

The Spinor Heat Kernel in Maximally Symmetric Spaces

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Abstract. The heat kernel $K(x, x', t)$ of the iterated Dirac operator on an N -dimensional simply connected maximally symmetric Riemannian manifold is calculated. On the odd-dimensional hyperbolic spaces K is a Minakshisundaram–DeWitt expansion which terminates to the coefficient $a_{(N-1)/2}$ and is exact. On the odd spheres the heat kernel may be written as an image sum of WKB kernels, each term corresponding to a classical path (geodesic). In the even dimensional case the WKB approximation is not exact, but a closed form of K is derived both in terms of (spherical) eigenfunctions and of a “sum over classical paths.” The spinor Plancherel measure $\mu(\lambda)$ and ζ function in the hyperbolic case are also calculated. A simple relation between the analytic structure of μ on H^N and the degeneracies of the Dirac operator on S^N is found.

1. Introduction

A maximally symmetric Riemannian manifold M of dimension N has an isometry group of maximum dimension $N(N + 1)/2$. M is also a constant curvature space, i.e., the Riemann tensor takes the form

$$R_{abcd} = k(g_{ad}g_{bc} - g_{ac}g_{bd}), \quad (1.1)$$

where k is a constant. The Ricci tensor and curvature scalar are given by $R_{ad} = k(N - 1)g_{ad}$, and $R = kN(N - 1)$. Moreover, M is necessarily isometric to one of the following spaces: a) Euclidean space R^N ($k = 0$); b) the sphere S^N of radius a ($k = 1/a^2$); c) the real projective space $P^N(R) = S^N/\sim$, where \sim is the antipodal points identification (k is the same as for S^N); d) the real hyperbolic space $H^N(R)$ of radius a ($k = -1/a^2$) (see ref. [24], vol. 1, p. 308). These spaces are all simply connected except for $P^N(R)$, which is doubly connected.

In the pseudo-Riemannian Lorentzian case [signature $(-, +, \dots, +)$] we have, similarly, that the maximally symmetric spacetimes are Minkowski spacetime M^N (zero curvature), de Sitter spacetime $(dS)_N$ (positive curvature), and anti-de Sitter spacetime $(AdS)_N$ (negative curvature). In the Euclidean approach to

quantum field theory in curved spacetime [28], time is Wick-rotated to imaginary time to make the path integral convergent, and the metric becomes a positive definite (Riemannian) metric. It has been pointed out by several authors [11, 13, 1, 16, 9] that S^N and H^N are the Euclidean sections appropriate to $(dS)_N$ and $(AdS)_N$, respectively. Therefore, the one-loop functional determinant on these spacetimes can be obtained from the ζ function on their Euclidean sections.

In ref. [2] Allen and Jacobson calculated the general form of the vector two-point function in maximally symmetric spaces (the scalar case is also discussed there). In this paper we would like to extend their results to the spinor case. In four dimensions the spinor two-point function has been calculated by Allen and Lütken [3]. Here we shall concentrate on the heat kernel K rather than the propagator. The heat equation for the iterated Dirac operator will be solved exactly on S^N and H^N . The two-point function and the zeta function can then be calculated from K via integral transforms in t (see Sects. 5 and 6).

There are two main steps in the construction of the heat kernel. The first is to identify and separate out the dependence on the spinor indices. To this end one considers the parallel spinor propagator U , a matrix in the spinor indices which parallel transports a spinor along a geodesic connecting two given points. By making the ansatz $K = Uf$, where f is a scalar function of the geodesic distance only, one can derive an equation for f . The second step is the use of the intertwining method (developed in ref. [5]) to reduce the problem from S^N to S^{N-2} (or $H^N \rightarrow H^{N-2}$). The induction procedure can be easily iterated, and one finds that the spinor heat kernel on H^N (or S^N), N odd, is obtained by applying a differential operator to the ordinary (scalar) heat kernel on the line (or on the circle with appropriate boundary conditions). This explains the exactness of the WKB approximation in the odd dimensional case. For N even one can similarly relate by a differential operator the heat kernels on S^N and S^2 . The equation on S^2 can then be solved exactly in terms of spinor spherical functions ϕ_n . By writing ϕ_n as a Mehler–Dirichlet integral one can relate the solutions on S^2 and S^1 by a pseudo-differential (fractional) operator and obtain a geometric representation of K , in analogy with the scalar case [5, 7].

The asymptotic form of the spherical eigenfunctions in the hyperbolic case allows one to obtain the spinor Plancherel measure $\mu(\lambda)$. This is the noncompact analogue of the spinor degeneracies on the N -sphere, i.e., it gives the spectral distribution of the eigenvalues of ∇^2 on H^N . For N odd $\mu(\lambda)$ is analytic in the λ -plane. For N even it is a meromorphic function with simple poles on the imaginary axis. The residues at these poles turn out to be proportional to the spinor degeneracies on S^N . This is a generalization of a result obtained recently for scalar fields [7, 21].

The plan of this paper is as follows. In Sect. 2 we write down the heat equation and an ansatz for its solution in terms of the parallel spinor propagator and a scalar function f . The equation satisfied by f is obtained. In Sect. 3 we solve this equation for N odd using the intertwining method. The even dimensional case is considered in Sect. 4. In Sect. 5 we obtain the spinor Plancherel measure and zeta function on the hyperbolic spaces. The two-point function is calculated in Sect. 6. In the appendix we construct a parallel vielbein on S^N using projective coordinates, and we calculate the covariant derivative of the parallel spinor propagator in geometric form.

2. The Heat Kernel and the Parallel Spinor Propagator

We begin by reviewing some basic facts about spinors (see, e.g., ref. [14], Appendix D). Let M be an N -dimensional orientable Riemannian manifold with vanishing second Stiefel–Whitney class. Then M admits a spinor structure and spinors can be defined globally on M . The Clifford algebra associated with the metric \mathbf{g} on M [signature $(+, +, \dots, +)$] is generated by N matrices Γ^a , $a = 1, \dots, N$, satisfying the anticommutation relations

$$\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}. \quad (2.1)$$

The dimension of these matrices is $2^{\lfloor \frac{N}{2} \rfloor}$, where $\lfloor \frac{N}{2} \rfloor = N/2$ for N even, $\lceil \frac{N}{2} \rceil = (N-1)/2$ for N odd. The $\frac{1}{2}N(N-1)$ matrices

$$\Sigma^{ab} \equiv \frac{1}{4}[\Gamma^a, \Gamma^b] \quad (2.2)$$

satisfy the $SO(N)$ commutation rules

$$[\Sigma^{ab}, \Sigma^{cd}] = \delta^{bc}\Sigma^{ad} - \delta^{ac}\Sigma^{bd} - \delta^{bd}\Sigma^{ac} + \delta^{ad}\Sigma^{bc}, \quad (2.3)$$

and generate $\text{Spin}(N)$, the double covering of $SO(N)$. The commutator of Σ^{ab} and Γ^c is then

$$[\Sigma^{ab}, \Gamma^c] = \delta^{bc}\Gamma^a - \delta^{ac}\Gamma^b. \quad (2.4)$$

Spinors are associated with orthonormal frames (vielbeins) of \mathbf{g} . Under a local frame rotation $A(x) = A^a_b(x) \in SO(N)$ a spinor transforms according to $\psi \rightarrow \psi' = S(A)\psi$, where $S(A) \in \text{Spin}(N)$ is determined by

$$S(A)^{-1}\Gamma^aS(A) = A^a_b\Gamma^b. \quad (2.5)$$

The covariant derivative of a spinor may be written, in a vielbein \mathbf{X}_a , as

$$\nabla_a\psi = \mathbf{X}_a\psi - \frac{1}{2}\omega_{abc}\Sigma^{bc}\psi, \quad (2.6)$$

where $\omega_{abc} \equiv \omega_{ab}{}^d\delta_{cd} = -\omega_{acb}$ are the (Levi–Civita) connection coefficients in the frame \mathbf{X}_a . These are given by

$$\omega_{abc} = \frac{1}{2}(C_{abc} - C_{acb} - C_{bca}) \quad (2.7)$$

in terms of the anolonomy coefficients,

$$[\mathbf{X}_a, \mathbf{X}_b] = C_{ab}{}^c\mathbf{X}_c, \quad (2.8)$$

the last index in $C_{ab}{}^c$ being lowered with the vielbein metric. The covariant derivative (2.6) can be generalized to higher order spinors and to tensor-spinors. It follows from (2.4) that the Γ matrices are covariantly constant, i.e.

$$\nabla_a\Gamma^b \equiv \omega_{ac}{}^b\Gamma^c - \frac{1}{2}\omega_{acd}[\Sigma^{cd}, \Gamma^b] = 0. \quad (2.9)$$

The commutation rule for the covariant derivatives [15]

$$[\nabla_a, \nabla_b]\psi = -\frac{1}{2}R_{cdab}\Sigma^{cd}\psi \tag{2.10}$$

implies the well known relation between the iterated Dirac operator, the spinor Laplacian, and the curvature scalar [15]

$$(\not{D})^2 \equiv (\Gamma^a \nabla_a)^2 = \delta^{ab} \nabla_a \nabla_b - \frac{R}{4}. \tag{2.11}$$

