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The Pfaffian Line Bundle

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Abstract. We analyze the holomorphic Pfaffian line bundle defined over an infinite dimensional isotropic Grassmannian manifold. Using the infinite dimensional relative Pfaffian, we produce a Fock space structure on the space of holomorphic sections of the dual of this bundle. On this Fock space, an explicit and rigorous construction of the spin representations of the loop groups LO_n is given. We also discuss and prove some facts about the connection between the Pfaffian line bundle over the Grassmannian and the Pfaffian line bundle of a Dirac operator.

1. Introduction

In this paper, we study the Pfaffian line bundle PF over the isotropic Grassmannian manifold of a Hilbert space. This line bundle, which was first defined in [21], is a unique holomorphic square root of the determinant line bundle over the Grassmannian. Here we will use the theory of the infinite dimensional relative Pfaffian developed in [13] and [16] to construct a Hilbert space $\mathscr{F}$ out of the space of holomorphic sections of the dual bundle of PF. If we use the space of square-integrable wave functions on the circle for the underlying Hilbert space, then $\mathscr{F}$ is interpreted as the Fock space of a Majorana fermion on the circle (with half the degrees of freedom of the Dirac Fock space, which arises from the corresponding construction for the determinant line bundle). The physical interpretation of this Pfaffian line bundle Fock space construction was speculated on in [26], and it serves as an example of the Fock space functor described in [24].

The Fock space $\mathscr{F}$ is isomorphic to the completion of an exterior algebra, but the Pfaffian line bundle approach reveals extra structure. The isotropic Grassmannian $\mathrm{Gr}_I(\mathscr{H})$ is a homogeneous space of the restricted orthogonal group $O_{\mathrm{res}}(\mathscr{H})$ associated to a real structure on the Hilbert space $\mathscr{H}$. We show that $\mathscr{F}$ carries a projective unitary representation of $O_{\mathrm{res}}(\mathscr{H})$, which is an analytic generalization of the representations described algebraically in [14]. In analogy to the Borel–Weil

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theorem in finite dimensions, this representation is seen to come from an action of $O_{\mathrm{res}}(\mathscr{H})$ on holomorphic sections of PF′ which covers its action on $\mathrm{Gr}_I(\mathscr{H})$. By embedding the loop groups LO_n in $O_{\mathrm{res}}(\mathscr{H})$, we obtain the spin representations of LO_n for all n. These representations were pointed out in [21]. We show here how to construct them explicitly using the relative Pfaffian as an analytic tool.

By applying the Pfaffian line bundle construction to the space $\mathscr{H} \oplus \mathscr{H}'$, where $\mathscr{H}'$ is the dual of $\mathscr{H}$, we can obtain by a pullback the determinant line bundle for $\mathscr{H}$. The determinant line bundle construction gives rise to representations of LU_n, which were described in detail in [21]. Obtaining the determinant line bundle by pulling back the Pfaffian line bundle corresponds, in terms of representations, to the embedding $LU_n \hookrightarrow LO_{2n} \hookrightarrow O_{\mathrm{res}}(\mathscr{H})$. The representations of LU_n arising from this embedding are exactly those obtained from the determinant line bundle directly.

Freed [12] has shown how to define a Pfaffian line bundle $\mathscr{K}$ over the moduli space of compact Riemann surfaces with spin structure. This $\mathscr{K}$ is a holomorphic square root of the Quillen determinant line bundle over moduli space [22, 7, 8], a structure which has been extremely important in string theory and conformal field theory (see [1, 11], for example). The moduli space of Riemann surfaces was connected to soliton theory by Krichever [18], a relation which was applied to string theory in [19]. A version of the Krichever map linking the moduli space to the Grassmannian and its determinant line bundle, which is the type of map we will consider here, appeared in [23, 25], and has also had applications to physics. In particular, this connection has been used to connect Virasoro algebras with the geometry of Riemann surfaces [5, 6, 17], which leads to the unification of the geometric and algebraic approaches to conformal field theory [2, 3, 15]. In our case, we would like use the Krichever map to think of the isotropic Grassmannian manifold as universal moduli space for once-punctured Riemann surfaces with spin structure. We show that the line bundles $\mathscr{K}$ and PF are related by a pullback by the Krichever map. $\mathscr{K}$ comes with a canonical hermitian structure and holomorphic section, which are relevant to the physical interpretation [1, 11]. One would like to relate these to PF, which has its own canonical hermitian structure. It is fairly clear how to choose a section on PF which gives rise to the canonical section of $\mathscr{K}$ under the pullback. Unfortunately, the canonical metrics on PF and $\mathscr{K}$ do not coincide, and we show that it is not possible to choose a metric on PF which pulls back correctly.

The basic objects of our discussion, the Grassmannian manifolds and the corresponding restricted unitary and orthogonal groups, are defined using the Hilbert–Schmidt norm, following [21]. One might ask if the Hilbert–Schmidt restriction could be relaxed to some other Schatten ideal I_p, or to the case of compact operators (which was the definition used in [23, 25]). This is a significant question if we want to consider higher dimensions, because for the group $\mathrm{Map}(X, G)$, where X is some d-dimensional manifold, the relevant operators lie in the class I_{d+1} (see [10, 20, 21]). Unfortunately, not much along the lines of the construction of this paper can be done for the class I_p, where $p > 2$. In [20], it was shown how to modify the definition of the determinant line bundle to extend to the $p > 2$ cases in such a way that DET has a hermitian structure which depends on regularized determinants. Because the transition functions for PF given below in Sect. 4 involve only finite dimensional matrices, it is trivial to do the same for the Pfaffian line bundle. The hermitian structure for the Pfaffian cases is just a positive square root of the hermitian structure given by [20], so that no extra analysis is required. Our construction here relies, however, on an inner product, or at least some metric structure, on the space of holomorphic sections of the line bundle. Because the regularized determinants do not satisfy multiplicative relations, one cannot use the methods presented here to obtain such structures. Furthermore, it is shown in [20] that the groups $\mathrm{Map}(X, G)$ admit only trivial cyclic extensions for $d > 1$, so that these cases would inherently behave very differently from the $d = 1$ case considered here.

This paper is organized as follows. In Sect. 2 we briefly review the definition of the Grassmannian manifold over a Hilbert space. In Sect. 3 we describe the construction of the determinant line bundle, and the formation of a Hilbert space out of the space of holomorphic sections. Section 4 contains the analogous constructions in the Pfaffian case. We define the line bundle PF by giving a trivialization such that the transition functions are Pfaffians of finite-dimensional matrices. In particular, this shows that the Pfaffian line bundle is holomorphic over the restricted Grassmannian. In Sect. 5 we describe in detail the Fock space arising from the space of holomorphic sections, and give an alternative definition of PF. We discuss the construction of the spin representation in finite dimensions in Sect. 6, as a prelude to the infinite dimensional case. In Sect. 7 we deal with the infinite dimensional case in detail, discussing the action of the restricted orthogonal group on PF, and the corresponding representations on the Fock space. We briefly describe some of the applications of these representations to loop groups in Sect. 8. In Sect. 9 we discuss the relations between the Pfaffian line bundle over the Grassmannian and the Pfaffian line bundle over moduli space and present some results connecting the two.

2. The Grassmannian Manifold

We start with a separable, polarized infinite dimensional Hilbert space, $\mathscr{H}$. By polarized, we mean simply that $\mathscr{H}$ comes with a decomposition into closed, infinite dimensional subspaces,

$$\mathscr{H} = \mathscr{H}_- \oplus \mathscr{H}_+ \,. \tag{2.1}$$

This Hilbert space is to be thought of as a one-particle Hilbert space, with subspaces consisting of negative and positive energy states, respectively. For the Pfaffian case, we will interpret $\mathscr{H}$ as the space of states for a single fermion moving on a circle, which we identify with the half-densities on the circle, subject to antiperiodic boundary conditions.

We define a subgroup of the unitary group of $\mathscr{H}$ for whose elements the off-diagonal parts are restricted to be Hilbert–Schmidt,

$$U_{\mathrm{res}}(\mathscr{H}) := \{g \in U(\mathscr{H})\colon P_- g P_+ ,\, P_+ g P_- \in I_2(\mathscr{H})\} \,, \tag{2.2}$$

where $P_\pm$ are the orthogonal projections onto $\mathscr{H}_\pm$. Note that this restriction on the off-diagonal parts of an element g implies that the diagonal parts, $P_+ g P_+$ and $P_- g P_-$, must be Fredholm, through the condition that g is invertible. The Grassmannian manifold is a homogeneous space of the restricted unitary group.

$$\begin{aligned} \mathrm{Gr}(\mathscr{H}) &:= \{W \subset \mathscr{H} : W = g\mathscr{H}_- \text{ for some } g \in U_{\mathrm{res}}(\mathscr{H})\} \\ &\cong U_{\mathrm{res}}(\mathscr{H})/U(\mathscr{H}_-) \times U(\mathscr{H}_+) \,. \end{aligned} \tag{2.3}$$

$\mathrm{Gr}(\mathscr{H})$ breaks up into connected components indexed by the index of the Fredholm operator $P_- g P_-$ on $\mathscr{H}_-$. If $\mathscr{H}_-$ and $\mathscr{H}_+$ are taken to have finite dimensions m and n, respectively, the definition of the Grassmannian reduces simply to $\mathrm{Gr}_m(\mathbb{C}^{m+n})$. One can form a nested set of finite dimensional submanifolds $\mathrm{Gr}^{(k)}(\mathscr{H})$ of $\mathrm{Gr}(\mathscr{H})$, with

$$\mathrm{Gr}^{(k)}(\mathscr{H}) \cong \bigcup_{m=1}^{2k-1} \mathrm{Gr}_m(\mathbb{C}^{2k}) , \tag{2.4}$$

and the union of these submanifolds is dense [21].

This definition of the Grassmannian of a Hilbert space is essentially that of [21]. As noted in the introduction, works relating the Grassmannian and determinant bundle to dynamical systems [23, 25] have used a broader definition, restricting off-diagonal terms to be compact operators where we have required Hilbert–Schmidt. The determinant and Pfaffian line bundles are still well-defined and holomorphic in the compact case, but they are not homogeneous, and one cannot introduce a Hilbert space structure on the space of holomorphic sections of the dual bundle in the manner described below.

To show that $\mathrm{Gr}(\mathscr{H})$ has the structure of a complex manifold [21], we define a set of coordinate charts as follows. Choose an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for $\mathscr{H}_+$, and a basis $\{e_k\}_{k=-\infty}^{-1}$ for $\mathscr{H}_-$. In our example of half-densities on the circle, we will take $e_{\pm k} = \exp\{\pm i(k - \frac{1}{2})\theta\}$. Think of this as a canonical basis fixed by our parametrization of the circle. For S a subset of the non-zero integers, let W_S be the subset of $\mathscr{H}$ given by the span of $\{e_k\}_{k \in S}$. We have $W_S \in \mathrm{Gr}(\mathscr{H})$ if and only if $S \in \mathscr{A}$, where

$$\mathscr{A} := \{S \subset \mathbb{Z}\backslash\{0\} \colon \mathrm{card}(S \cap \mathbb{Z}_+) < \infty,\ \mathrm{card}(\mathbb{Z}_- \backslash S) < \infty\} . \tag{2.5}$$

It is straightforward to check that any element of $\mathrm{Gr}(\mathscr{H})$ can in fact be written as the graph of some operator $A \in I_2(W_S, W_S^\perp)$, and conversely the graph of any such operator is an element of $\mathrm{Gr}(\mathscr{H})$. Thus we have an open cover of $\mathrm{Gr}(\mathscr{H})$ by sets

$$U_S := \{W \in \mathrm{Gr}(\mathscr{H}) \colon W = \mathrm{graph}(A), \text{ for } A \in I_2(W_S, W_S^\perp)\} , \tag{2.6}$$

each of which is isomorphic to $I_2(\mathscr{H}_+, \mathscr{H}_-)$. Moreover, the change of coordinate maps determined by these isomorphisms involve only determinants of finite dimensional submatrices. They are thus clearly holomorphic, which gives $\mathrm{Gr}(\mathscr{H})$ the structure of a complex manifold. We will generally identify the set of subspaces U_S with the set of maps $I_2(W_S, W_S^\perp)$, with no distinction of notation.

