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Algebraic Structures and Eigenstates for Integrable Collective Field Theories*, **

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Abstract. Conditions for the construction of polynomial eigen-operators for the Hamiltonian of collective string field theories are explored. Such eigen-operators arise for only one monomial potential $v(x)=\mu x^2$ in the collective field theory. They form a w_∞-algebra isomorphic to the algebra of vertex operators in $2d$ gravity. Polynomial potentials of orders only strictly larger or smaller than 2 have no non-zero-energy polynomial eigen-operators. This analysis leads us to consider a particular potential $v(x)=\mu x^2+g/x^2$. A Lie algebra of polynomial eigen-operators is then constructed for this potential. It is a symmetric 2-index Lie algebra, also represented as a subalgebra of $U(sl(2))$.

1. Introduction

Matrix models, i.e. quantum mechanics models with a $N \times N$ matrix as dynamical variable, were originally introduced as an approach to non-perturbative aspects of gauge theories [large-N limit of $su(N)$] [BIPZ, Co]. It was recently realized that they could be viewed as a natural regularization of string theory (in space-dimension $\leqq 2$) and thereby allowed a non-perturbative approach to it. This approach turned out to be extremely fruitful and has recently seen a lot of activity [BK, DS, GM, GMi, GKN].

The collective field method [JS, DJe] was applied to 1-dimensional matrix models as a natural description of the dynamics of the singlet sector (eigenvalues of the matrix). The resulting field theory was then extensively studied [P, G, K]. Perturbative computations were achieved [DJR 1, 2] and found in agreement with results from other approaches [Mo, DK]. On the other hand, it was shown that the

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collective field theory was classically [AJ 1] and quantum mechanically [AJ 2] Liouville-integrable [Li, Ar], in the sense that an infinite number of commuting (local) functionals of the dynamical field $\alpha(x,t)$ could be constructed. Of crucial importance in this theory is the existence of a w_∞-algebra [AJ 1, 2; MPZ, MS, AS, DDMW]. It is given, both at the classical and quantum level, by momenta of powers of the field $\alpha(x,t)$; the w_∞ algebraic structure of these functionals being triggered by the $U(1)\times U(1)$ Kac-Moody algebra structure of the field $\alpha(x,t)$.

An infinite-dimensional Lie-algebra was seen to play the role of a spectrum-generating algebra for the particular potential $v(x)=\mu x^2$ [AJ 2]. It gave an infinite sequence of discrete states. In the collective approach they appear as composite operators. In the continuum, conformal theory language, they are the discrete higher modes of a string in $1+1$ dimension [Po, KPo]. The intertwining w_∞-algebra is given in terms of operator products in this language [Wi, KPo].

Exact eigenstates were also constructed for the constant background $[v(x)=0]$ [Je, No]. They are given by character polynomials of $su(N)$, in a way strongly reminiscent of the free-fermion, infinite Grassmannian approach of the Kyoto school [JM]. Such a result is not surprising, given the close relations between matrix models, collective field theory and fermion theory [GK].

It must be understood that Liouville-integrability as defined above does not guarantee the existence of action-angle variables, or more loosely, of variables linearizing the equations of motion (but not necessarily action-angle stricto sensu – see for instance [OP, Pe]). This is only true in mechanics (finite number of degrees of freedom) [Li, Ar]. The question is therefore legitimate, whether physically meaningful angle-type variables or operators also exist for the collective field theory. As mentioned, it is already known that an algebra of polynomial eigen-operators for the collective-field Hamiltonian exists for a potential $v(x)=\pm x^2$ [AJ 2], but this remains yet an isolated construction.

We want to address here algebraic aspects of the collective field theory. They are directly related to the question of complete integrability. "Complete Integrability" now means for us the existence of an infinite algebra of eigen-operators of the original Hamiltonian and not simply a hierarchy of commuting Hamiltonians. In the same (formal) way quantum integrable systems are characterized by the existence of the quantum group structure which in turn leads to an exact construction of the complete spectrum of quantum states [SFT, Dr, Ji, F]. We shall see in fact that the algebraic structures arising in our particular theories are linear, not quadratic, and one can therefore consider these algebras as the simplest spectrum-generating structures [Wi, KPo].

Of course, we must put restrictions on the type of operators we are considering, since one can formally construct any kind of eigen-operator for any given potential, as will appear in the derivation. First of all, we shall restrict ourselves to collective theories with polynomial potentials v belonging to $\mathbb{C}(x, x^{-1})$. We then ask that the eigen-operators be polynomial (or possibly rational) in the dynamical field $\alpha(x,t)$ and the variable x itself. The reasons for this choice are that:

1. the conserved Hamiltonians themselves have this form;
2. $\alpha(x,t)$ is to be identified with the tachyon field, and we wish in ulterior constructions to define an action of these operators on the tachyon Fock space. The finite powers of α certainly exist as well-defined operators. Any additional operator has to be separately constructed.
3. In particular, the physical meaning of an infinite series $\sim \sum c_n \alpha^n$ is anyway not quite clear: in fact, strictly speaking, this operator would not belong to the vector

space generated by the w_∞-algebra. Inclusion of such operators requires an extension of the original w_∞-framework.

4. It is consistent to ask for polynomial dependence in x since we are only considering potentials $v(x)$ in $\mathbb{C}(x, x^{-1})$.

This restriction will then naturally reflect on which potentials $v(x)$ do lead to integrable collective field theories.

The plan of our paper is as follows: We first restrict ourselves to monomial potentials in the matrix models. We show that only $v(x)=\pm x^2$ has polynomial eigen-operators with non-zero eigenvalues. The algebra of these operators is a w_∞-algebra. Its two indices are interpreted as the energy (=eigenvalue under adjoint action of Hamiltonian) and momentum P_ϕ of the eigen-operator. This operator P_ϕ describes a translation-like invariance of the equations of motion [MPZ], but the corresponding quantum number is not conserved under the Lie bracket. This algebra is isomorphic to the (+)-algebra of vertex operators constructed in [Wi, KPo].

We then consider general polynomial potentials. We show that potentials containing only monomials of orders larger than 3 or smaller than 1, do not have polynomial eigen-operators with non-zero energy.

From the previous analysis, we are lead to the construction, on algebraic grounds, of a polynomial potential with polynomial eigen-operators: $v(x)=\mu x^2+\frac{g}{x^2}$. This construction has some features of unicity, closely related to properties of particular elements in the w_∞-algebra, which makes believe that it will be a more difficult task to construct other integrable polynomial potentials, although it cannot, of course, yet be ruled out. We then establish the structure of the algebra of polynomial eigen-operators: it is a 2-indices symmetric Lie algebra. It is described in terms of the enveloping algebra of $sl(2)$ as: $\left\{(J_3)^n(J_+)^{\frac{m}{2}}(J_-)^{b/2},\right.$ $n\in\mathbb{N}$, $b\in\mathbb{N}$, $\left.-b\leqq m\leqq 3b\right\}$, where $sl(2)=\{J^+, J^-, J^3\}$; such a representation with three indices is actually redundant.

