

On the Selection Rules for Spin-Lorentz Cobordisms

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Abstract. A recent result of Gibbons and Hawking on the existence of spin-Lorentz cobordisms is applied to the class of 3-manifolds that can support a non-negative scalar curvature metric. We call such manifolds admissible. It is shown how in general the existence of a spin-Lorentz cobordism restricts the number-mod-2 of certain prime manifolds to occur in the prime decomposition, which are explicitly listed in the case of admissible manifolds.

Introduction

Very recently, Gibbons and Hawking derived a necessary and sufficient condition for a closed, orientable 3-manifold, Σ , to spin-Lorentz bound, that is, to be the spacelike boundary of a Lorentzian 4-Manifold, M , that admits a $SL(2, C)$ -spin structure [2]. Their result was that the mod-2 Kervaire semi-characteristic, $\mathcal{U}(\Sigma)$, must be zero. Here, $\mathcal{U}(\Sigma)$ can itself be defined through the combination of cohomology groups:

$$\mathcal{S}(\Sigma) = \dim H^0(\Sigma, Z_2) + \dim H^1(\Sigma, Z_2), \quad (1)$$

$$\mathcal{U}(\Sigma) = \mathcal{S}(\Sigma) \bmod 2. \quad (2)$$

In Sect. 1 we want to reformulate this expression in terms of homology groups with integer coefficients and use it to derive a simple behaviour of $\mathcal{U}(\Sigma)$ under taking disjoint and connected sums. This is summarized in Corollary 1. In Sect. 2 we restrict attention to the so-called *admissible* 3-manifolds, which are characterized by admitting a metric of non-negative scalar curvature. Assuming a standard conjecture in 3-manifold theory to hold, we can give an explicit list of those prime manifolds whose occurrence is restricted mod 2 if their connected sum is required to spin-Lorentz bound, or if required to be spin-Lorentz cobordant to

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Table 1

$\pi_1(\Sigma)$	$H_1(\Sigma)$	$\mathcal{U}(\Sigma)$
Z_p	Z_p	$\frac{1}{2}(1 - (-1)^p)$
$D_{8n}^* \times Z_q$	$Z_2 \times Z_2 \times Z_q$	1
$D_{4(2n+1)}^* \times Z_q$	$Z_4 \times Z_q$	0
$T^* \times Z_q$	$Z_3 \times Z_q$	1
$O^* \times Z_q$	$Z_2 \times Z_q$	0
$I^* \times Z_q$	Z_q	1
$D_{2^k \cdot (2n+1)} \times Z_q$	$Z_{2^k} \times Z_q$	0
$T_{8 \cdot 3^j} \times Z_q$	$Z_{3^k} \times Z_q$	1
Z	Z	0
G_1	$Z \times Z \times Z$	0
G_2	$Z \times Z_2 \times Z_2$	0
G_3	$Z \times Z_3$	0
G_4	$Z \times Z_2$	1
G_5	Z	0
G_6	$Z_4 \times Z_4$	1

S^3 . This is summarized in Corollary 2 and Table 1. Although there is a way (not $1 - 1$) of counting 3-manifolds in which the non-admissible ones form most of all 3-manifolds (see e.g. [8]), they can be argued to be of less interest for applications in General Relativity [1, 8]. In that sense Corollary 2 seems to cover the general relativist’s demands so far.

1. Behaviour under Disjoint and Connected Sums

We start by noting that the first term in (1) simply counts the number, n , of connected components of Σ . Then, using standard theory of (co)homology (see e.g. [5]), one shows that for Σ having the connected components $\{\Sigma_1, \dots, \Sigma_n\}$

$$H^1(\Sigma, Z_2) \cong \bigoplus_{i=1}^n \{FH_1(\Sigma_i, Z) \otimes Z_2 \oplus \text{Tor}(H_1(\Sigma_i, Z), Z_2)\}, \tag{3}$$

where $\text{Tor}(G, Z_2)$ simply denotes the kernel of the “squaring” map on the abelian group $G: G \rightarrow G; g \mapsto 2g$, and FH_1 denotes the free part of the finitely generated abelian group $H_1(\Sigma_i, Z)$:

$$H_1(\Sigma_i, Z) = \left(\bigoplus_{k=1}^{f_i} Z \right) \oplus \left(\bigoplus_{l=1}^{t_i} Z_{p_{li}} \right) =: FH_1(\Sigma_i, Z) \oplus TH_1(\Sigma_i, Z). \tag{4}$$

