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Irrational Free Field Resolutions for $W(sl(n))$ and Extended Sugawara Construction

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Abstract. The existence of Miura-type free field realizations is established for the extended conformal algebras $W(sl(n))$ at irrational values of the screening parameter. The problem of the "closure" of the algebra is reduced to a finite dimensional quantum group problem. The structure of the Fock space resolution and the character formula are obtained for the irreducible modules. As graded vector spaces these modules are shown to be isomorphic to the space of $sl(n)$ singlets in $\widehat{sl}(n)$ affine level 1 modules. The isomorphism is given by the $\phi\beta\gamma$ free field realization of $\widehat{sl}(n)$.

1. Introduction

Certain classes of low dimensional field theories are exactly soluble due to the presence of infinite dimensional Lie algebras in these models. Besides their appearance in 2-dimensional conformal field theories or their 3-dimensional topological counterparts, also the massive, respectively, non-topological perturbations thereof are expected to carry remnants of this algebraic structure. $W(g)$ algebras are, besides the affine Kac Moody algebras $\hat{g}$, the second known class of infinite dimensional Lie algebras descending from simple finite dimensional ones g [1–4]. In general they are intrinsically non-linear in that the commutation relations close only on the enveloping algebra of the modes of the generating fields. This accounts for both the variety of applications, as well as certain difficulties in handling them.

In particular, the construction of realizations in terms of an underlying linear oscillator or affine algebra is non-trivial. The major obstruction lies in proving that the algebra of the proposed field generators closes. Associativity is then guaranteed by the associativity of the underlying oscillator or affine algebra. The existence of a realization turns out to be closely related to the structure of a characteristic Hilbert space $\mathscr{H}(g)$ associated with it. The space $\mathscr{H}(g)$ encodes the information about the operator product expansion of the proposed set of generating fields. If $\mathscr{H}(g)$ contains a sufficient number of independent states (w.r.t. some graduation), the operator product algebra is forced to close. Once the algebra is known to exist, the space $\mathscr{H}(g)$ can be identified with the vacuum representation space for generic central charge. In the first part of the present paper this strategy will be adopted to prove the existence of a free field realization for all members of the $W(sl(n))$ series. The required completeness property of the space $\mathscr{H}(g)$ will be traced back to a finite dimensional quantum group problem. This is achieved by employing a quantum group structure appearing in the multiple integrals of screening operators used in the Fock space construction of singular vectors [7, 8].

The procedure also leads to Fock space and Verma module resolutions for a class of "irrational" $W(sl(n))$ modules. The irreducible modules $\mathscr{L}(I(\Lambda_+, \Lambda_-))$ of that type are labelled by a pair of $sl(n)$-weights and an irrational parameter s_+^2, related to the central charge. Their characteristic property is that the embedding pattern of their singular vectors coincides with that of the underlying simple Lie algebra. The space $\mathscr{H}(g)$ is recovered as the singlet module $\mathscr{L}(I(0,0))$ and is characterized as the intersection of the kernels of the screening operators on the Fock module. In particular, a basis of $W(sl(n))$ is obtained in which the structure constants are polynomial in the central charge. In this basis the decoupling of nullfields, which may occur for special values of the central charge, as well as certain pathological features associated with it can be discussed systematically. In the second part of the paper the irrational $W(sl(n))$ modules are shown to be related to the extended Sugawara construction in level $k = 1$ affine algebras. As graded vector spaces, the modules $\mathscr{L}(I(\Lambda, 0))$ turn out to be isomorphic to the space of $sl(n)$ singlets in an affine level one module. The isomorphism can be made explicit by employing an infinite dimensional analogue of the Harish–Chandra theorem. Essentially it is given by the $\phi\beta\gamma$ free field realization of $\widehat{sl}(n)$. This leads to an infinite dimensional abelian subalgebra in the space of $sl(n)$ singlets [37].

The paper is organized as follows. In Sect. 2 extended conformal algebras are defined in relation to their highest weight representations. Sections 3 and 4 are devoted to the construction of the free field realization and the modules $\mathscr{L}(I(\Lambda_+, \Lambda_-))$ of irrational type. Section 5 discusses the pathological features at exceptional central charge and Sect. 6 deals with the extended Sugawara construction.

2. Definition of $W(g)$ Algebras and Highest Weight Representations

For technical reasons it is appropriate to define W-algebras as special meromorphic conformal field theories. Basically a meromorphic CFT (mCFT) or vertex operator algebra [14, 15] is an infinite dimensional Lie algebra which contains the Virasoro algebra as a distinguished subalgebra and for which all fields have integer or halfinteger conformal weight. In more detail, a mCFT consists of a (pre-)Hilbert space $\mathscr{H}$ and an assignment $|P\rangle \to P(z)$, which associates a unique field operator $P(z)$ to any state $|P\rangle$ in (a dense subspace of) $\mathscr{H}$. The Hilbert space $\mathscr{H}$ is a vacuum representation space of the Virasoro algebra, i.e. there exists a distinguished state $|L\rangle$ for which the modes L_n, $n \in \mathbb{Z}$ of the associated field $L(z)$ form a copy of the Virasoro algebra and which define a unique $su(1,1)$ invariant vacuum by $L_s|v\rangle = 0$, $s = 0, \pm 1$. The dense subspace $\underline{\kappa}$ is that of finite L_0 grade and for an element $|P\rangle$, the associated field operator satisfies $P(z)|0\rangle = e^{zL_{-1}}|P\rangle$ (*) as well as a number of additional conditions. The additional conditions force the spectrum of L_0 on $\mathscr{H}$ to be integer or halfinteger and in particular guarantee the injectivity of the assignment (*). Let $\mathscr{H}_\Delta = \{|P\rangle \in \mathscr{H}: L_0|P\rangle = \Delta|P\rangle\}$ denote the subspaces of fixed L_0-grade, where Δ is called the (conformal) weight of $|P\rangle$ or $P(z)$. The operators $P(z)$ are linear operators which map $\underline{\mathscr{H}}$ to infinite sums of elements in $\underline{\mathscr{H}}$ i.e. $P(z): \underline{\mathscr{H}} \to \bigoplus_\Delta \mathscr{H}_\Delta$. They are completely determined by their matrix elements $(P_{\Delta\Delta'})_{\Delta,\Delta'\in\mathbb{Z}_+}$, where $P_{\Delta\Delta'}: \mathscr{H}_\Delta \to \mathscr{H}_{\Delta'}$. The product $P(z)Q(w)$ exists (for $|z|>|w|$) if the series $\Sigma_\Delta\langle P^i|P_{\Delta\Delta_i}Q_{\Delta\Delta_k}|P^k\rangle$ is absolutely convergent for all $|P^i\rangle \in \mathscr{H}_{\Delta_i}, |P^k\rangle \in \mathscr{H}_{\Delta_k}$, which is the last condition stipulated. The associativity of the product is then guaranteed by the absolute convergence. For the product of two fields $P(z)$, $Q(z)$ of weights Δ_P, Δ_Q one has the series expansion

$$P(z)Q(w) = \sum_{k=-\Delta_P-\Delta_Q}^{\infty} (z-w)^k (P_{-k-\Delta_P}Q_{-\Delta_Q})(w), \quad |z|>|w| , \tag{2.1}$$

where $(P_{-k-\Delta_P}Q_{-\Delta_Q})(w)$ is the field corresponding to the state $P_{-k-\Delta_P}Q_{-\Delta_Q}|v\rangle$. In particular, $(P,Q)(z) := (P_{-k-\Delta_P}Q_{-\Delta_Q})(z)$ is a natural definition of the normal ordered product of both fields. The usual contour deformation argument then shows that (2.1) amounts to the specification of the Lie brackets $[P_m, Q_n]$. The Jacobi identity is implied by the associativity of the operator product expansion and hence is guaranteed whenever the product is well defined on $\mathscr{H}$. A convenient basis for the Lie algebra is obtained by decomposing the Hilbert space $\mathscr{H}$ w.r.t. the action of the $su(1,1)$ subalgebra of the Virasoro algebra generated by $\{L_{\pm 1}, L_0\}$. (For notational simplicity we will from now on drop the distinction between $\underline{\mathscr{H}}$ and $\mathscr{H}$.) The $su(1,1)$ highest weight states satisfy $L_1|P\rangle = 0$ and such states (or the corresponding fields) are called quasiprimary. The subspace of quasiprimary states in $\mathscr{H}$ will be denoted by $\hat{\mathscr{H}}$. The $su(1,1)$ descendences $L_{-1}^n|P\rangle$ of a basis in $\hat{\mathscr{H}}$ make up a basis of $\mathscr{H}$.

W-algebras are special mCFTs. The basic point is that one does not take all quasiprimary fields as the generators of the algebra but allows the use of normal ordered products to generate the algebra.

Definition. *A W-algebra of rank r is a meromorphic* CFT *which is generated by the operations ∂ and $\hat{\mathscr{N}}$ from r quasiprimary fields $W^1(z) = L(z), W^2(z), \ldots, W^r(z)$. The bilinearform on $\bigoplus\mathscr{H}_{\Delta_i}$, $1 \leqq i \leqq r$ induced by the Shapovalov form is nondegenerate.*

The last condition takes care of certain pathological features which may occur for special values of the central charge (cf. Sect. 5). $\mathscr{N}(\,,\,)$ is a $su(1,1)$ covariant normal ordering prescription. It differs from (,) (induced by (2.1)) by a finite number of derivative terms. The choice of the normal ordering is in principle irrelevant, but $\mathscr{N}(\,,\,)$ is a convenient one [15]. The basis $|\hat{W}^i\rangle$ is unique up to linear transformations in the sector $\bigoplus\bar{\mathscr{H}}_{\Delta_i}$, $1 \leqq i \leqq r$. A basis $\bar{W}^i$ is called a *Cartan basis* if its zero modes satisfy $[\bar{W}^i_0, \bar{W}^j_0] = 0$, $1 \leqq i,j \leqq r$. A drawback of the above definition is that it does not specify the commutator of arbitrary monomials in the modes W^i_n. To study the representation theory commutators of the type $[\mathscr{N}(W^{i_1} \ldots W^i_r)_n, W^{j_1}_{n_1} \ldots W^{j_k}_{n_k}]$ are needed, which cannot directly be traced back to the operator product expansion. The evaluation from the $[W^i_m, W^j_n]$ commutators on the other hand involves infinite sums of generators, whose convergence at intermediate stages is not guaranteed. This means that a regularization prescription is required. Clearly the detailed form of the regularization should be irrelevant and, if possible, direct reference to it should be avoided. In the case of the so-called Casimir algebras we shall therefore adopt the following, slightly stronger definition. To prepare this set

$$C_r = \{0 \leqq c \leqq r | c = r - 12\rho^2(s_+^2 + s_+^{-2} - 2), s_+^2 \text{ irrational}\} , \tag{2.2}$$

where ρ is the Weyl vector of g and r the rank. Further take $\mathscr{H}(g)$ to be the (completion of the) span of the lexicographically ordered states of the form

$$W^1_{-\nu_1} \dots W^r_{-\nu_r}|v\rangle, \quad \text{with}$$

$$\nu_i \in \mathrm{Par}(\Delta_i) := \{\nu = (n_1, \dots, n_l) \in \mathbb{Z}^l : n_j \geqq n_{j+1} \geqq \Delta_i, 1 \leqq j \leqq l, l \geqq 0\}. \tag{2.3}$$

Definition′. *For a complex Lie algebra g a Casimir algebra $W(g)$ is a W-algebra for which the weights of the generating fields coincide with the orders of the independent Casimirs of g. For $c \in C_r$ the algebra is of rank r and the L_0-graded highest weight module satisfying $W^i_n|v\rangle = 0$ iff $n > -\Delta_i$ is irreducible and coincides with $\mathscr{H}(g)$.*

The additional condition guarantees that any regularization prescription employed to evaluate the missing commutators yields the same answer which is moreover compatible with the parts directly fixed by the operator product expansion. In Definition′ weight spaces refer to the L_0-graduation only. For Casimir algebras with a Cartan basis one can introduce weight states as simultaneous eigenstates of $\bar{W}^i_0$. This leads to a $\mathbb{R}^r$-graduation in terms of Weyl invariant polynomials. We expect that every Casimir algebra possesses a Cartan basis. For $W(sl(r+1))$ $(r \leqq 4)$ this is the content of Proposition 4.2. Let $\Lambda_\pm \in P_+$ be dominant integral weights of g and $s_\pm$ be real parameters s.t. $s_+s_- = -1$. Fix a Weyl chamber, i.e. a set of simple roots $\{\alpha_1, \dots, \alpha_r\} \subset h^*$ in the dual of the Cartan subalgebra of g and set

$$x_i = s_+\alpha_i \cdot (\Lambda_+ + \rho) + s_-\alpha_i \cdot (\Lambda_- + \rho) , \tag{2.4}$$

where ρ is the Weyl vector and '$\cdot$' is the inner product in h^*, sometimes also denoted by $(,)$. Let $I^i(\Lambda_+, \Lambda_-) = I^i(x_1, \dots, x_r)$, $1 \leqq i \leqq r$ be polynomials of degree $i+1$ that generate the ring of Weyl invariant polynomials in $x_1, \dots, x_r$ (but not necessarily the standard basis obtained from the Casimir operators). A state $|I\rangle = |I(\Lambda_+, \Lambda_-)\rangle$ is called a highest weight vector for $W(g)$ if it satisfies

$$\bar{W}^i_n|I\rangle = \delta_{n,0} I^i(\Lambda_+, \Lambda_-)|I\rangle , \quad n \geqq 0 . \tag{2.5}$$