Before we begin our discussion, let us fix our notations and recall the essential features of collective string field theory. It is described by a Hamiltonian:

$$H=\int dx\{\tfrac{1}{2}\Pi_{,x}\phi(x)\Pi_{,x}-\tfrac{1}{6}\pi^2\phi^3+(v(x)-\mu)\phi\}\,. \tag{1.1}$$

The string field $\phi(x)$ is the continuum limit of the dynamical quantity

$$\phi(\{\lambda\})=\sum_{\lambda=1}^{N}\delta(x-\lambda_i(t)) \tag{1.2}$$

when $\lambda_i(t)$ are the eigenvalues of the matrix field $M(t)$, dynamical variable of the corresponding 1-dimensional hermitian matrix model.

$\Pi(x)$ is its canonical conjugate:

$$\{\Pi(x), \phi(y)\}=i\delta(x-y)\,. \tag{1.3}$$

$v(x)$ is the potential function of the matrix model; μ is the Fermi momentum, or equivalently a Lagrange multiplier implementing the normalization condition:

$$\int\phi(x)dx=N\,. \tag{1.4}$$

$\int dx$ is understood as $\oint dx$ acting on an analytic function, as it naturally arises when H is obtained from a canonical coadjoint construction such as described in [AJ 3]. It is therefore assumed that $\phi(x)$ is meromorphic in x. H, given in (1.1), is in fact the large-N limit of the collective Hamiltonian. Lower-order non-local terms also arise, and are described in [Je]. They will be ignored here, since we set ourselves in the $N \to \infty$ limit.

Introducing the $U(1) \times U(1)$ Kac-Moody current algebra:

$$\alpha_\pm(x) \equiv \partial_x \Pi \pm \pi\varphi(x); \qquad \{\alpha_\pm(x), \alpha_\pm(y)\} = \pm 2i\pi\delta'(x-y), \quad \{\alpha_+, \alpha_-\} = 0, \tag{1.5}$$

we define two classical w_∞-algebras:

$$h_m^n = \int \frac{\alpha_\pm^{m-n}}{m-n} x^{m-1} dx \tag{1.6}$$

with the well-known Poisson bracket relations [Ba 1, 2]:

$$\{h_{m_1}^{n_1}, h_{m_2}^{n_2}\} = 2i\pi\{(m_2-1)n_1 - (m_1-1)n_2\} h_{m_1+m_2-2}^{n_1+n_2}. \tag{1.7}$$

We assume in principle that $m-n \geqq 0$ and $m \geqq 1$, but the w_∞-algebra can be extended to negative powers using the definition of $\int dx$ as a contour integral $\oint dx$. We shall in particular allow $m \leqq 1$ in order to include potentials $v(x)$ with negative powers of x. A central charge $(1+m_1)\delta_{m_1+m_2+2}$ is then generated in (1.7).

The Hamiltonian H is now written as:

$$H = \int \frac{\alpha_+^3}{6} - \frac{a_-^3}{6} + (v(x) - \mu)(\alpha_+ - \alpha_-) dx, \tag{1.8}$$

an element of $w_\infty \oplus w_\infty$. The $+/-$ decoupling in both H and the w_∞-algebras allows us to consider solely from now on the $+$ (or $-$) part in the diagonalizing problem.

The quantum algebra reproduces exactly the classical one. We introduce the analogous H_m^n operators:

$$H_m^n = \int \frac{\alpha_\pm^{m-n}}{m-n} x^{m-1} dx.$$

Note that we have no normal-ordering convention. Introducing it would add central extensions and further linear deformations to the w_∞-algebra, turning it into a W_∞-algebra [PRS]. The non-normal-ordered operators, however, do not exhibit such deformations of the algebra, because in this case the reordering terms which generate such deformations exactly cancel by symmetry (see [AJ 2]).

Let us finally recall the form of the commuting Hamiltonians which guarantee the (weak) Liouville-integrability of (1.8).

Theorem 1.9. *The operators* $h^{(n)} \equiv \int dx \int d\alpha (\alpha^2 + 2v(x))^n$, $n \in \mathbb{N}$, *commute amongst themselves. In particular, H in* (1.8) *is* $h^{(1)}/2$.

Proof. It immediately follows from the Poisson structure (1.5) – after part-integration – for the classical quantities, and from the exactness of the identification between the classical Poisson algebra and the quantum Lie algebra, for the quantum operators. □

2. Monomial Potentials

We have restricted ourselves to polynomial eigen-operators [i.e. polynomial functionals of $\alpha(x,t)$]. They belong to the w_∞-algebra defined above, and can be expressed as:

$$\Theta = \sum_{p,q} C_p^q h_p^q, \tag{2.1}$$

p, q belonging to a finite set S of integers, and C_p^q being constant coefficients.

We shall now discuss in an exhaustive way the case of monomial potentials $v(x) = \frac{1}{2} g x^n$, $n \in \mathbb{N}^*$, $g \in \mathbb{R}$. The corresponding Hamiltonian reads:

$$H = \int \frac{\alpha_+^3}{6} + \frac{g}{2} x^{n} \alpha_+ + (\text{minus-term}). \tag{2.2}$$

The eigenvalue condition $[H, \Theta] = \varepsilon \Theta$ translates into a recursion relation

$$\frac{1}{4i\pi} \varepsilon C_p^q = 2p C_{p+1}^{q+2} + gn(q-p) C_{p-n+1}^{q-n}. \tag{2.3}$$

The study of this recursion relation leads us to the

Theorem 2.4. *Polynomial solutions of the eigenvalue equation* $[H, \Theta] = \varepsilon \Theta$ *with non-vanishing energy only exist for* $v(x) = \pm x^2$.

Proof. The recursion relation (2.3) relates 3 coefficients of the unknown operator Θ. These coefficients live on a 2-dimensional integer lattice (p, q) and take non-zero values only on a finite number of sites. For $\varepsilon \neq 0$, (1.3) gives the value of C_p^q as a linear combination of two coefficients sitting on lattice sites separated from (p, q), respectively, by lattice vectors $(1, 2)$ and $(n-1, n)$. For $n \neq 2$, these two vectors are non-colinear; one can therefore redefine the reference axis of the lattice in a way as to have $(p+1, q+2)$ and $(p-n+1, q-n)$ on the same axis, but not (p, q), namely by setting $\tilde{q} = (n+2)p - nq$.

Defining the new lattice-indexation as P (parallel to the lattice vector $(n+2, n)$), Q (perpendicular to this lattice vector), (2.3) now expresses values of coefficients $C(P, Q)$ as linear combination of $C(P', Q' \neq Q)$. Since only a finite number of C's are non-vanishing, there exists a value of Q beyond which all C vanish. By recursion, therefore, all C's on the other side of this limiting line also vanish; hence no solution exists to (2.3) with $\varepsilon \neq 0$. □

For $n = 2$, this argument is not valid, since all 3 coefficients are on the same line. This argument is not valid either when $\varepsilon = 0$. Indeed, the commuting hierarchy of Hamiltonians described by Theorem 1.9 gives us a set of polynomial eigen-operators for any monomial (even polynomial!) potential. However, in view of our original purpose in constructing such algebras of eigen-operators, having only operators which commute with the Hamiltonian is insufficient to define a quantum "integrable" system, and we do not intend to address here the question of finding all operators commuting with H, and their algebraic structure.

