \\
A\hat{\xi}^D &= \hat{\xi}^D A , \\
B\hat{\xi}^A &= \hat{\xi}^A B + q^{-1}\lambda \hat{\xi}^D C + q^{-1}\lambda \hat{\xi}^C D + \lambda^2 \hat{\xi}^B A - q\lambda(1 + q^{-1}\lambda)\hat{\xi}^D D , \\
B\hat{\xi}^B &= q^2 \hat{\xi}^B B , \\
B\hat{\xi}^C &= \hat{\xi}^C B - q\lambda \hat{\xi}^B D + q\lambda \hat{\xi}^B C + q^2 \lambda^2 \hat{\xi}^D B , \\
B\hat{\xi}^D &= q^2 \hat{\xi}^D B + q\lambda \hat{\xi}^B D , \\
C\hat{\xi}^A &= \hat{\xi}^A C + q\lambda \hat{\xi}^C A - q\lambda \hat{\xi}^D A + q^2 \lambda^2 \hat{\xi}^A D , \\
C\hat{\xi}^B &= q^2 \hat{\xi}^B C + q^3 \lambda \hat{\xi}^D B , \\
C\hat{\xi}^C &= q^2 \hat{\xi}^C C - q\lambda \hat{\xi}^B A + q^3 \lambda \hat{\xi}^A B - q^2 \lambda^2 \hat{\xi}^D D , \\
C\hat{\xi}^D &= \hat{\xi}^D C + q\lambda \hat{\xi}^B A + q^2 \lambda^2 \hat{\xi}^D D , \\
D\hat{\xi}^A &= q^2 \hat{\xi}^A D + q\lambda \hat{\xi}^D A , \\
D\hat{\xi}^B &= \hat{\xi}^B D , \\
D\hat{\xi}^C &= \hat{\xi}^C D + q\lambda \hat{\xi}^B A + q^2 \lambda^2 \hat{\xi}^D D , \\
D\hat{\xi}^D &= q^2 \hat{\xi}^D D .
\end{aligned} \tag{B.3}$$

These relations can be written in the compact form

$$X^i \hat{\xi}^j = \hat{R}_{\mathrm{II}}^{-1\,ij}{}_{kl} \hat{\xi}^k X^l . \tag{B.4}$$

Also we give the explicit form of $\xi\xi$ relations (2.12):

$$\begin{aligned}
(\xi^A)^2 &= (\xi^B)^2 = (\xi^D)^2 = 0 , \\
(\xi^C)^2 &= q\lambda \xi^B \xi^A .
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
\xi^A \xi^B &= -\xi^B \xi^A , & \xi^B \xi^C &= -q^{-2} \xi^C \xi^B - q\lambda \xi^D \xi^B , \\
\xi^A \xi^C &= -q^2 \xi^C \xi^A + q^3 \lambda \xi^D \xi^A , & \xi^B \xi^D &= -\xi^D \xi^B , \\
\xi^A \xi^D &= -\xi^D \xi^A , & \xi^C \xi^D &= -\xi^D \xi^C - q\lambda \xi^B \xi^A .
\end{aligned} \tag{B.6}$$

Using these relations along with the derivative algebra (5.2) one verifies that $d^2 = 0$. Conjugating (B.6) one finds commutation relations for the $\hat{\xi}$'s. Other relevant relations between the $\hat{\xi}$'s and operators discussed in Sect. 6 are listed in Appendix D.

C. Conjugate q-Derivatives

In this appendix we list some of the relations involving the conjugate derivatives. Their action on coordinates is determined by conjugating the action of the derivatives in (5.1). Hatted derivatives are then defined by normalizing so that $\hat{\partial}_i X^j = \delta_i^j + \cdots$. Writing

$$(\bar{\partial})_A = \overline{\partial_B}\,, \quad (\bar{\partial})_B = \overline{\partial_A}\,, \quad (\bar{\partial})_C = \overline{\partial_C}\,, \quad (\bar{\partial})_D = \overline{\partial_D}\,, \tag{C.1}$$

we get

$$\hat{\partial}_i = -q^{-4} g_{ki} g^{kj} (\bar{\partial})_j \tag{C.2}$$

or explicitly

$$\begin{aligned} \overline{\partial_A} &= -q^6 \hat{\partial}_B\,, \quad \overline{\partial_C} = -q^4 \hat{\partial}_C\,,\\ \overline{\partial_B} &= -q^2 \hat{\partial}_A\,, \quad \overline{\partial_D} = -q^4 \hat{\partial}_D\,. \end{aligned} \tag{C.3}$$