These states can be regarded as highest weight states w.r.t. a triangular decomposition induced by the $-\operatorname{ad} L_0$-grading on the modes of composite fields $W(g) = (W_+ \oplus W_0 \oplus W_-)(g)$. The corresponding highest weight module

$$V(I(\Lambda_+, \Lambda_-)) = \sum_{\nu_1, \dots, \nu_r \in \mathrm{Par}(1)} \mathbb{C}\bar{W}^1_{-\nu_1} \dots \bar{W}^r_{-\nu_r}|I\rangle \tag{2.6}$$

is called a Verma module for $W(g)$. Define a shifted action $w * \Lambda = w(\Lambda + \rho) - \rho$ of the Weyl group of g on h^*, so that $(w\alpha_i, \Lambda + \rho) = (\alpha_i, w^{-1} * \Lambda + \rho)$ for $w \in W$. It follows that $I(\Lambda_+, \Lambda_-)$ and hence the Verma module is invariant under the diagonal action of the Weyl group W of g,

$$I(w * \Lambda_+, w * \Lambda_-) = I(\Lambda_+, \Lambda_-), \quad w \in W , \tag{2.7}$$

and in particular, does not depend on the choice of the Weyl chamber.

The Verma module $V(I(\Lambda_+, \Lambda_-))$ provides a reducible representation of $W(g)$ if the central charge is in the interval $0 \leqq c \leqq r$. Irreducible highest weight representations $\mathscr{L}(I)$ should be obtained by dividing out the maximal singular submodule $SV(I(\Lambda_+, \Lambda_-))$,

$$\mathscr{L}(I) = V(I)/SV(I) . \tag{2.8}$$

The characteristic space $\mathscr{H}(g)$ can be identified with the singlet module $\mathscr{L}(I(0,0))$. The irreducible representation spaces $\mathscr{L}(I)$ are endowed with a unique nondegenerate hermitian bilinear form, which we normalize by $\langle I|I\rangle = 1$. With respect to this bilinear form the decomposition into quasiprimary and derivative states on $\mathscr{L}(I)$ may then be regarded as an orthogonal decomposition of $\mathscr{L}(I)$ into a quasiprimary sector $\hat{\mathscr{L}}(I)$ and its orthocomplement $\mathscr{L}(I) = \hat{\mathscr{L}}(I) \oplus \mathscr{L}(I)^{\perp}$. Let for $|P\rangle$ in $\mathscr{L}(I)$ denote $\mathscr{N}|P\rangle$ the projection onto the quasiprimary part.

These definitions do not, of course, imply the existence of the objects referred to. For low rank cases W-algebras can explicitly be constructed by solving the associativity condition. The complexity of the resulting commutation relations, however, hinders a direct access to the representation theory. Conversely, a free field realization provides a powerful tool to study the representation theory, but leaves the construction of the realization as the nontrivial task. The major obstruction constructing such a free field realization lies in proving that the proposed set of field generators closes. Associativity is then guaranteed by the associativity of the underlying oscillator algebra. A candidate for a free field realization of $W(sl(n))$ has been proposed by Fateev and Lukyanov to take the form of a generalized Miura transformation [3]. Despite sample calculations [30], considerations of the large N limit [35] and the argument [12][1], no conclusive proof of the existence of this realization seems to exist. We shall later argue that the problem of the closure of the algebra can be reduced to a finite dimensional quantum group problem, thereby establishing the existence of the free field realization. In preparation some results are needed on intertwining operators for the quantum groups $\mathscr{U}_q(sl(r+1))$.

3. q-Intertwiners

Let $\mathscr{U}_q(g)$ denote the standard q-deformation of the (enveloping algebra) of the complex simple Lie algebra g, with generators e_i, f_i, h_i, $1 \leqq i \leqq r$ [10, 11, 23]. Let $\mathscr{U}_q(g) = \mathscr{U}_q(n_+) \otimes \mathscr{U}_q(h) \otimes \mathscr{U}_q(n_-)$ be a triangular decomposition. The q-Verma module is defined as $M^q_\Lambda = \mathscr{U}_q(n_-) v_\Lambda$, where v_Λ is a (highest weight) vector s.t. $\mathscr{U}_q(n_+) v_\Lambda = 0$, $h_i \cdot v_\Lambda = (\Lambda, h_i) v_\Lambda$ (where the identifications $\mathscr{U}_q(h) \cong \mathscr{U}(h)$ and $h \cong h^*$ were used in the inner product). Suppose two q-Verma modules M^q_Λ and $M^q_{\bar{\Lambda}}$ to be given. The homomorphisms $M^q_\Lambda \to M^q_{\bar{\Lambda}}$ commuting with the action of $\mathscr{U}_q(g)$ on M^q_Λ and $M^q_{\bar{\Lambda}}$ are called q-intertwining operators. To describe the set of such intertwiners $\mathrm{Hom}_{U_q(g)}(M^q_\Lambda, M^q_{\bar{\Lambda}})$ some preparations are needed.

[1] The reasoning in [12] apparently is insensitive to the value of the central charge, which diminishes its conclusiveness. The closure of the OPE can be affected by the presence of nullfields in the sector $\mathscr{H}_\Delta$, $\Delta \leqq 2\max_i \Delta_i - 1$; which may happen for rational screening parameter. See [13] on this point for the case of coset realizations and Sect. 5 for the free field realizations of Fateev–Lukyanov-type

Lemma 3.1. *For q not a root of unity:*

a) *Given Λ, $\tilde{\Lambda} \in h^*$, there exists a 1–1 correspondence between elements of $\mathcal{U}_q(n_-)$ and* $\mathrm{Hom}_{\mathcal{U}_q(n_-)}(M^q_\Lambda, M^q_{\tilde{\Lambda}})$.
b) *There is a $1-1$ correspondence between elements of* $\mathrm{Hom}_{\mathcal{U}_q(g)}(M^q_\Lambda, M^q_{\tilde{\Lambda}})$ *and singular vectors in $M^q_{\tilde{\Lambda}}$ of weight Λ. Moreover, every such intertwiner is injective,* i.e. *defines an embedding $M^q_\Lambda \subsetneq M^q_{\tilde{\Lambda}}$.*

We include the proof to illustrate the concepts (see also [22]).

Proof. a) The maps $Q \in \mathrm{Hom}_{\mathcal{U}_q(n_-)}(M^q_\Lambda, M^q_{\tilde{\Lambda}})$ are completely determined by their action on v_Λ; so suppose that $Q(v_\Lambda) = x v_{\tilde{\Lambda}}$, for some $x \in \mathcal{U}_q(n_-)$. Then

$$Q(y v_\Lambda) = y Q(v_\Lambda) = y x v_{\tilde{\Lambda}}, \quad \forall y \in \mathcal{U}_q(n_-)\,. \tag{3.1}$$

Conversely, every $x \in \mathcal{U}_q(n_-)$ determines a map $Q \in \mathrm{Hom}_{\mathcal{U}_q(n_-)}(M^q_\Lambda, M^q_{\tilde{\Lambda}})$ by (3.1). Define now a representation ρ of $\mathcal{U}_q(n_-)$ on M^q_Λ by right multiplication

$$\rho(f_i)(y v_\Lambda) = -\, y f_i v_\Lambda, \quad i = 1 \ldots r\,, \tag{3.2}$$

and a "translation operator" $T^{\tilde{\Lambda}}_\Lambda$: $M_\Lambda \to M_{\tilde{\Lambda}}$ by $T^{\tilde{\Lambda}}_\Lambda(y v_\Lambda) = y v_{\tilde{\Lambda}}$. The map $x \to \rho(x) T^{\tilde{\Lambda}}_\Lambda$ then gives a 1–1 correspondence between $\mathcal{U}_q(n_-)$ and $\mathrm{Hom}_{\mathcal{U}_q(n_-)}(M^q_\Lambda, M^q_{\tilde{\Lambda}})$.

b) Requiring that Q in (3.1) intertwines also with $\mathcal{U}(n_+)$ amounts to $n_+(x v_{\tilde{\Lambda}}) = 0$, which means that $x v_{\tilde{\Lambda}}$ should be a singular vector in $M^q_{\tilde{\Lambda}}$ of weight Λ. Injectivity follows from (3.1). □

To find elements $x \in \mathcal{U}_q(n_-)$ which give rise to an intertwiner $Q \in \mathrm{Hom}_{\mathcal{U}_q(g)}(M^q_\Lambda, M^q_{\tilde{\Lambda}})$ one will try first to incorporate the intertwining property with $\mathcal{U}_q(h)$. This is to say that it suffices to consider elements in $\mathcal{U}_q(n_-)$ of isospin $\Lambda - \tilde{\Lambda}$. Every $\rho(f_i)$ in (3.2) should thus be accompanied by a change in the highest weight $\Lambda \to \tilde{\Lambda} = \Lambda + \alpha_i$. Defining $s_i = \rho(f_i) T^{\Lambda+\alpha_i}_\Lambda$, every polynomial in these "screening operators" will give rise to a map intertwining with $\mathcal{U}_q(h) \oplus \mathcal{U}_q(n_-)$. The subset of such polynomials that intertwines with the action of all of $\mathcal{U}_q(g)$ is in principle fixed by the remaining commutator $[e_i, s_j] = -\,\delta_{ij} \dfrac{q^{l_i} - q^{-l_i}}{q - q^{-1}} T^{\Lambda+\alpha_i}_\Lambda$. On a general monomial in the screening operators the action of e_i is (on M^q_Λ) given by

$$[e_i, s_{i_1} \ldots s_{i_n}] = -\sum_{j: i_j = i} \frac{q^{l_i - a_j} - q^{-l_i + a_j}}{q - q^{-1}} s_{i_1} \ldots \hat{s}_{i_j} \ldots s_{i_n} T^{\Lambda+\alpha_i}_\Lambda, \tag{3.3}$$

where $l_i = (\Lambda, \alpha_i)$, $a_j = a_{i_j i_{j+1}} + \ldots + a_{i_j i_n}$ and '^' denotes omission. One can check that $[e_i, (s_j)^{l_j+1}] = 0$ and $M_{r_i * \Lambda}$, so that

$$s_i^{l_i+1} \in \mathrm{Hom}_{\mathcal{U}_q(g)}(M^q_{r_i * \Lambda}, M^q_\Lambda) \tag{3.4}$$

provides a set of intertwiners. In the context of Lemma 3.1, the intertwiners (3.4) correspond to the singular vectors $f_i^{l_i+1} v_\Lambda$ of weight $r_i * \Lambda$ in M_Λ, which generate the maximal singular submodule of M^q_Λ [26]. Other intertwiners are more difficult to find from (3.3) directly. A systematic description is possible by means of a partial ordering on the Weyl group W of g. Recall that the Weyl group of $sl(r+1)$ is isomorphic to the symmetric group S_{r+1}. The generators r_i, $1 \leqq i \leqq r$ permute the

i^{th} and $i+1^{\text{th}}$ site and correspond to reflections in the simple roots α_i. An expression $w = r_{i_1} \ldots r_{i_l}$ for $w \in W$ is called reduced if it contains the minimal number of reflections required. In this case $l = l(w)$ is called length of $w \in W$. For $w_1, w_2 \in W$ write $w_1 \leftarrow w_2$ if $w_1 = r_\alpha w_2$ for some $\alpha \in \Delta_+$ and $l(w_1) = l(w_2) + 1$. The Bruhat ordering on W is then defined by: $w \prec \tilde{w}$ iff there exist $w_1, \ldots, w_k \in W$ s.t. $w \leftarrow w_1 \leftarrow \ldots \leftarrow w_k \leftarrow \tilde{w}$. In the appendix a number of related facts have been summarized. As in the undeformed case [18, 19] one then has

Proposition 3.2. *For $\Lambda \in P_+$, q not a root of unity:*

$$\dim \mathrm{Hom}_{\mathcal{U}_q(g)}(M^q_{\tilde{w}*\Lambda}, M^q_{w*\Lambda}) = \begin{cases} 1 & \text{if } \tilde{w} \preceq w \\ 0 & \text{otherwise}. \end{cases}$$

For $\tilde{w} \prec w$ let $Q_{\tilde{w}w}$ denote the intertwiner. This statement is equivalent to certain rearrangement identities in $\mathcal{U}_q(n_-)$ to which we will return later.