These conclusions are not modified by the inclusion of the central term $[h_p^{p-1}, h_q^{q-1}] = (p+1)\delta_{p+q+2}$ of the w_∞-algebra. This central term would simply *add* a supplementary equation to (2.3) defining the coefficient of $\mathbf{1} \equiv h_0^0$ in Θ, and not *modify* Eq. (2.3). Theorem 2.4 can therefore be immediately extended to negative-order monomials $v(x) = x^{-n}$.

We shall now describe the algebra of polynomial eigen-operators for $v(x)=\pm x^2$. Some features of this algebra were already described in [AJ 2].

The recursion relation becomes (reabsorbing $4i\pi$ into ε):

$$\varepsilon C_p^q = 2pC_{p+1}^{q+2} + 2(q-p)C_{p-1}^{q-2}. \tag{2.5}$$

We redefine

$$\begin{aligned} &p=K, \quad q=2+2K-N, \quad \text{leading to:} \\ &\varepsilon C_K^N = 2KC_{K+1}^N + 2(K+2-N)C_{K-1}^N. \end{aligned} \tag{2.6}$$

As expected, the recursion relation degenerates into decoupled 1-index relations for all values of N. Finiteness of the series (2.1) imposes that (2.6) be consistently truncated at 2 points, namely for $K \geqq K_{\max}$ and $K \leqq K_{\min}$. It follows from (2.6) that one has necessarily:

$$K_{\min}=1, \quad K_{\max}=N-3 \tag{2.7}$$

corresponding, respectively, to monomials in $\alpha \sim \int \frac{\alpha_+^{N-3}}{N-3}$ and $\int x^{N-4}\alpha_+$. Normalizing the coefficient of $\int \frac{\alpha_+^{N-3}}{N-3}$ to be 1, we obtain from (2.6) the first terms of the eigen-operator Θ as:

$$\Theta = H_1^{4-N} + \frac{\varepsilon}{2} H_2^{6-N} + \ldots . \tag{2.8}$$

Once N and ε are given, the operator Θ can be recursively constructed from (2.8) and (2.6).

The algebra of such eigen-operators can immediately be computed: Define $\Theta_{1,2} \equiv [\Theta_1, \Theta_2]$:

a) the energy is conserved by application of the Jacobi identity to (H, Θ_1, Θ_2), hence $\Theta_{1,2}$ is an eigen-operator with energy $\varepsilon_1+\varepsilon_2$.

b) From (2.8) and (2.6) it then follows that:

Proposition 2.9. *$\Theta(N,\varepsilon)$ form a Lie algebra defined to be:*

$$[\Theta(N_1,\varepsilon_1), \Theta(N_2,\varepsilon_2)] = \left\{\frac{\varepsilon_2}{2}(4-N_1) - \frac{\varepsilon_1}{2}(4-N_2)\right\} \Theta(N_1+N_2-6, \varepsilon_1+\varepsilon_2).$$

This algebraic relation is valid for any value of N and ε, independently of the finiteness of the formal series (2.8) giving Θ. We now implement the finiteness condition by first stating two propositions.

Proposition 2.10a. *C_K^N is given by a polynomial of order $K-1$ in ε.*

This is obvious from (2.6) and (2.8). □

Proposition 2.10b. *The truncation relation $C_K=0$ for $K=N-3$ is equivalent to a $(N-3)$-degree polynomial equation for ε.*

Indeed, rewriting (2.6) for $K=N-4, N-3$, gives:

- $\varepsilon C_{N-4} = 2(N-4)C_{N-3} - 4C_{N-5}$. This gives $C_{N-3}(\varepsilon)$ as a $(N-4)$ order polynomial.
- $\varepsilon C_{N-3} = -2C_{N-4}$. This is a consistency condition, written as a polynomial of order $N-3$ in ε. □

Hence for a fixed N, there exists $N-3$ values of ε such that the recursion relation (2.6) leads to a polynomial eigen-operator of eigenvalue ε.

We now explicitly construct these operators. Redefining the canonical dynamical variable $\alpha(x,t)$ as:

$$\tilde{\alpha}_{\pm}(x,t)=(\alpha(x,t)\pm x) \tag{2.11}$$

(the two signs are equally allowed and lead to two different sets of operators) induces a rewriting of the Hamiltonian in terms of canonically transformed generators $\tilde{H}(\tilde{\alpha})$ of the w_∞-algebra:

$$H=\tilde{H}_1^{-2}\pm 2\tilde{H}_2^0 \tag{2.12}$$

depending on the $\pm$ sign in (2.11).

The operator H_2^0 has the unique feature of stabilizing individually elements of w_∞ as $[H_2^0, H_n^m]\sim H_n^m$. It follows that generators $\tilde{H}_1^{-n}$ are natural eigen-operators of H with eigenvalues $\pm 2n$ ($n\geqq 0$ in order to have polynomial functionals of α). Since $\tilde{H}_{1(\pm)}^{-n}=\int\frac{(\alpha\pm x)^{n+1}}{n+1}dx$, comparison with (2.8) allows us to identify:

$$\tilde{H}_{1(\pm)}^{-n}=\Theta(N=n+4,\ \varepsilon=\pm 2n). \tag{2.13}$$

The algebraic relation (2.9) then leads to:

Proposition 2.14.

$$[\tilde{H}_{1(+)}^{(-n)},\tilde{H}_{1(-)}^{(-m)}]\equiv nm\Theta(n+m+2,2(n-m)).$$

This operator is therefore automatically polynomial in α. Since for each value of $N=n+m+2$ one has exactly $N-3=n+m-1$ allowed values of ε, and since $(n-m)$ takes precisely $(n+m-1)$ distinct values when $n+m$ is fixed and $(n\neq 0$, $m\neq 0)$, the set $\Theta(n+m+2,2(n-m))$ completely solves (2.3).

Finally, from (2.14) and (2.9), we end up with:

Theorem 2.15. *The algebra of polynomial eigen-operators for the potential $v(x)=-x^2$ is a w_∞-algebra defined by:*

- $$\Theta(n+m+2,2(n-m))\equiv B\left(\frac{n+m-2}{2},\frac{n-m}{2}\right)\equiv[\tilde{H}_{1(+)}^{(-n)},\tilde{H}_{1(-)}^{(-m)}],$$
- $$[B(J_1,m_1),B(J_2,m_2)]=(J_2m_1-J_1m_2)B(J_1+J_2-1,m_1+m_2).$$

(J,m) behave as angular momentum variables (J, j_3); indeed, it follows from the definition in (2.15) that $J\in\frac{1}{2}\mathbb{N}$ and $m=-J,-J+1\ldots+J$. The operators $B(J,m)$ are the quantum, continuum version of the "angle" variables used in [Pe] to solve exactly the Calogero-Perelomov integrable mechanics problem [Ca, OP]. A direct proof of the equivalence of the collective field theory at $v=x^2$ and the Calogero model is indeed given in [AJ 1].