3. The Determinant Line Bundle

Over the finite dimensional Grassmannians, the determinant line bundle can be defined by taking the bundle whose fiber over a subspace is the top exterior power of that subspace. In the infinite dimensional case, we will think of a trivialization of DET as given formally by the maps

$$A \mapsto \bigwedge_{k \in S} (1 + A)e_k , \tag{3.1}$$

where $A \in I_2(W_S, W_S^\perp)$. This would be a proper definition in finite dimensions, but the infinite wedge product is unfortunately ill-defined. However, suppose

$W \in U_S \cap U_R$ is given by graph(A) in U_S, and by graph(B) in U_R. The sets R and S can differ only by a finite number of elements, and on the subspace $W_{R \cap S}$ the maps A and B must agree. Furthermore, if we write $W = g\mathscr{H}_-$, then

$$\begin{aligned} \operatorname{index}(P_- g P_-) &= \operatorname{card}(S \cap \mathbb{Z}_{>0}) - \operatorname{card}(\mathbb{Z}_{<0} \backslash S) \\ &= \operatorname{card}(R \cap \mathbb{Z}_{>0}) - \operatorname{card}(\mathbb{Z}_{<0} \backslash R)\,, \end{aligned} \tag{3.2}$$

which implies that $\operatorname{card}(S \backslash R) = \operatorname{card}(R \backslash S)$. Thus the formal expression (3.1) can be interpreted as giving rise to the well-defined transition functions,

$$g_{RS}(A) := \frac{\bigwedge_{k \in S \backslash R}(1 + A)e_k}{\bigwedge_{k \in R \backslash S}(1 + A)e_k} = \det_{\substack{j \in R \backslash S \\ k \in S \backslash R}} \{\langle e_j, Ae_k \rangle\}\,. \tag{3.3}$$

As a function of A, this determinant is non-zero precisely when A corresponds to an element of $U_S \cap U_R$. Because they involve only determinants of finite-dimensional submatrices, the functions (3.3) are clearly holomorphic. The properties needed to define transition functions, namely that $g_{RS} g_{SR} = 1$ and $g_{RS} g_{ST} g_{TR} = 1$, follow immediately from the expression of g_{RS} as a ratio of wedge products. Therefore we can take the set of g_{RS}'s as the definition of a holomorphic line bundle DET over $\mathrm{Gr}(\mathscr{H})$. This definition does depend on the choice of basis for $\mathscr{H}$. As we mentioned earlier, in the case of half densities on the circle, we think of the elements $\exp\{\pm i(k - \frac{1}{2})\theta\}$ as a canonical choice of basis determined by the parametrization of the circle.

Let Γ_D be the space of holomorphic sections the dual bundle DET$'$ (the bundle DET itself has no non-zero holomorphic sections). We will create a Hilbert space out of Γ_D', the topological dual of this space. We start by defining a map, β: DET $\times$ DET $\to \mathbb{C}$. For $p, q \in$ DET, let $\beta(p, q)$ unless p, q both lie in DET$|_{U_S}$ for some S, and in this case set

$$\beta(p, q) = \bar{\lambda}_p \lambda_q \det_{W_S}(1 + A_p^* A_q)\,. \tag{3.4}$$

The notation here is that p corresponds to (A_p, λ_p) under DET$|_{U_S} \cong U_S \times \mathbb{C}$, and similarly for q. The determinant over W_S is a Fredholm determinant, which is well-defined because A_p and A_q are Hilbert–Schmidt.

Proposition 3.1. *β defines a map* DET $\times$ DET $\to \mathbb{C}$, *which is holomorphic in the second variable, antiholomorphic in the first, and linear and antilinear on the respective fibers.*

Proof. We first need to show that β is well-defined under the transition maps of the bundle DET. To see this, suppose we have points $p, q \in$ DET which lie above A_R and B_R, respectively, in U_R, and also above A_S and B_S in U_S. We want to show that

$$\det_{W_S}(1 + A_S^* B_S) = \overline{g_{SR}(A_R)}\, g_{SR}(B_R) \det_{W_R}(1 + A_R^* B_R)\,. \tag{3.5}$$

The fact that graph(A_S) = graph(A_R), and that graph(B_S) = graph(B_R), leads to some simple identities for various submatrices of the A's and B's. These can be used to prove the following fact,

$$\begin{aligned} (1 + A_S^* B_S P_S) = {} & (P_{S \cap R} - P_{S \cap R} A_R^* P_{S \backslash R} A_S^* P_{R \backslash S} + P_{S \backslash R} A_S^* P_{R \backslash S}) \\ & \times (1 + A_R^* B_R P_R)(P_{S \cap R} - P_{R \backslash S} B_S P_{S \backslash R} B_R P_{R \cap S} \\ & + P_{R \backslash S} B_S P_{S \backslash R})\,, \end{aligned} \tag{3.6}$$

where all these expressions are considered as operators on the full space $\mathscr{H}$. The proof is easy, but somewhat tedious. Each factor in (3.6) is of the form $1 + (\textit{trace class})$, so we can use the product rule when taking the determinant of each side. The determinant of the left-hand side just gives the determinant over W_S in the expression above, and the second term on the right-hand side gives the corresponding determinant over W_R. That leaves two determinants of expressions which differ from the identity only by operators of finite rank. It is easy to see that they reduce to the finite dimensional determinants which appear in the transition functions $g_{SR}(B_R)$ and $\overline{g_{SR}(A_R)}$.

This shows that β satisfies the correct transition law. The remaining step is to prove holomorphicity. Note first that β satisfies a hermitian property,

$$\beta(q, p) = \overline{\beta(p, q)}\,, \tag{3.7}$$

so that we really need only prove holomorphicity in the second variable. The linearity on the fibers is obvious. Fix a point $p \in \mathrm{DET}$, which lies over some open set U_S. It suffices for us to prove that on each open set U_R, the function

$$f(B) = \beta(p, (B, 1)_R) \tag{3.8}$$

is holomorphic, where $(B, 1)_R \in U_R \times \mathbb{C}$ refers to a point in $\mathrm{DET}|_{U_R}$.

Clearly, if U_R and U_S lie in different connected components of $\mathrm{Gr}(\mathscr{H})$, then f is identically zero, so we can assume that the two sets lie in the same component. Over $U_S \cap U_R$, we have

$$f(B) = \bar{\lambda}_p g_{SR}(B) \det_{W_S} (1 + A_p^* B_S)\,, \tag{3.9}$$

where B_S is the point in U_S corresponding to B. The g_{SR} appears when we transform the point $(B, 1) \in U_R \times \mathbb{C}$. We can rewrite this expression as follows,

$$\begin{aligned} f(B) &= \bar{\lambda}_p \det_{j,k \in S} \{\langle (1 + A_p)e_j, (1 + B_S)e_k \rangle\} \det_{\substack{j \in S \backslash R \\ k \in R \backslash S}} \{\langle e_j, Be_k \rangle\} \\ &= \bar{\lambda}_p \det_{\substack{j \in S \\ k \in R}} \{\langle (1 + A_p)e_j, (1 + B)e_k \rangle\}\,. \end{aligned} \tag{3.10}$$

We now observe that this last expression in fact gives a well-defined, explicit formula for f on all of U_R. It is holomorphic in B because of the absolute convergence of the expansion for the Fredholm determinant. □

Now that we have the map β, we can easily define a hermitian structure on DET. Given smooth sections of DET, τ_1, τ_2, we simply take

$$\langle \tau_1, \tau_2 \rangle (W) := \beta(\tau_1(W), \tau_2(W))\,. \tag{3.11}$$

We have already remarked that β satisfies a hermitian law. The smoothness of the hermitian structure follows immediately from the holomorphicity of β.

The map β provides more than a hermitian structure, however. With it, we also obtain an inner product on the space Γ'_D. This is done as follows. An element of Γ_D can be thought of simply as a holomorphic map $\mathrm{DET} \to \mathbb{C}$, which is linear on each fiber. Clearly, we can regard β as being an element of $\bar{\Gamma}_D \otimes \Gamma_D$, wher $\bar{\Gamma}_D$ is the natural complex conjugate space to Γ_D. Given two elements $\eta, \xi \in \Gamma'_D$, we define the pairing

$$\langle \eta, \xi \rangle_\beta := (\bar{\eta} \otimes \xi) \cdot \beta\,. \tag{3.12}$$

By $\bar{\eta}$ we mean the element of $\bar{\Gamma}'_D$ which is the conjugate of η, so that $(\bar{\eta} \otimes \xi) \in \bar{\Gamma}'_D \otimes \Gamma'_D$. The pairing (3.12) is continuous, by the continuity of the evaluation map, and it is hermitian by the hermitian property of β.

It is also, in fact, positive definite. Given an index set S, we write p_s for the point $(A = 0, \lambda = 1)_S$ in DET_{U_S}. We can define an element $\gamma_S \in \Gamma'_D$ by evaluation at p_S,

$$\gamma_S(\sigma) := \sigma(p_S) , \tag{3.13}$$

because $\sigma \in \Gamma_D$ is just a map $\mathrm{DET} \to \mathbb{C}$. Note that such elements of Γ'_D form an orthonormal set,

$$\langle \gamma_S, \gamma_R \rangle_\beta = \beta(p_S, p_R) = \delta_{RS} . \tag{3.14}$$

We can also use β to define elements $\chi_S \in \Gamma_D$ by

$$\chi_S(p) := \beta(p_S, p) . \tag{3.15}$$

For a general $\xi \in \Gamma'_D$, we see easily that

$$\langle \gamma_S, \xi \rangle_\beta = \xi(\chi_S) . \tag{3.16}$$

Proposition 10.1.5 of [21] showed that the algebraic span of the χ_S is dense in Γ_D. Therefore, if $\langle \gamma_S, \xi \rangle_\beta = 0$ for every $S \in \mathscr{A}$, then $\xi = 0$. This implies that the algebraic span of the γ_S's is dense in Γ'_D. The pairing is thus positive definite, and the γ_S's form an orthonormal basis.

Definition 3.2. *$\mathscr{F}_D$ is the completion of Γ'_D in the inner product $\langle \cdot , \cdot \rangle_\beta$.*

This Hilbert space gives the fundamental representation of $U_{\mathrm{res}}(\mathscr{H})$ [21]. We will define the corresponding Hilbert space for the Pfaffian line bundle in the next section.

4. The Pfaffian Line Bundle

The space of half-densities on the circle has a natural complex conjugation, which maps $\mathscr{H}_+$ to $\mathscr{H}_-$ and vice-versa. When $\mathscr{H}$ has such a real structure, we can define a submanifold of $\mathrm{Gr}(\mathscr{H})$ over which the determinant line bundle has a holomorphic square root.

In general, by a real structure we mean a complex anti-linear map J on $\mathscr{H}$ such that $J^2 = 1$ and $\langle Jx, Jy \rangle = \overline{\langle x, y \rangle}$. In addition, for a polarized Hilbert space we will assume that $J: \mathscr{H}_\pm \to \mathscr{H}_\mp$. In the case of half-densities, J is the natural complex conjugation, which takes e_k to e_{-k}. We shall always assume that we have chosen our basis to behave this way under J. With such a complex structure, we can define a symmetric bilinear form on $\mathscr{H}$,

$$(x, y) := \langle Jx, y \rangle . \tag{4.1}$$

Using this form, we define the isotropic Grassmannian by

$$\mathrm{Gr}_I(\mathscr{H}) := \{ W \in \mathrm{Gr}(\mathscr{H}) : JW = W^\perp \} . \tag{4.2}$$

The condition that $JW = W^\perp$ implies that $\mathrm{Gr}_I(\mathscr{H})$ is a submanifold of the zero index component of $\mathrm{Gr}(\mathscr{H})$. The spaces W_S lie in $\mathrm{Gr}_I(\mathscr{H})$ if and only if $S \in \mathscr{A}_I$, where

$$\mathscr{A}_I := \{ S \in \mathscr{A} : JS = S^c \} . \tag{4.3}$$

We also note that graph(A) lies in $\mathrm{Gr}_I(\mathscr{H})$, for $A \in I_2(W_S, W_S^\perp)$, $S \in \mathscr{A}_I$, if and only if

$$(x, Ay) = -(Ax, y) \tag{4.4}$$

for all $x, y \in W$. In other words, if $A = -JA^* J$. Such an operator is called skew, and we denote by $I_2^a(W_S, W_S^\perp)$ the space of skew Hilbert–Schmidt operators from W_S to $W_S^\perp$. We thus have a cover of $\mathrm{Gr}_I(\mathscr{H})$ by open sets $V_S \cong I_2^a(W_S, W_S^\perp)$. It is easy to check that $\mathrm{Gr}_I(\mathscr{H})$ falls into two connected components, depending on whether card$(S \cap \mathbb{Z}_+)$ is odd or even.

The finite dimensional analog of the isotropic Grassmannian is easily described. For a $2n$-dimensional complex vector space with real structure, the space of n-dimensional isotropic subspaces is just the homogeneous space O_{2n}/U_n, which consists of two simply connected copies of SO_{2n}/U_n. These finite-dimensional manifolds can be successively embedded to form a nested set of submanifolds of $\mathrm{Gr}_I(\mathscr{H})$ whose union is dense. The existence of a topological square root of the determinant line bundle over SO_{2n}/U_n follows from the fact that the Chern class of this line bundle is even. We can see this as follows. First of all, for $n = 2$, $SO_4/U_2 \cong \mathbb{C}P^1$. By writing out the transition functions, one sees immediately that the determinant line bundle in this case is just $L \otimes L$, where L is the tautological bundle over $\mathbb{C}P^1$. Thus the Chern class is even in this case (and the square root is obvious). Now consider the inclusion $\mathbb{C}P^1 \to SO_{2n}/U_n$. Because $\mathbb{C}P^1$ and SO_{2n}/U_n are connected and simply connected, this inclusion induces an isomorphism in H^2. Therefore, by naturality, the Chern class of the determinant line bundle over SO_{2n}/U_n is always even.