If q is not a root of unity, the Verma module M^q_Λ is reducible if $[(\Lambda + \rho, \check{\alpha}) - m]_{q_\alpha} = 0$, where $\check{\alpha} = 2\alpha/(\alpha, \alpha)$, $q_\alpha = q^{(\alpha,\alpha)}/2$ and $[n]_q = q^n - q^{-n}/(q - q^{-1})$. In this case there exists a q-singular vector $Q^{l_j}_\alpha \otimes v_\Lambda$ and an associated direct intertwiner $Q^{l_j}_\alpha = Q_{w\tilde{w}}$ with $w * \Lambda - \tilde{w} * \Lambda = (l_j + 1)\alpha$ and $l_j = (\Lambda, \alpha_j)$. We first give an enumeration of the intertwiners $Q^{l_j}_\alpha$. Let $\alpha \in \Delta_+$ be a positive root of height k, i.e. $\alpha = \alpha_i + \ldots + \alpha_{i+k-1}$ for $1 \leqq i \leqq r - k + 1$. To each inequivalent way of writing α as a simple root with a string of fundamental reflections applied to it, there exists an intertwining operator $Q^{l_j}_\alpha$, where $j = j(\alpha)$ is the index defined in (A.2). The associated presentation of the root α is given by

$$\alpha = r_{i_1} \ldots r_{i_{k-1}} \alpha_j, \quad r_\alpha = w r_j w^{-1} = r_{i_1} \ldots r_{i_{k-1}} r_j r_{i_{k-1}} \ldots r_{i_1}. \tag{3.5}$$

Modulo Weyl-equivalent forms this is explicitly

$$\alpha = \alpha_i + \ldots + \alpha_{i+k-1} = r_i \ldots r_{j-1} r_{i+k-1} \ldots r_{j+1} \alpha_j.$$

Thus, the set of direct intertwining operators is enumerated by

$$Q^{l_j}_{\alpha_i + \ldots + \alpha_{i+k}}, \quad j = i, \ldots, i + k. \tag{3.6}$$

In total these are $\sum_{\alpha \in \Delta_+} ht\alpha = \frac{1}{6} r(r+1)(r+2) = 2\rho^2$ operators.

The existence has been established in [43] by using an Ansatz in terms of single root space vectors only, corresponding to the form $r_\alpha = w r_j w^{-1}$ in (3.5). Explicitly

$$Q^{l_j}_\alpha = \sum_{s_1=0}^{l_{i_1}+1} \cdots \sum_{s_t=0}^{l_{i_t}+1} C_{s_1 \ldots s_t} f_{i_1}^{l_{i_1}+1-s_1} \ldots f_{i_t}^{l_{i_t}+1-s_t} f_j^{l_j+1} f_{i_t}^{s_t} \ldots f_{i_1}^{s_1}, \tag{3.7.a}$$

$$C_{s_1 \ldots s_t} = (-)^{s_1 + \ldots + s_t} C_t \binom{l_{i_1}+1}{s_1}_q \cdots \binom{l_{i_t}+1}{s_t}_q$$
$$\times \frac{[(\Lambda+\rho)(\bar{h}_{i_1})]_q}{([\Lambda+\rho)(\bar{h}_{i_1}) - s_1]_q} \cdots \frac{[(\Lambda+\rho)(\bar{h}_{i_t})]_q}{[(\Lambda+\rho)(\bar{h}_{i_t}) - s_t]_q}, \tag{3.7.b}$$

$$\bar{h}_{i_s} = \begin{cases} h_1 + \cdots + h_s & 1 \leqq s \leqq j-1 \\ h_{i+k-1} + \cdots + h_{j+k-s} & j \leqq s \leqq k-1 =: t \end{cases}, \tag{3.7.c}$$

where the subscript q refers to q-dimensions and q-binomials and C_t are constants.

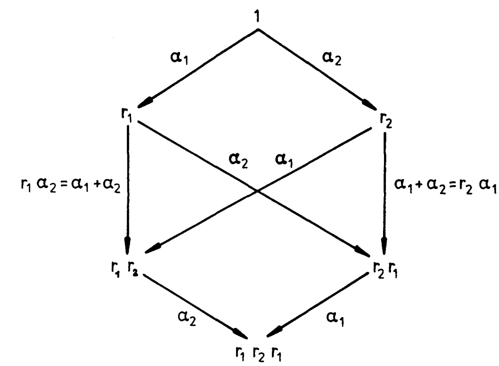

Fig. 1. Embedding diagram for irrational $\mathscr{W}(sl(3))$, $\mathscr{U}_q(Sl(3))$ and $\widehat{sl}(3)$ modules

By suitably combining the intertwiners, the irreducible modules L_Λ^q can now be described as the only nonvanishing cohomology class in a complex of q-Verma modules. The result is a resolution form-identical to the undeformed case (BGG-resolution [18, 19]). In particular, the resolution gives an exhaustive description of the singular submodules of M_Λ as well as their mutual embeddings. A systematic procedure to explicitly work out the embedding pattern of the singular modules is the following: First express the reflections corresponding to the positive roots in terms of the fundamental reflections r_i and likewise write down reduced expressions for the Weyl group elements. For given $w \in W$ then work out all $\tilde{w} \in W$ s.t. $\tilde{w} \leftarrow w$. Clearly the relation $\tilde{w} \leftarrow w$ in the Weyl group corresponds to direct embeddings of the modules, i.e. those for which there is no singular module M^q in M_Λ^q s.t. $M_{\tilde{w} * \Lambda}^q \subsetneq M^q \subsetneq M_{w * \Lambda}^q$. These direct embeddings will also be represented by an arrow, pointing to the submodule. The diagrams for $sl(3)$ and $sl(4)$ shown in Figs. 1, 2 have been obtained in this way. For $\Lambda_\pm$ not dominant integral, the number of singular vectors is less than $(r+1)!$ and the embedding diagrams (forming subdiagrams of the dominant integral ones) can be worked out similarly. Observe that for $w_1, w_2 \in W$ s.t. $l(w_1) = l(w_2) + 2$ the number of elements $w \in W$ s.t. $w_1 \leftarrow w \leftarrow w_2$ is either zero or two. In the latter case the quadruple $(w_1, w, \tilde{w}, w_2)$ is called a square and the embedding diagram is composed of such squares. In the BGG resolution to each arrow one assigns a sign $s(\tilde{w}w) = \pm 1$ s.t. for each square in the complex the products of signs equals -1. This can be done consistently throughout the diagram.

The nilpotency of the operators in the complex is then equivalent to certain compatibility relations for the intertwiners, which express the commutativity of the squares in the embedding diagram (see also Theorems 4.5, 4.6). There are two principle types of squares in the embedding diagram, which are shown in Fig. 3.a, b, where in 3.a the positive roots α, β are such that also $\alpha + \beta \in \Delta_+$. Other types of squares are obtained from them by reflection in the diagonals. The commutativity of squares of type b does not give rise to an integrability condition for the involved intertwiners, due to the invariance of the resolution under $\alpha_i \to \alpha_{r+1-i}$. Squares of type 3.a give rise to an integrability condition of the form

$$Q_{\alpha+\beta}^{l_j} Q_\alpha^{l_i} = Q_\alpha^{l_i + l_j} Q_\beta^{l_j}, \quad \alpha + \beta \in \Delta_+ . \tag{3.8}$$

These relations can be traced back to the $sl(3)$ case [43], which has been verified in [7].

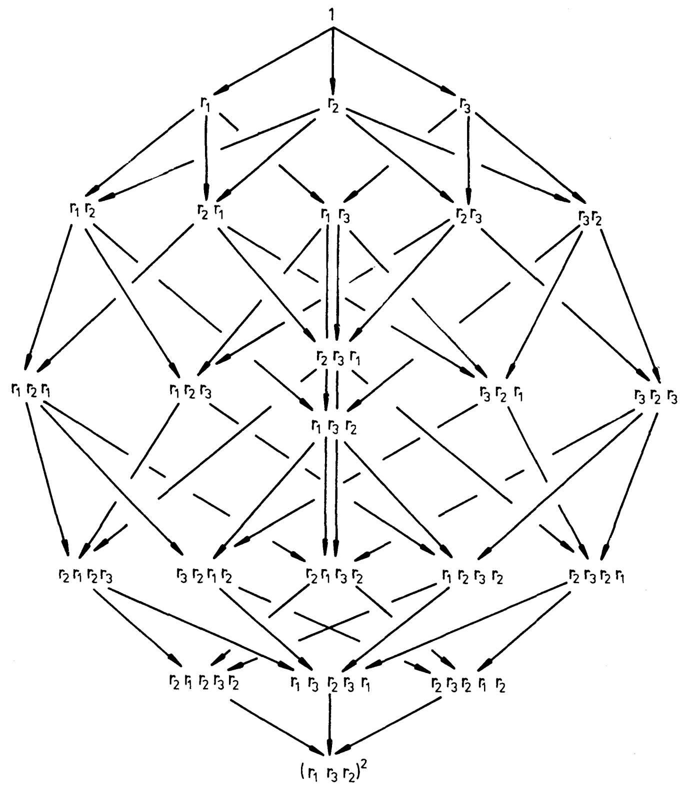

Fig. 2. Embedding diagram for irrational $\mathscr{W}(sl(4))$, $\mathscr{U}_q(Sl(4))$ and $\widehat{sl}(4)$ modules

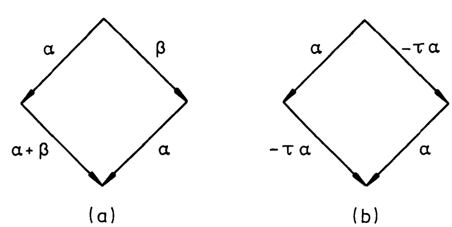

Fig. 3. Fundamental squares in the embedding diagrams

4. Realizations and Resolutions

4.1. Free Field Realization. Introduce r scalar fields $\phi^a(z)$

$$\phi^a(z) = q^a - ip^a \ln z + i \sum_{n \neq 0} \frac{1}{n} a_n^a z^{-n} ,$$

$$\phi^a(z)\phi^b(w) = -\delta^{ab} \ln(z-w) + \ldots \tag{4.1}$$

with modes having free oscillator commutation relations

$$[a_m^a, a_n^b] = m\delta^{ab}\delta_{n+m,0}, \quad [p^a, q^b] = -i\delta^{ab} . \tag{4.2}$$

For $\Lambda_+\Lambda_- \in h^*$ let $|\Lambda_+, \Lambda_-\rangle$ be a vector satisfying

$$\alpha \cdot a_n^a|\Lambda_+, \Lambda_-\rangle = 0, \quad n > 0\,,$$

$$(2s_0\alpha\cdot\rho + \alpha\cdot p)|\Lambda_+, \Lambda_-\rangle = x_\alpha(\Lambda_+, \Lambda_-)|\Lambda_+, \Lambda_-\rangle\,, \tag{4.3}$$

for some eigenvalue $x_\alpha(\Lambda_+, \Lambda_-)$. We choose normalizations s.t. $\Lambda_+{}^a\Lambda_-{}^a = \Lambda_+\cdot\Lambda_- = (\Lambda_+, \Lambda_-)$ is the bilinear form on h^*. The corresponding Fock space module is denoted by $F_{\Lambda_+\Lambda_-}$. In the enveloping algebra of the oscillator algebra (4.2) introduce r field operators $W^i(z)$ by means of a *symmetrized* Miura transformation

$$\tau[2s_0\partial_z + i\hat{h}_{r+1}\cdot\partial_z\phi][2s_0\partial_z + i\hat{h}_r\cdot\partial_z\phi]\ldots[2s_0\partial_z + i\hat{h}_1\cdot\partial_z\phi] =$$

$$= -\sum_{i=-1}^{r} W^i(z)(2s_0\partial)^{r-i}, \tag{4.4}$$

where $\alpha_j =: \hat{h}_{j+1} - \hat{h}_{j+2}(\hat{h}_{r+2} = \hat{h}_1)$ defines $\hat{h}_j$, $1 \leqq j \leqq r+1$, $2s_0 = s_+ + s_-$ and normal ordering shall be implicit. τ projects onto the sector invariant under the automorphism τ: $\alpha_i \to -\alpha_{r+1-i}$, $s_+ \to -s_+$ of the Dynkin diagram, which is implemented by the maximal element of the Weyl group (for simplicity we use the same symbol for the automorphism and the associated projection operator). This symmetrization turns out to be crucial in many respects (cf. Sects. 4.5, 6 and [37]). For the generators one finds, in particular, $W^{-1} = -1$, $W^0 = 0$ and

$$L(z) = W^1(z) = -\frac{1}{2}\partial_z\phi\cdot\partial_z\phi - 2is_0\rho\cdot\partial_z^2\phi \tag{4.5}$$

generates a Virasoro algebra of central charge $c = r - 48s_0^2\rho^2$. The fields W^i, $1 \leqq i \leqq r$ are of L_0-weight $i+1$, but in general neither primary nor quasiprimary relative to $L(z)$. By adding suitable normal ordered products of $W^{i-1}, \ldots, W^1$ to W^i one can try to promote W^i to a quasiprimary or primary field. Since $\tau L(z) = L(z)$ the invariance under τ is clearly a necessary condition for this to be possible. As there is no possible "counterterm," W^2 is always primary. For the other generators the projection onto quasiprimary or primary fields is nontrivial. The projection onto quasiprimary fields turns out to be unproblematic and will in the following often implicitly be assumed to be performed. The projection onto primary fields may fail for certain values of the central charge and will be discussed in Sect. 5. The main result to be proved in Sect. 4.3 is now simply

Theorem 4.1. (*Existence*) *The* (*quasiprimary projection of the*) *symmetrized Miura fields* $W^i(z)$ *generate a* $W(sl(r+1))$ *algebra in the sense defined. The structure constants are polynomial in the central charge.*

The characteristic Hilbert space will be a certain subspace $\mathscr{H}_{00}$ of F_{00}, which implies the bounds $0 \leqq c \leqq r$ on the accessible range of the central charge. (In a free field realization one has $0 \leqq \langle L, L\rangle = c/2$ and the upper bound is required for condition (2.3).) After the commutation relations have been reconstructed, the range of definition of the algebra can be extended to all values of the central charge, due to the polynomial form of the structure constants.