This algebra is remarkably similar to the algebra of $(+)$-type vertex operators for 2-dimensional gravity introduced in [KPo]. The similitude extends beyond the formal identification of the Lie brackets, once we describe the physical meaning of the two indices J and m in (2.15). m is clearly the energy eigenvalue for the operator $B(J,M)$. Interpretation of J requires a first

Lemma 2.16. *The classical equations of motion induced by the Hamiltonian H: $\alpha_{,t}=x-\alpha\alpha_{,x}$ have a translation-like invariance: $\delta\alpha=-x\partial_x\alpha+\alpha$.*

The proof of this lemma, formulated in [MPZ], is immediate. This translation-like invariance acts on the classical w_∞-algebra as follows:

Proposition 2.17.

$$\delta \cdot \int x^{m-1}\alpha^{m-n} = (2m-n)\int x^{m-1}\alpha^{m-n}$$

(obviously follows from partial integration once (2.16) is plugged into h_m^n). Hence one can formally define the associated "quantum" operator by its action on H_m^n, namely

Definition 2.18.

$$P_\phi \cdot H_m^n = (2m-n)H_m^n .$$

It must be emphasized that P_ϕ is *not* generated by the action of an operator inside w_∞, since its eigenvalues are not conserved under the Lie-bracket of w_∞. In fact, P_ϕ loses 4 units under the Lie bracket. Although P_ϕ is not an internal morphism of the algebra, it is not surprising that the eigen-operators of the Hamiltonian (which classically induces the equations of motion, and therefore generates a flow commuting with the P_ϕ-flow (2.16)) are also eigen-operators of P_ϕ.

Proposition 2.19.

$$P_\phi \cdot B(J,m) = 2(J+1)B(J,m) .$$

The quantity which is really conserved under the Lie bracket, however, is $P_\phi - 4 = 2(J-1)$. This is precisely twice the Liouville energy of the vertex operators in [KPo]. Since the Liouville mode ϕ in 2-dimensional gravity can be understood as a space-like variable, the Liouville energy is also the eigenvalue of a space-translation-like operator.

We therefore see a one-to-one equivalence between the collective theory w_∞-algebra (2.15) and the vertex operator algebra for the Liouville-2d gravity formulation. Let us finally mention that it is proposed to consider this w_∞-algebra structure to be the fundamental object in a first-principle approach to string theory [Wi, AJ 3].

3. Polynomial Potentials. General Case

The problem of classifying all integrable polynomial potentials is much more involved. The recursion relation generalizing (2.3) now contains as many terms as the potential has many monomials, but simple convexity-type arguments such as the one used to discard all but x^2-monomials, are available for particular classes of polynomial potentials.

We now prove the following

Theorem 3.1. *Polynomial potentials of the form* $v(x) = \sum_{k>2} a_k x^k$, *and* $v(x) = \sum_{k<2} b_k x^k$, *only have zero-energy polynomial eigen-operators.*

Proof. For a general polynomial (allowing negative powers of x) potentials of the form $v(x) = \sum_{k\in S} c_k x^k$, k belonging to a finite set S in $\mathbb{Z}$, the eigenvalue condition

applied to an operator defined by (2.1) reads:

$$\varepsilon C_p^q = 2pC_{p+4}^{q+2} + \sum c_k \cdot k(q-p)C_{p-k+1}^{q-k} \tag{3.2}$$

generalizing straightforwardly (2.3). When $\varepsilon \neq 0$, this gives the value of C_p^q at a site (p, q) of the lattice $\mathbb{Z} \times \mathbb{Z}$ as a well-defined linear combination of coefficients located at sites translated by the lattice vectors $(1, 2)$ and $(-k+1, -k)$ for all $k \in S$.

If $v(x)$ contains only monomials of order $k > 2$ or monomials of order $k < 2$, these lattice vectors are included in a sector of angle smaller than π, limited by the most external vectors $(1, 2)$ and $(-2, -3)$ (for $k > 2$) and $(1, 2)$ and $(0, -1)$ (for $k < 2$). We now shift the origin of the lattice to (p, q).

For $k > 2$, we define two half-planes with boundary $5p - 3q = -1$. The new origin of the lattice lies on the strictly positive side $(5p - 3q + 1 > 0)$, the extremity of all translation vectors (i.e. the lattice points defining the value of C at the origin point) lie on the negative side $(5p - 3q + 1 = -2k + 5 \leqq 0)$. Defining now a relabeling of the lattice points, as $\tilde{Q} = 5p - 3q$, we have shown that (3.2) gives a linear, well-defined for $\varepsilon \neq 0$, relation between the coefficient C at a given $(p, \tilde{Q}_0)$ and coefficients C with $\tilde{Q} < \tilde{Q}_0$.

For a polynomial eigen-operator having only a finite number of non-vanishing coefficients in (2.1), there exists a $\tilde{Q}_0$ such that all C vanish for $Q < \tilde{Q}_0$, and one C at least is non-zero for $Q = \tilde{Q}_0$. However, we see that $C(\tilde{Q}_0) = \sum q_n C(Q < \tilde{Q}_0) = 0$. Hence no polynomial eigen-operator exists with $\varepsilon \neq 0$.

For $k < 2$, the demonstration runs on similar lines: the half-plane boundary is here defined by the equation $3p - q - 1 = 0$. □

The theorem does not preclude zero-energy eigen-operators. Indeed, Theorem 1.9 guarantees the existence of a class of such operators for any potential $v(x)$, namely the hierarchy Hamiltonians $\int d\alpha \int dx(\alpha^2 + v(x))^n$.

If $v(x)$ contains a term x^2, the two translation vectors $(1, 2)$ and $(-k+1, -k)$, $k = 2$, become colinear. The translation sector is flat, and the convexity argument allowing the introduction of the new index $\tilde{Q}$ is not valid: the origin of the shifted lattice now *also* lies on the boundary of the half-plane defined by the externmost vectors $(1, 2)$ and $(-1, -2)$.

Finally, if $v(x)$ contains both terms of order $k > 2$ and $k < 2$, the translation-vector sector has an angle $\theta > \pi$ and the convexity argument collapses as well, the origin of the shifted lattice being now *inside* the negative half-plane.

The general study of these potentials, leading to non-convex recursion relations, is certainly much more involved since we lack the previous simple geometric arguments. We are, however, able to construct at least one explicit realization of a finite non-convex recursion relation, using a restricting hypothesis which will considerably simplify the study.

4. An Integrable Potential: $v(x) = \mu x^2 + \frac{g}{x^2}$

In order to obtain an integrable potential, we have to break from the structures of $v(x)$ described in Theorem 3.1. We now describe what appears to be the most natural construction of $v(x)$ on such lines.

1. Construction of the Potential. The mildest assumption in order to get an integrable potential is to allow an x^2-term. We accordingly formulate:

Hypothesis 1. The potential $v(x)$ contains a μx^2 term.