To show the existence of a holomorphic square root, in the infinite dimensional case, we turn to the transition functions. Over $\mathrm{Gr}_I(\mathscr{H})$, with the open cover we have just described, the restriction of the determinant line bundle is defined by the same transition functions g_{RS} as before, with S and R restricted to $\mathscr{A}_I$. Note, however, that because of the condition that $JW = W^\perp$, we have $k \in S \backslash R$ if and only if $-k \in R \backslash S$. Thus we can write,

$$\begin{aligned} g_{RS}(A) &= \det_{\substack{j \in R\backslash S \\ k \in S\backslash R}} \{\langle e_j, e_k \rangle\} \\ &= \det_{j,k \in S\backslash R} \{\langle e_{-j}, Ae_k \rangle\} \\ &= \det_{j,k \in S\backslash R} \{(e_j, Ae_k)\} , \end{aligned} \tag{4.5}$$

where $(\cdot\,,\cdot)$ is the bilinear form given by (4.1). This is the determinant of a skew-symmetric matrix. Because the determinant is non-zero when V_R intersects V_S, the skew matrices must be even dimensional in this case. These transition functions therefore have holomorphic square roots, given by the Pfaffian.

Hence we will define a line bundle PF over $\mathrm{Gr}_I(\mathscr{H})$ which has transition functions given by

$$h_{SR}(A) := \mathop{\mathrm{Pf}}_{j,k \in S\backslash R} \{(e_j, Ae_k)\} . \tag{4.6}$$

For the moment, we allow an arbitrary choice of orientation for the Pfaffians in these functions. We must check that the cocycle conditions, $h_{RS}h_{ST}h_{TR} = 1$, are satisfied. Because the Pfaffian is the square root of the determinant, we know these identities are satisfied up to sign. To check the sign, we note that the Pfaffian of a matrix and the Pfaffian of its inverse appear in these expressions with opposite orientations, where by opposite orientation we mean $e_{t_1} \wedge \ldots \wedge e_{t_{2n}}$ replaced by $e_{t_{2n}} \wedge \ldots \wedge e_{t_1}$. This involves a change of sign of $(-1)^n$. It is a simple fact about Pfaffians that when A is an invertible $2n$-dimensional matrix,

$$\mathrm{Pf}(A)\,\mathrm{Pf}(A^{-1}) = (-1)^n\,, \tag{4.7}$$

so that these signs always work out correctly.

Theorem 4.1. *As defined by the transition functions* (4.6), PF *is a holomorphic line bundle over the isotropic Grassmannian. Furthermore,* PF *is a square-root of the determinant line bundle in the sense that*

$$\mathrm{PF} \otimes \mathrm{PF} \cong \mathrm{DET}|_{\mathrm{Gr}_I(\mathscr{H})}\,. \tag{4.8}$$

Proof. The preceding discussion demonstrated the consistency of the transition functions. Because they involve only finitely many variables and the finite dimensional Pfaffian function is holomorphic, they are clearly holomorphic functions on the sets V_S. Because the square of the Pfaffian function is the determinant, the second property follows from comparing the transition functions (4.6) to the restrictions of the transition functions of DET to $\mathrm{Gr}_I(\mathscr{H})$ (4.5). □

We still have not specified the orientation with which the Pfaffians in the transition functions are to be defined, but we will show now that the condition (4.8) determines PF up to a holomorphic isomorphism, so that any consistent choice of orientation for the transition functions will give us an equivalent line bundle. Later, we will fix a particular choice of orientations for convenience.

Theorem 4.2. PF *is a unique holomorphic square-root of* DET *over* $\mathrm{Gr}_I(\mathscr{H})$.

Proof. We will show in general that holomorphic square roots of line bundles over $\mathrm{Gr}_I(\mathscr{H})$ are unique topologically and holomorphically. Because $\mathrm{Gr}_I(\mathscr{H})$ has a collection of finite dimensional submanifolds whose union is dense, as we mentioned above, a holomorphic line bundle on $\mathrm{Gr}_I(\mathscr{H})$ is specified completely by its restrictions to these submanifolds. Thus it suffices to prove that holomorphic square roots of holomorphic line bundles are unique on O_{2n}/U_n for each n, or rather SO_{2n}/U_n, since O_{2n}/U_n consists of two copies of SO_{2n}/U_n. Smooth complex line bundles on SO_{2n}/U_n are classified completely by Chern classes in $H^2(SO_{2n}/U_n, \mathbb{Z})$, which one can easily check to be isomorphic to $\mathbb{Z}$ for all n using the fibration $U_n \to SO_{2n} \to SO_{2n}/O_n$. This means at least that square roots are unique up to smooth isomorphism. Holomorphic line bundles are classified by elements of $H^1(SO_{2n}/U_n, \mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of non-vanishing holomorphic functions. The short exact sequence of sheaves,

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0\,, \tag{4.9}$$

where $\mathcal{O}$ is the sheaf of holomorphic functions, gives us the cohomology sequence

$$H^1(SO_{2n}/U_n, \mathcal{O}) \to H^1(SO_{2n}/U_n, \mathcal{O}^*) \to H^2(SO_{2n}/U_n, \mathbb{Z}) \to H^2(SO_{2n}/U_n, \mathcal{O})\,. \tag{4.10}$$

Because SO_{2n}/U_n is a simply-connected Kähler manifold, we can apply the results of [9] to see that $H^1(SO_{2n}/U_n, \mathcal{O}) = H^2(SO_{2n}/U_n, \mathcal{O}) = 0$. Thus we have the isomorphism $H^1(SO_{2n}/U_n, \mathcal{O}^*) \cong H^2(SO_{2n}/U_n, \mathbb{Z})$. We conclude that the holomorphic square root of a line bundle is unique. □

We note briefly that the determinant line bundle can be obtained directly from the Pfaffian line bundle construction. Suppose we are given a polarized Hilbert space $\mathscr{H}$ without any particular complex structure. Define a new Hilbert space $\mathscr{X} = \mathscr{H} \oplus \mathscr{H}'$. We give this space the polarization $\mathscr{X}_- = \mathscr{H}_- \oplus \mathscr{H}'_+, \mathscr{X}_+ = \mathscr{H}_+ \oplus \mathscr{H}'_-$. $\mathscr{X}$ has a natural complex conjugation which is compatible with this polarization, given by the canonical anti-linear map from a Hilbert space to its dual (i.e., $x \mapsto \langle x, \cdot \rangle$). There is a natural embedding of $\mathrm{Gr}(\mathscr{H})$ into $\mathrm{Gr}_I(\mathscr{X})$, and the bundle DET over $\mathrm{Gr}(\mathscr{H})$ is just given by the pullback of PF by this embedding.

Let Γ_P be the space of holomorphic sections of PF$'$. We construct an inner product on this space just as we did in the determinant case. We will define a map $\alpha\colon \mathrm{PF} \times \mathrm{PF} \to \mathbb{C}$, which is in some sense the square root of the map β we used earlier. First, note that for $B \in I_2^a(W_S, W_S^\perp)$, the map $(1 - P_S)BP_S$ on the full space $\mathscr{H}$ lies in $I_2^a(\mathscr{H})$, where P_S is the orthogonal projection on to the subspace W_S. This is easy to check,

$$\begin{aligned} J((1 - P_S)BP_S)^* J &= JP_S B^*(1 - P_S) J \\ &= (1 - P_S) JB^* JP_S \\ &= -(1 - P_S)BP_S . \end{aligned} \tag{4.11}$$

The relative Pfaffian is defined for operators in $I_2^a(\mathscr{H})$, and is a holomorphic square root of the relative Fredholm determinant [13]. For the rest of this paper we will use the notation

$$\underset{W_S}{\mathrm{Pf}}(A, B) := \mathrm{Pf}(P_S A(1 - P_S), (1 - P_S)BP_S) , \tag{4.12}$$

where $A \in I_2^a(W_S^\perp, W_S)$ and $B \in I_2^a(W_S, W_S^\perp)$. Note that

$$\begin{aligned} \underset{W_S}{\mathrm{Pf}}(JAJ, B)^2 &= \underset{\mathscr{H}}{\det}(1 + P_S A^* BP_S) \\ &= \underset{W_S}{\det}(1 + A^* B) , \end{aligned} \tag{4.13}$$

which is the expression which appears in the definition of β. It is clear that we should define the map α as follows. We set $\alpha(p, q) = 0$ unless p, q are both in $\mathrm{PF}|_{V_S}$ for some S, and in this case

$$\alpha(p, q) = \lambda_q \underset{W_S}{\mathrm{Pf}}(JA_p J, A_q) . \tag{4.14}$$

Proposition 4.3. *α defines a map* $\mathrm{PF} \times \mathrm{PF} \to \mathbb{C}$, *which is holomorphic in the second variable, antiholomorphic in the first, and linear and antilinear on the respective fibers.*

Proof. The proof follows almost entirely from the corresponding result for β proven in Proposition 3.1. The one point to check is the following. By taking the square root of the corresponding equation for β, we know that

$$\underset{W_S}{\mathrm{Pf}}(JA_S J, B_S) = \pm \overline{h_{RS}(A_S)} h_{RS}(B_S) \underset{W_R}{\mathrm{Pf}}(JA_R J, B_R) , \tag{4.15}$$

where $A_S \in V_S$ corresponds to $A_R \in V_R$, and likewise for the B's. We must check that the sign is positive. To fix the sign, we let $A = B$. Then we have

$$\underset{W_S}{\mathrm{Pf}}\,(JA_S\,J, A_S) = \pm\,|h_{RS}(A_S)|^2\,\underset{W_R}{\mathrm{Pf}}\,(JA_R\,J, A_R)\,. \tag{4.16}$$

The Pfaffians are both positive by continuity, since $\mathrm{Pf}(0,0) = 1$, and $\mathrm{Pf}(JAJ, A)^2 = \det(1 + A^*A) \geqq 1$. Thus the overall sign in Eq. (4.15) is $(+)$, as desired. □

This proposition tells us that $\alpha \in \bar{\Gamma}_P \otimes \Gamma_P$, and we can use it to define an inner product $\langle\cdot,\cdot\rangle_\alpha$ on Γ'_P, exactly as in the determinant case.

Definition 4.4. *$\mathscr{F}_P$ is the completion of Γ'_P in the inner product $\langle\cdot,\cdot\rangle_\alpha$.*

We obtain an orthonormal basis $\{\gamma_S\}$ for $\mathscr{F}_P$, indexed by $S \in \mathscr{A}_I$, in the same way as for $\mathscr{F}_D$.

5. The Fock Space

Define the positive energy Fock space $\mathscr{F}_+$ as the completion of the full exterior algebra of $\mathscr{H}_+$,

$$\bigwedge \mathscr{H}_+ := \bigoplus_{n=1}^{\infty} \wedge^n \mathscr{H}_+\,,$$

in the inner product

$$\langle x_1 \wedge \ldots \wedge x_n, y_1 \wedge \ldots \wedge y_m\rangle := \delta_{nm} \det\{\langle x_i, y_j\rangle\}\,. \tag{5.1}$$

It is clear that the elements of the form $e_{s_1} \wedge \ldots \wedge e_{s_n}$, for all sets of positive integers, form an orthonormal basis for $\mathscr{F}_+$. An element S of $\mathscr{A}_I$ is completely specified by the subset $S \cap \mathbb{Z}_+$. If we denote the elements of $S \cap \mathbb{Z}_+$ by $\{s_1, \ldots, s_n\}$, then we have the obvious isomorphism $\rho : \mathscr{F}_P \to \mathscr{F}_+$ given by

$$\rho(\gamma_S) := e_{s_1} \wedge \ldots \wedge e_{s_n}\,, \tag{5.2}$$

where $\{\gamma_S\}$ is the orthonormal basis for $\mathscr{F}_P$. This isomorphism is singled out by our choice of basis. There is no canonical way to relate the γ_S under change of basis of $\mathscr{H}$.