The Miura fields $W^i(z)$ do, however, not form a Cartan basis. A Cartan basis can be obtained as follows: Let Ω be the generator of the cyclic group Z_{r+1} acting by Ω: $(\alpha_1, \ldots, \alpha_r, -\theta) \to (\alpha_2, \ldots, \alpha_r, -\theta, \alpha_1)$ on the root system; where θ is the highest root. In terms of the fundamental reflections r_i, $1 \leqq i \leqq r$ of the Weyl group, Ω is given by the Coxeter element $\Omega = r_1 r_2 \ldots r_r$. The Dynkin automorphism τ is implemented by the maximal element of the Weyl group. Together Ω and τ generate a Coxeter subgroup of the Weyl group with relations

$$\Omega^{r+1} = 1, \quad \tau^2 = 1, \quad (\Omega\tau)^2 = 1 . \tag{4.6}$$

These are the defining relations of the dihedral group D_{r+1}, i.e. the symmetry group of a regular polygon ($r+1$-gon). Let $P[s_0, \alpha_i \cdot \partial_z \phi]$ be a (normal ordered) functional in $s_0, \alpha_i \cdot \phi$. By $\Omega P[s_0, \alpha_i \cdot \partial_z \phi] = P[s_0, (\Omega\alpha_i) \cdot \partial_z \phi]$ and $\tau P[s_0, \alpha_i \cdot \partial_z \phi] = P[-s_0, (\tau\alpha_i) \cdot \partial_z \phi]$ one has an induced action of the dihedral group. Let D_{r+1} denote the projector onto the dihedral invariant subsector and set

$$D_{r+1}[2s_0\partial_z + i\hat{h}_{r+1} \cdot \partial_z \phi][2s_0\partial_z + i\hat{h}_r \cdot \partial_z \phi] \ldots [2s_0\partial_z + i\hat{h}_1 \cdot \partial_z \phi]$$

$$= -\sum_{i=-1}^{r} D^i(z)(2s_0\partial)^{r-i} . \tag{4.7}$$

In particular, $D^{-1} = -1$, $D^0 = 0$, $D^1 = -\frac{1}{2}\partial_z \phi \cdot \partial_z \phi$. The fields are not quasiprimary relative to $L(z)$ in (4.5). Define

$$\bar{W}^i(z) = \mathcal{N} D^i(z) , \tag{4.8}$$

where $\mathcal{N}$ denotes the projection onto quasiprimary fields (i.e. $L_1 | \bar{W}^i \rangle = 0$).

Proposition 4.2. ($r \leqq 4$) *The fields $\bar{W}^i(z)$, $1 \leqq i \leqq r$ form a Cartan basis of $W(sl(r+1))$. The structure constants are polynomial in the central charge.*

We expect this to be correct in general. We will return to this statement in Sect. 3.4 when discussing the projection onto primary fields. See also [37] for the relation to infinite dimensional abelian subalgebras. The Cartan basis is the canonical basis to study the representation theory. In particular, one has free field realizations of highest weight vectors, Verma modules etc. The labels $I^i(\Lambda_+, \Lambda_-)$ are calculated explicitly as the eigenvalues of the zero modes $\bar{W}^i_0$ on $|\Lambda_+, \Lambda_-\rangle$. From the definition (4.8) it can be shown that I^i generate the ring of Weyl invariant polynomials in the variables x_i, defined in (2.4). In particular

$$\begin{aligned} I^1(\Lambda_+, \Lambda_-) &= \frac{1}{2} x_i (a^{-1})_{ij} x_j - 2s_0^2 \rho^2 \\ &= \frac{1}{2} s_+^2 (\Lambda_+, \Lambda_+ + 2\rho) + \rho^2 - (\Lambda_+ + \rho, \Lambda_- + \rho) \\ &\quad + \frac{1}{2} s_-^2 (\Lambda_-, \Lambda_- + 2\rho) , \end{aligned} \tag{4.9}$$

where a^{-1} is the inverse of the Cartan matrix. One has the isomorphisms

$$F_{w * \Lambda_+, \Lambda_-} \cong F_{\Lambda_+, w^{-1} * \Lambda_-}, \quad F^*_{\Lambda_+, \Lambda_-} \cong F_{-(\Lambda_+ + 2\rho), -(\Lambda_- + 2\rho)}, \tag{4.10}$$

as modules over the W-algebra (where $F^*_{\Lambda_+, \Lambda_-}$ denotes the dual of $F_{\Lambda_+ \Lambda_-}$ w.r.t. the standard inner product).

4.2. Resolutions and Character Formula. For $\lambda_+, \lambda_- \in h^*$ introduce the vertex operator

$$V_{\lambda_+,\lambda_-} : F_{\Lambda_+\Lambda_-} \to F_{\Lambda_+ + \lambda_+,\, \Lambda_- + \lambda_-}$$

$$V_{\lambda_+,\lambda_-} = \exp(is_+\lambda_+ \cdot \phi(z) + is_-\lambda_- \cdot \phi(z))\,, \tag{4.11}$$

where again normal ordering shall be implicit. The "screening operators" $V_i^+ = V_{-\alpha_i,0}$, $V_i^- = V_{0,-\alpha_i}$, $1 \leqq i \leqq r$ correspond to minus the simple roots. For any state $|P\rangle \in F_{\Lambda_+\Lambda_-}$ consider now the vectorspace M_P spanned by all states of the form

$$[\![V_{j_l}^+ \dots V_{j_1}^+ V_{i_k}^- \dots V_{i_1}^-]\!] |P\rangle$$

$$= \int_\Gamma dw_l \dots dw_1 \, dz_k \dots dz_1 \, V_{j_l}^+(w_l) \dots V_{j_1}^+(w_1) V_{i_k}^-(z_k) \dots V_{i_1}^-(z_1) |P\rangle\,, \tag{4.12}$$

where the contour is given in Fig. 4.a. The integrand is defined by analytic continuation from the region $0 < z_n < \dots < z_1$ on the real axis, where the integrand is taken to be real. The presence of a cut along 0, 1 will be seen later. The motivation for this choice of contours and a discussion of some of its properties can be found in [7]. The monodromy properties of the integrand can be exhibited by complete normal ordering. From this one deduces that the q-Serre relations are valid within the contour integrals $[\dots]$ [7], i.e. the relations

$$\begin{array}{ll} V_iV_iV_j - (q+q^{-1})V_iV_jV_i + V_jV_iV_i = 0 & \text{if } a_{ij} = -1 \\ V_iV_j - V_jV_i = 0 & \text{if } a_{ij} = 0\,, \end{array} \tag{4.13}$$

hold in the sense that inserting the r.h.s. into an arbitrary contour integral of M_P, causes it to vanish. Here V_i stands for either V_i^+ or V_i^- with $q = q_+ = e^{i\pi s_+^2}$ or $q = q_- = e^{i\pi s_-^2}$, respectively. Similarly Λ shall be shorthand for the respective of the weights Λ_+ or Λ_-. In the same sense the s_+ and s_- sectors decouple within $[\![\dots]\!]$ as the operators V_i^+ and V_i^- have no relative monodromy

$$V_i^+ V_j^- - V_j^- V_i^+ = 0, \quad 1 \leqq i,j \leqq r\,. \tag{4.14}$$

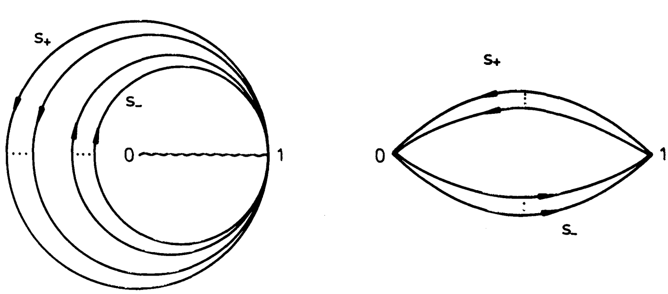

Fig. 4a, b Contour in multi-screening integrals. Deformed contour Fig. a

The significance of the operators $V_i^{\pm}$ for the W-algebra lies in the operator product expansions

$$W^i(z)\,V_j(w) = \partial_w\left(\frac{\tilde{W}^{i,j}(w)\,V_j(w)}{z-w}\right) + o(1)\,, \tag{4.15}$$

$$W^{i,j}(z)\,V_j(w) = o(1), \quad |z| > |w| \tag{4.16}$$

$$-\sum_{i=-1}^{r} \tilde{W}^{i,j}(z)(2s_0\partial_z)^{r-i} = \tau \prod_{k \neq j, j+1} [2s_0\partial_z + i\hat{h}_k\cdot\partial_z\phi]\,,$$

where, except for the deletion the order in the product is the same as in (4.4). They imply the following commutator (to be compared with the affine case [7].)

$$[W^i(z), [\![V_{i_1}\ldots V_{i_n}]\!]]|P\rangle = \sum_{k=1}^{n} \bar{c}_{ii_k}(z)\left(\frac{q^{-l_{i_k}-N+a_k} - q^{l_{i_k}+N-a_k}}{q-q^{-1}}\right)$$
$$\times [\![V_{i_1}\ldots \hat{V}_{i_k}\ldots V_{i_n}]\!]|P\rangle\,, \tag{4.17}$$

where

$$\bar{c}_{ij}(z) = \bar{W}^{i,j}(1)\,V_j(1)\,q^{l_j+\delta}\frac{q-q^{-1}}{z-1}\,, \tag{4.18}$$

$$a_k = a_{i_k i_{k+1}} + \ldots + a_{i_k i_n}, \qquad \delta = \sum_{l \neq k} a_{i_l i_k}\,,$$
$$l_i = (\Lambda, \alpha_i), \qquad (\oint \alpha_i\cdot\partial_z\phi)|P\rangle = (l_i + N)|P\rangle\,, \tag{4.19}$$

and '^' denotes omission. Equation (4.16) implies that the product in (4.18) is well defined and independent of the order. Comparing (4.17) with (3.3) one sees that $W^i(z)$ acts by commutation on $[\![V_{i_1}\ldots V_{i_n}]\!]|P\rangle, |P\rangle \in F_{0,\Lambda}$ or $F_{\Lambda,0}$ in a similar way as e_i does on $f_{i_1}\ldots f_{i_n}\cdot v_\Lambda$. Although the actions do not coincide, the r.h.s. of (4.17) vanishes whenever the r.h.s. of (3.3) does. To check this, it suffices to observe that in both cases the strings of type $f_{i_1}\ldots\hat{f}_k\ldots f_{i_n}$ with some fixed f_k removed have to cancel separately for each k, so that it is irrelevant whether their overall factor changes with k or does not. The relations (4.13), (4.17) therefore suggest the following result:

Proposition 4.3. *For fixed $\Lambda_- \in P_+$ there exists a 1–1 correspondence between elements in $\mathrm{Hom}_{\mathcal{U}_q(g)}(M^q_\Lambda, M^q_{\bar{\Lambda}})$, q not a root of unity and $\mathrm{Hom}_{\mathcal{W}(g)}(F^*_{\Lambda_+\Lambda_-}, F^*_{\bar{\Lambda}_+\Lambda_-})$, $c \in C_r$. In terms of the elements in $\mathcal{U}_q(n_-)$ associated to the $\mathcal{U}_q(g)$ intertwiners through Lemma 3.1.a) it reads*

$$f_{i_1}\ldots f_{i_n}\cdot v_{\Lambda_+} \to [\![V^+_{i_1}\ldots V^+_{i_n}]\!]_{\Lambda_+\Lambda_-}\,.$$

It is clear that every $\mathcal{U}_q(g)$ intertwiner will give rise to a $W(g)$ intertwiner. For example one can check from (4.17) that

$$Q^{l_i}_{\alpha_i} = Q_{r_i,1} = [\![(V^+_i)^{l^+_i+1}]\!]_\Lambda : F^*_{r_i * \Lambda_+,\Lambda_-} \to F^*_{\Lambda_+,\Lambda_-}\,. \tag{4.20}$$

provides a basic set of intertwiners. The point in Proposition 4.3 is the claimed 1–1 correspondence, which unfortunately requires a lot more work. Ultimately it is this 1–1 correspondence which guarantees the completeness property (2.3) in the definition of the W-algebra, and hence the existence of the free field realization. It is therefore necessary to prove this result *without* presupposing the closure of the W-algebra. This forces one to an indirect line of reasoning, which will be detailed in Sects. 4.3–4.5. Given this result Proposition 4.3 implies

Proposition 4.4. *For given $\Lambda_\pm$ one has*

$$\dim \mathrm{Hom}_{\mathscr{W}(g)}(F_{w*\Lambda_+\Lambda_-}, F_{\tilde{w}*\Lambda_+\Lambda_-}) = \begin{cases} 1 & \text{if } \tilde{w} \preceq w \\ 0 & \text{otherwise} . \end{cases}$$

For $\tilde{w} \preceq w$ let $Q_{w\tilde{w}}$ denote the intertwiner. Set

$$d^{(k)}_{w\tilde{w}} = \begin{cases} s(w\tilde{w}) Q_{w\tilde{w}} & \text{if } \tilde{w} \leftarrow w \\ 0 & \text{otherwise} , \end{cases} \tag{4.21}$$

with the sign pattern $s(w\tilde{w}) = \pm 1$ as in the BGG *resolution and consider $d^{(k)}_{w\tilde{w}}$ as the matrix elements of $d^{(k)}$: $F^{(k)}_{\Lambda_+\Lambda_-} \to F^{(k+1)}_{\Lambda_+\Lambda_-}$, $l(w) = k$, $l(\tilde{w}) = k+1$, with*

$$F^{(k)}_{\Lambda_+\Lambda_-} = \bigoplus_{\{w: l(w) = k\}} F_{w*\Lambda_+,\Lambda_-} . \tag{4.22}$$

The integrability conditions (3.8) are then equivalent to $d^{(k)} d^{(k+1)} = 0$. One obtains a Fock space resolution form-identical to that of the finite dimensional case (e.g. [27]).

Theorem 4.5. *Let $\mathscr{L}(I)$ be an irreducible $W(sl(r+1))$ module with highest weight state $|I\rangle = |I(\Lambda_+, \Lambda_-)\rangle$, $\Lambda_\pm \in P_+$ and $c \in C_r$. There exists a complex of Fock modules*

$$0 \to F^{(0)}_{\Lambda_+\Lambda_-} \xrightarrow{d^{(0)}} \dots \xrightarrow{d^{(t-1)}} F^{(t)}_{\Lambda_+\Lambda_-} \to 0$$

$$\text{s.t. } H^{(k)}_d(F) \cong \begin{cases} \mathscr{L}(I) & \text{if } k = 0 \\ 0 & \text{otherwise} . \end{cases}$$

In particular,

$$\begin{aligned} \mathscr{H}_{\Lambda_+\Lambda_-} &:= \tau \bigcap_{i=1}^{r} \mathrm{Ker}([\![(V_i^+)^{l_i^+ + 1}]\!] : F_{\Lambda_+\Lambda_-} \to F_{r_i*\Lambda_+,\Lambda_-}) \\ &= \tau \bigcap_{i=1}^{r} \mathrm{Ker}([\![(V_i^-)^{l_i^- + 1}]\!] : F_{\Lambda_+\Lambda_-} \to F_{\Lambda_+, r_i*\Lambda_-}) , \end{aligned} \tag{4.23}$$

provides the required Fock space model of the irreducible module $\mathscr{L}(I)$. Here $l_i^\pm = (\Lambda_\pm, \alpha)$, $t = |\Delta_+|$ and τ again projects onto the sector invariant under the Dynkin automorphism. This is necessary, for example, to avoid an overcounting of solutions. The equivalence of both characterizations of $\mathscr{H}_{\Lambda_+\Lambda_-}$ follows from the invariance under the diagonal action of the Weyl group. As a consistency check we note that, in particular, the highest weight state $|\Lambda_+, \Lambda_-\rangle$ of $F_{\Lambda_+\Lambda_-}$ solves Eqs. (4.23). This is to say that the corresponding type of multiple screening integrals have to vanish

$$[\![V_{i_1} \dots V_{i_n} (V_i)^{k_i^\mp}]\!]_{\Lambda_+\Lambda_-} |\Lambda_+, \Lambda_-\rangle = 0 \quad \text{iff } k_i^\pm > l_i^\pm . \tag{4.24}$$

Let $[\![\ldots]\!]_{z,1}$ denote the screening integral with the contour deformed to that shown in Fig. 4.b and consider only one of the screening sectors. Then

$$[\![V_{i_1} \ldots V_{i_n}(V_i)^{k_i} V_{\Lambda,0}(z)]\!] = g(q) \prod_{j=0}^{k_i-1} (q^{l_i} - q^{-l_i+2j})[\![V_{i_1} \ldots V_{i_n}(V_i)^{k_i} V_{\Lambda,0}(z)]\!]_{z,1} , \tag{4.25}$$

where $g(q)$ is a nonzero polynomial coming from the deformation of the contours $V_{i_1} \ldots V_{i_n}$. This is the multi-contour analogue of a transformation that appears also in the Hankel form of the Γ function [8]. Notice that (4.24) can be interpreted as turning the space $M_{|\Lambda_+\Lambda_-\rangle}$ into an irreducible $\mathscr{U}_q(sl(r+1))$-module. Lemma 4.8.c) in Sect. 4.4 guarantees that the Fock space resolution can be "lifted" to a Verma module resolution.

Theorem 4.6. *Let $\mathscr{L}(I)$ be an irreducible $W(g)$ module of highest weight $I(\Lambda_+, \Lambda_-)$, $c \in C_r$. There exists a resolution of $\mathscr{L}(I)$ in terms of Verma modules,* i.e. *a complex $(V(I), d)$ (with $d^{(k)}$ defined in terms of the canonical embeddings)*

$$0 \xleftarrow{d^{(0)}} V_I^{(0)} \xleftarrow{d^{(1)}} \ldots \xleftarrow{d^{(t)}} V_I^{(t)} \leftarrow 0$$

$$\text{s.t. } H_d^{(k)}(V) \cong \begin{cases} \mathscr{L}(I) & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\textit{with} \quad V_I^{(k)} = \bigoplus_{\{w: l(w)=k\}} V(I(w * \Lambda_+, \Lambda_-)) \quad \textit{and} \quad t = |\Delta_+| .$$

The character for the irreducible modules $\mathscr{L}(I)$ now follows from the so-called algebraic Lefschetz theorem. If the character of the Verma modules is known, one would expect that the character of the irreducible modules can be obtained by suitably cross-subtracting the dimensions of the singular submodules. Generally, the algebraic Lefschetz theorem states that also in infinite, one- or two-sided resolutions

$$\mathrm{Tr}_{L_\Lambda} \mathcal{O} = \sum_k (-)^k \mathrm{Tr}_{\mathscr{V}_\Lambda^{(k)}} \mathcal{O}^{(k)} , \tag{4.26}$$

where the sum is over all constituents $\mathscr{V}_\Lambda^{(k)}$ of the resolution and $\mathcal{O}^{(k)}$ satisfies $d^{(k)} \mathcal{O}^{(k+1)} = \mathcal{O}^{(k+1)} d^{(k)}$, $\mathcal{O}^{(0)}|_{L_\Lambda} = \mathcal{O}$. To apply this to the case at hand, with $\mathcal{O}^{(k)}$ induced by the L_0-graduation, an expression for the degree of the singular submodules is needed. From (4.9) one has

$$I^1(w * \Lambda_+, \Lambda_-) - I^1(\Lambda_+, \Lambda_-) = (\Lambda_+ - w * \Lambda_+, \Lambda_- + \rho) \tag{4.27}$$

and by (A.2) this equals

$$= \sum_{\alpha \in \Delta_+^{w^{-1}}} (\Lambda_+ + \rho, \alpha_{j(\alpha)})(\Lambda_- + \rho, \alpha)$$

$$= \sum_{k=1}^{l} (\Lambda_+ + \rho, \alpha_{i_k})(\Lambda_- + \rho, r_{i_1} \ldots r_{i_{k-1}} \alpha_{i_k}) \tag{4.28}$$

with $w = r_{i_1} \dots r_{i_l}$ a reduced expression. Equation (4.28) has a simple interpretation in terms of the embedding diagrams, in that the l.h.s. depends only on the set $\Delta_+^{w^{-1}}$ not on the different ways to write it as $\alpha \in \Delta_+^{\tilde{w}r_\alpha}$ for some $\tilde{w} \in W$, $w^{-1} \leftarrow \tilde{w}^{-1}$ and $\alpha \in \Delta_+$. By induction on l this means that any path of direct embeddings in the diagram – each step contributing a term to the sum in (4.28) – leads to the same answer, the degree of $V(I(w * \Lambda_+, \Lambda_-))$.

The (specialized) character of a Verma module of weight $I(\Lambda_+, \Lambda_-)$ is $\mathrm{ch}\,\mathscr{L}(I)(\tau) = (\phi(\tau))^{-r} e^{2\pi i \tau I^1(\Lambda_+ \Lambda_-)}$, where $\phi(\tau)$ is the Euler function $\prod_{n=1}^{\infty}(1-t^n)^{-1}$ for $t = e^{2\pi i \tau}$. Using Eq. (4.28) for the degree and the algebraic Lefschetz theorem, the character of the irreducible modules is found as

$$\mathrm{ch}\,\mathscr{H}_{\Lambda_+\Lambda_-}(\tau) = \frac{e^{2\pi i \tau I^1(\Lambda_+ \Lambda_-)}}{(\phi(\tau))^r} \sum_{w \in W} (-)^{l(w)} \times \exp\left(2\pi i \tau \sum_{\alpha \in \Delta_+^w} (\Lambda_+ + \rho, \alpha_{j(\alpha)})(\Lambda_- + \rho, \alpha)\right). \tag{4.29}$$

In the special case where $\Lambda_- = 0$ or $\Lambda_+ = 0$ this is conveniently rewritten in product form: Recall that for $\mu \in P_+$ the specialization of type μ of the formal exponential $F_\mu \colon \mathbb{C}[e(-\alpha_1), \dots, e(-\alpha_r)] \to \mathbb{C}[e^{2\pi i \tau}]$ is defined by $F_\mu(e(-\lambda)) = e^{2\pi i \tau(\lambda, \mu)}$ [28]. One may thus, for fixed Λ_+ in (4.29) think of $\Lambda_- + \rho$ as defining the specialization of the formal Weyl character. The denominator identity then implies product formulas for the principally specialized characters with $\Lambda_- = 0$:

$$\mathrm{ch}\,\mathscr{H}_{\Lambda,0}(\tau) = \frac{e^{2\pi i \tau I^1(\Lambda,0)}}{(\phi(\tau))^r} \prod_{\alpha \in \Delta_+} (1 - e^{2\pi i \tau(\Lambda + \rho, \alpha)}) ; \tag{4.30}$$

all this being for central charge $c = r - 48 s_0^2 \rho^2$ with irrational screening parameter s_+^2.

4.3. Existence of the Free Field Realization. This section is devoted to the proof of the results given before. In particular, a proof of the existence of the free field realization (Theorem 4.1) will be given, in the course of which the other desiderata will follow. We shall adopt the following

Strategy. The problem consists in showing that the operator product algebra of the proposed generating fields closes. This will be done by showing that the characteristic space $\mathscr{H}(g) = \mathscr{H}_{00}$ has the completeness property (2.3), i.e. that there are no other singular vectors than those implied by the relations $W^i_{n-\Delta_i}|v\rangle = 0$, $n > 0$. The study of singular vectors is in principle a representation theoretical task. Without the algebra known to close, one cannot talk about representation theory. However, any candidate for a singular vector has to be an eigenstate of L_0 and to be annihilated by the positive modes W^i_n, $n > 0$. In a Fock space realization one can find these candidates without knowing the commutation relations. By studying the structure of such "would be" singular vectors in $\mathscr{H}_{00}$ (or generally in modules $\mathscr{H}_{\Lambda_+\Lambda_-}$, $\Lambda_\pm \in P_+$) the question of the absence of additional singular vectors can be reduced to a finite dimensional quantum group problem. The solution of the latter is provided by the explicit construction of the intertwiners $Q_{\tilde{w}w} \in \mathrm{Hom}_{\mathscr{U}_q(g)}(M^q_{\tilde{w}*\Lambda}, M^q_{w*\Lambda})$ satisfying the integrability conditions (3.8).