This makes it possible, using the redefinition $\alpha \to (\alpha \pm i\sqrt{\mu}x)$, to rewrite

$$\tilde{H} = \tilde{H}_1^{-2} \pm i\sqrt{\mu}\tilde{H}_2^0 + (\bar{v}\text{-term}), \tag{4.1}$$

where ($\bar{v}$-term) means all remaining linear terms in α induced by $\int (v - \mu x^2)\alpha dx$. In this way, since H_2^0 is the stabilizing element of the w_∞-algebra, the general recursion relation for the coefficients C_p^q in (2.1) loses one term.

Hypothesis 2. There exists polynomial eigen-operators diagonalizing simultaneously $(\tilde{H}_1^{-2} + (\bar{v}\text{-term}))$ and H_2^0.

Although H_2^0 is a stabilizing term, this is a rather strong restriction, but it will ultimately considerably simplify the problem. Indeed, in many similar algebraic problems, the best way of diagonalizing a Hamiltonian is in fact to diagonalize a family of Hamiltonians, and the quantum inverse scattering method [SFT, F] relies precisely on such a formulation. In this particular case, we know that:

Proposition 4.2. *H_2^0 is diagonalized by linear combinations of operators in w_∞ with the same n-index* (obvious from (1.6)).

Proposition 4.3. *The operator* $(\tilde{H}_1^{-2} + \bar{v}\text{-term})$ *has polynomial eigen-operators* $h_\pm^{(n)}$,

$$h_\pm^{(n)} = \int dx \int d\tilde{\alpha}(\tilde{\alpha}_\pm^2 + \bar{v}(x))^n .$$

This is immediately obtained from the fact that this operator is exactly the collective field theory Hamiltonian for a potential $\bar{v} = (v - \mu x^2)$, and therefore, there exists an infinite set of commuting Hamiltonians according to Theorem 1.9.

Hypothesis 3. These Hamiltonians also diagonalize H_2^0.

This hypothesis, although again restrictive, is fortunately not empty and will provide us with a (unique) example of integrable potential.

Proposition 4.4. *The potential $\bar{v}(x) = g/x^2$ is the only one which fulfills Hypothesis 3.*

Proof. In order for the Hamiltonians $h^{(n)}$ to diagonalize H_2^0, they must be expanded as elements of the w_∞-algebra (1.6) with the same n-index. Since n is equal, up to a constant, to (degree of x) minus (degree of α), it follows that $\bar{v}(x)$ must be a monomial of order -2 to fulfill Hypothesis 3. □

The Hamiltonians $h_\pm^{(n)}$ have an eigenvalue $\pm\sqrt{\mu}2n\pi$ under the complete Hamiltonian for $v = \mu x^2 + g/x^2$. We shall usually disregard the factor $2\pi\sqrt{\mu}$ in the energy, except when explicitly required for need of a demonstration.

The use of the operators $h^{(n)}$ in the last hypothesis is actually almost unavoidable. If we do not want to solve directly the recursion relation (3.2) for the potential $\bar{v}(x)$, we have to use the eigen-operators known originally from Theorem 1.9. Moreover, in the (in principle) simplest case when $\bar{v}(x)$ is a monomial $\neq x^2$, Theorem 2.4 precludes the existence of *any* other diagonalizing polynomial operator. So does Theorem 3.1 when $\bar{v}(x)$ is a polynomial of order 3 and more, or 1 and less.

We see therefore that any deviation from this set of three hypotheses leads us necessarily to directly tackling the general, non-convex recursion relation (3.2) without any further help from the algebraic structure w_∞. In this sense, the potential $v(x) = \mu x^2 + \dfrac{g}{x^2}$ has unique features.

We now describe a naturally generated algebra of polynomial eigen-operators for $v(x)=\mu x^2+g/x^2$. Although it is not proved to be the full algebra of such operators, it exhibits nevertheless interesting structures which makes it worthy of investigation.

2. Algebra Associated to $v=\mu x^2+g/x^2$. The algebra of eigen-operators which we are going to construct is formally obtained by taking successive commutators of the Hamiltonians $h_{\pm}^{(n)}$ with themselves and with the successively generated operators. Our problem is to describe explicitly what this enveloping algebra is. We shall consider for the moment that the w_∞-algebra of generators $\oint x^{m-1}\alpha^{m-n}$ has no central terms. We have previously mentioned that such terms actually appear in commutators $[h_m^{m-1}, h_n^{n-1}]$ when $n+m=-2$, and we shall study their effect later.

We now define candidate eigen-operators as polynomials in α of the form:

$$B=\sum_{n=1}^{N}\int dx f_n(x)\frac{\alpha^n}{n}. \tag{4.5}$$

The eigenvalue condition $[H,B]=\varepsilon B$ is equivalent to the recursion relation:

$$2n\left(-\mu x^2+\frac{g}{x^2}\right)f_{n+1}=\varepsilon x f_n-2x\partial_x f_{n-1}. \tag{4.6}$$

This relation is simply the identification of the functional coefficients of α^n in the eigenvalue condition. The following proposition is now obvious:

Proposition 4.7. *The giving of f_1 and ε is necessary and sufficient to define the operator B.*

This now helps us to write the first orders in α^n of B:

$$B(\varepsilon)=\int dx f_1(x)\alpha(x)+\varepsilon\int dx\cdot x\frac{f_1(x)}{-\mu x^2+g/x^2}\frac{\alpha^2(x)}{2}+\ldots. \tag{4.8}$$

We now assume a particular form for $f_1(x)$. We already have an example given by the Hamiltonians $h_{\pm}^{(n)}$ in Proposition 4.3, for which:

$$f_1[h_{\pm}^{(n)}]=(-\mu x^2+g/x^2)^n. \tag{4.9}$$

In fact, the most general form of f_1 for eigen-operators obtained from $h_{\pm}^{(n)}$ is:

$$f_1=(-\mu x^2+g/x^2)^b(\mu x^2+g/x^2)^a;\qquad a,b\in\mathbb{N}. \tag{4.10}$$

This follows from

Proposition 4.11. *The eigen-operators $B^{(a,b)}(\varepsilon)$ defined under Proposition 4.7 by the energy ε and an initial function f_1 of the form* (4.10)*, form a closed Lie algebra. This algebra reads:*

$$[B^{(a_1,b_1)}(\varepsilon_1), B^{(a_2,b_1)}(\varepsilon_2)]=-(a_2\varepsilon_1-a_1\varepsilon_2)B^{(a_1+a_2-1,b_1+b_2)}(\varepsilon_1+\varepsilon_2)$$
$$-(b_2\varepsilon_1-b_1\varepsilon_2)B^{(a_1+a_2+1,b_1+b_1-2)}(\varepsilon_1+\varepsilon_2).$$

Proof. First of all, it follows from applying the Jacobi identity to the operators H, B_1, and B_2, that the l.h.s. is an eigen-operator with eigenvalue $\varepsilon_1+\varepsilon_2$.