Suppose now that M is a (nonflat) simply connected maximally symmetric space, i.e., $M = S^N$ or H^N . From (1.1) we have

$$R_{cdab}\Sigma^{cd} = -2k\Sigma_{ab} \tag{2.12}$$

(all indices are lowered or raised with the vielbein metric), and Eq. (2.10) becomes

$$[\nabla_a, \nabla_b]\psi = k\Sigma_{ab}\psi. \tag{2.13}$$

The spinor heat kernel with one point at the origin, $K(y, t) \equiv K(y_0, y, t)$, satisfies

$$\left(-\frac{\partial}{\partial t} + \not{D}^2\right)K(y, t) = 0. \tag{2.14}$$

with the initial condition

$$\lim_{t \rightarrow 0} K(y, t) = \mathbf{1} \delta_N(y). \tag{2.15}$$

Here t is the time parameter in the heat equation and we are suppressing all spinor indices. K and $\mathbf{1}$ are actually $2^{\lfloor \frac{N}{2} \rfloor} \times 2^{\lfloor \frac{N}{2} \rfloor}$ matrices, and δ_N is the invariant delta function on M . For the solution of (2.14) consider the following ansatz:

$$K(y, t) = U(y) f(\sigma, t), \tag{2.16}$$

where U is a matrix in the spinor indices and f is a scalar function of t and of the geodesic distance $\sigma = d(y_0, y)$. Plugging (2.16) in (2.14) and using (2.11) gives

$$\left(-U \frac{\partial f}{\partial t} + U(\nabla^a \nabla_a U f) + 2(n^a \nabla_a U) \frac{\partial f}{\partial \sigma} + (\nabla^a \nabla_a U) f - \frac{R}{4} U f\right) = 0, \tag{2.17}$$

where we used

$$\nabla_a f = (\nabla_a \sigma) \frac{\partial f}{\partial \sigma} \equiv n_a \frac{\partial f}{\partial \sigma}. \tag{2.18}$$

The vector field $n_a = \nabla_a \sigma$ at the point y is the unit tangent vector to the shortest geodesic $\gamma(t)$ between the origin y_0 and y . In Riemann normal coordinates $\{y^a\}$ based at y_0 we simply have $n_a(y) = y_a / \|y\|$, where $\|y\| \equiv (y^a y_a)^{1/2}$ is the length of γ and equals $d(y_0, y)$.

The Laplacian acting on f can be replaced by its radial part \square_N given by ($\partial_\sigma \equiv \partial/\partial\sigma$)

$$\nabla^a \nabla_a f = \square_N f = (\partial_\sigma^2 + (N - 1)B\partial_\sigma)f, \tag{2.19}$$

$$B = \begin{cases} \frac{1}{a} \cot\left(\frac{\sigma}{a}\right), & S^N, \\ \frac{1}{a} \coth\left(\frac{\sigma}{a}\right) & H^N. \end{cases} \tag{2.20}$$

We now observe that the terms containing the first order derivatives of f and U in (2.17) cancel out if U satisfies the parallel transport equation

$$\begin{cases} n^a \nabla_a U = 0, \\ U(y_0) = \mathbf{1}. \end{cases} \tag{2.21}$$

This is the so-called parallel propagator. Given a spinor ψ_0 at y_0 , $U(y)\psi_0$ is a spinor at the point y obtained by parallel transport of ψ_0 along $\gamma(t)$. By taking $\gamma(0) = y_0$, $\gamma(1) = y$, and tangent vector $\dot{\gamma}(t) = \dot{\gamma}^a(t)\mathbf{X}_a|_{\gamma(t)}$, we can rewrite Eq. (2.21) for $U(t) \equiv U(\gamma(t))$ as

$$\nabla_{\dot{\gamma}(t)} U(t) = 0, \tag{2.22}$$

i.e. explicitly

$$\begin{cases} \frac{dU(t)}{dt} - Q(t)U(t) = 0, \\ U(0) = \mathbf{1}, \end{cases} \tag{2.23}$$

$$Q(t) \equiv \frac{1}{2} \omega_{abc}(t) \Sigma^{bc} \dot{\gamma}^a(t). \tag{2.24}$$

The formal solution to this equation is

$$U(t) = \mathcal{P} \exp \int_0^t Q(\tau) d\tau, \tag{2.25}$$

where \mathcal{P} is the path-ordering operator

$$\mathcal{P}[Q(t_1)Q(t_2) \dots Q(t_n)] = Q(t_{j_1})Q(t_{j_2}) \dots Q(t_{j_n}), \tag{2.26}$$

with $t_{j_1} > t_{j_2} > \dots > t_{j_n}$. Thus, we have the series expansion

$$U(t) = \mathbf{1} + \int_0^t Q(\tau) d\tau + \frac{1}{2} \int_0^t \int_0^t \mathcal{P}[Q(\tau_1)Q(\tau_2)] d\tau_1 d\tau_2 + \dots \tag{2.27}$$

$$+ \frac{1}{n!} \int_0^t \dots \int_0^t \mathcal{P}[Q(\tau_1) \dots Q(\tau_n)] d\tau_1 \dots d\tau_n + \dots \tag{2.28}$$

The path-ordering operator is needed because in general the commutator $[Q(t), Q(t')] \neq 0$ for $t \neq t'$. A simple calculation gives

$$[Q(t), Q(t')] = \dot{\gamma}^a(t) \dot{\gamma}^d(t') \omega_{ab}{}^c(t) \omega_{dce}(t') \Sigma^{be}. \tag{2.29}$$

For special manifolds like $S^3 \simeq SU(2)$ and Lie groups we can use a left-invariant vielbein with constant connection coefficients and with $\dot{\gamma}^a(t) = y^a = \text{constant}$. It is then easy to show that the commutator (2.29) vanishes, so that \mathcal{P} in (2.25) can be omitted and the integration carried out explicitly. The results for S^3 and for a general Lie group are given in Eqs. (3.38) and (7.1) below.

On S^N (or H^N), $N \neq 3$, it is not possible to find a vielbein where ω_{abc} is constant. However, it is always possible to choose \mathbf{X}_a such that U takes a very simple form.

Given any orthonormal basis $\{\mathbf{T}_a\}$ in the tangent space at y_0 , let $\{\mathbf{X}_a\}$ be the vielbein obtained (at the point y) by parallel transport of $\{\mathbf{T}_a\}$ along $\gamma(t)$ (the shortest geodesic between y_0 and y). Then $\nabla_{\dot{\gamma}(t)}\mathbf{X}_b = 0$, i.e. $\omega_{abc}(t)\dot{\gamma}^a(t) = 0$ and $Q(t) = 0$ (see (2.24)). Therefore in a parallel vielbein equation (2.23) has (locally) the trivial solution $U(t) = \mathbf{1} \forall t$. From the global point of view a complication arises in the compact case of S^N , due to the fact that y_0 and y may be conjugate points, e.g. y_0 is the north pole and y the south pole. Then the statement $U(y) = \mathbf{1}$ will not be true at the south pole, where both U and the parallel vielbein are undefined. In the appendix a parallel vielbein will be constructed on S^N by using projective coordinates.

Let us now derive the equation satisfied by the scalar function f in (2.16). With U satisfying (2.21) Eq. (2.17) becomes $(\partial_t \equiv \partial/\partial t)$

$$\left(-\partial_t + \square_N - \frac{R}{4} + U^{-1}(\nabla^a \nabla_a U) \right) f = 0, \tag{2.30}$$

with the initial condition

$$\lim_{t \rightarrow 0} f(\sigma, t) = \delta_N(\sigma), \tag{2.31}$$

where $\delta_N(\sigma)$ is the radial invariant delta function on M [7]. We now need to calculate the spinor Laplacian acting on U . Define the matrix V_a by

$$\nabla_a U \equiv V_a U, \tag{2.32}$$

i.e., $V_a = (\nabla_a U)U^{-1}$. Using (2.13) gives the following integrability condition on V_a :

$$\nabla_a V_b - \nabla_b V_a - [V_a, V_b] = k \Sigma_{ab}. \tag{2.33}$$

As observed in ref. [11], the simplest solution to this equation is $V_a = \frac{i\sqrt{k}}{2} \Gamma_a$, which, however, does not satisfy the parallel propagator equation (2.21). Thus, we look for a solution of (2.33) of the form

$$V_a = A \Sigma_{ab} n^b, \tag{2.34}$$

where A is a scalar function of the geodesic distance to be determined. Notice that the parallel propagator equation is identically satisfied by (2.34). Using (2.19) it is easy to show that [2]

$$\nabla_a n_b = B(g_{ab} - n_a n_b). \tag{2.35}$$

Inserting (2.34) in (2.33) and using (2.35) gives the following system of equations for the function A :

$$\begin{cases} A^2 - 2AB - k = 0, \\ A' - AB + A^2 = 0, \end{cases} \tag{2.36}$$

where $A' \equiv \partial A / \partial \sigma$. There are two solutions of (2.36). On S^N

$$A = \begin{cases} -\frac{1}{a} \tan(\sigma/2a), \\ \frac{1}{a} \cot(\sigma/2a). \end{cases} \tag{2.37}$$

On H^N

$$A = \begin{cases} \frac{1}{a} \tanh(\sigma/2a), \\ \frac{1}{a} \coth(\sigma/2a). \end{cases} \tag{2.38}$$

The only acceptable solutions are those that are regular at the origin (where they vanish). Therefore we take

$$A = \begin{cases} -\frac{1}{a} \tan(\sigma/2a), & S^N, \\ \frac{1}{a} \tanh(\sigma/2a), & H^N. \end{cases} \tag{2.39}$$

With this solution for A (2.34) solves both the integrability condition (2.33) and the parallel transport equation (2.21). An alternative proof of Eqs. (2.34) and (2.39) is given in the appendix.