The action of these operators on the coordinates is given by the $\hat{R}$-matrix equation

$$\hat{\partial}_i X^j = \delta_i^j + \hat{R}_{\mathrm{II}}^{-1\,jk}{}_{il} X^l \hat{\partial}_k\,. \tag{C.4}$$

Explicitly we have

$$\begin{aligned}
\hat{\partial}_A A &= 1 + q^2 A\hat{\partial}_A + q^3 \lambda D\hat{\partial}_C\,,\\
\hat{\partial}_A B &= B\hat{\partial}_A\,,\\
\hat{\partial}_A C &= C\hat{\partial}_A + q^2\lambda^2 D\hat{\partial}_A + q^3\lambda B\hat{\partial}_C\,,\\
\hat{\partial}_A D &= q^2 D\hat{\partial}_A\,,\\[6pt]
\hat{\partial}_B A &= A\hat{\partial}_B\,,\\
\hat{\partial}_B B &= 1 + q^2 B\hat{\partial}_B - q\lambda D\hat{\partial}_C + q\lambda C\hat{\partial}_C + q\lambda D\hat{\partial}_D + \lambda^2 A\hat{\partial}_A\,,\\
\hat{\partial}_B C &= q^2 C\hat{\partial}_B - q\lambda A\hat{\partial}_C + q\lambda A\hat{\partial}_D\,,\\
\hat{\partial}_B D &= D\hat{\partial}_B + q\lambda A\hat{\partial}_C\,,\\[6pt]
\hat{\partial}_C A &= q^2 A\hat{\partial}_C\,,\\
\hat{\partial}_C B &= B\hat{\partial}_C + q^{-1}\lambda D\hat{\partial}_A\,,\\
\hat{\partial}_C C &= 1 + q^2 C\hat{\partial}_C + q\lambda A\hat{\partial}_A\,,\\
\hat{\partial}_C D &= D\hat{\partial}_C\,,\\[6pt]
\hat{\partial}_D A &= A\hat{\partial}_D + q\lambda D\hat{\partial}_B\,,\\
\hat{\partial}_D B &= q^2 B\hat{\partial}_D - q\lambda(1 + q^{-1}\lambda) D\hat{\partial}_A + q^2\lambda^2 B\hat{\partial}_C + q^{-1}\lambda C\hat{\partial}_A\,,\\
\hat{\partial}_D C &= C\hat{\partial}_D - q\lambda A\hat{\partial}_A + q^3\lambda B\hat{\partial}_B - q^2\lambda^2 D\hat{\partial}_C + q^2\lambda^2 D\hat{\partial}_D\,,\\
\hat{\partial}_D D &= 1 + q^2 D\hat{\partial}_D + q\lambda A\hat{\partial}_A + q^2\lambda^2 D\hat{\partial}_C\,.
\end{aligned} \tag{C.5}$$

These relations are consistent with the expressions for hatted derivatives in terms of unhatted ones (6.12) and the action of unhatted derivatives (5.1). Among themselves the $\hat{\partial}$'s satisfy the same algebra (5.2) as the ∂'s. This is compatible with the conjugation rules (C.3).