Now in order to apply Proposition 4.7, we must compute the linear term in α on the l.h.s. of (4.10). For two general operators of the form (4.5), respectively, defined as:

$$B_1 = \int f_1 \alpha + f_2 \frac{\alpha^2}{2} + \ldots, \qquad B_2 = \int g_1 \alpha + g_2 \frac{\alpha^2}{2} + \ldots, \tag{4.12}$$

the commutation relation (1.5) for α implies that the linear term of $[B_1, B_2]$ reads:

$$(-\partial_x f_1 \cdot g_2' + \partial_x g_1 \cdot f_2'). \tag{4.13}$$

Inserting now (4.10) and (4.8) into (4.13) and using the fact that $x\partial_x(\mu x^2 \pm g/x^2) = \mu x^2 \mp g/x^2$ leads us immediately to (4.11). □

We now need to establish the set of values for a, b, and ε obtained by the repeated action of $h_{\pm}^{(n)} \equiv B^{(0,n)}(\pm n)$. In the first place, commutators $[h_+, h_-]$ and $[h_-, h_-]$ obviously vanish since h_+ and h_- are canonical transformations of original commuting Hamiltonians for $g/x^2 = v(x)$. We then prove

Lemma 4.14.

$$\{[B^{(0,n_1)}(+n_1), B^{(0,n_2)}(-n_2)],\ n_1, n_2 \in \mathbb{N}\}$$
$$\equiv \{B^{(1,b)}(\varepsilon),\ b \in \mathbb{N},\ \varepsilon \in \mathbb{Z},\ |\varepsilon| \leqq b,\ \varepsilon = b, b-2 \ldots -b\}.$$

Proof. From (4.11), these commutators only generate $B^{(1,n_1+n_2-2)}(n_1-n_2)$ for n_1, n_2 strictly positive integers. For a fixed $b = n_1 + n_2 - 2 \geqq 0$, n_1 can take all values from 1 to $b+1$ and respectively, n_2 goes from $b+1$ to 1. Hence $\varepsilon = n_1 - n_2$ goes from b to $b-2\ldots$ to $-b$. □

This leads us to consider the set $Q = \{(b, \varepsilon) \in \mathbb{Z} \times \mathbb{N},\ b \geqq |\varepsilon|,\ \varepsilon = b,\ b-2 \ldots -b\}$. It is a square sublattice of $\mathbb{Z} \times \mathbb{Z}$, limited by $\varepsilon = \pm b$ with lattice spacing 2. We now prove

Lemma 4.15. *For any $a_1, a_2 \in \mathbb{N}$, the commutator $[B^{(a_1,b_1)}(\varepsilon_1), B^{(a_2,b_2)}(\varepsilon_2)]$, where (b_1, ε_1) and (b_2, ε_2) belong to Q, yields only B-operators with (b, ε) inside Q.*

Proof. Consider the commutation relation (4.11). Such a commutator yields two terms:

1) One has $a = a_1 + a_2 - 1$; $(b, \varepsilon) = (b_1, \varepsilon_1) + (b_2, \varepsilon_2)$. The lattice Q is closed under addition in $\mathbb{Z} \times \mathbb{Z}$ since:

$$|\varepsilon_1 + \varepsilon_2| \leqq |\varepsilon_1| + |\varepsilon_2| \leqq b_1 + b_2, \qquad \forall b_1, b_2, \varepsilon_1, \varepsilon_2 \text{ in } Q.$$

Hence the first term in the commutator is inside Q.

2) The other term has $a = a_1 + a_2 + 1$; $(b, \varepsilon) = (b_1, \varepsilon_1) + (b_2 - 2, \varepsilon_2)$. The closure argument applies here unless $\varepsilon_2 = \pm b_2$, and since one also has $(b, \varepsilon) = (b_1 - 2, \varepsilon_1) + (b_2, \varepsilon_2)$, it again applies unless $\varepsilon_1 = \pm b_1$. Two different cases are yet to be discussed:

a) $\varepsilon_1 = \pm b_1$, $\varepsilon_2 = \pm b_2$: in this case the coefficient in front of the term vanishes (see (4.11)).

b) $\varepsilon_1 = \pm b_1$, $\varepsilon_2 = \pm b_2$. One can then rewrite, respectively:

$$\text{either} \quad (b, \varepsilon) = (b_1, b_1 - 2) + (b_2 - 2, -b_2 + 2) \in Q \quad \text{by closure property},$$

$$\text{or} \quad (b, \varepsilon) = (b_1, -b_1 + 2) + (b_2 - 2, b_2 - 2) \in Q \quad \text{by closure property}.$$

This demonstration holds unless $b_1 = 0 = \varepsilon_1$; then the coefficient in (4.11) vanishes anyway. □

The natural counterpart of (4.15) is:

Lemma 4.16. *The repeated adjoint action of* $B^{(0,b)}(\varepsilon)$, $(b,\varepsilon)\in Q$, *generates all* $\{B^{(a,b)}(\varepsilon),\ a\in\mathbb{N},(b,\varepsilon)\in Q\}$.

Proof. Assume we have constructed $B^{(0,b)}(\varepsilon)$, $(b,\varepsilon)\in Q$ (we shall soon prove this assumption). Lemma 4.16 now follows by recursion:
1) $a=0$ is true by assumption.
2) Once (4.16) is valid for $a\in\mathbb{N}$ up to a_0, we act by $B^{(0,b_0)}(\varepsilon_0)$ on $B^{(1,b_1)}(\varepsilon_1)$. From (4.11) we get two terms:
a) $a_0\varepsilon\cdot B^{(a_0-1,b_0+b_1)}(\varepsilon_1+\varepsilon_0)$: already constructed by recursion hypothesis.
b) $(b_1\varepsilon_0-b_0\varepsilon_1)B^{(a_0+1,b_0+b_1-2)}(\varepsilon_1+\varepsilon_0)\simeq B^{(a_0+1,b)}(\varepsilon)$.

To get any operator with non-zero energy, use $\varepsilon_0=0$, $b_0=b+2-|\varepsilon|\neq 0$ (automatically since $(b,\varepsilon)\in Q$), $b_1=|\varepsilon_1|=|\varepsilon|\neq 0$ and thus $(b_1\varepsilon_0-b_0\varepsilon)\neq 0$. To get $\varepsilon=0$, use $b_0=b_1=\dfrac{b}{2}+1$ (for $\varepsilon=0$, b is necessarily even-positive, in Q) and $\varepsilon_0=-\varepsilon_1=1$. □

We have left as an assumption the existence of $B^{(0,b)}(\varepsilon)$. We now prove:

Lemma 4.17. $B^{(0,b)}(\varepsilon)$, $(b,\varepsilon)\in Q$, $\varepsilon\neq 0$, *are polynomial eigen-operators generated by* $\{B^{(0,n)}(\pm n)\}$.

