On the Classification of Quasihomogeneous Functions

Maximilian Kreuzer [1, *] **and Harald Skarke** [2, **]

[1] Theory Division, CERN, CH-1211 Geneva 23, Switzerland
[2] Institut für Theoretische Physik, Technische Universität Wien, Wiedner Hauptstrasse 8–10, A-1040 Wien, Austria

Abstract. We give a criterion for the existence of a non-degenerate quasihomogeneous polynomial in a configuration, i.e. in the space of polynomials with a fixed set of weights, and clarify the relation of this criterion to the necessary condition derived from the formula for the Poincaré polynomial. We further prove finiteness of the number of configurations for a given value of the singularity index. For the value 3 of this index, which is of particular interest in string theory, a constructive version of this proof implies an algorithm for the calculation of all non-degenerate configurations.

1. Introduction

Recently, a particular class of singularities [1, 2], namely singularities of holomorphic quasihomogeneous functions, have been found useful for the classification of superconformal field theories (SCFT) with particular significance for the case of $N = 2$ superconformal symmetry [3, 4] due to a non-renormalization theorem. The requirement of conformal invariance implies quasihomogeneity of degree 1 for the superpotential

$$W(\lambda^{n_i}\Phi_i) = \lambda^d W(\Phi_i) \tag{1}$$

in the effective Lagrangian description, with the scaling dimensions of the chiral superfields Φ_i translating into the weights $q_i = n_i/d$ of the variables X_i of a holomorphic function $W(X_i)$. The local algebra of this function, i.e. the quotient of the polynomial ring by the ideal generated by the gradients $\partial_j W(X_i)$, is isomorphic to the operator product algebra of chiral primary states [4]. In order that this

---

* e-mail: kreuzer@cernvm.cern.ch
** e-mail: hskarke@email.tuwien.ac.at

algebra be finite we will concentrate on isolated singularities, i.e. require that the origin be the only solution to the equation $dW = 0$. We will consider only quasihomogeneous holomorphic functions with positive weights $q_i > 0$, which are automatically polynomials.

In the mathematical literature isolated singularities have been classified up to 3 variables and for low values of the modality or of the dimension of the local algebra [2]. A general tool for the investigation of degeneracy is the Poincaré polynomial $P(t) = \mathrm{tr}\, t^{dJ}$ [2, 5], where the trace extends over a basis of the local algebra and $J = \sum_i q_i X_i \partial_i$ gives the weight of a basis monomial, i.e. the coefficients μ_a of t^{dq_a} in this polynomial are the multiplicities of the weights q_a. For any non-degenerate quasihomogeneous function this polynomial can be calculated from

$$P(t) = \prod_i \frac{1 - t^{d - n_i}}{1 - t^{n_i}} . \tag{2}$$

Thus a necessary condition for non-degeneracy is that this expression is a polynomial with non-negative coefficients. Note that the Poincaré polynomial only depends on the set of weights and not on the particular form of W. We call the set of all polynomials which are quasihomogeneous with respect to the weights $q_i = n_i/d$ a configuration

$$\mathfrak{C}_{(n_1, \dots, n_I)}[d] . \tag{3}$$

A configuration is said to be degenerate if it has only degenerate members. A final guidance of our interest comes from the relation between the singularity index

$$\hat{c} = \sum_i (1 - 2q_i) \tag{4}$$

and the central charge c in the Virasoro subalgebra of the corresponding $N = 2$ SCFT, which can be shown to be $c = 3\hat{c}$ [4].

In this note we will supply tools for the classification of non-degenerate configurations with a given singularity index. In Sect. 2 we start with some definitions and examples and then give a criterion for non-degeneracy of a configuration. We also show how this criterion is related to the necessary condition that the r.h.s. of (2) is a polynomial (PP-condition). In Sect. 3 we prove that for any given value of the singularity index there is only a finite number of non-degenerate configurations. Then we consider the cases of low integer index $\hat{c} \leq 3$ more explicitly. In the conclusion we summarize our results and briefly comment on their implications for the classification of SCFTs.