Since an element of PF corresponds to an element of $\mathscr{F}_P$ by the evaluation map, we can ask how the isomorphism acts on the elements of PF. For example, suppose we take a point $(A, \lambda)_S$ in $\mathrm{PF}|_{V_S}$. Considering the point $(A, \lambda)_S$ as an element of $\mathscr{F}_P$, we can expand

$$\begin{aligned}(A, \lambda)_S &= \sum_{R \in \mathscr{A}_I} \langle \gamma_R, (A, \lambda)_S\rangle_\alpha \gamma_R \\ &= \sum_{R \in \mathscr{A}_I} \alpha(p_R, (A, \lambda)_S)\gamma_R \\ &= \sum_{R \in \mathscr{A}_I} \lambda h_{RS}(A)\gamma_R\,. \end{aligned} \tag{5.3}$$

Here and elsewhere, we will use the same notation $(A, \lambda)_S$ to refer to a point of PF and a point in $\mathscr{F}_P$ via the evaluation map. We also adopt the convention that $h_{RS}(A) = 0$ if $V_R \cap V_S$ is empty. Equation (5.3) implies that

$$\begin{aligned} \rho((A, \lambda)_S) &= \lambda \sum_{R \in \mathscr{A}_I} h_{RS}(A) \rho(\gamma_R) \\ &= \lambda \sum_{R \in \mathscr{A}_I} h_{RS}(A) e_{r_1} \wedge \ldots \wedge e_{r_n} . \end{aligned} \tag{5.4}$$

Let S_0 denote the index set $\mathbb{Z}_- \in \mathscr{A}_I$, and for convenience let V_0 denote V_{S_0}. We can define a map ϕ from $V_0 \cong I_2^a(\mathscr{H}_-, \mathscr{H}_+)$ to the completion of $\wedge^2 \mathscr{H}_+$ by

$$\phi(A) := \frac{1}{2} \sum_{i,j \in \mathbb{Z}_+} (e_{-i}, Ae_{-j}) e_i \wedge e_j . \tag{5.5}$$

This map is well-defined, because

$$\| \phi(A) \|_{\wedge^2 \mathscr{H}_+} = \frac{1}{2} \| A \|_2 . \tag{5.6}$$

Because the space $\mathscr{F}_+$ has an exterior algebra structure, we can take the exponential of an element. It in fact follows from the definition of the finite dimensional Pfaffian that

$$e^{\phi(A)} = \sum_{v \subset \mathbb{Z}_+} \operatorname*{Pf}_{1 \leqq i,j \leqq n} \{(e_{-v_i}, Ae_{-v_j})\} e_{v_1} \wedge \ldots \wedge e_{v_n} . \tag{5.7}$$

Hence, if we orient the transition functions so that

$$h_{RS_0}(A) = \operatorname*{Pf}_{1 \leqq i,j \leqq n} \{(e_{-r_i}, Ae_{-r_j})\} , \tag{5.8}$$

then the map ρ has a particularly nice form

$$\rho((A, \lambda)_{S_0}) = \lambda e^{\phi(A)} . \tag{5.9}$$

Because ρ is a Hilbert space isomorphism, this expression implies

$$\operatorname*{Pf}_{\mathscr{H}_-} (JAJ, B) = \langle e^{\phi(A)}, e^{\phi(B)} \rangle . \tag{5.10}$$

This result, which is quite simple to prove directly, was first pointed out in [21].

We can develop a formula similar to (5.9) for points lying over an arbitrary subspace V_S, provided we choose the appropriate orientations for transition functions. First we introduce the action of a Clifford algebra on $\mathscr{F}_+$. Because of the exterior algebra structure, we can let an element of $\mathscr{F}_+$ act on $\mathscr{F}_+$ by exterior multiplication, or by interior multiplication. For $j > 0$, let χ_j be the operator which acts as the sum of exterior multiplication by e_j plus interior multiplication by e_j. Thus, if ω is a form such that $e_j \wedge \omega \neq 0$, then we have

$$\chi_j(\omega) = e_j \wedge \omega, \quad \chi_j(e_j \wedge \omega) = \omega . \tag{5.11}$$

The χ's are clearly self-adjoint, and they satisfy the anticommutation relations,

$$\{\chi_j, \chi_k\} = \delta_{jk} . \tag{5.12}$$

Thus they form an infinite dimensional Clifford algebra which acts unitarily on $\mathscr{F}_+$.

Let σ_j be the element of $U_{\mathrm{res}}(\mathscr{H})$ whose sole effect is to interchange the basis elements e_j and e_{-j}. As above, we identify the index sets $S \in \mathscr{A}_I$ with sets of integers $\{s_1, \ldots, s_k\} = S \cap \mathbb{Z}$, arranged in increasing order. For $A \in V_0$, we define a new map $A^S\colon W_S \to W_S^\perp$ by

$$A^S := (\sigma_{s_1} \ldots \sigma_{s_k}) A (\sigma_{s_1} \ldots \sigma_{s_k}) . \tag{5.13}$$

This map A^S lies in V_S. In fact, the pairing of A to A^S gives an isomorphism between V_0 and V_S. Now we can make the following proposition.

Proposition 5.1. *We can choose orientations for the transition functions h_{RS} so that for $A \in V_0$, we have*

$$\rho((A^S, \lambda)_S) = \lambda \chi_{s_1} \ldots \chi_{s_k} e^{\phi(A)} , \tag{5.14}$$

where $(A^S, \lambda)_S$ represents a point in $\mathscr{F}_P$ by the evaluation map.

Proof. We have already noted that ρ has the general form

$$\rho((A^S, \lambda)_S) = \lambda \sum_R h_{RS}(A^S) e_{r_1} \wedge \ldots \wedge e_{r_n} . \tag{5.15}$$

The function $h_{RS}(A^S)$ can be reduced as follows,

$$\begin{aligned} h_{RS}(A^S) &= \pm \operatorname*{Pf}_{i,j \in S\backslash R} \{(e_i, A^S e_k)\} \\ &= \pm \operatorname*{Pf}_{i,j \in S\backslash R} \{(e_i, (\sigma_{s_1} \ldots \sigma_{s_k}) A (\sigma_{s_1} \ldots \sigma_{s_k}) e_k)\} \\ &= \pm \operatorname*{Pf}_{i,j \in S_0\backslash T} \{(e_i, A e_k)\} , \end{aligned} \tag{5.16}$$

where $T = (\sigma_{s_1} \ldots \sigma_{s_k}) R$ (the σ's act on $\mathscr{A}_I$ by interchanging j and $-j$). This last expression is just the transition function $h_{TS_0}(A)$, so that

$$\rho((A^S, \lambda)_S) = \pm \lambda \sum_T h_{TS_0}(A) e_{r_1} \wedge \ldots \wedge e_{r_n} . \tag{5.17}$$

The sign of course depends on the orientations we choose for the Pfaffians. Now we simply observe that $T = (\sigma_{s_1} \ldots \sigma_{s_k}) R$ implies that

$$e_{r_1} \wedge \ldots \wedge e_{r_n} = \pm \chi_{s_1} \ldots \chi_{s_k} (e_{t_1} \wedge \ldots \wedge e_{t_m}) . \tag{5.18}$$

Thus we have

$$\begin{aligned} \rho((A^S, \lambda)_S) &= \pm \chi_{s_1} \ldots \chi_{s_k} \sum_{T \in \mathscr{A}_I} h_{TS_0}(A) (e_{t_1} \wedge \ldots \wedge e_{t_m}) \\ &= \pm \chi_{s_1} \ldots \chi_{s_k} e^{\phi(A)} . \end{aligned} \tag{5.19}$$

We can fix the orientation so that the sign is positive as follows. Suppose that $S\backslash R = \{q_1, \ldots, q_m\}$, with the q's in increasing order, $q_1 < \ldots < q_m$. Define $\varepsilon_{RS} = \pm 1$ by the equation,

$$\chi_{s_1} \ldots \chi_{s_k} (e_{t_1} \wedge \ldots \wedge e_{t_m}) = \varepsilon_{RS} e_{r_1} \wedge \ldots \wedge e_{r_n} . \tag{5.20}$$

The transition function h_{RS} involves a Pfaffian over the vector space which is the span of $\{e_{q_1}, \ldots, e_{q_m}\}$. We now specify the orientation on this vector space by the top form,

$$\varepsilon_{RS} e_{q_m} \wedge \ldots \wedge e_{q_1} , \tag{5.21}$$

where ε_{RS} is determined from Eq. (5.20). It is straightforward to check that these choices of orientation imply the condition (5.14). □

The composition of the evaluation map $\mathrm{PF} \to \mathscr{F}_P$ with the map $\rho: \mathscr{F}_P \to \mathscr{F}_+$ gives us a map from PF to $\mathscr{F}_+$. The fiber of PF over any point of $\mathrm{Gr}_I(\mathscr{H})$ maps to a ray in $\mathscr{F}_+$. Thus we obtain a map τ from $\mathrm{Gr}_I(\mathscr{H})$ to the projective space $P(\mathscr{F}_+)$, given by

$$\tau(A^S) := [\chi_{s_1} \dots \chi_{s_k} e^{\phi(A)}] . \tag{5.22}$$

As a projective space, $P(\mathscr{F}_+)$ has a tautological holomorphic line bundle $\mathscr{L}$, whose fiber over a ray is the ray itself.

Proposition 5.2. PF *is the pullback by the map* τ *of the tautological line bundle* $\mathscr{L}$ *over* $P(\mathscr{F}_+)$.

Proof. It is simple to check that the trivialization with which we have defined PF is recovered from the maps $\eta_s: V_S \to \mathscr{L}$ given by

$$\eta_s: (A^S, \lambda)_S \mapsto (\tau(A^S), \rho((A^S, \lambda)_S)) , \tag{5.23}$$

where ρ was given by (5.14). □

This proposition connects the definition of PF in Sect. 4 with the definition used in [21].

6. The Representation in Finite Dimensions

Because the discussion of the restricted orthogonal group and its representations on sections of PF′ in the following section is somewhat dense with analytical details, we give here a brief discussion of the situation in finite dimensions. See [21] for a more thorough treatment of the finite dimensional case.

The basic philosophy of the construction of irreducible representations on the space of holomorphic sections of a homogeneous line bundle comes from the Borel–Weil theorem [9]. As an example of this theorem, let $X = \mathbb{C}^n$, and consider the irreducible representation of U_n on the k^{th} exterior power $\wedge^k X'$. The ray defined by the vector $\Omega := \alpha_1 \wedge \dots \wedge \alpha_k \in \wedge^k X'$, where $\{\alpha_j\}$ is a basis for X', is invariant under the subgroup $U_k \times U_{n-k}$ of U_n. By considering complexifications, we see that the orbit of Ω defines a holomorphic map $U_n/U_k \times U_{n-k} \to P(\wedge^k X')$. Associated to such a map is a complex line bundle L and a map $\wedge^k X \to \Gamma(L')$. This line bundle L is just the determinant line bundle over $\mathrm{Gr}_k(X)$, and the map $\wedge^k X \to \Gamma(L')$ is an isomorphism.

In Sect. 7, we essentially give the analog of this Borel–Weil construction for a generalization of the infinite dimensional wedge representations which were described algebraically in [14]. In finite dimensions this amounts to a construction of the spin representation of SO_{2n}. Let X be a finite $2n$-dimensional complex vector space, with real structure J and decomposed into $X_- \oplus X_+$. Let PF_X be the Pfaffian line bundle over the even component of $\mathrm{Gr}_I(X)$. We will sketch the construction of the spin representation on F, the even degree subspace of the full exterior algebra $\bigwedge X_+$. Then we will show that the action on F, which is isomorphic to the space of holomorphic sections of PF'_X, comes from a holomorphic action on PF_X. The extension of the representation to both components of the Pfaffian line bundle is simple and will be done for the infinite dimensional case in the next section.

Given $g \in SO_{2n}$, let

$$g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.1}$$

with respect to the decomposition $X_- \oplus X_+$, i.e. a is a map from $X_- \to X_-$, etc. Let U_g be the subset of V_0 consisting of the skew maps $A: X_- \to X_+$ for which $(a + bA)$ is invertible. U_g is just the subset of V_0 whose image under the action of g on $\mathrm{Gr}_I(X)$ still lies in V_0. This action takes $A \in U_g$ to

$$A^g := (c + dA)(a + bA)^{-1} . \tag{6.2}$$

We will see in Sect. 7 that the elements of F of the form $e^{\phi(A)}$, where $A \in U_g$, span F. Since we want the action of g on F to cover the action on $\mathrm{Gr}_I(X)$, by Proposition 5.1 we would like to set

$$g \cdot e^{\phi(A)} = \mu_g(A) e^{\phi(A^g)} , \tag{6.3}$$

where μ_g is a numerical factor which depends holomorphically on A. For the representation to be unitary, we see from the form (5.10) of the inner product on F that we need to have

$$\mu_g(A)^2 = \det(a + bA) . \tag{6.4}$$

This requirement is the key to the difference between the spin representation in finite and infinite dimensions. For, in the finite dimensional case, Eq. (6.4) has two solutions,

$$\mu_g(A)^2 = \pm [\det a]^{1/2} \operatorname{Pf}(a^{-1}b, A) . \tag{6.5}$$

It is not possible to make a global choice of sign here for all of SO_{2n}, because $\pi_1(SO_{2n}) = \mathbb{Z}_2$. We can, however, define a projective representation which comes from an honest unitary representation of Spin_{2n}, the simply connected double cover of SO_{2n}. The difference in infinite dimensions is that the square root of the determinant of a becomes ill-defined. We have to define the numerical factor μ in a more involved way, resulting in a cyclic extension of the group rather than a double cover. The fundamental group of the infinite dimensional group $SO_{\mathrm{res}}(\mathscr{H})$, which will be defined in the next section, is no longer $\mathbb{Z}_2$.

To continue the finite dimensional discussion, define the group Spin_{2n} to consist of pairs (g, μ_g), such that $g \in SO_{2n}$ and $\mu_g: U_g \to \mathbb{C}$ satisfies Eq. (6.4). We have seen that Spin_{2n} acts on F, but have not yet shown that this comes from an action on PF_X. Let $i: \mathrm{PF}_X \to F$ be the inclusion, as described at the end of Sect. 5, consisting of the composition of the evaluation map and ρ. We need to see that the action preserves the image of i, and, moreover, that the resulting action on PF_X is holomorphic. Let $\xi = (g, \mu_g)$, and for index sets R, T, let U_g^{RT} be the set of all $B \in V_R$ such that g maps the subspace graph(B) into V_T. All that we need do is find a holomorphic function f_ξ on U_g^{RT} such that the action (6.3) corresponds under i to

$$\xi \cdot (B, 1)_R = (g \cdot B, f_\xi(B))_T , \tag{6.6}$$

where $B \in U_g^{RT}$. Suppose that B corresponds to a point $C \in U_g$. Then one can easily check from (6.3) and the definition of i that we would have

$$\xi \cdot (B, 1)_R = (g \cdot B, \mu_g(C) h_{TS_0}(C^g) h_{s_0 R}(B)) . \tag{6.7}$$

Thus we will be finished if we can extend the holomorphic function,

$$f(B) = \mu_g(C) h_{TS_0}(C^g) h_{S_0 R}(B) \tag{6.8}$$

(C depends holomorphically on B), from $U_g^{RT} \cap U_g$ to all of U_g^{RT}. Since the complement of U_g is defined by the vanishing of a holomorphic function, namely $\det(a + bA)$, in finite dimensions we can simply apply the Riemann extension theorem.