Proof. Lemma 4.14 proved directly the existence of $B^{(a=1)}$ for (b,ε) in Q. We now apply again $B^{(0,1)}(\pm 1)$; according to (4.11):

$$[B^{(0,1)}(\pm 1),B^{(1,b)}(\varepsilon)]=\pm 1B^{(0,b+1)}(\varepsilon\pm 1)+(\varepsilon-b(\pm 1))B^{(2,b+1-2)}(\varepsilon\pm 1). \tag{4.18}$$

Hence we explicitly construct:

$$-B^{(0,b+1)}(\varepsilon+1)+(\varepsilon-b)B^{(2,b-1)}(\varepsilon+1)\equiv B^{+}(\varepsilon),$$
$$-B^{(0,b+1)}(\varepsilon-1)+(-\varepsilon-b)B^{(2,b-1)}(\varepsilon-1)\equiv B^{-}(\varepsilon).$$

For $\varepsilon\neq b$, let us consider $B^{+}(\varepsilon)-B^{-}(\varepsilon+2)$. We get,

$$B^{+}(\varepsilon)-B^{-}B^{-}(\varepsilon+2)=(\varepsilon+1)B^{(2,b-1)}(\varepsilon+1), \tag{4.19}$$

$$B^{(0,b+1)}(\varepsilon+1)=-B^{+}(\varepsilon)+\frac{\varepsilon-b}{\varepsilon+1}(B^{+}(\varepsilon)-B^{-}(\varepsilon+2)). \tag{4.20}$$

Hence for all values of $\varepsilon+1$ except $\pm(b+1)$ and 0, $B^{(0,b+1)}(\varepsilon+1)$ is obtained as a linear combination of successive commutators of $B^{(0,n)}(\pm n)$. Finally, we have identified in (4.8) the Hamiltonians $h_{\pm}^{(n)}$ as the eigen-operators $B^{(0,n)}(\pm n)$. □

We now prove directly the

Lemma 4.21. *Operators* $B^{(0,n)}(0)$ *are polynomial eigen-operators of* H.

Proof. The recursion relation (4.6) for zero-eigenvalued operators reads:

$$2n\left(-\mu x^2+\frac{g}{x^2}\right)f_{n+1}=-2x\partial_x f_{n-1}. \tag{4.22}$$

Application of the operator $\left[\frac{x}{-\mu x^2+g/x^2}\partial_x\right]$ on a function of the form $\left(-\mu x^2+\frac{g}{x^2}\right)^b \left(\mu x^2+\frac{g}{x^2}\right)^a$ leads to $b\left(-\mu x^2+\frac{g}{x^2}\right)^{b-2}\left(\mu x^2+\frac{g}{x^2}\right)^a + a\left(-\mu x^2+\frac{g}{x^2}\right)^b\left(\mu x^2+\frac{g}{x^2}\right)^{a-1}$. Hence it decreases the global degree by 1 without creating negative degrees if b initially is even. Successive applications of (4.22) on an initial function f_1 with $b=0$ leads to 0 after $(a+1)$ steps. Therefore, the recursion relation (4.22) leads to a polynomial operator $B^{(0,n)}(0)$. □

We can now state the major

Theorem 4.23. *The operators* $\{B^{(a,b)}(\varepsilon);\ a\in\mathbb{N};\ (b,\varepsilon)\in Q\}$ *form a closed algebra of polynomial eigen-operators for the potential* $\mu x^2+g/x^2$. *The algebra structure is:*

$$[B^{(a_1,b_1)}(\varepsilon_1), B^{(a_2,b_2)}(\varepsilon_2)] = (a_1\varepsilon_2 - a_2\varepsilon_1)B^{(a_1+a_2-1,\,b_1+b_2)}(\varepsilon_1+\varepsilon_2) + (b_1\varepsilon_2 - b_2\varepsilon_1)B^{(a_1+a_2+1,\,b_1+b_2-2)}(\varepsilon_1+\varepsilon_2).$$

Proof. This follows obviously from (4.17) and (4.21), when $a_1=0$; and from (4.16) recursively, when $a_1\neq 0$. □

The commuting Hamiltonians constructed in Theorem 1.9 are here $B^{(n,0)}(0)$. Now the 3-index representation is in fact very much redundant, although it was maintained until now for practical purposes. From (4.9) and (4.7) one has in fact

Proposition 4.24. *For all allowed values of* a, b, ε,

$$B^{(a+2,b)}(\varepsilon) - B^{(a,b+2)}(\varepsilon) = 4\mu g B^{(a,b)}(\varepsilon).$$

It follows that the values of a are actually reduced to $a=0,1$. The algebra (4.23) reduces to a 2-index symmetric Lie algebra:

Corollary 4.25. *The set of linearly independent eigen-operators can be chosen as* $\{B^{(0,b)}(\varepsilon), B^{(1,b)}(\varepsilon),\ (b,\varepsilon)\in Q\}$. *The algebra reads:*

$$\begin{aligned}
[B^0(b_1,\varepsilon_1), B^0(b_2,\varepsilon_2)] &= (b_1\varepsilon_2 - b_2\varepsilon_1)B^1(b_1+b_2-2, \varepsilon_1+\varepsilon_2),\\
[B^0(b_1,\varepsilon_1), B^1(b_2,\varepsilon_2)] &= (b_1\varepsilon_2 - \varepsilon_1(b_2+1))B^0(b_1+b_2, \varepsilon_1+\varepsilon_2)\\
&\quad + 4\mu g(b_1\varepsilon_2 - \varepsilon_1 b_2)B^0(b_1+\ b_2-2, \varepsilon_1+\varepsilon_1), \qquad (4.25)\\
[B^1(b_1,\varepsilon_1), B^1(b_2,\varepsilon_2)] &= ((b_1+1)\varepsilon_2 - \varepsilon_1(b_2+1))B^1(b_1+b_2, \varepsilon_2+\varepsilon_2)\\
&\quad + 4\mu g(b_1\varepsilon_2 - b_1\varepsilon_1)B^1(b_1+b_2-2, \varepsilon_1+\varepsilon_2)b_2.
\end{aligned}$$

The above algebra generalizes the w_∞-algebra of the oscillator potential. It will be interesting to address the question of a physical interpretation of this algebra in terms of discrete states of some 2-dimensional theory.

3. Effect of Central Terms. We have now constructed an algebra of eigen-operators defined by the commutation relations (4.25). The addition of the correct central terms in the w_∞-algebra does not modify sensibly this demonstration. Such terms arise only when computing commutators of linear terms in α. In particular, operators defined as in (4.5) get a central term $\sim \oint f_1\partial_x g_1$ in their Lie bracket. The induced changes are as follows:

(1) in (4.5), B acquires a term $f_0\cdot\mathbf{1}$,

(2) in (4.6), one must add a recursion relation for f_0:

$$\varepsilon f_0 = \oint f_1\cdot(\mu x - g/x^3). \qquad (4.26)$$

For $\varepsilon=0$, however, we prove that:

Proposition 4.27.

$$\oint f_1(\mu x-g/x^3)dx=0.$$

Proof. Inside our initial algebra (4.11), $\varepsilon=0$ implies b even. Hence

$$\oint f_1(\mu x-g/x^3)dx=\oint(\mu x^2-g/x^2)^b(\mu x^2+g/x^2)^a(\mu x-g/x^3)dx,$$

$$\text{(part-integration)}=\oint\frac{b}{a+1}(\mu x^2-g/x^2)^{b-1}(\mu x^2+g/x^2)^{a+1}(\mu x+g/x^3)$$

$$=\oint\frac{b}{a+1}(\mu x^2-g/x^2)^{b-2}(\mu x^2+g/x^2)^{a+2}(\mu x-g/x^3)dx$$

$$=\ldots 0 \quad \text{after reaching} \quad b=0. \quad \square$$

Hence (4.26) consistently determines f_0 for $\varepsilon\neq 0$. Proposition 4.7 is not modified. Proposition 4.11 could only be modified by a central extension, but the Jacobi identity applied to B_1, B_2 and H implies:

$$[H,[B_1,B_2]]=(\varepsilon_1+\varepsilon_2)[B_1,B_2] \tag{4.28}$$

since B_1, B_2 are now exact eigenstates of H; from (4.25) it appears that no extra central term can be generated in the *exact* commutators of the *exact* eigen-operators. Since all further derivations follow from (4.11), we conclude that the central extension of the w_∞-algebra does not modify the 2-index symmetric algebra of eigen-operators for $\mu x^2+g/x^2$.