One can also find the algebra of hatted with unhatted derivatives. This is given by the $\hat{R}$-matrix equation

$$\partial_i \hat{\partial}_j = \hat{R}_{\mathrm{II}}^{-1}{}^{lk}{}_{ji} \hat{\partial}_k \partial_l \,. \tag{C.6}$$

This yields the explicit relations

$$\begin{aligned}
\partial_A \hat{\partial}_A &= q^2 \hat{\partial}_A \partial_A \,, \\
\partial_B \hat{\partial}_A &= \hat{\partial}_A \partial_B + q^3 \lambda \hat{\partial}_C \partial_C \,, \\
\partial_C \hat{\partial}_A &= \hat{\partial}_A \partial_C \,, \\
\partial_D \hat{\partial}_A &= q^2 \hat{\partial}_A \partial_D + q^2 \lambda^2 \hat{\partial}_A \partial_C + q^3 \lambda \hat{\partial}_C \partial_A \,, \\[1ex]
\partial_A \hat{\partial}_B &= \hat{\partial}_B \partial_A - q\lambda \hat{\partial}_C \partial_C + q\lambda \hat{\partial}_D \partial_C + q\lambda \hat{\partial}_C \partial_D + \lambda^2 \hat{\partial}_A \partial_B \,, \\
\partial_B \hat{\partial}_B &= q^2 \hat{\partial}_B \partial_B \,, \\
\partial_C \hat{\partial}_B &= q^2 \hat{\partial}_B \partial_C + q\lambda \hat{\partial}_C \partial_B \,, \\
\partial_D \hat{\partial}_B &= \hat{\partial}_B \partial_D - q\lambda \hat{\partial}_C \partial_B + q\lambda \hat{\partial}_D \partial_B \,, \\[1ex]
\partial_A \hat{\partial}_C &= q^2 \hat{\partial}_C \partial_A + q\lambda \hat{\partial}_A \partial_C \,, \\
\partial_B \hat{\partial}_C &= \hat{\partial}_C \partial_B \,, \\
\partial_C \hat{\partial}_C &= q^2 \hat{\partial}_C \partial_C \,, \\
\partial_D \hat{\partial}_C &= \hat{\partial}_C \partial_D + q^{-1} \lambda \hat{\partial}_A \partial_B \,, \\[1ex]
\partial_A \hat{\partial}_D &= \hat{\partial}_D \partial_A + q\lambda \hat{\partial}_A \partial_D - q\lambda \hat{\partial}_A \partial_C \,, \\
\partial_B \hat{\partial}_D &= q^2 \hat{\partial}_D \partial_B + q^2 \lambda^2 \hat{\partial}_C \partial_B + q^3 \lambda \hat{\partial}_B \partial_C \,, \\
\partial_C \hat{\partial}_D &= \hat{\partial}_D \partial_C + q^{-1} \lambda \hat{\partial}_A \partial_B \,, \\
\partial_D \hat{\partial}_D &= q^2 \hat{\partial}_D \partial_D + q^2 \lambda^2 \hat{\partial}_C \partial_D + q^2 \lambda^2 \hat{\partial}_D \partial_C \\
&\quad - q^2 \lambda^2 \hat{\partial}_C \partial_C + q\lambda \hat{\partial}_B \partial_A - q\lambda (q^{-1}\lambda + 1) \hat{\partial}_A \partial_B \,.
\end{aligned} \tag{C.7}$$

These relations may be verified using the expressions for the hatted derivatives in terms of unhatted ones (6.12) and the algebra of hatted derivatives (5.2).

Finally the algebra of the conjugate derivatives with the Lorentz generators is given by (5.3) and (5.4) with hatted derivatives replacing the unhatted ones.

D. Relations for Scalar Operators

In this section we list some relations involving the Lorentz scalar operators discussed earlier. These formulas are useful in checking the properties of the conjugate derivatives and differentials.

First, for the derivatives acting on the Minkowski length L we have

$$\partial_i L = q^{-2} L \partial_i + g_{ij} X^j ,$$
$$\hat{\partial}_i L = q^2 L \hat{\partial}_i + q^{-2} g_{ij} X^j . \tag{D.1}$$

The action of the Laplacians on the coordinates is

$$\Delta X^i = q^{-2} X^i \Delta + q^2 g^{ij} \partial_j ,$$
$$\hat{\Delta} X^i = q^2 X^i \hat{\Delta} + q^{-2} g^{ij} \hat{\partial}_j . \tag{D.2}$$