7. The Restricted Orthogonal Group

We were able to realize the manifold $\mathrm{Gr}(\mathscr{H})$ as a homogeneous space for the restricted unitary group, given by (2.2). We can do the same for the submanifold $\mathrm{Gr}_I(\mathscr{H})$, using the restricted orthogonal group. By definition, any element of $\mathrm{Gr}(\mathscr{H})$ has the form $g\mathscr{H}_-$, for some $g \in U_{\mathrm{res}}(\mathscr{H})$. To define an element of $\mathrm{Gr}_I(\mathscr{H})$, we require further that an operator g preserve the bilinear form on $\mathscr{H}$, i.e. that g commute with the conjugation operator J. We thus define

$$O_{\mathrm{res}}(\mathscr{H}) := \{g \in U_{\mathrm{res}}(\mathscr{H}) : Jg = gJ\} . \tag{7.1}$$

To write $\mathrm{Gr}_I(\mathscr{H})$ as a homogeneous space, we need to know when two different elements of $O_{\mathrm{res}}(\mathscr{H})$ act on $\mathscr{H}_-$ to give the same subspace. It is clear that $g\mathscr{H}_- = gh\mathscr{H}_-$ for $h \in U(\mathscr{H}_-) \times U(\mathscr{H}_+)$, since such an h preserves H_- and $\mathscr{H}_+$. The combination gh will be an element of $O_{\mathrm{res}}(\mathscr{H})$, however, only if $[J, h] = 0$. This condition is satisfied only if h is of the form (u, JuJ), for $u \in U(\mathscr{H}_-)$. Therefore we can write

$$\mathrm{Gr}_I(\mathscr{H}) = O_{\mathrm{res}}(\mathscr{H})/U(\mathscr{H}_-) , \tag{7.2}$$

where $U(\mathscr{H}_-)$ acts on the right by its embedding $u \mapsto (u, JuJ)$.

The group $O_{\mathrm{res}}(\mathscr{H})$ splits into two connected components. For any $g \in O_{\mathrm{res}}(\mathscr{H})$, $\mathrm{index}(\mathscr{P}_- g P_-) = 0$. The components are determined by whether the kernel and cokernel of $P_- g P_-$ have even or odd dimension. The identity component forms a group, which we will label $SO_{\mathrm{res}}(\mathscr{H})$. Because $\mathrm{Gr}_I(\mathscr{H})$ is a homogeneous space with respect to the orthogonal group, we would expect to have at least a projective action of $SO_{\mathrm{res}}(\mathscr{H})$ on the line bundle. We will define this projective action by taking a central extension, $\widetilde{SO}_{\mathrm{res}}(\mathscr{H})$, of $SO_{\mathrm{res}}(\mathscr{H})$. We will show that there is a unitary representation of $SO_{\mathrm{res}}(\mathscr{H})$ on the space $\mathscr{F}_+$, which comes from an action on PF. At the end of this section we will show how to apply these results to the full group, $O_{\mathrm{res}}(\mathscr{H})$.

The first step is to understand the action of $SO_{\mathrm{res}}(\mathscr{H})$ on $\mathrm{Gr}_I(\mathscr{H})$. Suppose a point of $\mathrm{Gr}_I(\mathscr{H})$ is represented by $A \in V_S$. We want to find the point which corresponds to $g(\mathrm{graph}(A))$, for $g \in SO_{\mathrm{res}}(\mathscr{H})$. If we assume that this point lies in V_R, then for some $B \in V_R$ and some isomorphism $q : W_S \to W_R$, we must have

$$g\begin{pmatrix} 1 \\ A \end{pmatrix} = \begin{pmatrix} 1 \\ B \end{pmatrix} q . \tag{7.3}$$

Take $S \doteq R = S_0$, where, as before, S_0 is the set of all negative integers. If we let

$$g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7.4}$$

with respect to the decomposition $\mathscr{H}_- \oplus \mathscr{H}_+$, then we see from Eq. (7.3) that $q = (a + bA)$, and that $B = (c + dA)(a + bA)^{-1}$. Let U_g be the set of all $A \in V_0$ such that $(a + bA)$ is invertible. For $A \in U_g$, define

$$A^g := (c + dA)(a + bA)^{-1}, \tag{7.5}$$

as in Sect. 6.

Because $SO_{\mathrm{res}}(\mathscr{H})$ preserves the two connected components of $\mathrm{Gr}_I(\mathscr{H})$, it will also preserve the decomposition of $\mathscr{F}_+$ into the subspaces of even and odd degree. We will first consider the action on the space of even degree, which we will denote by $\mathscr{F}_+^{\text{even}}$.

Lemma 7.1. *For any open set $U \subset V_0$, the elements of the form $e^{\phi(A)}$ for $A \in U$ span $\mathscr{F}_+^{\text{even}}$.*

Proof. We can assume that U is a neighborhood of $A = 0$, because we can always translate. That is, suppose U is a neighborhood of the point A_0, and we want to expand x. If we can expand

$$e^{\phi(A_0)}x = \sum_A \lambda_A e^{\phi(A)}, \tag{7.6}$$

with sum over A's lying in a neighborhood of $A = 0$, then we will have

$$x = \sum_A \lambda_A e^{\phi(A + A_0)}, \tag{7.7}$$

with $A + A_0$ lying in a neighborhood of A_0.

Given a basis element $e_{r_1} \wedge \dots \wedge e_{r_k}$ of $\mathscr{F}_+^{\text{even}}$, with k even, we can always find an $A \in V_0$ so that

$$\phi(A) = \kappa(e_{r_1} \wedge e_{r_2} + \dots + e_{r_{k-1}} \wedge e_{r_k}), \tag{7.8}$$

and with κ sufficiently small we can find such a point in any neighborhood of the identity. Exponentiating gives

$$e^{\phi(A)} = \kappa(e_{r_1} \wedge \dots \wedge e_{r_k}) + \text{lower degree terms}.$$

Thus we can recover all of the basis elements, and hence span all of $\mathscr{F}_+^{\text{even}}$. $\square$

Because of this lemma, we can define the action for each g by specifying it on elements of $\mathscr{F}_+^{\text{even}}$ of the form $e^{\phi(A)}$ for all $A \in U_g$. The action will preserve the inner product, so that it will automatically extend to all of $\mathscr{F}_+^{\text{even}}$. If the action is to cover the action of $\widetilde{SO}_{\mathrm{res}}(\mathscr{H})$ on $\mathrm{Gr}_I(\mathscr{H})$, then we will have to have

$$g \cdot e^{\phi(A)} = \mu_g(A) e^{\phi(A^g)}, \tag{7.9}$$

where μ_g is some numerical factor depending holomorphically on A. The inner product which the action must preserve is

$$\langle e^{\phi(A)}, e^{\phi(B)} \rangle = \operatorname*{Pf}_{\mathscr{H}_-}(JAJ, B). \tag{7.10}$$

If we transform A and B by g, then the inner product becomes

$$\langle e^{\phi(A^g)}, e^{\phi(B^g)}\rangle = \underset{\mathscr{H}_-}{\mathrm{Pf}}\,(JA^gJ, B^g)\,. \tag{7.11}$$

Recall that A^g was defined so that

$$\begin{pmatrix} 1 \\ A^g \end{pmatrix} = g\begin{pmatrix} 1 \\ A \end{pmatrix}(a + bA)^{-1}\,. \tag{7.12}$$

We can thus evaluate

$$\begin{aligned}\langle e^{\phi(A^g)}, e^{\phi(B^g)}\rangle^2 &= \underset{\mathscr{H}_-}{\det}\,(1 + (A^g)^*B^g) \\ &= \det\left[(a + bA)^{*^{-1}}\begin{pmatrix} 1 \\ A \end{pmatrix}^* g^*g\begin{pmatrix} 1 \\ B \end{pmatrix}(a + bB)^{-1}\right] \\ &= \frac{\det(1 + A^*B)}{\det(a + bB)(a + bA)^*}\,. \end{aligned} \tag{7.13}$$

This means we would like to choose the factor $\mu_g(A)$ so that

$$\overline{(\mu_g(A)}\mu_g(B))^2 = \det(a + bB)(a + bA)^*\,. \tag{7.14}$$

In the finite dimensional discussion of Sect. 6, this amounted to choosing one of two square roots. In infinite dimensions it is not possible to choose any square root. We are thus led to a more complicated procedure, which ends up giving a central extension of $SO_{\mathrm{res}}(\mathscr{H})$, as follows.

The reason (7.14) has no solution is because there is no infinite dimensional Pfaffian analogous to the Fredholm determinant. We have only the relative Pfaffian. It turns out that we should define the numerical factor as a function of two variables,

$$\mu_g(A, B) := \underset{\mathscr{H}_+}{\mathrm{Pf}}\,(A - B, (a + bA)^{-1}b)\,. \tag{7.15}$$

This relative Pfaffian is well-defined because both arguments are skew Hilbert–Schmidt operators. This is obvious for the first argument, $A - B$. The second argument is Hilbert–Schmidt because b is Hilbert–Schmidt, and we see that it is skew as follows. The fact that $gJg^*J = 1$ implies that bJA^*J is skew, and the fact that A is skew implies that $bAJB^*J$ is also skew. The difference of these two terms is

$$b(Ja^*J - AJb^*J) = bJ(a + bA)^*J\,, \tag{7.16}$$

so that $bJ(a + bA)^*J$ is skew,

$$bJ(a + bA)^*J = -(a + bA)Jb^*J\,. \tag{7.17}$$

By applying $(a + bA)^{-1}$ on the left and $(a + bA)^{*^{-1}}$ on the right, we get

$$(a + bA)^{-1}b = -Jb^*(a + bA)^{*^{-1}}J\,, \tag{7.18}$$

so that $(a + bA)^{-1}b$ is skew.

With μ depending on two variables, we will now try to replace the relation (7.9) with something like

$$g \cdot e^{\phi(A)} := \mu_g(B, A)e^{\phi(A^g)}\,. \tag{7.19}$$

Here B is an extra parameter, the choice of which will need to be included in our extension of $SO_{\text{res}}(\mathscr{H})$. We will also need to revise Eq. (7.14). Taking the square of μ_g gives

$$\begin{aligned}\mu_g(A, B)^2 &= \det(1 - (A - B)(a + bA)^{-1} b) \\ &= \det(1 - b(A - B)(a + bA)^{-1}) ,\end{aligned} \tag{7.20}$$

(we can switch the order of b and $(A - B)(a + bA)^{-1}$ because both are Hilbert–Schmidt). We can simplify

$$1 - b(A - B)(a + bA)^{-1} = (a + bB)(a + bA)^{-1} ,$$

so that

$$\mu_g(A, B)^2 = \det(a + bB)(a + bA)^{-1} . \tag{7.21}$$

We cannot extract the determinant of $(a + bB)$ from this expression, but the combination $(a + bB_1)(a + bB_2)^*$ does have a determinant. Therefore, we can revise Eq. (7.14) by a constant factor which depends on the extra parameter,

$$(\overline{\mu_g(A, B_2)}\mu_g(A, B_1))^2 = \frac{\det(a + bB_1)(a + bB_2)^*}{\det(a + bA)^*(a + bA)} . \tag{7.22}$$

We will take care of the extra factor (the denominator) through the following extension of $SO_{\text{res}}(\mathscr{H})$.

Definition 7.2. *$\widetilde{SO}_{\text{res}}(\mathscr{H})$ is the group whose elements consist of triples*

$$(g, A, \lambda) \in SO_{\text{res}}(\mathscr{H}) \times I_2^a(\mathscr{H}_-, \mathscr{H}_+) \times \mathbb{C}^\times , \tag{7.23}$$

such that $A \in U_g$ and

$$|\lambda|^2 = \sqrt{\det(a + bA)^*(a + bA)} . \tag{7.24}$$

We identify two triples (g, A, λ) and (g, B, λ') when

$$\lambda' = \lambda\mu_g(B, A) . \tag{7.25}$$

The multiplication is given by

$$(g_1, A_1, \lambda_1)\cdot(g_2, A_2, \lambda_2) = (g_3, A_3, \lambda_3) , \tag{7.26}$$

where $g_3 = g_1 g_2$, $A_3 \in U_{g_3} \cap U_{g_2}$ is chosen so that $A_3^{g_2} \in U_{g_1}$, and

$$\lambda_3 = \lambda_1\lambda_2\mu_{g_1}(A_1, A_3^{g_2})\mu_{g_2}(A_2, A_3) . \tag{7.27}$$

This extension is referred to as $\text{Spin}^{\mathbb{C}}(\mathscr{H})$ in [21]. Because its definition is somewhat involved, so we will explain where the various parts come from. The requirement on the $|\lambda|$ given by Eq. (7.24) appears so that the denominator of Eq. (7.22) will be cancelled off. The identification (7.25) simply equates two elements of the group which for an action of the form (7.19) should be identical. The multiplication laws (7.26) and (7.27) also follows directly from the action (7.19).