Introduce the vector space $V_\pi(I(\Lambda_+, \Lambda_-))$ generated by the modes of the fields $W^i(z)$ in $F^*_{\Lambda_+\Lambda_-}$

$$V_\pi(I(\Lambda_+, \Lambda_-)) = \sum_{\nu_1, \ldots, \nu_r \in \mathrm{Par}(1)} \mathbb{C}\, W^1_{-\nu_1} \ldots W^r_{-\nu_r} |\Lambda_+, \Lambda_-\rangle^* \tag{4.31}$$

but strictly to be regarded as carrying only a (non-invariant) action of the oscillator algebra. The labels $I^i(\Lambda_+, \Lambda_-) = I^i(-(\Lambda_+ + \rho), -(\Lambda_- + \rho))$ are as before defined as the eigenvalues of the zero modes W^i_0 on $|\Lambda_+, \Lambda_-\rangle^*$, with I^1 given by (4.9). The relations $F_{w*\Lambda_+, \Lambda_-} \cong F_{\Lambda_+, w^{-1}*\Lambda_-}, F^*_{\Lambda_+, \Lambda_-} \cong F_{-(\Lambda_+ + 2\rho), -(\Lambda_- + 2\rho)}$ now hold in the sense that both sides lead to the same vector space $V_\pi(I(\Lambda_+, \Lambda_-))$.

4.4. π-Singular Vectors. In the present context, define a π-singular vector of $F_{\Lambda_+\Lambda_-}$ as an eigenstate of L_0, annihilated by the positive modes W^i_n, $n > 0$, $1 \leqq i \leqq r$ (with π mnemotechnical for the Fock space projection). It is not part of the definition that these states are expressible in terms of W^i_{-n}, $n > 0$ modes. The (positive) difference of the weight to $I^1(\Lambda_+, \Lambda_-)$ is called degree and the vector space generated by it via (4.31) a π-singular subspace. The following proposition gives the well-known Fock space construction of π-singular vectors [2].

Proposition 4.7. *For $\alpha \in \Delta_+$, $\Lambda_\pm \in P$, $m_\pm$ positive integers, there exists a π-singular vector s in $F^*_{\Lambda_+\Lambda_-}$ at degree m_+m_- whenever $x_\alpha = s_+m_+ + s_-m_-$. The Weyl invariants $I^i(x_1, \ldots, x_r)$, $x_i = x_{\alpha_i}$ separate different singular vectors* (i.e. $I(x) \neq I(\tilde{x})$ *implies* $s(x) \neq s(\tilde{x})$).[2]

In principle, the π-singular vectors of a given Fock module and their descendence pattern can be obtained from an iterated application of Proposition 4.7 and analysis of the multiplicities of the Kac determinant. Due to the somewhat indirect criterion in the condition of 4.7, this is feasible only in simple cases. For s^2_+ irrational, the iteration can be solved in terms of the Weyl group of $sl(r+1)$ and provides a rudimentary form of Proposition 4.4.

Proposition 4.4′. *For s^2_+ irrational, the π-singular subspaces of $F^*_{\Lambda_+\Lambda_-}$, $\Lambda_\pm \in P^+$ (provided by Prop. 4.7) are grouped into disjoint sets $V_\pi(I(w*\Lambda_+, \Lambda_-))$ labelled by elements of the Weyl group W. Their descendence pattern is induced by the Bruhat ordering on W, i.e. $V_\pi(I(w*\Lambda_+, \Lambda_-)) \subset F^*_{w*\Lambda_+, \Lambda_-}$ is a (set of) singular vector space(s) of $F^*_{\tilde{w}*\Lambda_+, \Lambda_-}$* iff *$w \preceq \tilde{w}$.*

Proof. For s^2_+ irrational one can parametrize $x_\alpha(\Lambda_+, \Lambda_-) = s_+(\Lambda_+ + \rho, \alpha) + s_-(\Lambda_- + \rho, \alpha)$ and has $x_\alpha(\Lambda_+, \Lambda_-) = x_\alpha(\tilde{\Lambda}_+, \tilde{\Lambda}_-)$ iff $\Lambda_\pm = \tilde{\Lambda}_\pm\ (*)$. Suppose the Proposition 4.7 implies the existence of a π-singular vector in $F_{\Lambda_+\Lambda_-}$ for some $\alpha \in \Delta_+$, $\Lambda_\pm \in P$. Its degree is $(\Lambda_+ + \rho, \alpha)(\Lambda_- + \rho, \alpha) = (\Lambda_+ - r_\alpha * \Lambda_+, \Lambda_- + \rho)$ so that by (4.27) the Fock space labels are given by $(r_\alpha * \Lambda_+, \Lambda_-)$ or $(\Lambda_+, r_\alpha^{-1} * \Lambda_-)$. By iteration it follows, in particular, that the labels $\tilde{\Lambda}_\pm$ of all π-singular vectors obtained in this way are integral weights. Every integral weight is Weyl equivalent to one and only one dominant integral weight. Thus, every such π-singular vector of $F^*_{\Lambda_+\Lambda_-}$, $\Lambda_\pm \in P_+$ is labeled by some element of the Weyl group. Further, inside a fixed Weyl chamber, the map $x_i \to I^i$ is invertible (e.g. [16]). Together with $(*)$ this means that different Weyl group elements correspond to different singular

[2] The converse is not correct in general

vectors (but still one element might lable a number of them). To obtain the descendence pattern of these π-singular vectors, fix $\Lambda_- \in P_+$ and set $\tilde{\Lambda}_+ = w * \Lambda_+$ for some $w \in W$, $\Lambda_+ \in P_+$. The condition $(\alpha, \tilde{\Lambda}_+ + \rho) > 0$ is equivalent to $w^{-1}\alpha > 0$ (or $\alpha \in \Delta_+^{w^{-1}r_\alpha}$, see Appendix) and by (A.1) one has $l(r_\alpha w) > l(w)$. Equation (A.2) implies, in particular, that the degree $(\Lambda_+ - w * \Lambda_+, \Lambda_- + \rho)$ is strict monotonically increasing with $l(w)$. If $l(r_\alpha w) = l(w) + 1$, the relation $r_\alpha w \leftarrow w$ therefore corresponds to a direct descendence of vector spaces; i.e. those for which there is no π-singular vector space $V_\pi \subset F^*$ s.t. $F^*_{\tilde{w} * \Lambda_+, \Lambda_-} \subset F^* \subset F^*_{w * \Lambda_+, \Lambda_-}$ as proper submodules. Thus, $V_\pi(I(w * \Lambda_+, \Lambda_-)) \subset F^*_{w * \Lambda_+ \Lambda_-}$ is a (set of) π-singular vector space(s) of $F^*_{\tilde{w} * \Lambda_+, \Lambda_-}$, whenever $w \preceq \tilde{w}$. For the converse let $V_\pi(I(r_\alpha * \Lambda_+, \Lambda_-))$ be a (set of) singular subspace(s) of $F^*_{\tilde{w} * \Lambda_+, \Lambda_-}$ and take $l(r_\alpha w) = l(w) + k$. By (A.4) there exists a sequence $\beta_k, \ldots, \beta_1 \in \Delta_+$ s.t. $r_\alpha w = r_{\beta_k} \ldots r_{\beta_1} w$, $l(r_{\beta_i} \ldots r_{\beta_1} w) = l(w) + i$, $1 \leqq i \leqq k$. This means $r_\alpha w \prec w$. $\square$

To proceed with the general discussion, a number of points should be emphasized. First, the above Proposition 4.4′ does not exclude that there are several π-singular vectors labeled by the same Weyl group element, nor does it guarantee that all π-singular vectors can be found in this way. Second, the π-singular vectors and the corresponding vector spaces have been introduced as pure Fock space concepts and are in general not known to be elements of $V_\pi(I(\Lambda_+, \Lambda_-))$. In particular $\tilde{w} \leftarrow w$ does not imply so far that $V_\pi(I(w * \Lambda_+, \Lambda_-))$ is a subspace of $V_\pi(I(\tilde{w} * \Lambda_+, \Lambda_-))$. Suppose further momentarily that the $W(sl(r+1))$ algebra and its free field realization (4.4) are known to exist. Let π: $V \to V_\pi$ be the linear map defined by the free field realization (if irrelevant, we will sometimes drop the labels $I(\Lambda_+, \Lambda_-)$). In general π will be a projection: A π-singular vector either has a pre-image in the Verma module (which is then annihilated by W^i_n, $n > 0$) or it does not, in which case the mapping π must be singular. For s_+^2 irrational, such situations can partially be excluded. As indicated above, the mapping from the Weyl invariant polynomials $I^i(x_1, \ldots, x_r)$ to the pairs $(\Lambda_+, \Lambda_-) \in h^* \times h^*$ is 1–1 inside a fixed Weyl chamber. For given $\Lambda_- \in P_+$, the weights $\tilde{\Lambda}_+ \in h^*$ can therefore be used to label the π-singular vectors.

Lemma 4.8. *For fixed $\Lambda_- \in P_+$, s_+^2 irrational:*
a) *π-singular vectors exist only for $\tilde{\Lambda}_+ \in W * \Lambda_+$, the Weyl orbit of Λ_+ in h^*.*
b) *The π-singular vectors labeled by $r_i * \Lambda_+$, $\Lambda_+ \in P_+$ are unique and are elements of $V_\pi(I(\Lambda_+, \Lambda_-))$. In particular $V_\pi(I(r_i * \Lambda_+, \Lambda_-))$, $1 \leqq i \leqq r$ are subspaces of $V_\pi(I(\Lambda_+, \Lambda_-))$.*
c) *If the $W(sl(r+1))$ algebra and its free field realization* (4.4) *are known to exist, the singular vector labeled by $r_i * \Lambda_+$, $\Lambda_+ \in P_+$ in V is unique and is mapped onto the corresponding π-singular vector in V_π.*

This follows from the determinant of the bilinear form on $V_\pi(I(\Lambda_+, \Lambda_-))$ and its multiplicities. Using the basis (4.31) the determinant for V_π is evaluated from the *oscillator* algebra (i.e. without pre-supposing the closure of the W-algebra) [4, 31]. The result is, up to a non-zero factor, given by

$$\det \mathscr{S}^N(I(\Lambda_+, \Lambda_-)) = \prod_{\alpha \in \Delta_+} \prod_{\{m_+ m_- \leqq N\}} [x_\alpha - s_+ m_+ - s_- m_-]^{P_r(N - m_+ m_-)}, \tag{4.32}$$

where $x_\alpha = s_+(\alpha, \Lambda_+ + \rho) + s_-(\alpha, \Lambda_- + \rho)$ and $P_r(N)$ is the number of partitions of N into parts of r colours. Notice that the r.h.s. is invariant under the diagonal

action of the Weyl group in $x_\alpha(\Lambda_+, \Lambda_-)$. The determinant therefore depends only on the invariants $I(\Lambda_+, \Lambda_-)$, as anticipated by the notation.
Proof of Lemma 4.8. a) Vanishing of $\det \mathscr{S}^{(N)}$ is a necessary condition for the existence of at least one vector at grade N, which is either π-singular or π-co-singular. As in the proof of Proposition 4.4′ one concludes that for s_+^2 irrational, the Fock space labels of the π-(co-)singular vector(s) are given by $(r_\alpha * \Lambda_+, \Lambda_-)$ or $(\Lambda_+, r_\alpha^{-1} * \Lambda_-)$. The Fock module generated by (each of) it may again contain π-(co-)singular vectors. Upon iteration, one sees that for fixed $\Lambda_- \in P_+$, the Fock space labels of π-(co-)singular vectors are constrained to the Weyl orbit $W * \Lambda_+$ of $\Lambda_+ \in h^*$.
b) Generally the number of π-(co-)singular vectors that appear at grade N can not exceed the order to which $\det \mathscr{S}^{(N)}$ vanishes. Further, the π-singular vectors associated with grades for which the determinant vanishes for the first time (in order of increasing N) are known to be π-singular (not π-co-singular) and to be elements of V_π [31]. Clearly the Weyl reflections corresponding to the simple roots yield the singular vectors of lowest possible grade. Take therefore $N = m_+ m_- = (\Lambda_+ + \rho, \alpha_{i_j})(\Lambda_- + \rho, \alpha_{i_j})$, $1 \leqq j \leqq k$ to be the grade at which the determinant vanishes for the first time with multiplicity $1 \leqq k \leqq r$. The Fock space construction provides k π-singular vectors at that degree, which are thus known to be unique and to be elements of V_π. To extend this to the remaining π-(co-)singular vectors labeled by $r_i * \Lambda_+$, $i \notin \{i_1, \ldots, i_k\}$, it suffices to show that the Fock space construction of the $r_i * \Lambda_+$ π-singular vectors does not depend on the relative size of the Dynkin labels. For fixed i one can choose a basis in r-dimensional Euclidean space s.t. $\alpha_i = \sqrt{2} e_i$ and $r_i * \Lambda_+ - \Lambda_+ = -\frac{1}{\sqrt{2}}(l_i + 2) e_i$, if e_i, $1 \leqq i \leqq r$ denotes an orthonormal basis. Thus only the i^{th} Dynkin label enters the Fock space construction of the π-singular vector $r_i * \Lambda_+$.
c) This follows from b) and the fact that the Fock space operator defining the π-singular vectors labelled by $r_i * \Lambda_+$ is injective (cf. Eq. (4.24)). □