Acting on the Minkowski length the Laplacians give

$$\Delta L = q^{-4} L \Delta + q^{-2} E + (q^2 + 1) ,$$
$$\hat{\Delta} L = q^4 L \hat{\Delta} + \hat{E} + q^{-2}(q^{-2} + 1) . \tag{D.3}$$

The action of E and $\hat{E}$ on coordinates is

$$E X^i = q^{-2} X^i E + X^i + q\lambda L g^{ij} \partial_j ,$$
$$\hat{E} X^i = q^2 X^i \hat{E} + X^i - q\lambda L g^{ij} \hat{\partial}_j , \tag{D.4}$$

and on the length is

$$E L = q^{-2} L E + (q^2 + 1) L ,$$
$$\hat{E} L = q^2 L \hat{E} + (q^{-2} + 1) L . \tag{D.5}$$

The algebra of the derivatives with E and $\hat{E}$ is

$$\partial_i E = q^{-2} E \partial_i + \partial_i + q^{-1} \lambda g_{ij} X^j \Delta ,$$
$$\partial_i \hat{E} = \hat{E} \partial_i + \hat{\partial}_i . \tag{D.6}$$

Conjugation gives the algebra with hatted derivatives:

$$\hat{\partial}_i \hat{E} = q^2 \hat{E} \hat{\partial}_i + \hat{\partial}_i - q\lambda g_{ij} X^j \hat{\Delta} ,$$
$$\hat{\partial}_i E = E \hat{\partial}_i + \partial_i . \tag{D.7}$$

Now we turn to the relations including ξ's. First, with the scalars, L, E, Δ and Λ

$$L \xi^i = q^{-2} \xi^i L , \quad E \xi^i = \xi^i E , \quad \Delta \xi^i = q^2 \xi^i \Delta , \quad \Lambda \xi^i = \xi^i \Lambda . \tag{D.8}$$

The operator W has the following action on the coordinates:

$$W X^i = X^i W + q^{-1} \lambda \xi^i L . \tag{D.9}$$

With derivatives it obeys

$$\partial_i W = W \partial_i + g_{ij}(q^{-2} \xi^j + q\lambda X^j d - q^{-3} \lambda \xi^j E) . \tag{D.10}$$

The operator W has the following commutation relations with ξ^i:

$$W \xi^i = -q^{-2} \xi^i W . \tag{D.11}$$

The operators U and V commute with coordinates as

$$U X^i = X^i U , \quad V X^i = q^{-2} X^i V + \xi^i \Lambda + q\lambda U g^{ij} \partial_j . \tag{D.12}$$

Now we list several relations involving the exterior derivative d. With the derivatives d obeys

$$\partial_i d = q^2 d \partial_i - q^{-1} \lambda g_{ij} \xi^j \Delta \tag{D.13}$$

and with the scalar operators

$$d\Delta = q^{-2}\Delta d\,, \quad dL = W + Ld\,, \quad dW = -Wd\,,$$
$$dE = d + q^{-2}Ed + q^{-3}\lambda W\Delta\,. \tag{D.14}$$

The Laplacian Δ with these quantities W and U obeys

$$\Delta W = W\Delta + q^4 d\,, \quad \Delta U = U\Delta + \Lambda d\,. \tag{D.15}$$

Several relevant relations with $\hat{\xi}$'s are:

$$\begin{aligned} L\hat{\xi}^i &= q^2\hat{\xi}^i L\,, \\ \hat{W}X^i &= X^i\hat{W} - q\lambda L\hat{\xi}^i\,, \\ \hat{d}L &= q^{-2}\hat{W} + L\hat{d}\,. \end{aligned} \tag{D.16}$$

Finally we list some useful summation relations:

$$\begin{aligned} g^{ij}g_{ik}\partial_j X^k &= (q+q^{-1})^2 + q^{-4}E\,, \\ g_{ij}\xi^i EX^j &= q^{-2}WE + q^3\lambda Ld + W\,, \\ g_{ij}X^i dX^j &= (q^2+1)Ld + q^2 W\,, \\ g_{ij}\xi^i\xi^j &= 0\,. \end{aligned} \tag{D.17}$$

These are the relations needed in Sect. 6.

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