The Laplacian acting on U is found to be

$$\nabla^a \nabla_a U = -\frac{1}{4} A^2 (N - 1) U, \tag{2.40}$$

and it is easy to check that (2.11) is equivalent to the following differential equation for A

$$A' + (N - 1)AB - \frac{1}{2}(N - 2)A^2 + \frac{R}{2(N - 1)} = 0, \tag{2.41}$$

which is satisfied by (2.39). From (2.30) and (2.40) we obtain the following equation for the scalar function f :

$$(-\partial_t + L_N)f = 0, \tag{2.42}$$

$$L_N = \square_N - \frac{R}{4} - \frac{1}{4}(N - 1)A^2. \tag{2.43}$$

We shall now solve this equation by using the intertwining operator method.

3. The Intertwining Method

The basic idea is to develop an induction procedure which reduces the problem on the N -sphere to a problem on the $(N - 2)$ -sphere (or $H^N \rightarrow H^{N-2}$), and then iterate this from S^N to S^1 , for N odd, and to S^2 , for N even. Some insight is provided by the

scalar case, discussed in ref. [5]. The heat operator is then simply $(-\partial_t + \square_N)$, and one shows that e.g. on S^N the operator $\mathcal{O} = \frac{1}{\sin \theta} \partial_\theta$, $\theta = \sigma/a$, satisfies

$$\square_N \mathcal{O} = \mathcal{O} \left(\square_{N-2} + \frac{N-2}{a^2} \right), \tag{3.1}$$

i.e., *intertwines* the radial Laplacian on S^N with the radial Laplacian on S^{N-2} plus a constant. Similarly, the operator $\frac{1}{\sinh x} \partial_x$, $x = \sigma/a$, intertwines the radial Laplacians on H^N and H^{N-2} with constant $\left(\frac{2-N}{a^2}\right)$. Let us, for simplicity, normalize the radius a of the space to 1, so that θ and x represent the geodesic distance (from the origin) on S^N and H^N , respectively. Written explicitly, the operator L_N on S^N takes the form

$$L_N = \partial_\theta^2 + (N-1) \cot(\theta) \partial_\theta - \left(\frac{N-1}{2}\right)^2 - \frac{N-1}{4 \cos^2 \theta/2}, \tag{3.2}$$

with an analogous relation on H^N . We look for an operator D such that

$$L_N D = D L_{N-2}, \tag{3.3}$$

and we make the ansatz (on S^N)

$$D = \frac{b}{\sin \theta} (\partial_\theta + g(\theta)), \tag{3.4}$$

where g is a function of θ to be determined and b is a constant. When this is substituted in (3.3) and the terms of like derivatives are equated, there result the following two equations for g :

$$g'(\theta) = 1 / \left(4 \cos^2 \frac{\theta}{2} \right), \tag{3.5}$$

$$g'' - \frac{g}{2 \cos^2 \frac{\theta}{2}} + (N-3) \cot(\theta) \left[\left(g' - \frac{1}{4 \cos^2 \frac{\theta}{2}} \right) - \frac{1}{\sin \theta \cos \theta} \left(g - \frac{1}{2} \tan \frac{\theta}{2} \right) \right], \tag{3.6}$$

where $g' = \partial_\theta g$. The unique solution to these equations is

$$g(\theta) = \frac{1}{2} \tan \frac{\theta}{2}. \tag{3.7}$$

Requiring that D relate the delta functions on S^N and S^{N-2} fixes $b = -1/(2\pi)$. Thus, we find that the operator

$$D = -\frac{1}{2\pi \sin \theta} \left(\partial_\theta + \frac{1}{2} \tan \frac{\theta}{2} \right) = \frac{1}{2\pi} \cos \frac{\theta}{2} \frac{\partial}{\partial \cos \theta} \circ \left(\cos \frac{\theta}{2} \right)^{-1} \tag{3.8}$$

satisfies (3.3) and relates f_N to f_{N-2} . A simpler way of showing this is to rewrite L_N in (3.2) as

$$L_N = \cos \frac{\theta}{2} \left(D_{\cos \theta}^{(N/2-1, N/2)} - \frac{N^2}{4} \right) \circ \left(\cos \frac{\theta}{2} \right)^{-1}, \quad (3.9)$$

where $D_x^{(a,b)}$ is the differential operator for the Jacobi polynomials $P_n^{(a,b)}(x)$, given in Eq. (4.19) below. From (3.9) and the relation [19]

$$D_x^{(a,b)} \partial_x = \partial_x (D_x^{(a-1, b-1)} + a + b) \quad (3.10)$$

it follows immediately that $D \propto \cos \frac{\theta}{2} \partial_{\cos \theta} \left(\cos \frac{\theta}{2} \right)^{-1}$ satisfies Eq. (3.3).

In the hyperbolic case we find similarly

$$D = -\frac{1}{2\pi \sinh x} \left(\partial_x - \frac{1}{2} \tanh \frac{x}{2} \right) = -\frac{1}{2\pi} \cosh \frac{x}{2} \frac{\partial}{\partial \cosh x} \circ \left(\cosh \frac{x}{2} \right)^{-1}. \quad (3.11)$$

Iteration of (3.3) gives

$$L_N D^{\frac{N-1}{2}} = D^{\frac{N-1}{2}} L_1, \quad N \text{ odd}, \quad (3.12)$$

$$L_N D^{\frac{N-2}{2}} = D^{\frac{N-2}{2}} L_2, \quad N \text{ even}. \quad (3.13)$$

The odd-dimensional case is elementary, since the operator $-\partial_t + L_1 = -\partial_t + \partial_\theta^2$ (or $-\partial_t + \partial_x^2$) can be inverted at once. On H^N the solution for f_N , Eq. (2.42), is obtained by applying the differential operator $D^{(N-1)/2}$ to the heat kernel of $(-\partial_t + \partial_x^2)$ on the line. Thus, we obtain the following result:

Theorem 3.1. *The heat kernel of the iterated Dirac operator on H^N , N odd, is given by*

$$\hat{K}_N(y, t) = U(y) \cosh \frac{x}{2} \left(\frac{-1}{2\pi} \frac{\partial}{\partial \cosh x} \right)^{\frac{N-1}{2}} \circ \left(\cosh \frac{x}{2} \right)^{-1} \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}}, \quad (3.14)$$

where $U(y)$ is the parallel spinor propagator from the origin y_0 to the point y , and x is the geodesic distance between y_0 and y .

In the compact case f_N is given by an ‘‘image sum’’ over ‘‘indirect’’ geodesics on the sphere and the operator $D^{(N-1)/2}$ should be applied to either the periodic or antiperiodic propagator $K_1^{(\pm)}$ on the circle,

$$K_1^{(\pm)}(\theta, t) = (4\pi t)^{-1/2} \sum_{-\infty}^{+\infty} (\pm 1)^n e^{-(\theta + 2\pi n)^2/4t}. \quad (3.15)$$

To find the appropriate boundary conditions on S^1 we shall solve (2.42) in terms of eigenfunctions. We define the *spinor spherical functions* ϕ_n as the eigenfunctions of L_N that are regular at the origin where they are normalized to one,

$$L_N \phi_n = -\lambda_n^2 \phi_n, \quad \phi_n(0) = 1. \quad (3.16)$$

From Eq. (3.9) and the differential equation for the Jacobi polynomials [19],

$$D_x^{(a,b)} P_n^{(a,b)}(x) = -n(n+a+b+1) P_n^{(a,b)}(x), \quad (3.17)$$

we obtain $\lambda_n = n + \frac{N}{2}$ and

$$\phi_n(\theta) = \frac{\text{vol}(S^N)}{\pi d_n} \cos \frac{\theta}{2} \left(\frac{1}{2\pi} \frac{\partial}{\partial \cos \theta} \right)^{\frac{N-1}{2}} \circ \left(\cos \frac{\theta}{2} \right)^{-1} \cos \left[\left(n + \frac{N}{2} \right) \theta \right] \quad (3.18)$$

$$= \frac{n! \Gamma\left(\frac{N}{2}\right)}{\Gamma\left(n + \frac{N}{2}\right)} \cos \frac{\theta}{2} P_n^{\left(\frac{N}{2}-1, \frac{N}{2}\right)}(\cos \theta), \quad n = 0, 1, \dots \quad (3.19)$$

Here $\text{vol}(S^N) = 2\pi^{\frac{N+1}{2}} / \Gamma\left(\frac{N+1}{2}\right)$ is the volume of the N -sphere and d_n are the degeneracies of \mathcal{V}^2 on S^N (without the spin factor $2^{\lfloor \frac{N}{2} \rfloor}$). They can be obtained from the relation

$$\int_0^\pi |\phi_n(\theta)|^2 (\sin \theta)^{N-1} d\theta = \frac{\text{vol}(S^N)}{\text{vol}(S^{N-1}) d_n}, \quad (3.20)$$

which gives

$$d_n = \frac{2(n + N - 1)!}{n!(N - 1)!}, \quad (3.21)$$

in agreement with ref. [12]. [Equations (3.21) and (3.19) are valid also for N even, see Sect. 4.] The eigenfunction expansion of f_N is now

$$f_N(\theta, t) = \frac{1}{\text{vol}(S^N)} \sum_{n=0}^\infty d_n \phi_n(\theta) e^{-t\lambda_n^2}. \quad (3.22)$$

Notice that the spherical functions (and f_N) are antiperiodic, $\phi_j(\theta + 2\pi n) = (-1)^n \phi_j(\theta)$, and vanish at $\theta = \pi$ (the south pole). Thus, although the parallel propagator U is undefined at the south pole, the heat kernel $K = U f_N$ is well defined and vanishes there. The antiperiodicity of the ϕ_n 's is then required in order for the heat kernel to be regular everywhere.