4. Relation with sl(2). It is easier to work here with the 3-index redundant representation. This 3-index algebra (4.23) can be interpreted as a subalgebra of the enveloping algebra of $sl(2)$, at least at the classical level, and allowing negative integer powers of the generators. Introducing the classical Poisson $sl(2)$ algebra as:

$$\begin{aligned} sl(2)=\{J_3,J_+,J_-\}; \qquad \{J_3,J_\pm\}=\pm J_\pm, \\ \{J_+,J_-\}=-2J_3; \end{aligned} \tag{4.29}$$

we now prove the

Proposition 4.30. *The classical Poisson-bracket algebra corresponding to* (4.23) *is isomorphic to the subalgebra of* $U(sl(2))=\mathcal{G}$:

$$\mathcal{G}\equiv\{J_3^a J_+^{\varepsilon+b/2}J_-^{b/2};\ a\in\mathbb{N};(b,\varepsilon)\in Q\}.$$

Proof. From the definition in (4.30), we compute directly the Poisson bracket of two generators of $\mathcal{G}$ as:

$$\begin{aligned} &\{J_3^{a_1}J_+^{\varepsilon_1+\frac{b_1}{2}}J_-^{\frac{b_1}{2}},J_3^{a_2}J_+^{\varepsilon_2+\frac{b_2}{2}}J_-^{\frac{b_2}{2}}\} \\ &=\left(a_1\left(\varepsilon_2+\frac{b_2}{2}\right)-a_1\frac{b_2}{2}\right)J_3^{a_1+a_2-1}J_+^{\varepsilon_1+\varepsilon_2\frac{b_1+b_2}{2}}J_-^{\frac{b_1+b_2}{2}}-(1\leftrightarrow 2) \\ &\quad+2\left(\varepsilon_1+\frac{b_1}{2}\right)\left(\frac{b_2}{2}\right)J_3^{a_1+a_2+1}J_+^{\varepsilon_1+\varepsilon_2+\frac{b_1+b_2-2}{2}}J_-^{\frac{b_1+b_2-2}{2}}-(1\leftrightarrow 2) \\ &=(a_1\varepsilon_2-a_2\varepsilon_1)J_3^{a_1+a_2-1}J_+^{\varepsilon_1+\varepsilon_2-\frac{b_1+b_2}{2}}J_-^{\frac{b_1+b_2}{2}} \\ &\quad+(\varepsilon_1 b_2-\varepsilon_2 b_1)J_3^{a_1+a_2+1}J_+^{\varepsilon_1+\varepsilon_2+\frac{b_1+b_2-2}{2}}J_-^{\frac{b_1+b_2-2}{2}}. \quad \square \end{aligned}$$

The occurrence of $Sl(2)$ is not surprising in a problem concerning a potential $v(x)=\mu x^2+g/x^2$, which has an associated $Sl(2)$ symmetry [DFF]. The energy of an eigen-operator is understandably obtained as the difference between the exponents of J^+ and J^- while the spin is identified as $a+\varepsilon+b\geqq 0$. One notices that the set of exponents allowed for J^+ and J^- is asymmetrical since $b/2\in\frac{1}{2}\mathbb{N}$ and $\varepsilon+b/2=\frac{3b}{2}$, $\frac{3b}{2}-2\ldots-\frac{b}{2}$. The redundancy of indices is associated here to the existence of the Casimir operator $J_3^2-J^+J^-$ which all but reduces (4.30) to the 2-index symmetric algebra.

5. Limits of the Algebra. It is finally interesting to study, in the light of the results in Chap. 2, the two monomial limits of the potential $v(x)=\mu x^2+g/x^2$.
a) $\mu\to 0$. The potential becomes a "non-integrable" monomial according to Theorem 2.4. We recall that the energy eigenvalues are here normalized by a factor $2\sqrt{\mu}$, hence they become 0 and the theorem is valid (it only mentions non-vanishing eigenvalues).
b) $g\to 0$. The potential becomes μx^2. In this case, it is clear from (4.10) that a and b are totally redundant variables and the only meaningful quantity is $(a+b)$. Accordingly, the algebra (4.23) reduces to a 2-index algebra:

$$[B^{(b_1)}(\varepsilon_1), B^{(b_2)}(\varepsilon_2)]=(b_2\varepsilon_1-b_1\varepsilon_2)B^{(b_1+b_2-1)}(\varepsilon_1+\varepsilon_2). \tag{4.31}$$

Reinterpreting the indices b_i and the energies ε_i leads to understanding this algebra as the integer-spin subalgebra of the full eigen-operator algebra described in (2.15). Due to the fact that the form (4.10) automatically leads to even powers of x for a, b integers, one cannot obtain the half-integer spin subset. In order to get it, one should, in particular, allow half-integer indices for the hierarchy Hamiltonians $h_{\pm}^{(n)}$. Although formally correct, this approach, however, induces infinite series of powers of α when $g\neq 0$, and thus goes beyond our restricted definition of eigen-operators inside the w_∞-algebra, and beyond the span of this present study.

Note. We have noticed that from Propositions 4.24 and 4.30, we can in fact identify the algebra (4.25):

Proposition 4.32. *The algebra* (4.25) *is the classical limit of the extended* $W_\infty(c)$ *algebras (for* $c=4\mu g$*).*

These extended algebras were constructed in [PRS 2] precisely as sub-algebras of the enveloping algebra of $sl(2)$ quotiented by the ideal generated by $J^2-:J^+J^-:=c$. Their classical limit is identified as a subalgebra of the symplectic algebra on an $sl(2)$-coadjoint orbit defined by the quadratic equation $z^2-xy=4\mu g$.

This orbit is either a cone ($g=0$, corresponding to a pure w_∞ algebra as we have seen) or a 2-sheet hyperboloid ($\mu g<0$) or a 1-sheet hyperboloid ($\mu g>0$). It may therefore be that the most relevant underlying fundamental algebra for string theory is not the particular cone-symplectomorphism algebra w_∞, but the more general quadric-symplectomorphism algebras $w_\infty(c)$.

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