The second bracket, containing the cyclic groups of finite order, is called the torsion part (TH_1). Its representation as a direct sum of cyclic groups of order p_{li} is certainly not unique since $Z_p \oplus Z_q \cong Z_{pq}$ for p, q any pair of coprime integers. Uniqueness can be achieved by the requirement that all p_{li} be powers of prime numbers. In any case, the number of even p_{li} ’s is easily seen to be invariant. We will call it the even-torsion-rank τ_i , and f_i the free-rank. The latter one, usually just called the rank, is equal to the dimension of the first DeRahm cohomology group, that is, to the dimension of the space of harmonic 1-forms on Σ . The significance of τ_i is that only a cyclic group of even order has non-trivial elements that square to

the identity, so that one has

$$\text{Tor } H_1(\Sigma_i, Z_2) \cong \bigoplus_{i=1}^{\tau_i} Z_2. \tag{5}$$

Using (5) and (4) in (3) gives

$$\mathcal{U}(\Sigma) = \left(\sum_{i=1}^n \mathcal{U}(\Sigma_i) \right) \text{ mod } 2 = \left(n + \sum_{i=1}^n (f_i + \tau_i) \right) \text{ mod } 2. \tag{6}$$

Next, we assume that Σ is connected and given by the connected sum of n manifolds $\{\Sigma_1, \dots, \Sigma_n\}$. It is well known (see e.g. [5]) that the fundamental group of this connected sum is given by the free product of the fundamental groups of the individual factors. The first homology group with integer coefficients, being just the abelianisation of the fundamental group, is therefore just the direct sum of the individual ones. If we again denote by f_i and τ_i the free- and even-torsion-rank of the Σ_i 's, we have

$$\mathcal{U}(\Sigma) = \left(1 + \sum_{i=1}^n (t_i + \tau_i) \right) \text{ mod } 2 = \left(1 + \sum_{i=1}^n (1 + \mathcal{U}(\Sigma_i)) \right) \text{ mod } 2. \tag{7}$$

We call a manifold *nuclear*, if its characteristic \mathcal{U} vanishes and *non-nuclear* otherwise. Note that a manifold is non-nuclear if and only if it is spin-Lorentz-cobordant to the 3-sphere. We can now use Eqs. (6) and (7) to state a corollary to the theorem of Gibbons and Hawking.

Corollary 1. A) *The disjoint sum of closed orientable 3-manifolds is nuclear (respectively non-nuclear), if and only if there is an even (respectively odd) number of non-nuclear connected components.*

B) *The connected sum of 3-manifolds is nuclear (respectively non-nuclear), if and only if the number of nuclear factors is odd (respectively even).* \square

2. Restriction to Admissible Manifolds

From now on, we take the word manifold to also imply connected, closed and orientable. Since any manifold can be uniquely decomposed into a finite connected sum of so-called prime manifolds, we can calculate the characteristic $\mathcal{U}(\Sigma)$ once we know it for the prime factors.

A list of all prime manifolds is, however, not known and we shall therefore restrict attention to admissible prime manifolds, already characterized in the introduction. For this we look at the following three classes of prime 3-manifolds: The first class comprises all manifolds of finite fundamental group. Amongst those are the elliptic spaces, S^3/G , where G is any finite subgroup of $SO(4)$ that acts freely via the action induced from the standard orthogonal action of $SO(4)$ on S^3 . It is at present an undecided conjecture that any 3-manifold of finite fundamental group is homeomorphic to such an elliptic space. Besides the Poincaré conjecture, this involves a conjecture stating that the groups listed are all the freely acting ones, and that these can act only via the orthogonal action (up to conjugation by any homeomorphism). This is known to hold for most but not all of the groups. See [7] for more information on this. We shall now assume this conjecture to hold, that is, we assume that any manifold of finite fundamental group is homeomorphic to an elliptic space S^3/G . There is a countable infinity of

such groups which have the conventional names: $Z_p, D_{4m}^* \times Z_q, T^* \times Z_q, O^* \times Z_q, I^* \times Z_q, D_{2^{k \cdot (2n+1)}}^* \times Z_q$, and $T_{8 \cdot 3^j}^* \times Z_q$, where the range of the integers p, m, k, n, j, q is $m, k \geq 2; p, k, j, q \geq 1$, where q has to be coprime to the order of the group the Z_q is multiplied with. In particular, q has to be odd. D_{4m}^*, T^*, O^* , and I^* denote $SU(2)$ -double covers of the dihedral, tetrahedral, octahedral, and icosahedral groups of orders $4m, 24, 48$, and 120 , respectively. $D_{2^{k \cdot (2n+1)}}^*$ and $T_{8 \cdot 3^j}^*$ can be understood as extensions of $Z_{2^{k-2}}$ and $Z_{3^{j-1}}$ by $D_{4(2n+1)}^*$ and T^* , respectively. Presentations of these groups can be found in [6]. Together they comprise all 3-manifolds of constant, positive curvature.