Using (3.18) and (3.8) we can rewrite (3.22) as

$$f_N(\theta, t) = D^{\frac{N-1}{2}} \frac{1}{\pi} \sum_{n=0}^\infty \cos \left[\left(n + \frac{N}{2} \right) \theta \right] e^{-t\lambda_n^2}. \quad (3.23)$$

The sum over n can now be taken to run from $\cos(\frac{1}{2}\theta)$ rather than $\cos\left(\frac{N}{2}\theta\right)$.

Indeed it is easy to see that the two series differ by terms that give identically zero when acted upon by the operator $D^{(N-1)/2}$. Using the Poisson summation formula we find

$$f_N(\theta, t) = D^{\frac{N-1}{2}} \frac{1}{\pi} \sum_{n=0}^\infty \cos \left[\left(n + \frac{1}{2} \right) \theta \right] e^{-t(n+1/2)^2} = D^{\frac{N-1}{2}} K_1^{(-)}(\theta, t), \quad (3.24)$$

where $K_1^{(-)}$ was defined in (3.15). Thus, the scalar part of K is obtained by applying the differential operator $D^{(N-1)/2}$ [where D is given in (3.8)] to the antiperiodic propagator on the circle. We have obtained.

Theorem 3.2.

The heat kernel of the iterated Dirac operator on S^N , N odd, is given by

$$K_N(y, t) = U(y) \sum_{n=-\infty}^{+\infty} (-1)^n f_{dp}(\theta + 2\pi n, t), \tag{3.25}$$

where U is the parallel spinor propagator (2.25), θ is the geodesic distance from the given point y to the origin y_0 (the north pole), and the “direct-path” $n = 0$ term is

$$f_{dp}(\theta, t) = \cos \frac{\theta}{2} \left(\frac{1}{2\pi} \frac{\partial}{\partial \cos \theta} \right)^{\frac{N-1}{2}} \circ \left(\cos \frac{\theta}{2} \right)^{-1} \frac{e^{-\theta^2/4t}}{(4\pi t)^{1/2}}. \tag{3.26}$$

Thus, K_N is given in terms of intrinsic geometric objects. From (3.25) we see that (the scalar part of) the spinor heat kernel on S^N , N odd, is an image sum of WKB kernels, each term in the sum over n corresponding to a classical path (geodesic). The direct path term is a Minakshisundaram expansion [26] which terminates to the coefficient $a_{(N-1)/2}$

$$f_{dp}(\theta, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \left(\frac{\theta}{\sin \theta} \right)^{\frac{N-1}{2}} e^{-\theta^2/4t} \left(1 + \sum_{k=1}^{(N-1)/2} a_k(\theta) t^k \right). \tag{3.27}$$

This result can also be obtained by calculating the (massless) spinor ζ -function of \mathbb{P}^2 on S^N . For both even and odd N , $\zeta_N(z)$ may be written as a finite sum of Riemann–Hurwitz functions, see ref. [7] Eqs. (11.93)–(11.94). In the odd dimensional case the fact that ζ_N has only a finite number of poles implies that the spinor heat kernel expansion must terminate, in agreement with (3.27). Our result here is more general since we can compute the finite coefficients $a_k(\theta)$ (from (3.26)) and not just the coincidence limits $a_k(0)$. The actual calculation is, of course, complicated. The simple examples of S^3 and S^5 are considered below. On S^3 we can use the isomorphism $S^3 \simeq SU(2)$ to work in a left-invariant vielbein defined everywhere. The parallel spinor propagator can be easily evaluated in this frame, with the result

$$U(\theta, \mathbf{n}) = \exp\left(\frac{i}{2} \theta \mathbf{n} \cdot \boldsymbol{\sigma}\right) = \mathbf{1} \cos \frac{\theta}{2} + i \mathbf{n} \cdot \boldsymbol{\sigma} \sin \frac{\theta}{2}, \tag{3.28}$$

where $\boldsymbol{\sigma}$ are the ordinary Pauli matrices and we are using canonical coordinates $y = \text{Exp}(\theta \mathbf{n})$, with $\theta = d(y_0, y)$ and $\|\mathbf{n}\| = 1$. Theorem 3.2 gives

$$f_{dp}(\theta, t) = (4\pi t)^{-\frac{3}{2}} \frac{\theta}{\sin \theta} \left(1 - t \frac{\tan \theta/2}{\theta} \right) e^{-\theta^2/4t}. \tag{3.29}$$

This result was obtained long ago by Altaie and Dowker [4]. Thus

$$a_1^{(S^3)}(\theta) = -\frac{\tan(\theta/2)}{\theta}, \quad a_k^{(S^3)} = 0, \quad k \geq 2. \tag{3.30}$$

Notice that U given by (3.28) is antiperiodic, so K can also be written as

$$K_{S^3}(\theta, \mathbf{n}, t) = \sum_{n=-\infty}^{+\infty} U(\theta + 2\pi n, \mathbf{n}, t) f_{dp}(\theta + 2\pi n, t). \tag{3.31}$$

On the five-sphere we obtain the following heat kernel coefficients:

$$a_1^{(S^5)} = -\frac{2}{\theta^2} + \frac{2}{\theta \sin \theta} - \frac{4 \tan \theta/2}{\theta}, \tag{3.32}$$

$$a_2^{(S^5)}(\theta) = \frac{3 \tan^2 \theta/2}{\theta^2}, \tag{3.33}$$

and $a_k^{(S^5)} = 0$ for $k \geq 3$. The coincidence limits can be checked by remembering the expression for a_1 and a_2 in terms of the curvature. Since the trace over the spinor indices is not included here, we have from ref. [6], p. 172,

$$a_1(0) = -\frac{R}{12}, \tag{3.34}$$

$$a_2(0) = \frac{1}{720} \left[\frac{5}{2} R^2 - \frac{7}{2} R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} \right]. \tag{3.35}$$

From (3.34) and (3.35) we find

$$a_1^{(S^3)}(0) = -\frac{1}{2}, \quad a_2^{(S^3)}(0) = 0, \tag{3.36}$$

$$a_1^{(S^5)}(0) = -\frac{5}{3}, \quad a_2^{(S^5)}(0) = \frac{3}{4}. \tag{3.37}$$

These values agree with those obtained by taking the limit $\theta \rightarrow 0$ in the finite expressions given above.

In the hyperbolic case since H^N is noncompact there is only one geodesic connecting two given points. Therefore in (3.14) we only have the “direct path” term, which is an exact Minakshisundaram expansion terminating at $a_{(N-1)/2}$. The corresponding expression may be obtained by replacing $\theta \rightarrow ix$ and $t \rightarrow -t$ in (3.27), and multiplying by $(-1)^{N/2}$.

4. The Even Dimensional Case

In order to apply (3.13) we need to solve for the heat kernel on S^2 (or H^2). The solutions of $L_2 \phi_n = -\lambda_n^2 \phi_n$, $\lambda_n = n + 1$, normalized to $\phi_n(0) = 1$ are given by

$$\phi_n(\theta) = \cos \frac{\theta}{2} P_n^{(0,1)}(\cos \theta) \tag{4.1}$$

(see (3.19)). They are antiperiodic and vanish at $\theta = \pi$. The eigenfunction expansion of f_2 can be written down immediately, but it is more instructive to first write ϕ_n as a Mehler–Dirichlet integral. There are two (fractional) integral representations that are useful. Let us prove that

$$P_n^{(a,0)}(\cos \theta) = \sqrt{\frac{2}{\pi}} \frac{n! \Gamma(a)}{\Gamma(a+n+1)} \partial_1^{1/2} \Big|_{+\cos \theta} C_{2n+1}^a \left(\cos \frac{\theta}{2} \right), \tag{4.2}$$

where C_n^a is a Gegenbauer polynomial, and the fractional derivative is defined by [27, 7]

$$\partial_{1+\cos\theta}^{1/2} f = \frac{1}{\sqrt{\pi}} \int_{\theta}^{\pi} \frac{\partial f / \partial \phi}{\sqrt{\cos\theta - \cos\phi}} d\phi, \tag{4.3}$$

provided $f(\pi) = 0$. In our case $C_{2n+1}^a\left(\cos\frac{\theta}{2}\right)$ is an odd polynomial in $\cos\frac{\theta}{2}$ and therefore it vanishes at $\theta = \pi$. To prove (4.2) we start from the explicit form of the Gegenbauer polynomials [25] to write

$$C_{2n+1}^a\left(\cos\frac{\theta}{2}\right) = \sum_{m=0}^n \frac{(-1)^m \Gamma(a+2n-m+1) 2^{n-m+1/2}}{m!(2n-2m+1)! \Gamma(a)} (1+\cos\theta)^{n-m+1/2}. \tag{4.4}$$

Then we apply the operator $\partial_{1+\cos\theta}^{1/2}$ and use the following rule for fractional differentiation of a power

$$\partial_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{4.5}$$

(valid for any α and for $\beta > -1$, see ref. [27] p. 67). By shifting the sum to $k = n - m$ and using the duplication formula for the Γ function we obtain

$$\begin{aligned} \partial_{1+\cos\theta}^{1/2} C_{2n+1}^a\left(\cos\frac{\theta}{2}\right) &= \sqrt{\frac{\pi}{2}} \frac{(-1)^n}{\Gamma(a)} \sum_{k=0}^n (-1/2)^k \frac{\Gamma(a+n+k+1)}{(n-k)!(k!)^2} \\ &\quad \times (1+\cos\theta)^k. \end{aligned} \tag{4.6}$$