2. Non-Degeneracy Criterion

A necessary condition for a quasihomogeneous polynomial to be non-degenerate is that every variable X occurs either in the form X^α or $X^\alpha Y$. We will use a graphic representation in which every variable is represented by a dot, and a term of the form $X^\alpha Y$ is indicated by an arrow from X to Y. We will sometimes say "X points at Y." If no arrow originates from a variable X, then there is a term of the form X^α in the polynomial. Note that, as compared to the book by Arnold et al. [2], we omit
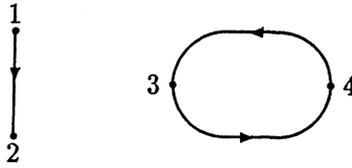
Fig. 1. Graphic representation of $X_1^{\alpha_1}X_2 + X_2^{\alpha_2} + X_3^{\alpha_3}X_4 + X_4^{\alpha_4}X_3$
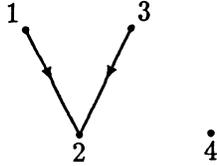

Fig. 2. Graphic representation of the degenerate polynomial $X_1^{\alpha_1}X_2 + X_2^{\alpha_2} + X_3^{\alpha_3}X_2 + X_4^{\alpha_4}$

Fig. 3. Dressing of the graph in Fig. 2: (a) by the link $X_1^p X_3^q$ and (b) by the pointer $X_1^p X_3^q X_4$

the "loops" indicating such terms[1]. For example, a polynomial of the form $X_1^{\alpha_1}X_2 + X_2^{\alpha_2} + X_3^{\alpha_3}X_4 + X_4^{\alpha_4}X_3$ is shown in Fig. 1.

A possible danger for the non-degeneracy of the polynomial arises when two or more arrows end at the same point, such as in the polynomial $X_1^{\alpha_1}X_2 + X_2^{\alpha_2} + X_3^{\alpha_3}X_2 + X_4^{\alpha_4}$, depicted in Fig. 2. The reason for the problem is that when we calculate dW/dX_i and set X_2 to 0 we are left with only one equation for the two variables X_1 and X_3. Therefore, we have to introduce extra terms in W that yield terms in X_1 and X_3 in dW even when all other variables are set to zero, as in $X_1^{\alpha_1}X_2 + X_2^{\alpha_2} + X_3^{\alpha_3}X_2 + X_4^{\alpha_4} + \varepsilon X_1^p X_3^q$ or in $X_1^{\alpha_1}X_2 + X_2^{\alpha_2} + X_3^{\alpha_3}X_2 + X_4^{\alpha_4} + \varepsilon X_1^p X_3^q X_4$. The graphic representation of these polynomials is shown in Fig. 3. We use dashed lines and pointers originating from dashed lines to indicate such extra terms. We shall call the graph without these additional lines "skeleton graph." It should be noted that all weights can be calculated only with the knowledge of the skeleton. As we have already mentioned, a necessary condition for the existence of a non-degenerate quasihomogeneous function is that the r.h.s. of (2) is actually a polynomial. For configurations with up to three variables, but not in general, this

---

[1] In our graphical representation of polynomials the important issue will be the pointer structure, to which "self pointers" do not contribute (see below). Thus, in particular for complicated structures, we think it is more transparent to omit this redundant information

condition is also sufficient and identical to the condition for the existence of the terms represented by dashed lines [2].

Definition. We call a variable X a root if the polynomial W contains a term X^a. A monomial $Y^a Z$ is called a pointer at Z. a is called the exponent of X or Y, respectively. We recursively define a link between two expressions, which may themselves be variables or links, as a monomial depending only on the variables occurring in these expressions. A link may further be linear in an additional variable Z, which does not count as a variable of the link. In this case we say that the link points at Z, thus extending the previous definition of a pointer. Of course, a specific monomial occurring in W can have more than one interpretation as a link or pointer. Given W, we call any graph (not necessarily the maximal one) whose lines allow the above interpretation in terms of monomials in W a graphic representation of W.

Theorem 1. *For a configuration a necessary and sufficient condition for non-degeneracy is that it has a member which can be represented by a graph with:*
1. Each variable is either a root or points at another variable.
2. For any pair of variables and/or links pointing at the same variable Z there is a link joining the two pointers and not pointing at Z or any of the targets of the sublinks which are joined.

Before proving this theorem, we shall illustrate the ideas on which it is based with some examples.