Since by (A.3) each Weyl group element w lies in the image of at least one fundamental reflection w.r.t. the Bruhat ordering, at least one of the π-singular vectors labeled by $w \in W$ is from a) known to be an element of V_π. Part b) implies that if there are π-singular vectors in addition to that described by Proposition 4.4′, they have to appear at the same grades as the ones covered by the Fock space construction. If, in fact, there is a 1–1 correspondence between π-singular vectors and elements of the Weyl group, a nontrivial consistency condition arises: The mappings given by composition of the canonical embeddings $\iota_{w\tilde{w}}$: $V_\pi(I(w * \Lambda_+, \Lambda_-)) \to V_\pi(I(\tilde{w} * \Lambda_+, \Lambda_-))$ have to define commutative diagrams for each of the fundamental squares introduced before. Conversely if all of these compatibility conditions can independently be shown to hold, the mentioned 1–1 correspondence follows: By (A.3) uniqueness of the highest weight state $|\Lambda_+, \Lambda_-\rangle^*$ implies that if all the squares with Weyl group elements of length 2 at the top are given to be commutative, the singular vectors corresponding to length 2 elements have to be unique and to be elements of V_π. Induction in the length gives the uniqueness of all π-singular vectors and by a) no other π-singular vectors exist. The diagram summarising the descendence pattern of the π-singular vectors in $F^*_{w * \Lambda_+, \Lambda_-}$, $w \in W$ turns into an embedding diagram for the π-singular subspaces $V_\pi(I(w * \Lambda_+, \Lambda_-))$. Finally, if the $W(sl(r+1))$ algebra and the free field realization are known to exist, the canonical projection π: $V \to V_\pi$ is non-singular at all grades, which implies Theorem 4.6.

For brevity we shall refer to the condition that all squares of the descendence/embedding diagram form commutative diagrams w.r.t. composition of the canonical embeddings, as the "integrability condition" for the embedding diagram. They will later be seen to form a sufficient condition for the existence of a W-algebra and for V_π to be a $W(sl(r+1))$ module.

4.5. π-Intertwining Operators. In the present context define a π-intertwining operator for $V_\pi(I(\Lambda_+, \Lambda_-))$ as a map

$$Q_{\tilde{w}w}\colon F^*_{\tilde{w}*\Lambda_+,\Lambda_-} \to F^*_{w*\Lambda_+,\Lambda_-}, \quad \tilde{w} \preceq w \tag{4.33}$$

s.t. the image $Q_{\tilde{w}w}|\tilde{w}*\Lambda_+, \Lambda_-\rangle^*$ is (one of) the π-singular vector(s) labelled by w in Proposition 4.4′. If the free field realization were already known to exist, these operators would become proper intertwiners of the $W(sl(r+1))$ algebra. Notice that because of $(V_i^\pm)^* = V_i^\pm$ the π intertwiners remain unchanged when taking the dual, although we shall adopt the convention $Q^*_{w\tilde{w}} = Q_{\tilde{w}w}$ for $\tilde{w} \preceq w$. The definition guarantees in particular that a set of direct π-intertwiners exists for each positive root $\alpha \in \Delta_+$ acting on $F^*_{w*\Lambda_+,\Lambda_-}$ s.t. $\alpha \in \Delta_+^{w^{-1}r_\alpha}$,

$$Q_\alpha^{l^+_{j(\alpha)}}\colon F^*_{r_\alpha w*\Lambda_+\Lambda_-} \to F^*_{w*\Lambda_+\Lambda_-}\,. \tag{4.34}$$

Here $l^+_{j(\alpha)}$ is the (for given $w \in W$) unique Dynkin index paired to the positive root α in (4.28), (A.2). Again we will drop the $\pm$ labels whenever possible. $Q_\alpha^{l_j}$ will be realized as a polynomial in multi-screening integrals of the form (4.12). The integrability conditions for the π-intertwiners coincide with that of the q-intertwiners (3.8). In particular, the commutativity of squares of type b does not give rise to an integrability condition for the involved intertwiners. This is because if the free field realization of the W-algebra exists at all in the required form, the states in $V_\pi(I(\Lambda_+, \Lambda_-))$ and its singular submodules are τ invariant. Thus, the oscillator algebra may be supplemented by s_0 and only the τ invariant sector needs to be considered.

A first consequence of the integrability relations (3.8) is that the π-intertwining operators obtained from the $\mathscr{U}_q(n_+)$ operators are unique, well defined and non-vanishing: The discussion following Lemma 4.8 together with the relations (3.8) implies that the set of π-singular vectors labeled by $w \in W$ in Proposition 4.4′ contains one element only. The unique π-singular vector labeled by $w \in W$ can be expressed in a variety of different, but mutually consistent ways as products of direct intertwiners acting on $|w*\Lambda_+, \Lambda_-\rangle^*$. In particular one can choose

$$Q_{w,1}|w*\Lambda_+, \Lambda_-\rangle^* = Q^{l_{i_k}}_{\alpha_{i_k}} \ldots Q^{l_{i_1}}_{\alpha_{i_1}}|w*\Lambda_+, \Lambda_-\rangle^*\,, \tag{4.35}$$

where $w \leftarrow r_{i_k} \leftarrow \ldots \leftarrow r_{i_1} \leftarrow 1$. This is because by (A.3) each $w \in W$ lies in the image of at least one path in the embedding diagram consisting of fundamental reflections alone. A contour deformation of the type leading to Eq. (4.24) shows that these states are nonvanishing. For the operators $Q^{l_i}_{r_i}$ the intertwining property is known from (4.20). Alternatively, one can in this case deform the contour in Fig. 4 to that used in [21], where the intertwining property is manifest. Thus, $Q_{w,1}$ is a well defined, nonvanishing π-intertwiner for all $w \in W$. By considering squares in the embedding diagram which contain a single π-intertwiner $Q_\alpha^{l_j}$ with $ht\alpha = k$ and others of height smaller than k, one obtains by induction on k that all $Q_\alpha^{l_j}$ are well defined and non-vanishing. By Lemma 4.8.b) the π-singular vectors they generate are unique, are elements of $V_\pi(I(\Lambda_+, \Lambda_-))$ and by a) no others exist. This also implies that there is in fact a 1–1 correspondence between singular vectors in $M^q_{\Lambda_+}$ and π-intertwiners. In summary one arrives at:

Proposition 4.4″. *For fixed $\Lambda_- \in P_+$, q not a root of unity: There exists a bijective map from the set of singular vectors in M^q_Λ of weight $w * \Lambda$ to π-intertwining operators $r_\alpha w \leftarrow w$*

$$Q^{l^+_{j(\alpha)}}_\alpha : F^*_{r_\alpha w * \Lambda_+, \Lambda_-} \to F^*_{w * \Lambda_+, \Lambda_-}$$

given by

$$f_{i_1} \dots f_{i_n} \cdot v_{\Lambda_+} \to [\![V^+_{i_1} \dots V^+_{i_n}]\!]_{\Lambda_+ \Lambda_-} .$$

Here $\{v \in M^q_\Lambda \mid h_i \cdot v = ((\Lambda - \lambda, h_i)v\}$ is the subspace of M^q_Λ of weight λ. Moreover introduce the $(l_1 + 1, \dots, l_r + 1)$ graduation on M^q_Λ, i.e. set $\deg f_i = l_i + 1$, $1 \leqq i \leqq r$. In this graduation, the degree of a π-singular vector of weight $w * \Lambda$ in M^q_Λ coincides with the degree of the singular vector $Q_{w,1}|w * \Lambda_+, \Lambda_-\rangle^*$ in $V_\pi(I(\Lambda_+, \Lambda_-))$. For the simple roots Eq. (4.20) provides the explicit form of the π-intertwiners. As a last step in exploiting the integrability conditions (3.8) for the π-intertwiners we arrive at the following criterion:

Proposition 4.9. *For fixed $\Lambda_- \in P^+$, the integrability conditions* (3.8) *for the operators $Q^{l^+_i}_\alpha$ on the $\mathscr{U}_q(n_-)$ module $M^q_{\Lambda_+}$, $\Lambda_+ \in P_+$ are a sufficient condition for the existence of the free field realization* (4.4) *of the $W(sl(r+1))$ algebra with $\mathscr{H}_{\Lambda_+ \Lambda_-}$ forming an irreducible $W(sl(r+1))$ module.*

Proof. The π-singular vectors $Q_{w,1}|w * \Lambda_+, \Lambda_-\rangle^*$ to $w \in W$ are already known to be unique, well defined and non-vanishing and no other π-singular vectors exist. Further they are elements of $V_\pi(\Lambda_+, \Lambda_-)$ which generate π-singular vector spaces of $V_\pi(I(\Lambda_+, \Lambda_-))$ with the embedding diagram induced by the Bruhat ordering. All states in the quotient V_π/SV_π are thus known to be expressible in terms of lexicographically ordered creation modes of the Miura fields (4.4). The ordering may be taken as in (2.1). Now consider the singlet case, i.e. $\Lambda_\pm = 0$. Generally, $\prod_{\alpha \in \Delta_+}(1 - e^{2\pi i(\rho, \alpha)}) = \prod_{i=1}^r (\prod_{k=1}^{\Delta_i - 1}(1 - e^{2\pi i r k}))$, where $\Delta_i - 1$ are the exponents of g. The character formula (4.30) thus shows that $\mathscr{H}(sl(r+1)) \cong \mathscr{H}_{00}$ as graded linear spaces, so that condition (2.3) in the definition of Sect. 2 is satisfied. This can be used to reconstruct the operator product algebra of normal ordered products of $W^i(z)$ and their derivatives. In particular all fields which appear in the operator product algebra can be expressed in terms of normal ordered products of the fields $W^i(z)$ and their derivatives. Picking a regularization prescription to regularize infinite sums, the commutator of normal ordered regularized sums of W^i_n modes closes on regularized sums of such modes. On any state of a highest weight representation only a finite number of terms contribute and the regulator can be removed. The W-algebra is thus known to close on the closure of the universal enveloping algebra of the modes W^i_n w.r.t. the topology induced by taking matrix elements in $\mathscr{H}_{00}$. After projecting the generators onto quasiprimary fields (see also Sect. 5) the postulates for a $W(g)$-algebra given in Sect. 2 are satisfied. For general $\Lambda_\pm \in P^+$, this W-algebra has a well defined action on the vector space $\mathscr{H}_{\Lambda_+ \Lambda_-} \cong V_\pi/SV_\pi$. The "would be" representation theory of this subsection becomes the proper representation theory and the space $\mathscr{H}_{\Lambda_+ \Lambda_-}$ exists as irreducible $W(sl(r+1))$ module. □

In principle one would expect the integrability conditions (3.8) also to be a necessary condition for the existence of the free field realization. However, without the conditions (3.8) to be given, it seems to be difficult to show that the intertwiners are well defined and non-vanishing, so that the question for (3.8) can not be asked properly.

5. Pathologies at Exceptional Central Charge

In this section we discuss two pathological features of Casimir algebras which may occur for special values of the central charge. The first is that the rank of a Casimir algebra is actually a discontinuous function of the central charge. For a finite number of points the rank will be smaller than that of the underlying finite dimensional Lie algebra. The second pathology is that the number of independent primary fields may be smaller than the rank for another finite set of exceptional points.