Using [19]

$$P_n^{a,0}(\cos\theta) = (-1)^n F\left(-n, a+n+1, 1, \frac{1+\cos\theta}{2}\right) \tag{4.7}$$

and formula (22) p. 40 of ref. [25], we easily establish the validity of Eq. (4.2). We now use the symmetry property of the Jacobi polynomials

$$P_n^{(a,b)}(x) = (-1)^n P_n^{(b,a)}(-x), \tag{4.8}$$

to obtain

$$\phi_n(\theta) = \sqrt{\frac{2}{\pi}} \frac{(-1)^n}{n+1} \cos\frac{\theta}{2} \partial_{1-\cos\theta}^{1/2} C_{2n+1}^1\left(\cos\frac{\pi-\theta}{2}\right). \tag{4.9}$$

But [19]

$$C_n^1(\cos\phi) = \frac{\sin[(n+1)\phi]}{\sin\phi}, \tag{4.10}$$

and finally we get

$$\phi_n(\theta) = \sqrt{\frac{2}{\pi}} \cos\frac{\theta}{2} \partial_{1-\cos\theta}^{1/2} \left(\cos\frac{\theta}{2}\right)^{-1} \frac{\sin[(n+1)\theta]}{n+1}. \tag{4.11}$$

The fractional operator $\partial_{1-\cos\theta}^{1/2}$ is now defined with the boundary point at $\theta = 0$, i.e.

$$\partial_{1-\cos\theta}^{1/2} f = \frac{1}{\sqrt{\pi}} \int_0^\theta \frac{d\phi \partial_\phi f}{\sqrt{\cos\phi - \cos\theta}}, \tag{4.12}$$

provided $f(0) = 0$. By noting that the degeneracy factor is precisely $d_n = 2(n + 1)$ we can write the eigenfunction expansion of f_2 as

$$f_2(\theta, t) = \frac{1}{4\pi} \sum_{n=0}^\infty d_n \phi_n(\theta) e^{-t(n+1)^2} \tag{4.13}$$

$$= \cos \frac{\theta}{2} \left(\frac{1}{2\pi} \partial_{1-\cos\theta} \right)^{\frac{1}{2}} \left(\cos \frac{\theta}{2} \right)^{-1} \frac{1}{\pi} \sum_{n=1}^\infty \sin(n\theta) e^{-tn^2}. \tag{4.14}$$

The operator acting on the sum in (4.14) could be identified with the fractional power of order 1/2 of the operator D in Eq. (3.8). The problem with (4.14) is that the sum is not a Jacobi theta function and can not be “inverted” in terms of elementary functions. In the odd dimensional case (with a cosine instead of a sine) the sum is inverted in terms of $\exp(-\theta_n^2/4t)$, $\theta_n \equiv \theta + 2\pi n$. Using the Poisson summation formula in the present case gives $\exp(-\theta_n^2/4t)$ times an error function $\text{Erf}(i\theta_n/2\sqrt{t})$, which is not easy to handle. We shall now use a different fractional representation of $P_n^{(1,0)}$.

It is proved in ref. [18] that for $a > b > -\frac{1}{2}$ one has the following integral representation of $P_n^{(a,b)}(\cos\theta)$:

$$\begin{aligned} P_n^{(a,b)}(\cos\theta) &= \frac{2^{a-2b+3/2} \Gamma(n+a+1)}{\sqrt{\pi n!} \Gamma(a-b) \Gamma(b+\frac{1}{2})} \left(\sin \frac{\theta}{2} \right)^{-2a} \left(\cos \frac{\theta}{2} \right)^{-2b} \\ &\quad \times \int_0^{\theta/2} d\psi \sin \psi (\cos 2\psi - \cos \theta)^{b-1/2} \\ &\quad \times \int_0^\psi d\phi \cos [2(n+\rho)\phi] (\cos \phi - \cos \psi)^{a-b-1}, \end{aligned} \tag{4.15}$$

where $\rho = (a + b + 1)/2$. It is not difficult to rewrite this in terms of fractional operators

$$P_n^{(a,b)}(\cos\theta) = \frac{1}{\sqrt{\pi n!}} \Gamma(n+a+1) 2^{a-2b+1/2} M^{(a,b)} \frac{\sin[(n+\rho)\theta]}{n+\rho}, \tag{4.16}$$

$$M^{(a,b)} \equiv \left(\sin \frac{\theta}{2} \right)^{-2a} \left(\cos \frac{\theta}{2} \right)^{-2b} \partial_{1-\cos\theta}^{1/2-b} \partial_{1-\cos\theta/2}^{b-a}, \tag{4.17}$$

where ∂_x^α is defined by the Riemann–Liouville integral [27, 7]. Equation (4.16) is a consequence of the following fractional commutation:

$$D_{\cos\theta}^{(a,b)} M^{(a,b)} f = M^{(a,b)} (\partial_\theta^2 + \rho^2) f, \tag{4.18}$$

where

$$D_x^{(a,b)} = (1 - x^2) \partial_x^2 + [b - a - (a + b + 2)x] \partial_x \tag{4.19}$$

is the differential operator for the Jacobi polynomials. Equation (4.18) can be proved by using the rules of fractional calculus given in ref. [27]. It holds provided f satisfies

$$f(\theta = 0) = 0, \quad \partial_{1-\cos\theta/2}^{b-a} f|_{\theta=0} = 0. \tag{4.20}$$

From (4.16) and (4.8) we obtain another fractional integral representation for the spherical functions (4.1)

$$\phi_n(\theta) = -\sqrt{\frac{2}{\pi}} \left(\cos \frac{\theta}{2}\right)^{-1} \partial_{1+\cos\theta}^{-1/2} \circ \cos \frac{\theta}{2} \partial_{\cos\theta} \frac{\cos[(n+1)\theta]}{n+1}, \tag{4.21}$$

i.e. explicitly

$$\phi_n(\theta) = \frac{\sqrt{2}}{\pi} \left(\cos \frac{\theta}{2}\right)^{-1} \int_{\theta}^{\pi} \frac{\cos \frac{\phi}{2} \sin[(n+1)\phi]}{\sqrt{\cos\theta - \cos\phi}} d\phi. \tag{4.22}$$

Using this in the eigenfunction expansion (4.13) we obtain:

Theorem 4.1. *The heat kernel of \mathbb{V}^2 on S^2 is $K^{(S^2)} = Uf_2$, where U is the parallel spinor propagator and f_2 may be written as a “sum over classical paths”*

$$f_2(\theta, t) = -\left(\cos \frac{\theta}{2}\right)^{-1} (2\pi\partial_{1+\cos\theta})^{-1/2} \circ \cos \frac{\theta}{2} \partial_{\cos\theta} K_1^{(+)}(\theta, t) \tag{4.23}$$

$$= \frac{\sqrt{2}}{(4\pi t)^{3/2}} \sum_{n=-\infty}^{+\infty} \int_{\theta}^{\pi} \frac{\cos \frac{\phi}{2} (\phi + 2\pi n)}{\sqrt{\cos\theta - \cos\phi}} e^{-(\phi + 2\pi n)^2/4t} d\phi. \tag{4.24}$$

The heat kernel on S^N , N even, is given by Uf_N , where $f_N = D^{(N-2)/2} f_2$ and D is the differential operator (3.8).

We can check our result (4.24) by computing the coincidence limit of the heat kernel coefficients in the asymptotic expansion

$$f_2(0, t) \simeq (4\pi t)^{-1} \sum_{k=0}^{\infty} a_k t^k, \tag{4.25}$$

valid for $t \rightarrow 0$. Only the $n = 0$ term in (4.24) must be retained in this limit, the $n \neq 0$ terms being “exponentially small” when compared to the direct path contribution. The a_k should coincide with the values obtained from the spinor zeta function [7]

$$\zeta_{S^2}(z) = 2\zeta_R(2z - 1), \tag{4.26}$$

(ζ_R is the Riemann zeta function) through the relations [7]

$$\text{res}_{S^2}(z)|_{z=1} = a_0, \quad \zeta_{S^2}(-k) = (-1)^k k! a_{k+1}. \tag{4.27}$$

Thus, we expect $a_0 = 1$ and

$$a_k = \frac{(-1)^{k-1} \zeta_{S^2}(1-k)}{(k-1)!} = -\frac{|B_{2k}|}{k!}, \tag{4.28}$$

where B_k are the Bernoulli numbers [19].

In order to prove this we take the coincidence limit ($\theta = 0$) in the $n = 0$ term in (4.24),

$$f_2^{dp}(0, t) = (4\pi t)^{-3/2} \int_0^\pi \phi \cot \frac{\phi}{2} e^{-\phi^2/4t} d\phi, \tag{4.29}$$

and use

$$\phi \cot \frac{\phi}{2} = 2 \left(1 - \sum_{k=1}^\infty \frac{|B_{2k}| \phi^{2k}}{(2k)!} \right), \tag{4.30}$$

valid for $|\phi/2| < \pi$ [19]. Then we change variable to $x^2 = \phi^2/4t$, and integrate term by term by replacing the upper limit of integration $\pi/\sqrt{4t}$ with infinity. Using [19]

$$\int_0^\infty x^{2k} e^{-x^2} dx = 2^{-k-1} \sqrt{\pi} (2k-1)!! , \tag{4.31}$$

and

$$\frac{(2k-1)!!}{(2k)!} = \frac{1}{2^k k!} , \tag{4.32}$$

we obtain as $t \rightarrow 0$

$$f_2^{dp}(0, t) \simeq (4\pi t)^{-1} \left(1 - \sum_{k=1}^\infty \frac{|B_{2k}| t^k}{k!} \right), \tag{4.33}$$

in complete agreement with the result from ζ -function theory. The values of a_1 and a_2

$$a_1 = -\frac{1}{6}, \quad a_2 = -\frac{1}{60}, \tag{4.34}$$

can also be checked by using Eqs. (3.34)–(3.35).