Example 1. A polynomial of the form

$$X_1^{\alpha_1} X_2 + X_2^{\alpha_2} + X_2 X_3^{\alpha_3} + X_4^{\alpha_4} X_5 + X_5^{\alpha_5} + X_5 X_6^{\alpha_6} + X_7^{\alpha_7} + \varepsilon_1 X_1^{p_1} X_3^{p_3} X_7 + \varepsilon_2 X_4^{p_4} X_6^{p_6} X_7 \tag{5}$$
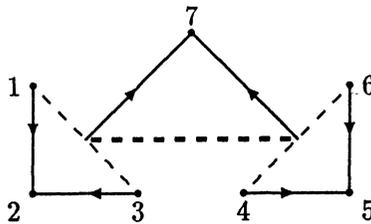

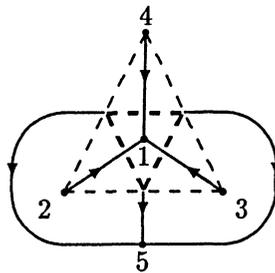**Fig. 4.** The pointers $(13)\to 7$ and $(46)\to 7$ require the link $((13)(46))$ drawn in boldface

Fig. 5. The monomial $X_2^p X_3^q X_4^r$ provides all three links $((23)(34))$, $((34)(42))$, and $((42)(23))$ required by the pointers $(23)\to 5$, $(34)\to 5$, and $(42)\to 5$

is degenerate, as one can see by calculating dW and setting X_2, X_5, and X_7 to 0. By adding the "missing link" $\varepsilon_3 X_1^{q_1} X_3^{q_3} X_4^{q_4} X_6^{q_6}$ we obtain non-degeneracy. The translation into our graphic language is given by Fig. 4.

Example 2. Fig. 5 shows the graphic representation of a polynomial where more than two arrows end at the same point. Distinct "links for links" are realized by the same monomial.

Example 3. $W = X_1^3 + (X_2^2 + X_3^2 + X_4^2) X_1 + \varepsilon X_2 X_3 X_4$. Note that the last monomial fulfills three tasks at once: pointer from X_2 and X_3 to X_4, pointer from X_2 and X_4 to X_3 and pointer from X_3 and X_4 to X_2.

Proof of Theorem 1. A) "Necessary": Calculating dW/dX_i yields n equations in n variables. Non-degeneracy means that they can be fulfilled only by the trivial solution. Obviously, a necessary condition is that by setting k variables to 0 no more than k equations can be fulfilled identically. Starting with $k = n - 1$, we find that for each X_i monomials in this variable must occur in dW/dX_j, which means that each variable has to be a root or has to point at another variable. (This was the first condition of the theorem.)

Let us now consider $k = n - 2$, i.e. we set to 0 all variables except two (which we shall call X_1 and X_2). According to the above, monomials $X_1^{\alpha_1}$ and $X_2^{\alpha_2}$ must occur. There are two possibilities: either these monomials occur in different equations, meaning that there are at least two equations that are not automatically fulfilled by setting all variables except X_1 and X_2 to zero, or they both occur in the same equation. In the latter case a necessary condition for non-degeneracy is the occurrence of a monomial in both variables in one of the other equations, i.e. either one of them points at the other or there must be a link between them.

We proceed inductively in $l = n - k$: We construct the graph by adding just the lines that we need in each step. Assume we have all the lines up to level $l - 1$. Suppose there are two links pointing at the same variable Z. Let them have $l_1 < l$ and $l_2 < l$ variables, of which they have q variables in common, and $l = l_1 + l_2 - q$ variables together. Setting all other variables to zero, the links and pointers we have drawn until now correspond to $l_1 + l_2 - q - 1 = l - 1$ equations, as $\partial W / \partial Z$ and q additional equations are double counted due to the overlap in the variables. So we need one more equation in the l variables, i.e. we need the link implied by the second condition of the criterion.