We can easily see that $\widetilde{SO}_{\text{res}}(\mathscr{H})$ is a cyclic extension of $SO_{\text{res}}(\mathscr{H})$. Suppose we are given two elements lying over g, which are represented by triples (g, A, λ) and (g, B, λ'). The equivalence relation (7.25) tells us that

$$(g, B, \lambda') = (g, A, \lambda'\mu_g(A, B)) . \tag{7.28}$$

Using Eqs. (7.22) and (7.24), we see that the scalar factor relating the two elements lies on the unit circle,

$$\left| \frac{\lambda'}{\lambda} \mu_g(A, B) \right|^2 = 1 , \tag{7.29}$$

so that our extension is cyclic. We will compute the Lie algebra cocyle corresponding to this extension in the next section.

Theorem 7.3. *For an element $\xi = (g, A, \lambda) \in \widetilde{SO}_{\mathrm{res}}(\mathscr{H})$, and for $B \in U_g$, let*

$$\xi \cdot e^{\phi(B)} = \lambda \mu_g(A, B) e^{\phi(B^g)} . \tag{7.30}$$

This action extends linearly to define a unitary action of $\widetilde{SO}_{\mathrm{res}}(\mathscr{H})$ on $\mathscr{F}_+^{\mathrm{even}}$.

Proof. First we check that Eq. (7.30) is compatible with the multiplication laws (7.26) and (7.27). For $j = 1, 2, 3$, let $\xi_j = (g_j, A_j, \lambda_j) \in \widetilde{SO}_{\mathrm{res}}(\mathscr{H})$, such that $\xi_1 \xi_2 = \xi_3$. Given $B \in U_{g_2}$ such that $B^{g_2} \in U_{g_1}$, we have

$$\begin{aligned} \xi_1 \cdot (\xi_2 \cdot e^{\phi(B)}) &= \xi_1 \cdot (\lambda_2 \mu_{g_2}(A_2, B) e^{\phi(B^{g_2})}) \\ &= \lambda_1 \lambda_2 \mu_{g_1}(A_1, B^{g_2}) \mu_{g_2}(A_2, B) e^{\phi(B^{g_1 g_2})} \\ &= \lambda_3 \left(\frac{\mu_{g_1}(A_1, B^{g_2}) \mu_{g_2}(A_2, B)}{\mu_{g_1}(A_1, A_3^{g_2}) \mu_{g_2}(A_2, A_3)} \right) e^{\phi(B^{g_3})} . \end{aligned} \tag{7.31}$$

We need to show that the factor is parentheses is equal to $\mu_{g_3}(A_3, B)$. The square of the numerator is given by

$$\begin{aligned} (\mu_{g_1}(A_1, B^{g_2}) \mu_{g_2}(A_2, B))^2 &= \det(a_1 + b_1 B^{g_2})(a_1 + b_1 A_1)^{-1} \\ &\quad \times \det(a_2 + b_2 B)(a_2 + b_2 A_2)^{-1} . \end{aligned} \tag{7.32}$$

Now, since

$$\begin{aligned} (a_1 + b_1 B^{g_2})(a_2 + b_2 B) &= a_1(a_2 + b_2 B) + b_1(c_2 + d_2 B) \\ &= a_3 + b_3 B , \end{aligned} \tag{7.33}$$

we can write Eq. (7.32) as

$$(\mu_{g_1}(A_1, B^{g_2}) \mu_{g_2}(A_2, B))^2 = \det(a_3 + b_3 B)(a_2 + b_2 A_2)^{-1}(a_1 + b_1 A_1)^{-1} . \tag{7.34}$$

Similarly, we can write

$$(\mu_{g_1}(A_1, A_3^{g_2}) \mu_{g_2}(A_2, A_3))^2 = \det(a_3 + b_3 A_3)(a_2 + b_2 A_2)^{-1}(a_1 + b_1 A_1)^{-1} . \tag{7.35}$$

Taking the ratio of Eqs. (7.34) and (7.35), we have

$$\left(\frac{\mu_{g_1}(A_1, B^{g_2}) \mu_{g_2}(A_2, B)}{\mu_{g_1}(A_1, A_3^{g_2}) \mu_{g_2}(A_2, A_3)} \right)^2 = \det(a_3 + b_3 A_3)(a_3 + b_3 A_3)^{-1} , \tag{7.36}$$

so that

$$\frac{\mu_{g_1}(A_1, B^{g_2})\mu_{g_2}(A_2, B)}{\mu_{g_1}(A_1, A_3^{g_2})\mu_{g_2}(A_2, A_3)} = \pm\, \mu_{g_3}(A_3, B)\,. \tag{7.37}$$

To fix the sign, we simply let $B = A_3$, and see that both sides are equal to 1. This proves that

$$\xi_1 \cdot (\xi_2 \cdot e^{\phi(B)}) = \xi_3 \cdot e^{\phi(B)}\,. \tag{7.38}$$

The next step is to check that the action preserves the inner product. For $\xi = (g, A, \lambda)$, we have

$$\langle \xi \cdot e^{\phi(B_1)}, \xi \cdot e^{\phi(B_2)} \rangle = |\lambda|^2 \overline{\mu_g(A, B_1)}\mu_g(A, B_2) \langle e^{\phi(B_1^g)}, e^{\phi(B_2^g)} \rangle\,. \tag{7.39}$$

As usual we first work with the square,

$$\begin{aligned}\langle e^{\phi(B_1^g)}, e^{\phi(B_2^g)} \rangle^2 &= \det(1 + B_1^{g*} B_2^g) \\ &= \det \begin{pmatrix} 1 \\ B_1^g \end{pmatrix}^* \begin{pmatrix} 1 \\ B_2^g \end{pmatrix}.\end{aligned} \tag{7.40}$$

By the definition of B^g, we have

$$\begin{pmatrix} 1 \\ B^g \end{pmatrix} = g \begin{pmatrix} 1 \\ B \end{pmatrix} (a + bB)^{-1}\,, \tag{7.41}$$

so that we get

$$\begin{aligned}\langle e^{\phi(B_1^g)}, e^{\phi(B_2^g)} \rangle^2 &= \det \left[g \begin{pmatrix} 1 \\ B_1 \end{pmatrix} (a + bB_1) \right]^* \left[g \begin{pmatrix} 1 \\ B_2 \end{pmatrix} (a + bB_2)^{-1} \right] \\ &= \frac{\det(1 + B_1^* B_2)}{\det(a + bB_1)^*(a + bB_2)} \\ &= \frac{\langle e^{\phi(B_1)}, e^{\phi(B_2)} \rangle^2}{\det(a + bB_1)^*(a + bB_2)}\,.\end{aligned} \tag{7.42}$$

Putting this together with Eqs. (7.22) and (7.24), we find that

$$\langle \xi \cdot e^{\phi(B_1)}, \xi \cdot e^{\phi(B_2)} \rangle^2 = \pm \langle e^{\phi(B_1)}, e^{\phi(B_2)} \rangle^2\,. \tag{7.43}$$

If B_1 is set equal to B_2, we see that the sign is positive.

We have now checked the behavior of the action (7.30) for elements of the form $e^{\phi(B)}$, where B lies in an open set of V_0. Lemma 7.1 tells us, however, that these elements span all of $\mathscr{F}_+^{\text{even}}$. Since the action on these elements preserves the inner product, it clearly extends to all of $\mathscr{F}_+^{\text{even}}$. □

The initial form of the action of $\widetilde{SO}_{\text{res}}(\mathscr{H})$ on $\mathscr{F}_+^{\text{even}}$ was defined so as to correspond to an action of $\widetilde{SO}_{\text{res}}(\mathscr{H})$ on PF_{even}, the Pfaffian line bundle restricted to the even component of $\text{Gr}_I(\mathscr{H})$. We will now show how $SO_{\text{res}}(\mathscr{H})$ acts on PF_{even} holomorphically. Given $\xi = (g, A, \lambda)$, let U_g^{RT} denote the set of all $B \in V_R$ such that $g(\text{graph}(B))$ lies in V_T. For each R and T, we need to find a holomorphic function f_ξ on U_g^{RT} such that

$$\xi \cdot (B, 1)_R = (B', f_\xi(B))_T\,, \tag{7.44}$$

where $B' \in V_T$ is the point $g(\mathrm{graph}(B))$. From Theorem 7.3 we know already that such a function exists when $R = T = S_0$, and is given by $\lambda\mu_g(A, B)$ in this case.

Lemma 7.4. *Given R and T, and for a fixed $\xi = (g, A, \lambda) \in \widetilde{SO}_{\mathrm{res}}(\mathscr{H})$, there is a holomorphic function f_ξ on U_g^{RT} such that*

$$\xi \cdot (B, 1)_R = \xi(B', f_\xi(B))_T , \tag{7.45}$$

for $B \in U_g^{RT}$.

Proof. Suppose that $B \in U_g^{RT}$ corresponds to a point $C \in U_g$. Using Eq. (7.30) and the transition functions for PF, we have

$$\begin{aligned}\xi \cdot (B, 1)_R &= \xi \cdot (C, h_{S_0 R}(B))_{S_0} \\ &= (C^g, \lambda\mu_g(A, C) h_{S_0 R}(B))_{S_0}) \\ &= (B', \lambda\mu_g(A, C) h_{TS_0}(C^g) h_{S_0 R}(B))_{S_0}) .\end{aligned} \tag{7.46}$$

The lemma will be proven if we can find an extension to all of U_g^{RT} of the function

$$f_0(B) = \lambda\mu_g(A, C) h_{TS_0}(C^g) h_{S_0 R}(B))_{S_0}) , \tag{7.47}$$

which is well-defined and holomorphic on $U_g^{RT} \cap U_g$. For, if such an extension exists, it is unique, and by continuity the action of ξ on elements of PF must obey the formula (7.45).

The extension of f_0 can in fact be written out explicitly. Let

$$g = \begin{pmatrix} a_{RT} & b_{RT} \\ c_{RT} & d_{RT} \end{pmatrix} , \tag{7.48}$$

with respect to the decomposition $W_R \oplus W_R^\perp \to W_T \oplus W_T^\perp$, e.g. a_{RT} maps $W_R \to W_T$, etc. Choose some $D \in U_g^{RT}$, and let

$$f(B) = \kappa \operatorname*{Pf}_{W_R} (D - B, (a_{RT} + b_{RT} D)^{-1} b_{RT}) . \tag{7.49}$$

Because the set U_g^{RT} is defined precisely by the condition that $(a_{RT} + b_{RT} D)$ be invertible, this function is holomorphic on all of U_g^{RT}. For the moment, κ and D are arbitrary.

Because ξ preserves the inner product of $\mathscr{F}_+^{\mathrm{even}}$, we know that

$$|f_0(B)|^2 = \frac{\mathrm{Pf}_{W_R}(JBJ, B)}{\mathrm{Pf}_{W_T}(JB'J, B')} . \tag{7.50}$$

By manipulations similar to those done to obtain Eq. (7.22) for μ_g, we have

$$|f(B)|^2 = |\kappa|^2 \left[\frac{\det_{W_R}(a_{RT} + b_{RT} B)(a_{RT} + b_{RT} B)^*}{\det_{W_R}(a_{RT} + b_{RT} D)(a_{RT} + b_{RT} D)^*} \right]^{1/2} . \tag{7.51}$$

The arguments used in Eq. (7.42) can be repeated to give

$$\sqrt{\operatorname*{det}_{W_R} (a_{RT} + b_{RT} B)(a_{RT} + b_{RT} B)^*} = \frac{\mathrm{Pf}_{W_R}(JBJ, B)}{\mathrm{Pf}_{W_T}(JB'J, B')} . \tag{7.52}$$

Combining Eqs. (7.51) and (7.52), we have

$$|f(B)|^2 = \frac{|\kappa f_0(B)|^2}{\det_{W_R}(a_{RT} + b_{RT}D)(a_{RT} + b_{RT}D)^*} \,. \tag{7.53}$$

Because there are no non-constant holomorphic functions on $\mathrm{Gr}_J(\mathscr{H})$, if the norms of two holomorphic functions differ by a constant, the functions themselves differ only by a constant. Thus, by choosing κ appropriately, we have $f = f_0$ on $U_g^{RT} \cap U_g$. □

Theorem 7.5. *$\widetilde{SO}_{\mathrm{res}}(\mathscr{H})$ acts on the even component of* PF. *The action is holomorphic and linear on each fiber.*

Proof. The form of the action is given by (7.45). Lemma 7.4 tells us that the functions f_ξ and holomorphic, so the action is holomorphic. □

We have now worked out the results for the even components of $O_{\mathrm{res}}(\mathscr{H})$ and PF. We can apply them to the full spaces by means of a simple trick. We enlarge the Hilbert space $\mathscr{H}$ slightly, and embed PF in the even component of the Pfaffian line bundle for the new Hilbert space. Let

$$\hat{\mathscr{H}} := \mathscr{H} \oplus \mathbb{C}^2 \,, \tag{7.54}$$

which we decompose as $\hat{\mathscr{H}}_\pm = \mathscr{H}_\pm \oplus \mathbb{C}$. We define the complex structure $\hat{J}$ for $\hat{\mathscr{H}}$ by

$$\hat{J}(x, \lambda) := (Jx, \bar{\lambda}) \,, \tag{7.55}$$

for $(x, \lambda) \in \mathscr{H}_\pm \oplus \mathbb{C}$. Starting with the pair $\hat{\mathscr{H}}, \hat{J}$, we construct a line bundle $\hat{\mathrm{PF}}$ and the corresponding Fock space $\hat{\mathscr{F}}_P \cong \hat{\mathscr{F}}_+$.