The hyperbolic spaces can be handled in a similar way. (The spherical functions and the eigenfunction expansion of the heat kernel will be considered in the next section.) The basic difference from the compact case is that the Riemann–Liouville integral becomes a Weyl integral, with the boundary point at infinity [27, 7]. The geometric representation of the heat kernel on H^2 is $K^{(H^2)} = U\hat{f}_2$,

$$\hat{f}_2(x, t) = \frac{\sqrt{2} \left(\cosh \frac{x}{2} \right)^{-1}}{(4\pi t)^{3/2}} \int_x^{+\infty} \frac{y \cosh \frac{y}{2} e^{-y^2/4t}}{\sqrt{\cosh y - \cosh x}} dy . \tag{4.35}$$

On H^N we have $K^{(H^N)} = U\hat{f}_N, \hat{f}_N = D^{(N-2)/2} \hat{f}_2$, where D is the operator (3.11).

5. The Plancherel Measure and the ζ -function on H^N

In the hyperbolic case the spectrum of L_N is continuous and the spherical functions satisfy

$$L_N \phi_\lambda = -\lambda^2 \phi_\lambda \tag{5.1}$$

(λ is an arbitrary real number), with the normalization $\phi_\lambda(0) = 1$. Letting $z = -\sinh^2 \frac{x}{2}$ and $\phi_\lambda = (1-z)^{1/2} \tilde{\phi}_\lambda$ we obtain an hypergeometric equation for $\tilde{\phi}_\lambda$

$$\left\{ z(1-z)\partial_z^2 + \left[\frac{N}{2} - (N+1)z \right] \partial_z - \lambda^2 - \frac{N^2}{4} \right\} \tilde{\phi}_\lambda = 0. \tag{5.2}$$

The solution for ϕ_λ is then found to be

$$\phi_\lambda(x) = \cosh \frac{x}{2} F \left(i\lambda + \frac{N}{2}, -i\lambda + \frac{N}{2}, \frac{N}{2}, -\sinh^2 \frac{x}{2} \right). \tag{5.3}$$

The spectral distribution of the eigenvalues is given by the *Plancherel measure* $\mu(\lambda)$, which is completely determined by the asymptotic form of the spherical functions at infinity. For $x \rightarrow \infty$ we have

$$\phi_\lambda(x) \simeq C(\lambda)e^{i\lambda x - \rho x} + C(-\lambda)e^{-i\lambda x - \rho x}, \tag{5.4}$$

where ρ is a (positive) constant, and $C(\lambda)$ is the Harish–Chandra function [21]. In terms of $C(\lambda)$ the Plancherel measure is [20, 7]

$$\mu(\lambda) = [C(\lambda)C(-\lambda)]^{-1} = |C(\lambda)|^{-2}, \tag{5.5}$$

where the last equality holds only for λ real. In the scalar case the Plancherel measure is known on any noncompact Riemannian symmetric space, where it may be written as a product over the positive roots of the space [20]. The explicit form of μ on the real, complex, and quaternion hyperbolic spaces may be found e.g. in ref. [7].

In order to calculate the spinor Plancherel measure from (5.4)–(5.5) we use the following functional transformation for F [19]:

$$\begin{aligned} F(\alpha, \beta, \gamma, z) &= \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} (-1/z)^\beta F(\beta, \beta + 1 - \gamma, \beta + 1 - \alpha, 1/z) \\ &+ \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} (-1/z)^\alpha F(\alpha, \alpha + 1 - \gamma, \alpha + 1 - \beta, 1/z). \end{aligned} \tag{5.6}$$

Using this in (5.3) and letting $x \rightarrow \infty$ gives

$$\phi_\lambda(x) \simeq \frac{\Gamma(N/2)\Gamma(2i\lambda)2^{N-1-2i\lambda}}{\Gamma(i\lambda)\Gamma(i\lambda + N/2)} e^{i\lambda x - \rho x} + (\lambda \rightarrow -\lambda), \tag{5.7}$$

where $\rho = (N - 1)/2$. Comparing with (5.4) and using (5.5) we obtain

$$\mu(\lambda) = \frac{2^{4-2N}}{(\Gamma(N/2))^2} \cosh(\pi\lambda) \left| \Gamma \left(i\lambda + \frac{N}{2} \right) \right|^2. \tag{5.8}$$

In the odd dimensional case $\mu(\lambda)$ reduces to a polynomial:

$$\mu(\lambda) = \frac{\pi}{2^{2(N-2)}(\Gamma(N/2))^2} \prod_{j=\frac{1}{2}}^{(N-2)/2} (\lambda^2 + j^2), \quad N \text{ odd}. \tag{5.9}$$

In the even dimensional case we get

$$\mu(\lambda) = \frac{\pi\lambda \coth(\pi\lambda)}{2^{2(N-2)}(\Gamma(N/2))^2} \prod_{j=1}^{(N-2)/2} (\lambda^2 + j^2), \quad N \text{ even}, \quad (5.10)$$

where the product is omitted for $N = 2$. The analytic structure of μ is different in the two cases. For N even μ defines a meromorphic function in the complex λ -plane with simple poles on the imaginary axis at $\lambda = \pm i(n + N/2)$, $n = 0, 1, \dots$. It is easy to calculate the residues at these poles, and we find the following relation:

$$\frac{\text{res } \mu|_{\lambda=i(n+N/2)}}{\text{res } \mu|_{iN/2}} = \frac{(n + N - 1)!}{n!(N - 1)!} = \frac{d_n}{d_0}, \quad (5.11)$$

where

$$d_n = \frac{(n + N - 1)!}{n!(N - 1)!} 2^{\lfloor \frac{N}{2} \rfloor + 1} \quad (5.12)$$

are the degeneracies of the iterated Dirac operator on the N -sphere [12]. Thus, the singular points of the Plancherel measure determine the spectrum of \not{V}^2 on S^N . A similar result holds in the scalar case [21, 7], where it may be proved for any pair of dual Riemannian symmetric spaces [22]. It is natural to conjecture that the spinor result generalizes to any symmetric pair admitting a spinor structure.¹

In odd dimensions $\mu(\lambda)$ is analytic and from (5.9) we obtain a relation similar to (5.11)

$$\frac{\mu(i(n + N/2))}{\mu(iN/2)} = \frac{d_n}{d_0}. \quad (5.13)$$

The eigenfunction expansion of the spinor heat kernel can be written as

$$K^{(H^N)}(y, t) = U(y) c_N \int_0^{+\infty} \phi_\lambda(x) e^{-t\lambda^2} \mu(\lambda) d\lambda, \quad (5.14)$$

where the constant $c_N = 2^{N-3} \Gamma(N/2) / \pi^{N/2+1}$ is determined by requiring that (5.14) agree with the Minakshisundaram–De Witt expansion for $t \rightarrow 0$. The zeta function of the operator $\text{Tr}[-\not{V}^2 + m^2]$, defined as a Mellin transform of $\text{Tr}[K(y_0, t)e^{-tm^2}]$, is given by

$$\begin{aligned} \zeta_N(z, m) &= \frac{1}{\Gamma(z)} \int_0^{+\infty} dt t^{z-1} \text{Tr} K^{(H^N)}(y_0, t) e^{-tm^2} \\ &= 2^{\lfloor \frac{N}{2} \rfloor} c_N \int_0^{+\infty} \frac{\mu(\lambda) d\lambda}{(\lambda^2 + m^2)^z}. \end{aligned} \quad (5.15)$$

Here a mass m for the spinor field has been inserted in order to avoid the infrared divergence at the lower limit of integration. (In the compact case the ζ -function of \not{V}^2 is well defined because the spectrum does not include zero.) The integral (5.15) converges for $\text{Re } z > N/2$ and is defined by analytic continuation for the other values of z . For N odd the integrations are elementary. Defining numbers $a_{k,N}$ by

$$\prod_{j=1/2}^{(N-2)/2} (\lambda^2 + j^2) \equiv \sum_{k=0}^{(N-1)/2} a_{k,N} \lambda^{2k}, \quad (5.16)$$

¹ For example it should hold on the complex projective spaces $P^6(C)$, $P^{10}(C)$, \dots , but not on $P^4(C)$, $P^8(C)$, \dots

and using¹ Eq. 3.251, n.2 of ref. [19], we obtain

$$\zeta_N(z, m) = \frac{2^{\lfloor \frac{N}{2} \rfloor} m^{1-2z}}{(4\pi)^{N/2} \Gamma(N/2)} \sum_{k=0}^{(N-1)/2} a_{k, N} m^{2k} B\left(k + \frac{1}{2}, z - k - \frac{1}{2}\right), \tag{5.17}$$

where $B(x, y)$ is Euler’s beta function. For N even > 2 we define $b_{k, N}$ by

$$\prod_{j=1}^{(N-2)/2} (\lambda^2 + j^2) = \sum_{k=0}^{(N-2)/2} b_{k, N} \lambda^{2k}, \tag{5.18}$$

and $b_{0, 2} = 1$. Using the identity

$$\coth(\pi\lambda) = 1 + 2/(e^{2\pi\lambda} - 1), \tag{5.19}$$

we find

$$\begin{aligned} \zeta_N(z, m) = & \frac{2^{\lfloor \frac{N}{2} \rfloor}}{(4\pi)^{N/2} \Gamma(N/2)} \sum_{k=0}^{(N-2)/2} b_{k, N} \left[m^{2k+2-2z} B(k+1, z-k-1) \right. \\ & \left. + 4 \int_0^\infty \frac{\lambda^{2k+1} (\lambda^2 + m^2)^{-z}}{e^{2\pi\lambda} - 1} d\lambda \right]. \end{aligned} \tag{5.20}$$