B) "Sufficient": We show that no degenerate configuration can fulfill our conditions. Let $W = \sum M_\mu \varepsilon^\mu$ represent a degenerate configuration, where the M_μ are monomials in X_i, and let $\partial_i = \partial/\partial X_i$ and $\partial_\mu = \partial/\partial \varepsilon^\mu$. Further, for any choice of ε^μ there are non-vanishing solutions $X_i(\varepsilon)$ to the equations $\partial_i W = 0$. As these equations are polynomial, we can choose an open set of $(\varepsilon^1, ..., \varepsilon^k)$ and a solution $(X_1(\varepsilon), ..., X_n(\varepsilon))$ depending smoothly on the ε^μ in this set. Quasihomogeneity implies that $W(X_i(\varepsilon)) \equiv 0$. Therefore,

$$\frac{d}{d\varepsilon^\mu} W = \sum_i \partial_i W \partial_\mu X_i + \partial_\mu W = M_\mu = 0, \tag{6}$$

i.e. every monomial in W has to vanish. We now choose a point ε in this set for which $X_1(\varepsilon) \neq 0, ..., X_l(\varepsilon) \neq 0$, but $X_{l+1} \equiv ... \equiv X_n \equiv 0$ in some neighbourhood of ε (we are, of course, free to choose the labels for the X_i). Differentiating $\partial_i W$ with

respect to ε^μ we find

$$\frac{d}{d\varepsilon^\mu} \partial_i W = \sum_{j \leq l} \partial_i \partial_j W \partial_\mu X_j + \partial_i M_\mu = 0. \tag{7}$$

Due to quasihomogeneity $\sum_{j \leq l} q_j X_j \partial_i \partial_j W = (1 - q_i) \partial_i W = 0$, i.e. the rank of the rectangular matrix $\partial_i \partial_j W$ with $j \leq l$ is less than l. Thus there are at least $n - l + 1$ independent vectors $c_i^{(m)}$ with $\sum_{i \leq n} c_i^{(m)} \partial_i \partial_j W = 0$ and hence $\sum_{i \leq n} c_i^{(m)} \partial_i M_\mu = 0$. As all monomials M_μ have to vanish, only one of the derivatives $\partial_i M_\mu$ can be non-vanishing for a given μ. Thus the sum $\sum_i c_i^{(m)} \partial_i M_\mu$ has at most one non-vanishing contribution, and, as $c_i^{(m)}$ can be 0 for all m for at most $l - 1$ values of i, all variables $X_1, \ldots, X_l$ and all links among these variables have to point at a subset of at most $l - 1$ variables X_j with $j > l$. The resulting double pointer cannot be completely resolved, as the required link would again have to point at the same set of $l - 1$ variables and thereby generate a new double pointer. $\quad\square$

Lemma 1. *The necessary condition for non-degeneracy that the expression (2) for the Poincaré polynomial is a polynomial (we refer to this as the PP-condition) is equivalent to the criterion of Theorem 1 if one omits the requirement that all exponents in the link monomials have to be non-negative.*

Proof. $\prod (1 - t^{d - n_i})/(1 - t^{n_i})$ is a polynomial if all zeros in the denominator, counted according to their multiplicities, are matched by zeros in the numerator, i.e. if the set of all multiples of $1/n_i$ between 0 and 1 is a subset of the multiples of $1/(d - n_i)$, even when multiplicities are taken into account. The relaxed condition on the links, which we are referring to, is that the number theoretic condition $\sum p_i n_i = d$ (for non-pointing links) or $\sum p_i n_i = d - n_k$ (for a link pointing at X_k), where the sum runs over all X_i joined by the link, is fulfilled without requiring that the p_i all be non-negative. This is equivalent to the condition that the greatest common divisor of these n_i divides d or some $d - n_k$. To show the equivalence of these two conditions we first note that the PP-condition implies that each X_i has to be a pointer or a root, as $1/n_i$ must be a multiple of some $1/(d - n_k)$. In this context a root is to be considered as a pointer at itself.