Let γ denote the representation of $O_{\mathrm{res}}(\mathscr{H})$ on $\mathbb{C}^2$ given by

$$\gamma(g) := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } g \text{ lies in the identity component of } O_{\mathrm{res}}(\mathscr{H}); \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{otherwise} \,. \end{cases} \tag{7.56}$$

This gives us a homomorphism

$$O_{\mathrm{res}}(\mathscr{H}) \hookrightarrow SO_{\mathrm{res}}(\hat{\mathscr{H}}) \,, \tag{7.57}$$

which maps

$$g \mapsto \hat{g} := (g, \gamma(g)) \,. \tag{7.58}$$

Using this map we make the following definition.

Definition 7.6. *The extension $\tilde{O}_{\mathrm{res}}(\mathscr{H})$ is the pullback to $O_{\mathrm{res}}(\mathscr{H})$ of the extension $\widetilde{SO}_{\mathrm{res}}(\hat{\mathscr{H}})$ by the map* (7.58).

This map (7.58) also defines an embedding of $\mathrm{Gr}_J(\mathscr{H})$ in the even component of $\mathrm{Gr}_J(\hat{\mathscr{H}})$, by

$$g\mathscr{H} \mapsto \hat{g}\hat{\mathscr{H}} \,. \tag{7.59}$$

There is a corresponding map of index sets $S \mapsto \hat{S}$,

$$\hat{S} := \begin{cases} (S, +), & \text{if} \dim(S \cap \mathbb{Z}_+) \text{ is even;} \\ (S, -), & \text{if} \dim(S \cap \mathbb{Z}_+) \text{ is odd ,} \end{cases} \tag{7.60}$$

where the $+$ and $-$ refer to a basis $\{\zeta_\pm\}$ for $\mathbb{C}^2$. We can use this to describe the embedding of PF in $\hat{\mathrm{PF}}_{\mathrm{even}}$ very simply.

$$(A, \lambda)_S \mapsto (A, \lambda)_{\hat{S}} , \tag{7.61}$$

where the map A is extended from W_S to $W_{\hat{S}}$ by zero. Recall that the elements of PF can be identified with elements of the Hilbert space $\mathscr{F}_+$. The embedding (7.61) leads to the following map $\mathscr{F}_+ \to \hat{\mathscr{F}}_+^{\mathrm{even}}$,

$$\begin{aligned} \omega_{\mathrm{even}} &\mapsto \omega_{\mathrm{even}} , \\ \omega_{\mathrm{odd}} &\mapsto \zeta_+ \wedge \omega_{\mathrm{odd}} . \end{aligned} \tag{7.62}$$

This map is a Hilbert space isomorphism.

Theorem 7.7. *The action of* $O_{\mathrm{res}}(\mathscr{H})$ *on* $\mathrm{Gr}_I(\mathscr{H})$ *is covered by a holomorphic action of* $\tilde{O}_{\mathrm{res}}(\mathscr{H})$ *on* PF, *and this gives rise to a unitary representation of* $\tilde{O}_{\mathrm{res}}(\mathscr{H})$ *on* $\mathscr{F}_+ (\cong \mathscr{F}_P)$.

Proof. Given the embeddings and isomorphisms defined above, this is a straightforward generalization of Theorem 7.3 and Theorem 7.5. □

8. Loop Groups

Let $\mathscr{H}^{(n)}$ be the space $\mathscr{H} \otimes \mathbb{C}^n$, where $\mathscr{H}$ is the Hilbert space of square-integrable half-densities on S^1. An element of $\mathscr{H}^{(n)}$ can be thought of as a vector valued function

$$f(\theta) = (f_1(\theta), \ldots , f_n(\theta)) , \tag{8.1}$$

such that $f(\theta + 2\pi) = -f(\theta)$. This space has a natural basis given by $e^k_{\pm j}$ with $k = 1, \ldots , n$ and $j \in \mathbb{Z}_+$, where e^k_j corresponds to the function for which $f_l(\theta) = \delta_{lk} \exp\{ \pm i(j - \frac{1}{2})\theta\}$ $\mathscr{H}^{(n)}$ has its natural complex conjugation

$$J f(\theta) := \overline{f(\theta)} . \tag{8.2}$$

We can construct an isomorphism $\mathscr{H}^{(n)} \cong \mathscr{H}$ by mapping $e^k_{\pm j} \mapsto e_{\pm(n(j-1)+k)}$, which clearly preserves the action of J on the two spaces.

The group LO_n of smooth loops in O_n acts naturally on $\mathscr{H}^{(n)}$. For $\gamma \in LO_n, f \in \mathscr{H}^{(n)}$, we set

$$(\gamma f)_i(\theta) := \sum_j \gamma_{ij}(\theta) f_j(\theta) . \tag{8.3}$$

Because γ is real-valued, we clearly have $J\gamma J = \gamma$.

Proposition 8.1. *Equation* (8.3) *defines an embedding* $LO_n \hookrightarrow O_{\mathrm{res}}(\mathscr{H}^{(n)})$ (*and hence in* $O_{\mathrm{res}}(\mathscr{H})$).

Proof. We have already noted that $J\gamma J = \gamma$. For the remainder of the proof, the corresponding result for unitary matrices done in [21] applies directly. □

The pullback through this embedding of the extension $\tilde{O}_{\mathrm{res}}(\mathscr{H})$ gives us cyclic extensions $\widetilde{LO}_n$ of LO_n. To determine what these extensions are we first compute the Lie algebra cocycle of the extension $\tilde{O}_{\mathrm{res}}(\mathscr{H})$. The Lie algebra $o_{\mathrm{res}}(\mathscr{H})$ of $O_{\mathrm{res}}(\mathscr{H})$ is given by

$$o_{\mathrm{res}}(\mathscr{H}) = \{\eta \in \mathscr{L}(\mathscr{H}) : \eta^* = -\eta, J\eta J = \eta, P_+ \eta P_- \text{ and } P_- \eta P_+ \text{ are Hilbert–Schmidt}\} .$$

It is easy to verify that an element of $o_{\mathrm{res}}(\mathscr{H})$ must have the form

$$\eta = \begin{pmatrix} v & w \\ -w^* & JvJ \end{pmatrix},$$

with respect to the decomposition $\mathscr{H}_- \oplus \mathscr{H}_+$, where $v^* = -v$, and $w \in I_2^a(\mathscr{H}_+, \mathscr{H}_-)$.

To compute the Lie algebra cocycle, we need to find a cross section of the extension $\tilde{O}_{\mathrm{res}}(\mathscr{H})$ over a neighborhood of the identity. For our neighborhood, we will use the set U consisting of all $g \in O_{\mathrm{res}}(\mathscr{H})$ for which a is invertible, when g is written in the usual form (7.4). The cross section $U \to \tilde{O}_{\mathrm{res}}(\mathscr{H})$ is

$$g \mapsto \tilde{g} := (g, 0, [\det a^* a]^{1/4}) . \tag{8.4}$$

We are looking for as map $c : U \times U \to \mathbb{C}$ such that

$$\tilde{g}_1 \tilde{g}_2 = c(g_1, g_2) \tilde{g}_3 , \tag{8.5}$$

when $g_1 g_2 = g_3$. This map is given by

$$c(g_1, g_2) = \left[\frac{\det a_1^* a_1 \det a_2^* a_2}{\det a_3^* a_3} \right]^{1/4} \operatorname*{Pf}_{\mathscr{H}_+} (-c_2 a_2^{-1}, a_1^{-1} b_1) . \tag{8.6}$$

Given such a map c, the general formula for the Lie algebra cocyle of the extension is

$$\omega(\eta_1, \eta_2) = D_1 D_2 c(\eta_1, \eta_2) - D_1 D_2 c(\eta_2, \eta_1) , \tag{8.7}$$

where $D_1 D_2 c : o_{\mathrm{res}}(\mathscr{H}) \times o_{\mathrm{res}}(\mathscr{H}) \to \mathbb{C}$ is the mixed second partial derivative of c at the identity. We compute that for $\eta_1, \eta_2 \in o_{\mathrm{res}}(\mathscr{H})$,

$$\begin{aligned} c(e^{\eta_1}, e^{\eta_2}) &= \frac{\mathrm{Pf}(w_2^*, w_1)}{[\det(1 - w_2^* w_1 - w_1^* w_2)]^{1/4}} + \mathscr{O}(\eta_1^2, \eta_2^2) \\ &= \exp\left\{ -\frac{1}{2} \operatorname{Tr} w_2^* w_1 \right\} \exp\left\{ \frac{1}{4} \operatorname{Tr}(w_2^* w_1 + w_1^* w_2) \right\} + \cdots \\ &= 1 + \frac{1}{4} \operatorname{Tr}(w_1^* w_2 - w_2^* w_1) + \ldots . \end{aligned} \tag{8.8}$$

Thus we read off that

$$\omega(\eta_1, \eta_2) = \frac{1}{2} \operatorname{Tr}(w_1^* w_2 - w_2^* w_1) . \tag{8.9}$$

We can think of this cocycle as being the Chern class of the unit circle bundle in PF, which is identified with $\tilde{O}_{\mathrm{res}}(\mathscr{H})/U(\mathscr{H}_-)$ [21].

The extensions $\widetilde{LO}_n$ will be completely determined by their Lie algebra cocycles, which we can now specify.

Proposition 8.2. *The extension $\widetilde{LO}_n$ given by the pullback of $\tilde{O}_{\mathrm{res}}(\mathscr{H})$ corresponds to the Lie algebra cocycle*

$$S(\alpha, \beta) = \frac{i}{4\pi} \int_0^{2\pi} \langle \alpha(\theta), \beta'(\theta) \rangle_{so_n} d\theta \,, \tag{8.10}$$

where $\alpha, \beta \in Lso_n$, and $\langle X, Y \rangle_{so_n} = -\operatorname{tr} XY$.

Proof. Let the Fourier decompositions of α and β be

$$\alpha(\theta) = \sum_{k \in \mathbb{Z}} \alpha^{(k)} e^{ik\theta} \,,$$
$$\beta(\theta) = \sum_{k \in \mathbb{Z}} \beta^{(k)} e^{ik\theta} \,. \tag{8.11}$$

We simply compute the induced cocycle

$$\begin{aligned} S(\alpha, \beta) &= \omega(\eta_\alpha, \eta_\beta) \\ &= \frac{1}{2} \operatorname{Tr}_{\mathscr{H}_+} (\eta_\alpha^* P_- \alpha_\beta - \eta_\beta^* P_- \eta_\alpha) \\ &= \frac{1}{2} \sum_{k, m > 0} \operatorname{tr}(\alpha^{(-k-m+1)*} \beta^{(-k-m+1)} - \beta^{(-k-m+1)*} \alpha^{(-k-m+1)}) \\ &= -\frac{1}{2} \sum_{k > 0} k \operatorname{tr}(\alpha^{(k)} \beta^{(-k)} - \beta^{(k)} \alpha^{(-k)}) \\ &= -\frac{1}{2} \sum_{k \in \mathbb{Z}} k \operatorname{tr} \alpha^{(k)} \beta^{(-k)} \\ &= \frac{i}{4\pi} \int_0^{2\pi} \langle \alpha(\theta), \beta'(\theta) \rangle_{so_n} d\theta \,. \quad \square \end{aligned}$$

Theorem 8.3. *The loop groups, LO_n act on $\mathscr{F}_P$ by irreducible projective unitary representations, corresponding to the cyclic extensions defined above.*

Proof. The only aspect of this theorem which has does not follow immediately from our previous results is the irreducibility of the representations. The proof or irreducibility can be taken virtually unchanged from the corresponding proof for LU_n in [21], so we will not repeat the arguments here. $\square$

We mentioned in Sect. 4 that the determinant line bundle can be pulled back from the Pfaffian line bundle defined over a larger Hilbert space $\mathscr{X} = \mathscr{H} \oplus \mathscr{H}'$. The embedding of the two line bundles corresponds to an embedding of $U_{\mathrm{res}}(\mathscr{H})$ in $O_{\mathrm{res}}(\mathscr{X})$. The subgroup LU_n of $U_{\mathrm{res}}(\mathscr{H})$ maps to LO_{2n} under this embedding, precisely by the canonical map $U_n \subsetneq O_{2n}$. Thus, we see that if we apply the above results for LO_{2n} to LU_n via the embedding $U_n \subsetneq O_{2n}$, we obtain the same extensions and projective representations which were obtained in [21] using the determinant line bundle construction.