The terms containing the beta functions give the meromorphic part, with poles at

$$z = \frac{N}{2}, \frac{N}{2} - 1, \dots, 1. \tag{5.21}$$

The other terms are analytic in the z -plane. For example on H^4 we obtain

$$\begin{aligned} \zeta_4(z, m) = & \frac{a^{2z-4}}{2\pi^2} \left\{ \frac{(a^2 m^2)^{1-z}}{2(z-1)} + \frac{(a^2 m^2)^{2-z}}{2(z-1)(z-2)} \right. \\ & \left. + 2 \int_0^\infty \frac{\lambda(\lambda^2 + 1) d\lambda}{(e^{2\pi\lambda} - 1)(\lambda^2 + a^2 m^2)^z} \right\}, \end{aligned} \tag{5.22}$$

where the radius a has been reinstated. The derivative of ζ_4 at zero gives the one-loop functional determinant for a spinor field in anti-de Sitter space, and can be used to calculate the spinor effective potential and stress-energy tensor in this spacetime [10].

6. The Two-Point Function

The massive spinor Green’s function with one point at the origin, $G(y) \equiv G(y_0, y)$, is a solution of

$$(\not{V}^2 - m^2)G(y) = -\delta_N(y), \tag{6.1}$$

with the appropriate boundary conditions to be specified below. G is related to the heat kernel by

$$G(y) = \int_0^{+\infty} K(y, t) e^{-tm^2} dt. \tag{6.2}$$

From Eq. (2.16) we have

$$G(y) = U(y)g_N(\sigma) , \tag{6.3}$$

$$g_N(\sigma) = \int_0^{+\infty} f_N(\sigma, t)e^{-tm^2} dt , \tag{6.4}$$

where σ is the geodesic distance from y_0 to y (hereby the radius $a = 1$), and f_N is the scalar function multiplying U in Eqs. (3.14) and (3.25), for N odd, and is given by Theorem 4.1 and Eq. (4.35), for N even. It is, perhaps, more instructive to solve directly Eq. (6.1) with the boundary conditions appropriate to the compact and the noncompact case. On S^N we define $g_N(\theta) = \cos\frac{\theta}{2}h_N(\theta)$ and change variable to $z = \cos^2\frac{\theta}{2}$, to obtain an hypergeometric equation for h_N

$$\left\{ z(1-z)\partial_z^2 + \left[\frac{N}{2} + 1 - (N+1)z \right] \partial_z - \frac{N^2}{4} - m^2 \right\} h_N(z) = 0 , \tag{6.5}$$

where we assume $\sigma \neq 0$. The boundary conditions require that for $N \geq 2$, h_N have only one singular point at the origin $\theta = 0$ (north pole), i.e. in the coincidence limit. In particular, the Green's function should be regular at the south pole ($\theta = \pi$). This uniquely determines h_N up to an overall factor,

$$h_N(z) = c_N F\left(\frac{N}{2} + im, \frac{N}{2} - im, \frac{N}{2} + 1, z\right) . \tag{6.6}$$

To obtain the constant c_N we require that the $\theta \rightarrow 0$ singularity have the same strength as in flat space, namely as $\theta \rightarrow 0$ g_N should approach g_N^{flat} , where

$$g_N^{\text{flat}}(\theta) = \begin{cases} \Gamma\left(\frac{N}{2} - 1\right) / [4\pi^{N/2} \theta^{N-2}] , & N \neq 2 , \\ -\frac{1}{2\pi} \ln \theta , & N = 2 . \end{cases} \tag{6.7}$$

From ref. [25] we have near $z = 1$ and for $\alpha + \beta - \gamma > 0$,

$$F(\alpha, \beta, \gamma, z) \approx \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1-z)^{\gamma-\alpha-\beta} . \tag{6.8}$$

From this we find the value of c_N for $N \neq 2$:

$$c_N = \frac{\Gamma\left(\frac{N}{2} + im\right)\Gamma\left(\frac{N}{2} - im\right)}{2^N \pi^{N/2} \Gamma\left(\frac{N}{2} + 1\right)} . \tag{6.9}$$

For $N = 2$ we use Eq. (2), p. 74 of ref. [17] (with $l = 0$), which shows the required logarithmic singularity of $F(1 + im, 1 - im, 2, z)$ near $z = 1$, and gives the same value (6.9) (with $N = 2$) for c_2 . Thus we obtain the following result, valid for any N :

Theorem 6.1. *The spinor Green’s function (6.1) on S^N is given by (6.3) with*

$$g_N(\theta) = \frac{\Gamma\left(\frac{N}{2} + im\right)\Gamma\left(\frac{N}{2} - im\right)}{2^N \pi^{N/2} \Gamma\left(\frac{N}{2} + 1\right)} \cos \frac{\theta}{2} F\left(\frac{N}{2} + im, \frac{N}{2} - im, \frac{N}{2} + 1, \cos^2 \frac{\theta}{2}\right). \tag{6.10}$$

Consider now the case of H^N . As before, let x be the geodesic distance from y_0 to y (in units of the radius a), and let $g_N(x) = \cosh \frac{x}{2} h_N(x)$. Then h_N is a solution of Eq. (6.5) with $z = \cosh^2 \frac{x}{2}$ and $m^2 \rightarrow -m^2$. It is useful to change variable to $u = 1/z$, and define

$$p(u) = u^{-N/2 - m} h_N(u). \tag{6.11}$$

The function $p(u)$ satisfies the hypergeometric equation with

$$\alpha = \frac{N}{2} + m, \quad \beta = m, \quad \gamma = 2m + 1. \tag{6.12}$$

Two linearly independent solutions are [17]

$$p_1(u) = F(\alpha, \beta, \gamma, u), \tag{6.13}$$

$$p_2(u) = u^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, u). \tag{6.14}$$

The boundary conditions in the hyperbolic case require that the Green’s function g_N (or h_N) fall off as fast as possible for $x \rightarrow \infty$, and have the same strength $x \rightarrow 0$ singularity as in flat space [2, 3]. The first condition selects $p(u) \propto p_1(u)$ and the second condition fixes the overall coefficient in p . We obtain:

Theorem 6.2. *The spinor Green’s function (6.1) on H^N is given by (6.3) with*

$$g_N(x) = \frac{\Gamma\left(\frac{N}{2} + m\right)\Gamma(m)}{2^N \pi^{N/2} \Gamma(2m + 1)} \times \left(\cosh \frac{x}{2}\right)^{1-N-2m} F\left(\frac{N}{2} + m, m, 2m + 1, \cosh^{-2} \frac{x}{2}\right). \tag{6.15}$$

We end this section with a few remarks. First of all we note that the spinor propagators and heat kernels on the maximally symmetric spacetimes $(dS)_N$ and $(AdS)_N$ (signature $(-, +, \dots, +)$) coincide, for spacelike separation, with the corresponding quantities on S^N and H^N , respectively. In the timelike case θ and x are imaginary and the hypergeometric functions in (6.10) and (6.15) have a branch cut, since the argument $z > 1$. In this case the Feynman propagator is obtained as the limiting value $G_N(\sigma + i0)$ above the cut [2, 3].

The second observation concerns the Green’s function D to the first order Dirac operator

$$(\not{\Psi} + m)D = -\delta_N, \tag{6.16}$$

which is related to G by [15]

$$D = (\Psi - m)G . \tag{6.17}$$

We shall evaluate D and compare our results with those obtained by Allen and Lütken in the four dimensional case. Using Eqs. (2.18), (2.32), (2.34), (2.39), (6.3) and

$$\Gamma^a \Sigma_{ab} = \frac{1}{2}(N - 1)\Gamma_b \tag{6.18}$$

in (6.17) we obtain

$$D(y) = n^a \Gamma_a U(y) \left[\frac{\partial}{\partial \sigma} + \frac{1}{2}(N - 1)A(\sigma) \right] g_N(\sigma) - mU(y)g_N(\sigma) . \tag{6.19}$$

In order to evaluate the first term we rewrite the operator in square brackets in (6.19) as

$$-\frac{1}{2} \sin \theta \left(\cos^2 \frac{\theta}{2} \right)^{-\frac{N-1}{2}} \partial_{\cos^2 \frac{\theta}{2}} \circ \left(\cos^2 \frac{\theta}{2} \right)^{\frac{N-1}{2}}$$

for S^N , and as

$$-\frac{1}{2} \sinh x \left(\cosh^{-2} \frac{x}{2} \right)^{\frac{N+3}{2}} \partial_{\cosh^{-2} \frac{x}{2}} \circ \left(\cosh^{-2} \frac{x}{2} \right)^{-\frac{N-1}{2}}$$

for H^N . Then we take g_N from (6.10) and (6.15), and use ref. [17], p. 102, Eqs. (22) (for S^N) and (21) (for H^N) with $n = 1$, to obtain the following result:

Theorem 6.3. *The spinor Green's function (6.16) on S^N is given by*

$$D(y) = -\frac{\Gamma\left(\frac{N}{2} + im\right)\Gamma\left(\frac{N}{2} - im\right)}{2^N \pi^{N/2} \Gamma\left(\frac{N}{2} + 1\right)} \left\{ mU(y) \cos \frac{\theta}{2} F\left(\frac{N}{2} + im, \frac{N}{2} - im, \frac{N}{2} + 1, \cos^2 \frac{\theta}{2}\right) + \frac{N}{2} n^a \Gamma_a U(y) \sin \frac{\theta}{2} F\left(\frac{N}{2} + im, \frac{N}{2} - im, \frac{N}{2}, \cos^2 \frac{\theta}{2}\right) \right\} . \tag{6.20}$$

The spinor Green's function (6.16) on H^N is given by

$$D(y) = -\frac{\Gamma\left(\frac{N}{2} + m\right)\Gamma(m + 1)}{2^N \pi^{N/2} \Gamma(2m + 1)} \left(\cosh \frac{x}{2} \right)^{-N-2m} \times \left\{ n^a \Gamma_a U(y) \sinh \frac{x}{2} F\left(\frac{N}{2} + m, m + 1, 2m + 1, \cosh^{-2} \frac{x}{2}\right) + U(y) \cosh \frac{x}{2} F\left(\frac{N}{2} + m, m, 2m + 1, \cosh^{-2} \frac{x}{2}\right) \right\} . \tag{6.21}$$

For $N = 4$ one can easily show that (6.20) coincides with Eqs. (3.9), (3.13) and (3.14) of ref. [3], and that (6.21) coincides with Eqs. (3.9), (3.15) and (3.16) of [3].

7. Conclusions

In this paper we have obtained a geometric representation of the spinor heat kernel in maximally symmetric spaces. By squaring the Dirac operator the heat equation is solved by the simple ansatz $K = Uf$, where U is the parallel spinor propagator and f a scalar function. It is not clear, at the moment, whether this simplification is due to the maximal symmetry of the manifold or holds for other homogeneous spaces as well. A possible generalization to rank-one symmetric spaces and to compact Lie groups is presently under investigation.

For groups there is the interesting conjecture [8] that the spinor heat kernel expansion terminates to the coefficient a_μ , where μ is the number of positive roots of the group. The parallel spinor propagator on a group G can be written down explicitly in a left-invariant vielbein as

$$U(y) = \exp(-y^a Q_a/2), \tag{7.1}$$

where $\{y^a\}$ are canonical coordinates, and

$$Q_a = -\frac{1}{2} f_{abc} \Sigma^{bc} \tag{7.2}$$

are the generators of the spin representation of G (f_{abc} are the structure constants). Since a compact Lie group is topologically the same as a product of odd spheres (apart from torsion they have the same cohomology) the conjecture formulated above seems quite plausible, in view of the results obtained in this paper. We defer this problem to our future work.

A. Appendix

In this appendix we calculate the covariant derivative of the parallel spinor propagator U on S^N . The basic idea, due to Higuchi [23], is to work in a vielbein which is parallel along each geodesic emanating from the north pole. As proved in Sect. 2, the parallel transport equation satisfied by U has the trivial solution $U \equiv \mathbf{1}$ in this frame, so that the covariant derivative of U reduces to

$$\nabla_a U = -\frac{1}{2} \omega_{abc} \Sigma^{bc} U \tag{A.1}$$

(see Eq. (2.6)). Then we need to calculate the connection components in this vielbein and reexpress the right-hand side of (A.1) in geometric form, so that it is valid in any frame. Let us first construct a parallel vielbein on S^N . Consider projective coordinates $\{x^a\}$ based at the south pole (SP), i.e., define a mapping $S^N - \{\text{SP}\} \rightarrow R^N$ by associating to the point $x \in S^N$ the intersection of the line $\text{SP} - x$ with the plane $z^{N+1} = 0$, where $(z^1, \dots, z^N, z^{N+1}) = (z^a, z^{N+1})$ are cartesian coordinates in R^{N+1} . Then

$$x^a = z^a / (1 + z^{N+1}), \quad a = 1, \dots, N, \tag{A.2}$$

with $(z^{N+1})^2 + \sum_{a=1}^N (z^a)^2 = 1$, and conversely

$$z^a = 2x^a / (1 + x^2), \tag{A.3}$$

$$z^{N+1} = (1 - x^2) / (1 + x^2), \tag{A.4}$$

where $x^2 \equiv \sum_{a=1}^N (x^a)^2$. The north pole (NP) corresponds to $x^a = 0$ ($\Rightarrow z^a = 0$, $z^{N+1} = 1$). A simple calculation gives the metric in projective coordinates

$$g_{ab}(x) = \left\langle \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right\rangle \Big|_x = \frac{4\delta_{ab}}{(1+x^2)^2}. \tag{A.5}$$

A vielbein of this metric is

$$\mathbf{X}_a|_x = \frac{1}{2}(1+x^2) \frac{\partial}{\partial x^a} \Big|_x. \tag{A.6}$$

The vielbein components of the Levi-Civita connection (2.7) are given by

$$\omega_{abc}(x) = x_c \delta_{ab} - x_b \delta_{ac} = \omega_{a[bc]}(x), \tag{A.7}$$

where all indices are raised and lowered with the vielbein metric, e.g. $x_a = \delta_{ab}x^b$. We shall now prove that the vielbein $\{\mathbf{X}_a\}$ is parallel along each geodesic $\gamma_y(t)$, with $\gamma_y(0) = \text{NP}$ and $\gamma_y(1) = y \in S^N - \{\text{SP}\}$. Let us first derive the equation of such geodesic. The connection components in the coordinate frame $\partial/\partial x^a$ are

$$\omega_{ab}{}^c(x) = -\frac{2}{1+x^2}(x_b \delta_a^c + x_a \delta_b^c - x^c \delta_{ab}) = \omega_{(ab)}{}^c(x). \tag{A.8}$$

The equation for $\gamma_y(t) = \{x^a(t)\}$ reads

$$\frac{d^2 x^c}{dt^2} - \frac{2}{1+x^2} \left(2x_a(t) \frac{dx^a}{dt} \frac{dx^c}{dt} - x^c(t) \frac{dx^a}{dt} \frac{dx_a}{dt} \right) = 0, \tag{A.9}$$

with the boundary conditions

$$x^a(0) = 0, \quad x^a(1) = y^a, \tag{A.10}$$

$\{y^a\}$ being the coordinates of the point y . The unique solution to (A.9) and (A.10) is

$$x^a(t) = y^a \tan(bt)/y, \tag{A.11}$$

where

$$y \equiv \left[\sum_{a=1}^N (y^a)^2 \right]^{1/2}, \tag{A.12}$$

$$b \equiv \arctan(y). \tag{A.13}$$

The tangent vector to $\gamma_y(t)$ may be written as

$$\partial_t = \frac{dx^a}{dt} \frac{\partial}{\partial x^a} \Big|_{\gamma_y(t)} = \frac{y^a}{y} b [1 + \tan^2(bt)] \frac{\partial}{\partial x^a} \Big|_{\gamma_y(t)} = 2b \frac{y^a}{y} \mathbf{X}_a \Big|_{\gamma_y(t)}, \tag{A.14}$$

where (A.6) was used. Thus ∂_t has constant (t -independent) components with respect to \mathbf{X}_a . From (A.14) and (A. 7) it easily follows that \mathbf{X}_a is parallel along $\gamma_y(t)$,

$$\nabla_{\partial_t} \mathbf{X}_a|_{\gamma_y(t)} = 0. \tag{A.15}$$

We can now express the geodesic distance on a great circle, $\theta = d(\text{NP}, y)$, in terms of projective coordinates. From the definition of Riemannian distance and from (A.14) we have

$$\theta \equiv \int_0^1 \sqrt{\langle \partial_t, \partial_t \rangle} dt = 2b, \tag{A.16}$$

and remembering (A.13) we find

$$\theta = 2\arctan(y), \quad y = \tan\left(\frac{\theta}{2}\right). \quad (\text{A.17})$$

We also notice, from (A.11), the following relation between projective and Riemann normal coordinates on S^N :

$$y^a = \frac{\tan \theta/2}{\theta} y_{\text{RNC}}^a, \quad (\text{A.18})$$

with $\sum (y_{\text{RNC}}^a)^2 = \theta^2$.

The unit vector field $n_a = \nabla_a \theta$ has vielbein components

$$n_a(y) = \mathbf{X}_a \theta = \frac{1}{2}(1 + y^2) \frac{\partial \theta}{\partial y^a} = \frac{y_a}{y}, \quad (\text{A.19})$$

where (A.17) was used. The spinor covariant derivative (2.6) takes the form (using (A.7))

$$\nabla_a \psi|_y = \mathbf{X}_a \psi|_y - \Sigma_{ab} y^b \psi(y), \quad (\text{A.20})$$

and the parallel transport operator reduces to

$$n^a \nabla_a = n^a \mathbf{X}_a, \quad (\text{A.21})$$

proving that $U = \mathbf{1}$ in this frame. The covariant derivative of U is then

$$\begin{aligned} \nabla_a U|_y &= \nabla_a \mathbf{1} = -\Sigma_{ab} y^b \mathbf{1} \\ &= -y \Sigma_{ab} n^b \mathbf{1} \\ &= -\tan \frac{\theta}{2} \Sigma_{ab} n^b U(y). \end{aligned} \quad (\text{A.22})$$

This is now written in geometric form and is valid in any frame. Another derivation of Eq. (A.22) may be found in Sect. 2.

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