There is a problem, however, if two variables X_i and X_j point at the same variable X_k. Then the multiples of $1/(n_i \cap n_j)$ are contained only once and have to be taken care of by some further variable X_l with $n_i \cap n_j$ dividing $d - n_l$. We thus recover the requirement of the links implied by the non-degeneracy criterion except for the positivity of the exponents in the monomial. If l cannot be chosen as i or j, i.e. if $n_i \cap n_j$ does not divide d, this link is a pointer at X_l. The roots, considered as pointers at themselves, and the links which are no pointers, i.e. can be considered as pointing at one of their variables, do not imply additional links. This is because the corresponding n_i (or $n_i \cap n_j$ for pointers) are divisors of d. In the simplest case, for example, where n_i and n_j divide $d - n_i$, the missing ratio $1/(n_i \cap n_j)$ is a multiple of $1/(d - n_j)$ and thus acts like a pointer at X_j with the additional feature that the denominator divides d. In this way the missing numbers can be passed on backwards along any pointers with ever smaller denominators until they find their match in a free multiple of some $1/(d - n_k)$ with no other pointer at X_k or until $n_i \cap \ldots \cap n_k = 1$. Proceeding in the same way with the additional overlaps which may arise due to the additional pointers in each step, we indeed find equivalence of the PP-condition and the relaxed link criterion. $\quad\square$

To illustrate this equivalence we use the following example by B. M. Ivlev of a degenerate configuration fulfilling the PP-condition [2],

$$\mathbb{C}_{(1,24,33,58)}[265],\tag{8}$$

with the corresponding skeleton polynomial

$$X^{265}+XY^{11}+XZ^8+ZU^4.\tag{9}$$

To fulfill the criterion of Theorem 1 we would need a link between Y and Z, which has to point at U as neither d nor any $d-n_i$ is a multiple of $3=24\cap33$ except for $d-n_U=207=24p+33q$. This equation, however, does not have a solution with both p and q positive, which explains why (2) is a polynomial although the configuration is degenerate. It would be interesting to find out if it is possible to construct an $N=2$ SCFT with these weights.

3. On Configurations with a Given Index

Lemma 2. *If a non-degenerate configuration contains n variables X_i with a given weight $q\in(1/3,1/2)$, then it also contains at least n variables Y_j of weight $\bar{q}=1-2q$.*

Proof. Let $W=W(X_i,Y_j,Z_k)$ with weight$(X_i)=q$, weight$(Y_j)=\bar{q}$, and weight$(Z_k)\neq q,\bar{q}$. We calculate dW and set Y_j and Z_k to 0. Non-degeneracy implies that at least n equations for the X_i must remain. Because $q\in(1/3,1/2)$, these equations must be quadratic in the X_i, i.e. of weight $2q=1-\bar{q}$. Therefore, they must come from dW/dY_i, implying that the number of Y_i's is at least n. $\square$

We call variables trivial if they correspond to terms X^2. Trivial variables have weights $q=1/2$ and therefore do not contribute to $\hat{c}$, nor to the local algebra, as they can be eliminated by $\partial W/\partial X=0$.

Corollary 1. *For every non-degenerate configuration $\hat{c}$ is greater than or equal to 1/3 times the number of non-trivial variables.*

Proof. Grouping variables with $q\in(1/3,1/2)$ together with $\bar{q}$, we have $1-2q+1-2\bar{q}=2q>2/3$; the contributions of all other non-trivial variables (with weights $\leq1/3$) are at least $1-2/3=1/3$. $\square$

Theorem 2. *Given a positive rational number $\hat{c}$, there is only a finite number of non-degenerate configurations whose index is $\hat{c}$.*

Proof (indirect). Suppose there is an infinite sequence of configurations with index $\hat{c}$. Due to the above corollary there is only a finite number of skeleton graphs that can realize $\hat{c}$, so there must be an infinite subsequence corresponding to just one graph. Considering a specific exponent, we can find either a subsequence for which this exponent is constant or one for which this exponent goes monotonically to infinity. Doing this for every exponent, we end up with a sequence of polynomials corresponding to the same skeleton graph, for which $n-m$ exponents are constant while m exponents tend monotonically to infinity. Of course, the "limit configuration," which contains m variables of weight 0, also has index $\hat{c}$. We denote the weights in the lth member of the sequence by $q_i^{(l)}$, $i\in\{1,...,n\}$ with $\lim_{l\to\infty}q_i^{(l)}=0$ for $i\in\{1,...,m\}$. We will now show that the index of a member of such a