9. The Pfaffian Line Bundle Over Moduli Space

The determinant line bundle of Quillen [22], which involves the determinant lines of Dirac operators, has been studied extensively. Its differential geometry [7] and holomorphic structure [8] have been worked out. As mentioned in the introduction, the construction has proven extremely useful in string theory and conformal field theory. Freed has also given a construction of a Pfaffian line bundle which is a square root of the Quillen determinant line bundle [12]. We will attempt here to relate his Pfaffian line bundle to our construction. We start by outlining his construction briefly.

Let $\mathscr{M}_g$ be the moduli space of Riemann surfaces of genus g together with a choice of spin structure. An element of $\mathscr{M}_g$ is thus a pair (Σ, S), with S a holomorphic line bundle on Σ such that

$$S^2 \cong T^*\Sigma^{1,0} . \tag{9.1}$$

Given such a pair, we define the Dirac operator D to be the $\bar{\partial}$ operator acting on sections of S,

$$D := \bar{\partial}_S : \Omega^{0,0}(S) \to \Omega^{0,1}(S) . \tag{9.2}$$

Note that because of the relation (9.1), we have a natural bilinear pairing

$$\Omega^{0,0}(S) \otimes \Omega^{0,1}(S) \to \mathbb{C} , \tag{9.3}$$

given by integration over Σ. This pairing will play the role of the bilinear form $(\cdot\,,\cdot)$ on $\mathscr{H}$, and its existence is the reason for requiring S to be a spin bundle. The papers dealing with the determinant line bundle and the Krichever map use a more general moduli space, involving an arbitrary line bundle [2, 5, 6, 15, 25].

It is a special case of the results of [8] that the complex (9.2) varies holomorphically over $\mathscr{M}_g$. Given a point $m \in \mathscr{M}_g$, we can choose a finite-dimension subspace V of the bundle $\Omega^{0,0}(S)$ which varies holomorphically and for which $\operatorname{Ker} D \subset V$ at the point m. Then V necessarily contains $\operatorname{Ker} D$ in some neighborhood U of m. Over this neighborhood, we define the holomorphic line bundle

$$\mathscr{K}_U := \wedge V' ,$$

where $\wedge$ designates the highest exterior power.

These locally defined line bundles can be patched together as follows. Suppose that $V_1 \subset V_2$ are finite dimensional subbundles of $\Omega^{0,0}(S)$ defined on open sets $U_1 \subset U_2$. Let $X = V_2/V_1$. By construction, D has no kernel when restricted to X. Via the pairing (9.3), we see that D corresponds to an element of $\omega_D \in X' \otimes X'$, given by

$$\omega_D(\alpha, \beta) = \int_\Sigma \alpha \otimes D\beta . \tag{9.4}$$

We see through integration by parts that ω_D is skew, and so belongs to $\wedge^2 X'$. Let

$$\Omega_D := \frac{1}{r!}\,\omega_D^r\,, \tag{9.5}$$

where $2r = \operatorname{rank} X$ (we know that the rank of X is even because $D: X \to X'$ is both invertible and skew-symmetric). Ω_D gives us an isomorphism

$$\mathscr{K}_{U_1} \to \mathscr{K}_{U_2}\,, \tag{9.6}$$

defined over U_1, which is

$$s \mapsto s \wedge \Omega_D\,. \tag{9.7}$$

Because the complex (9.2) varies holomorphically, the form Ω varies holomorphically over U_1. Thus the patching maps defined by (9.7) are holomorphic, and the result is a holomorphic line bundle $\mathscr{K}$. A canonical holomorphic section, which vanishes where D has a kernel, is easily obtained. We can define $\Omega_D \in \wedge V'$ for any subbundle V, just as above, and we define the section locally by this form. The line bundle $\mathscr{K}$ and its canonical section are the square roots of the determinant line bundle and its holomorphic section.

To make contact with the Pfaffian over a Grassmannian manifold, we must enlarge the moduli space to specify a coordinate patch and describe a variant of the Krichever map [18]. Krichever's original map was from a moduli space to the space of solutions to the KdV equations. The equivalent map to the Grassmannian, which we use, was introduced in [23, 25]. Let $\hat{\mathscr{M}}_g$ be the space consisting of a pair $(\Sigma, S) \in \mathscr{M}_g$ together with a local coordinate patch z on Σ. By local coordinate patch we mean an invertible holomorphic map from an open set of Σ to an open neighborhood of the unit disk in $\mathbb{C}$. $\hat{\mathscr{M}}_g$ can be given the structure of an infinite dimensional complex manifold [4]. Denote by π the natural projection $\hat{\mathscr{M}}_g \to \mathscr{M}_g$.

Let $\mathscr{H}$ be the Hilbert space we have mentioned previously, the space of square-integrable function on S^1 with anti-symmetric boundary conditions. For each point $(\Sigma, S, z) \in \hat{\mathscr{M}}_g$, we can define a map from $\Omega^{0,0}(S) \to \mathscr{H}$, as follows. In our coordinate patch, $\sqrt{dz}$ gives a trivialization of S. For and $\omega \in \Omega^{0,0}(S)$ we can find a function f on an open neighborhood of the unit disk in $\mathbb{C}$ so that locally ω has the form

$$\omega = f\sqrt{dz}\,. \tag{9.8}$$

We define the map $\Omega^{0,0}(S) \to \mathscr{H}$ to be

$$\omega \mapsto \phi_\omega(\theta) = f(e^{i\theta})e^{i\theta/2}\,. \tag{9.9}$$

Note that for $v, \omega \in \Omega^{0,0}(S)$, the product $v \otimes \omega$ can be regarded as a form in $\Omega^{1,0}(\Sigma)$, so that $\bar{\partial}(v \otimes \omega)$ is a volume form on Σ. Let O be the disk $|z| < 1$. By Stokes' theorem we have

$$\begin{aligned}\int_{\Sigma - O} \bar{\partial}(v \otimes \omega) &= \oint_{\partial O} \phi_v(\theta)\phi_\omega(\theta)\,d\theta \\ &= (\phi_v, \phi_\omega)_{\mathscr{H}}\end{aligned} \tag{9.10}$$

(recall that $(x, y)_{\mathscr{H}} = \langle Jx, y \rangle_{\mathscr{H}}$).

Proposition 9.1. *For* $\hat{m} = (\Sigma, S, z) \in \hat{\mathcal{M}}_g$, *let* $W_{\hat{m}}$ *be the subspace of* $\mathcal{H}$ *which is the closure of the set,*

$$\{f \in \mathcal{H} : f = \phi_\omega \text{ for some } \omega \in \Gamma(S|_{\Sigma - O})\}\,, \tag{9.11}$$

where $\Gamma(S|_{\Sigma - O})$ *is the space of holomorphic sections of* S *over* $\Sigma - O$. *Then the assignment* $\hat{m} \mapsto W_{\hat{m}}$ *defines a map*

$$k : \hat{\mathcal{M}}_g \hookrightarrow \mathrm{Gr}_I(\mathcal{H})\,, \tag{9.12}$$

which is injective and holomorphic.

Proof. The fact that k defines an injective map to the Grassmannian was proven in [25] (in greater generality), and the holomorphicity of this map has been discussed in [5, 15]. Therefore we will only demonstrate that the image of k lies in $\mathrm{Gr}_I(\mathcal{H})$ for our particular moduli space. Fix $\hat{m}$ and let $W = W_{\hat{m}}$. We will show that the projection $P_- : W \to \mathcal{H}_-$ has index zero, and that W is isotropic with respect to $(\cdot\,,\cdot)$.

To prove the first, let Σ_0 be the patch where the coordinate z is defined, and let Σ_∞ denote the set $\Sigma \backslash \{z = 0\}$. For any Riemann surface we have an exact sequence,

$$0 \to H^0(\Sigma, S) \to \Gamma(S|_{\Sigma_0}) \oplus \Gamma(S|_{\Sigma_\infty}) \to \Gamma(S|_{\Sigma_0 \cap \Sigma_\infty}) \to H^1(\Sigma, S) \to 0\,, \tag{9.13}$$

where Γ denotes the space of holomorphic sections, and $H^*(\Sigma, S)$ is the cohomology of Σ with values in the sheaf of holomorphic sections of S. From the Dolbeault theorem, we have

$$H^0(\Sigma, S) \cong \mathrm{Ker}\, D, \quad H^1(\Sigma, S) \cong \mathrm{Cok}\, D\,. \tag{9.14}$$

It is also clear that

$$\begin{aligned} \Gamma(S|_{\Sigma_0}) &\cong \mathcal{H}_+^{\mathrm{an}}\,, \\ \Gamma(S|_{\Sigma_\infty}) &\cong W^{\mathrm{an}}\,, \\ \Gamma(S|_{\Sigma_0 \cap \Sigma_\infty}) &\cong \mathcal{H}^{\mathrm{an}}\,, \end{aligned} \tag{9.15}$$

where the superscript an refers to the subspaces of analytic functions. The map

$$W^{\mathrm{an}} \oplus \mathcal{H}_+^{\mathrm{an}} \to \mathcal{H}^{\mathrm{an}}\,, \tag{9.16}$$

appearing in the exact sequence above takes $(f, g) \mapsto f - g$. Its kernel is $\mathrm{Ker}\, P_- : W^{\mathrm{an}} \to \mathcal{H}_-^{\mathrm{an}}$, which is just $\mathrm{Ker}\, P_-$, since all functions in this kernel are analytic. Similarly, the cokernel of the map (9.16) is just $\mathrm{Cok}\, P_-$. From the sequence (9.13) we thus obtain the exact sequence

$$0 \to \mathrm{Ker}\, D \to W \to \mathcal{H}_- \to \mathrm{Cok}\, D \to 0\,. \tag{9.17}$$

From this exact sequence, we read off that P_- is Fredholm with index zero, since the Dirac operator D has index zero. We note that the arguments which show that the complex (9.2) varies holomorphically over $\mathcal{M}_g$ can be applied to see that (9.17) varies holomorphically over $\hat{\mathcal{M}}_g$.

To prove the isotropy of W, we use integration by parts and Stokes' theorem. Suppose that $f, g \in W$, and let $f = \phi_v, g = \phi_\omega$. We have already noted that

$$(f, g) = \int_{\Sigma - O} \bar{\partial}(v \otimes \omega)\,. \tag{9.18}$$

Since v and ω are holomorphic, this is equal to 0. Hence, W is isotropic. $\square$

This type of Krichever map is essential to the Fock space functor construction of [24]. We use it to give the relationship between the two Pfaffian line bundles in the following theorem. The corresponding result for the determinant line bundle is well known [5, 15].

Theorem 9.2. *With* π *the projection* $\hat{\mathscr{M}}_g \to \mathscr{M}_g$, *and* k *the map defined in Proposition* 9.1, *we have the holomorphic isomorphism*

$$\pi^* \mathscr{K} \cong k^* \mathrm{PF}' . \tag{9.19}$$

Proof. The proof is easy if we use a slightly different definition of the Pfaffian line bundle. Given a point $W \in \mathrm{Gr}_I(\mathscr{H})$, we can choose a finite dimensional subspace $V \subset W$ such that $\operatorname{Ker} P_- \subset V$ in a neighborhood U of W. Then we define PF locally by

$$\mathrm{PF}_U := U \times \bigwedge V' . \tag{9.20}$$

These definitions are patched together using the bilinear form $(\cdot, \cdot)$ and the map P_-, which is skew with respect to the pairing, in exactly the same way we patched $\mathscr{K}$ using the pairing (9.3) and the Dirac operator. It is easy to check that this definition is equivalent to the one given in Sect. 4. The parallel to the construction of $\mathscr{K}$ is obvious. All that is needed to complete the isomorphism is the exact sequence (9.17), which varies holomorphically over $\hat{\mathscr{M}}_g$. $\square$

This theorem relates the two bundles PF and $\mathscr{K}$. The latter bundle has a canonical holomorphic section, which is thought of as the Pfaffian of the Dirac operator. Its defining property is that it vanishes precisely when the Dirac operator has zero eigenvalues, and the order of vanishing is equal to half the number of zero modes (the zero modes come in pairs because D is always skew symmetric). It is easy to pick out the section of PF′ which has this property when pulled back by k, because zero modes of D correspond to elements of $W \cap \mathscr{H}_+$. The holomorphic section γ_{S_0} of PF′, in the notation of the first few sections, pulls back to the canonical section of $\mathscr{K}$. This is very natural, because in the Fock space interpretation of the space of holomorphic sections of PF′, γ_{S_0} corresponds to the vacuum state.

Unfortunately, the situation with regard to metrics is not so clear. It would be nice to be able to relate the metrics on the line bundles $\mathscr{K}$ over the moduli spaces for different genera to a single metric coming from PF, but this is impossible. The metric on $\mathscr{K}$ is independent of any local coordinate patches. By choosing an arbitrarily small coordinate patch on a higher genus surface, we can bring the corresponding point in $\mathrm{Gr}_I(\mathscr{H})$ arbitrarily close to the point $\mathscr{H}_-$, which corresponds to the sphere. Thus no smooth hermitian structure on PF can be pulled back to give the canonical metrics on $\mathscr{K}$ over moduli spaces for different genera.

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