The second class just contains the 3-handle, $S^1 \times S^2$, and the third class contains all manifolds of the form R^3/G , where G is a freely, properly discontinuously acting subgroup of the 3-dimensional Euclidean group, E_3 , with compact, orientable quotient space. There are six such groups, which, in standard notation [9], we denote by G_1, \dots, G_6 . G_1 is just $Z \times Z \times Z$ with the 3-torus as resulting quotient manifold, and G_2, G_3, G_4, G_5, G_6 are certain extensions of the groups $Z_2, Z_3, Z_4, Z_6, Z_2 \times Z_2$ by G_1 , respectively. Presentations of these can be found in [9]. It follows that the latter five quotient manifolds are finitely covered by the 3-torus. Together they comprise all flat 3-manifolds.

It is known that a manifold admits a metric of everywhere positive scalar curvature if and only if its prime decomposition consists entirely of elliptic spaces and handles [3]. Moreover, any connected sum in which any other prime-manifold appears and which allows for a non-negative scalar curvature metric must in fact be flat and therefore homeomorphic to one of the six manifolds $R^3/G_1, \dots, R^3/G_6$. Up to the conjecture mentioned, the manifolds listed are therefore sufficient to generate (via connected sums) those manifolds that admit metrics with nowhere negative scalar curvature. These metrics are required in a variety of applications, like 1) for the existence of maximal spatial slicings of spacetimes [8], 2) for the transition (nucleation) from Euclidean to Lorentzian spacetimes [1], and 3) to ensure the absence of harmonic spinors [4].

Since the fundamental group of S^3/G or R^3/G is just G , we easily obtain the first homology by abelianisation and thus the characteristic $\mathcal{U}(\Sigma)$. The result is summarized in Table 1. Here the manifolds are sufficiently characterized by their fundamental group, whereby $\pi(\Sigma) = Z$ corresponds to $\Sigma = S^1 \times S^2$.

We now call the nuclear (respectively non-nuclear) elliptic spaces \mathcal{N}_E (respectively $\bar{\mathcal{N}}_E$), the nuclear (respectively non-nuclear) flat spaces \mathcal{N}_F (respectively $\bar{\mathcal{N}}_F$) and have, by inspection from Table 1,

$$S^3/G \in \begin{cases} \mathcal{N}_E, & \Leftrightarrow G \in \{Z_{2^p}, D_{4(2n+1)}^* \times Z_q, O^* \times Z_q, D_{2^{k \cdot (2n+1)}}^* \times Z_q\} \\ \bar{\mathcal{N}}_E, & \Leftrightarrow G \in \{Z_{2n+1}, D_{8n}^* \times Z_q, T^* \times Z_q, I^* \times Z_q, T_{8 \cdot 3^k}^* \times Z_q\}, \end{cases}$$

$$R^3/G \in \begin{cases} \mathcal{N}_F, & \Leftrightarrow G \in \{G_1, G_2, G_3, G_5\} \\ \bar{\mathcal{N}}_F, & \Leftrightarrow G \in \{G_4, G_6\}. \end{cases}$$

Given the validity of the conjecture mentioned above, and making essential use of the theorem of Gromov and Lawson, we have

Corollary 2. *An admissible manifold Σ is nuclear (respectively non-nuclear), if and only if its prime decomposition contains either an odd (respectively even) number of manifolds from $\mathcal{N}_E \cup \{S^1 \times S^2\}$, or is given by a single prime-manifold from \mathcal{N}_F (respectively $\bar{\mathcal{N}}_F$). \square*

References

1. Gibbons, G.W., Hartle, J.: *Phys. Rev. D* **42**, 2458 (1990)
2. Gibbons, G.W., Hawking, S.W.: Selection rules for topology change. *Commun. Math. Phys.* **148**, 345–352 (1992)
3. Gromov, M., Lawson, H.B.: *I.H.E.S. Publicationes Mathematicae* **58** (1983)
4. Lichnerowicz, A.: *Comp. Rend. Acad. Sci. Paris, Sér. A–B* **257**, 7–9 (1963)
5. Maunder, C.R.F.: *Algebraic topology*. Cambridge: Cambridge University Press 1980
6. Orlik, P.: *Seifert manifolds*. *Lecture Notes in Mathematics*, Vol. 291. Berlin, Heidelberg, New York: Springer 1972
7. Thomas, C.B.: *Elliptic structures on 3-manifolds*. *London Math. Soc. Lecture Note Series* 103. Cambridge: Cambridge University Press (1986); also: *Bull. London Math. Soc.* **20**, 65–67 (1988)
8. Witt, D.: *Topological obstructions to maximal slices*. UCSA Preprint, University of California, Santa Barbara (1987)
9. Wolf, J.A.: *Spaces of constant curvature*. Berkeley, CA: Publish or Perish 1977

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