sequence is in fact smaller than the index of the limit sequence, in contradiction with the assumed constance of $\hat{c}$. Consider a specific l and let $\varepsilon = \max q_i^{(s)}$ with $i \leq m$ and $s \geq l$. We define the intervals $I_k = (2^{-k}\varepsilon, 2^{-k+1}\varepsilon]$. By A_k we denote the number of points with $q^{(l)} - q^{(\infty)} \in I_k$, by B_k we denote the number of points with $q^{(\infty)} - q^{(l)} \in I_k$. Note that $B_1 = 0$, as for any variable $q^{(\infty)} = 0$ or the possible target variable has $q < \frac{1}{2}$. Now consider all $\sum_{i=1}^{k} B_i$ points for which $q^{(\infty)} - q^{(l)} > 2^{-k}\varepsilon$. Setting all other variables to zero in dW/dX_i, we see that we need at least as many equations, coming from points with $q^{(l)} - q^{(\infty)} > 2^{-k+1}\varepsilon$, as these equations are at least quadratic in the non-zero variables. We thus have $\sum_{i=1}^{k} B_i \leq \sum_{i=1}^{k-1} A_i$, i.e. $\sum_{i=1}^{k} (A_i - B_{i+1}) \geq 0$ and

$$\hat{c}^{(\infty)} - \hat{c}^{(l)} = 2\sum (q^{(l)} - q^{(\infty)}) > 2\varepsilon \sum_i (2^{-i} A_i - 2^{-i+1} B_i) = 2\varepsilon \sum_i 2^{-i}(A_i - B_{i+1}) \geq 0.$$

(10)

This is the contradiction we were looking for. $\square$

Theorem 3. *Given a rational number r, one can find a positive number ε such that no number in the interval $(r, r + \varepsilon)$ is the index of a non-degenerate quasihomogeneous polynomial; i.e. the accumulation points in this set of indices are all approached from below.*

Proof. Like the previous theorem with an infinite sequence of configurations with decreasing indices $\hat{c}^{(l)} \to r$ instead of constant index. $\square$

Theorem 2 has given us valuable theoretical information, but does not help us in explicitly finding all configurations of a given index. Especially, for the case $\hat{c} = 3$, which is most important for string theory, one would like to have a way of constructing normal forms for all possible configurations. We will now formulate and prove a lemma which makes it possible to write a computer program which calculates explicitly all configurations with $\hat{c} = 3$.

Lemma 3. *For a non-degenerate quasihomogeneous polynomial with $\hat{c} = 3$ the number of exponents $\alpha_i > 18$ or $\alpha_i > 84$ is smaller than 3 or 2, respectively. These limits cannot be improved, as is seen from the polynomials $X_1^3 + X_2^{18} + X_3^{18} + X_4^{18}$ and $X_1^3 + X_2^7 + X_3^{84} + X_4^{84}$.*

Proof. Let us assume that there are three exponents $\alpha_i > 18$, corresponding to variables with weights $q_1, q_2, q_3 < 1/18$, contributing $1 - 2q_i$ to $\hat{c}$. If there are also variables with exponents $\alpha = 2$, pointing at variables with $q_i < 1/18$, they add $1 - 2(1/2(1 - q_i)) = q_i$ to $\hat{c}$. The total contribution of the variables considered so far is therefore between $3(1 - 2/18) = 3 - 1/3$ and 3. It is impossible to complete this to a polynomial with $\hat{c} = 3$. This proves the first assertion.

Assume now that there are terms $X_i^{a_i}$ or $X_i^{a_i} Y^i$ with $a_i > 84$, $i = 1, 2$ in the potential, thus $q_i < 1/84$. The contribution $\hat{c}_{12}$ of these variables to $\hat{c}$ fulfills $2 - \frac{1}{21} < \hat{c}_{12} = 2 - 2q_1 - 2q_2 < 2$. A "partner variable" Z_i pointing at X_i with exponent 2, i.e. a term $X_i Z_i^2$, can be disregarded as it effectively just doubles a_i (if $a_1 = a_2$ both possible partner variables might point to the same X_i). According to Lemma 2 there have to be 2 or 3 more variables in order to make up for the difference to a total $\hat{c}$ of 3.

Case 1. 2 additional variables U_i with weights r_i. If one of their exponents is 2, it has to point at a Z_i, since U_1 and U_2 have to contribute more than 1. Then one can explicitly calculate the contribution of X_1, X_2, Z_1 (Z_2), U_1 to be $\frac{5}{2} - \frac{3}{2}q_1 - (2)q_2$. $\hat{c} = 3$ would then require $r_2 = \frac{1}{4} - \frac{3}{4}q_1 - (\frac{1}{2})q_2$, which cannot be satisfied. The exponents of U_1, U_2 also have to be less than 7, since $\frac{1}{3} + 1 - \frac{2}{7} = \frac{22}{21}$. Enumeration of all relevant singularities in 2 variables shows that the smallest possible contribution to $\hat{c}$ larger than 1 is 22/21, thus the total $\hat{c}$ cannot be 3. If one or both U_i point at some X_i or Z_i, $\hat{c}$ can only be enlarged. So we finally have to consider the contributions with $\hat{c} = 2 - 2(r_1 + r_2) \leqq 1$. Pointers at X_j would not make any difference, as the decrease in r_i would be $r_i q_j$, which is not sufficient to reach $\hat{c} = 3$. So let U_1 point at Z_j. This makes $1 - 2r_1$ larger than $\frac{2}{3} - \frac{1}{252}$. Thus U_2 cannot point at a Z_k and its exponent has to be 3, so U_2 cannot make up for the difference to $\hat{c} = 3$.

Case 2. 3 additional variables U_i with weights r_i. We split this case according to the number of exponents equal to 2. If all exponents are larger than 2 this implies $r_i \leqq \frac{1}{3}$, and hence $r_i \geqq \frac{13}{42} = \frac{1}{3}(1 - \frac{1}{14})$, i.e. all exponents have to be 3. These variables thus may only point at a variable with weight less than 1/14, i.e. at X_i. Examining all cases there are 10 infinite series of polynomials with $\hat{c} = 3$:

$$X^a + Y^{6a} + XZ^2 + U^3 X + V^3 X + W^3 \qquad \text{and} \quad Y^{6a} \rightarrow Y^{3a} + YT^2,$$

$$X^a + Y^{4a} + XZ^2 + U^3 X + V^3 X + W^3 Y \qquad \text{and} \quad Y^{4a} \rightarrow Y^a + YT^2,$$

$$X^a + Y^{2a} + XZ^2 + U^3 X + V^3 X + W^3 X \qquad \text{and} \quad Y^{2a} \rightarrow Y^a + YT^2,$$

$$X^a + Y^a + X(U^3 + V^3 + W^3 + Z^2 + T^2) \qquad \text{and} \quad XW^3 \rightarrow YW^3,$$

$$X^a + Y^a + X(U^3 + V^3 + W^3) + Y(Z^2 + T^2) \quad \text{and} \quad XW^3 \rightarrow YW^3.$$

For none of these polynomials, however, do the links which would be necessary to make them non-degenerate exist for $a > 84$. In the first case, for example, it is impossible to have a link between Z and U, since the weights of these two variables add up to $\dfrac{5}{6}\left(1 - \dfrac{1}{a}\right)$.

Now let the first exponent be equal to 2. U_1 must not point at Z_i (otherwise $\hat{c} > 3$), so it points at U_2 and the exponents of U_2 and U_3 have to be 3 as above. If the latter variables do not point further the total $\hat{c}$ stays below 3. They might only point at X_1, because otherwise we would violate $\hat{c} \leqq 3$. This, however, does not help either, because the decrease of r_2 and/or r_3 is sufficiently compensated by the increase of r_1 to keep $\hat{c}$ below 3. The proof is completed by the observation that there cannot be 2 exponents of the U's equal to 2, because one of these variables would then have to point at a Z_i, which would increase $\hat{c}$ beyond 3 for an isolated configuration. $\square$

Lemma 3, together with Corollary 1, implies an algorithm for the calculation of all configurations with $\hat{c} = 3$, as the only unconstrained exponent in each skeleton graph can be calculated (and, of course, has to turn out integer to yield a solution). It is straightforward to further reduce the number of possibilities using Lemma 2. Consider, for example, a skeleton graph with n points of which i have an exponent larger than or equal to a. If p of these i variables are pointed at by a variable with exponent 2, this implies

$$\hat{c} \geqq (n - i - p)/3 + i(1 - 2/a) + p/a. \qquad (11)$$

If $n + 2i > 3\hat{c} + p$ this is equivalent to

$$a \leq 3(2i - p)/(n + 2i - 3\hat{c} - p). \tag{12}$$

As $\partial a/\partial p$ is always positive, and obviously $p \leq i$ and $p \leq n - i$, we obtain the bounds $a \leq 3i/(n + i - 3\hat{c})$ or $a \leq (3i - n)/(i - \hat{c})$ for $i \leq n/2$ or $i \geq n/2$, respectively. It is also straightforward to check that not more than i variables can have an exponent larger than $a = 2i/(i - \hat{c})$.

For completeness we state the following results referring to $\hat{c} = 1$ or 2. They can be proved with the same methods as above.

Lemma 4. *Any configuration with $\hat{c} = 1$ corresponds to weights $(1/3, 1/3, 1/3)$, $(1/3, 1/6)$ or $(1/4, 1/4)$. For a polynomial with $\hat{c} = 2$ there is at most one exponent greater than 12 and no exponent can be greater than 42.*

4. Conclusion

We have given a criterion for the non-degeneracy of a configuration which requires the check of a recursive condition concerning the existence of certain monomials consistent with quasihomogeneity, and have introduced a convenient graphic representation for these monomials. The PP-condition is equivalent to this condition except for positivity of the exponents. We have also shown that for a given singularity index the number of non-degenerate configurations is finite and that such a value cannot be approximated from above by non-degenerate configurations. Finally, applications of these results provide the necessary ingredients for explicit calculations of all configurations at least for low values of $\hat{c}$.

Our results, in particular, imply an algorithm for the calculation of all configurations with $\hat{c} = 3$, which is straightforward to implement due to the recursive structure of the condition of Theorem 1. According to Lemma 2 only polynomials in up to $3\hat{c}$ non-trivial variables have to be considered. As the weights of all variables are already determined by the skeleton graph, one only has to investigate a reasonable number of such graphs, which can be constructed recursively. The crucial point is that the combinations of exponents which have to be checked can be restricted to a finite number even for these skeletons, as has been done in Lemma 3 for the particularly interesting case $\hat{c} = 3$. Non-degeneracy can then be checked in a second step.

Such a construction would be an extension of the work by Candelas et al. [6] on Calabi-Yau manifolds in weighted projective spaces [7]. The connection between these manifolds and the construction of $N = 2$ SCFT from non-degenerate quasihomogeneous functions has been discussed by Greene et al. in [8]. The formulae for the calculation of the Hodge numbers from the scaling dimensions of the superfields have later been supplied by Vafa [9] and rederived by methods of algebraic geometry in [10]. The authors of [6] have implemented 30 polynomials fulfilling the criterion of Theorem 1 and have constructed some 6000 Calabi-Yau manifolds in weighted $\mathbb{P}_4$. As the number of skeleton graphs already grows faster than 2^n for polynomials in n variables, a complete construction along these lines, however, appears to be difficult for larger numbers of variables.

References

1. Arnold, V.I.: Singularity theory. London Mathematical Society Lecture Note Series, Vol. **53**. Cambridge: Cambridge Univ. Press 1981
2. Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of differentiable maps, Vol. I. Basel-Boston: Birkhäuser 1985
3. Martinec, E.: Phys. Lett. B **217**, 431 (1989)
   Vafa, C., Warner, N.: Phys. Lett. B **218**, 51 (1989)
4. Lerche, W., Vafa, C., Warner, N.P.: Nucl. Phys. B **324**, 427 (1989)
5. Bourbaki, N.: Lie groups and Lie algebras. Paris: Hermann 1971, Chap. V, Sect. 5.1
6. Candelas, P., Lynker, M., Schimmrigk, R.: Nucl. Phys. B **341**, 383 (1990)
7. Dolgachev, I.: In: Carell, J. (ed.) Group actions and vector fields. Lecture Notes in Mathematics. Vol. **956**. Berlin, Heidelberg, New York: Springer 1982
   Dimca, A.: J. Reine Ang. Math. **366**, 458 (1986)
8. Greene, B.R., Vafa, C., Warner, N.P.: Nucl. Phys. B **324**, 371 (1989)
9. Vafa, C.: Mod. Phys. Lett. A **4**, 1169 (1989); Superstring Vacua. HUTP-89/A057 preprint
10. Roan, S.S.: Int. J. Math. **1**, 211 (1990)

Communicated by A. Connes