The structure constants of a Casimir algebra are functions in the central charge and depend on the choice of basis fields. In the basis of Miura fields one infers that the structure constants are polynomial in the c. This is a consequence of the definition of normal ordering (induced by (2.1)) and the τ-invariance. In particular, this allows to extend the range of definition of the commutation relations from C_r to all values of c and also renders the projection onto quasiprimary fields unproblematic. By Proposition 4.2 the same holds for the Cartan basis (4.8), which is the preferred basis to study the representation theory. In contrast, the structure constants in a basis of primary fields will be complicated roots of rational functions. Call a basis *regular* if the structure constants are polynomial in c. Besides the practical advantages, such bases can be used to calculate the rank of a Casimir algebra. Recall from Sect. 2 that the rank of a Casimir algebra is the minimal number of quasiprimary fields required to generate the mCFT. For $c \in C_r$ one has the expected rank, i.e. rank $W(g) = r = \operatorname{rank} g$. For a finite set of c values the rank of the algebra may actually be smaller than r. At these values one or more of the generating fields becomes composite, i.e. some linear combination of generators decouples from all conformal blocks. Consider the commutators $[P^i_m, P^j_n]$ of the quasiprimary fields of weight Δ. The coefficients of the c-number term form a matrix D_Δ which yields a metric on the corresponding vector space. The vanishing of the determinant of this metric gives a criterion for the decoupling.

Lemma 5.1. *For a Casimir algebra in a regular basis the rank is given by $r - s$, where s is the number of $1 \leqq i \leqq r$ for which $\det D_{\Delta_i}$ vanishes.*

As a non-trivial example consider the $sl(4)$ case and set $L = W^1$, $W = W^2$, $V = W^3$. The Cartan basis is given by $\bar{L} = L$, $\bar{W} = W$, $\bar{V} = V + \frac{1}{8}\Lambda + \frac{c-3}{200}\partial^2 L$, where $\Lambda = \mathcal{N}(L, L) = (L, L) - \frac{3}{10}\partial^2 L$. From the explicit form of the commutation relations in this basis [9] one finds

$$\det D_2 = \frac{1}{2}c \,,$$

$$\det D_3 = \frac{1}{30}c(c+7) \,,$$

$$\det D_4 = \frac{1}{1200}c(c+2)(c+7)(7c+114) \,. \tag{5.1}$$

This means that the algebra is of rank 3 except for $c = 0, -2, -7, -114/7$, where it is of rank 0, 2, 1, 2, respectively. Notice that these points lie in the $W(sl(4))$-minimal spectrum.

A second source of exceptional c-values is the projection onto primary fields. The use of primary fields gives a realization-independent way to determine the structure constants of W-algebras of low rank by explicitly solving the associativity condition. This allows also the investigation of W-algebras for which the weights of the generating fields do not coincide with the exponents of some Lie algebra. The principle can be summarized as follows (cf. [33, 34] and references therein): Starting with the Virasoro algebra, one adds a number of fields $\underline{W}^i$ *primary* relative to it with weight Δ_i, normalized as $\langle \underline{W}^k, \underline{W}^k \rangle = c/k$. Further one uses the $su(1, 1)$ covariant normal ordering $\mathcal{N}$. This allows one to write down an $su(1, 1)$ covariant Ansatz for the commutator of any two fields $\underline{W}^k$, with only a few structure constants undetermined. Imposing the Jacobi identity (on a computer) gives a set of algebraic equations for the structure constants. These equations turn out to have either none, a finite number, or a 1-parameter family of solutions (with the parameter corresponding to the central charge). In this way, the resulting algebraic structure (if any) is uniquely determined by the weights of the generating primary fields and the postulated covariance properties.

For the $sl(r + 1)$ series, the method has been applied to $sl(3)$ and $sl(4)$ [33, 34] and confirms that the Jacobi identity generically has a 1-parameter family of solutions. The structure constants are roots of rational functions of the central charge. For a certain finite set of exceptional values of c, the structure constants are ill defined due to the presence of poles. These poles indirectly signal two different types of defects. For some of the singular c-values the algebra turns out to be not of maximal rank as described before. For another set of c-values the number of independent primary fields is smaller than the rank and hence smaller as the number taken as input for the calculation.

Consider again the $sl(4)$ case for illustration. Let $\underline{W}^k$ denote the projections of the Cartan field generators onto primary fields, normalized s.t. $\langle \underline{W}^k, \underline{W}^k \rangle = c/(k + 1)$. In the above $sl(4)$ example one finds

$$\underline{W} = \sqrt{\frac{10}{c + 7}}\, \bar{W},$$

$$\underline{V} = \sqrt{\frac{300(5c + 22)}{(7c + 114)(c + 7)(c + 2)}} \left(\bar{V} - \frac{7c + 114}{40(5c + 22)} \Lambda \right). \tag{5.2}$$

We have verified explicitly that the transformation $\bar{W}^k \rightsquigarrow \underline{W}^k$ leads to the commutation relations given in [7]. For the c values $\{-2, -\frac{22}{5}, -7, -\frac{114}{7}\}$ the basis transformation is singular which introduces a corresponding set of singularities in the structure constants. Three of these singular points can (a posteriori) be removed by relaxing the normalization condition, but at $c = -22/5$ the projection onto primary fields fails. Notice that this point does not lie in the $W(sl(4))$-minimal spectrum. Algebraically the pole at $c = -\frac{22}{5}$ arises from

$$[L_m, \Lambda_n] = (3m - n)\Lambda_{m+n} + \frac{1}{6}\left(c + \frac{22}{5} \right) m(m^2 - 1) L_{m+n}\,. \tag{5.3}$$

From $\det D_\Delta \neq 0$ for $\Delta = 2, 3, 4$ one also verifies that $\Lambda(z)$ is not a null field at this point and thus does not decouple from representation spaces or correlation functions. Quite generally, in the commutator of L_n with some normal ordered composite field, the coefficients of the fields on the right-hand side will be polynomials in the central charge. At the zeros of these polynomials, the linear system, defined by the elimination of unwanted higher pole terms in the operator product expansion, degenerates and the projection onto primary fields no longer exists. The contribution of the subleading terms in the primary projections of the Cartan generators leads to a central term which is a rational function of c. The final normalization to $\langle \underline{W}^k, \underline{W}^k \rangle = c/k$ introduces the remaining poles found in [33, 34] as well as the square roots in the structure constants. It is often useful not to insist on this normalization, in which case one is left with a set of singular points C_r^* arising from the projection onto quasiprimary fields.

In terms of the characteristic space $\mathscr{H}(sl(r+1)) = \mathscr{H}_{00}$ in the definition of Sect. 2 one is faced with the following situation: Even for the part of the c-spectrum where the algebra is of maximal rank, the attempt of a decomposition of $\mathscr{H}_{00}$ into irreducible Virasoro modules might fail for certain values of the central charge. The known proofs that a basis of primary fields can always be chosen in conformal field theory explicitly assume unitarity [32]. Conversely, this seems to imply that at the exceptional points only non-unitary representations of the W-algebra can exist.

Beyond that we can only offer the following tentative partial result: The set C_r^* of exceptional points of the $W(sl(r+1))$ algebra arising from the projection onto primary fields satisfies

$$C_r^* \subset \left\{ c = 1 - 24{s_0}^2 \,|\, s_+^2 = \frac{p}{q}, \quad p, q \text{ coprime}, \min(p, q) \leqq r \right\}. \tag{5.4}$$

The following argument tries to model the exceptional situation $c \in C_r^*$ within the framework of the free field realization. Let a $W(sl(r+1))$ algebra be given, in the sense of the definition in Sect. 2, with generic central charge. Let $\mathscr{H}_{00} = \mathscr{H}(sl(r+1))$ denote the Fock space construction of its characteristic space and $\mathscr{H}_{00} = \bigoplus_{m_+ m_-} \mathscr{H}_{\underline{m_+ m_-}}$. The sum is over some set of $sl(2)$-weights $\mu_\pm = m_\pm/\sqrt{2} \in P$, with $1/\sqrt{2}$ the fundamental weight of $sl(2)$. Each of these irreducible representations can be obtained as the unique non-vanishing cohomology class in a complex of Fock spaces of a single boson ϕ (see [21] and references therein). The complex is defined in terms of intertwining operators of the form $[\![(V)^{m+1}]\!]$, with $V = e^{-is_+\sqrt{2}\phi}$. The free field realization of the Virasoro algebra is $L(z) = -\frac{1}{2}\partial\phi\partial\phi - i\sqrt{2}s_0\partial^2\phi$ with $c = 1 - 24{s_0}^2$. Any *fixed* of these representations can be regarded as a subspace of a single boson sector in $\mathscr{H}_{00}$: Let $\phi = (\phi_1, \ldots, \phi_r)$ denote the Euclidean vector of bose fields from which $\mathscr{H}$ is constructed. One can choose a basis in root space for which $\alpha_i \cdot \phi = \sqrt{2}\phi_i$, so that the $W(sl(r+1))$ screening operator V_i may be used to define a complex for the Virasoro algebra. The Fock vacuum for the boson ϕ_i may contain modes of all other $\phi_j, j \neq i$. The Fock space resolution of $\mathscr{H}_{00}$ may (a posteriori) be defined in terms of intertwiners $[\![(V_i)^{m_i+1}]\!]$, $1 \leqq i \leqq r$, $0 \leqq m_i \leqq n_i$ only. Here n_i denotes the maximal "power" of V_i required. Clearly, $\max_i(n_i + 1)$ is the maximal difference in the levels of two singular modules labelled by w, $r_\alpha * w$ s.t. $r_\alpha * w \leftarrow w$. By (A.5) one has $\max_i(n_i + 1) = r$.

Now consider the limit, where c approaches one of the exceptional points in C_r^*. By assumption, the different irreducible Virasoro representations in $\bigoplus_{m_+ m_-} \mathscr{H}_{m_+, m_-}$ can then no longer be matched to reconstruct $\mathscr{H}_{00}$ (because the "Clebsch Gordon" coefficients appearing in the decomposition develop zeros/poles). But this means that at least for one $1 \leqq i \leqq r$ and $0 \leqq m_i \leqq n_i$ the singular vector $[\![(V_i^+)^{m_i^+ + 1}]\!] | r_i * \mu_+, \mu_- \rangle^*$, $m_i^+ = (\mu_+, \alpha_i)$ is either ill defined or vanishes. From (4.24) or by direct evaluation of the integral against a basis of symmetric polynomials [21] one finds: For $s_+ > 0$ the state is always well defined and vanishes iff $s_+^2 = p/q$, $q \leqq m_i^+ + 1$. Here the screening operators of the s_+ sector were used, but the same has to hold on the s_- sector, as also the invariance under the diagonal action of the Weyl group $F_{w * \mu_+, \mu_-} \cong F_{\mu_+, w^{-1} * \mu_-}$ must not fail in the complex, to allow the reconstruction of $\mathscr{H}_{00}$. In summary, one concludes that a necessary condition for the failure of the attempt to reconstruct $\mathscr{H}_{00}$ from its irreducible Virasoro components is that $s_+^2 = p/q$, $\min(p, q) \leqq \max_i(n_i + 1) = r$. The relevant parametrization of the central charge is that of (one of) the single boson Fock space(s) where the reconstruction fails. As the vacuum for ϕ_i is itself a nontrivial Fock state, the usual positivity bound does not apply.

6. Extended Sugawara Construction

The irrational Miura-type realization of $W(sl(r + 1))$ is closely related to affine Kac–Moody algebras at level $k = 1$. These are particularly relevant for applications, for example to KdV-type hierarchies.

Let $\hat{g}$ denote a simply laced affine Lie algebra and $L_{\hat{\lambda}}^k$ the irreducible module of affine weight $\hat{\lambda} = (\lambda, k)$, $\lambda \in P_+$. Any such module can be decomposed w.r.t. the horizontal subalgebra. For level $k = 1$ modules it reads [28]

$$L_{\hat{\lambda}}^1 = \bigoplus_{\Lambda \in P_+ \cap (Q + \lambda)} L^1(\lambda | \Lambda) \otimes L_\Lambda . \tag{6.1}$$

Here, $\hat{\lambda} = (\lambda, 1)$ is an integrable weight of $\hat{g}$, Q is the root lattice of g and L_Λ the irreducible g-module to $\Lambda \in P_+$. $L^1(\lambda | \Lambda)$ are subspaces of g singlets. Let $L[x] = v^{-2} \sum_a (x^a x^a)(z)$, $v^2 = k + \check{h}$ be the usual Sugawara operator with $x^a(z)$ a linear basis of the current algebra. For the (homogeneously specialized) character of $L_1(\lambda | \Lambda)$ one has [28]

$$\operatorname{ch} L^1(\lambda | \Lambda) = \operatorname{Tr}_{L^1(\lambda | \Lambda)} e^{2\pi i \tau (L_0)} = \frac{e^{2\pi i \tau \cdot \frac{1}{2}(\Lambda, \Lambda)}}{\phi(\tau)^r} \prod_{\alpha \in \Delta_+} (1 - e^{2\pi i \tau (\Lambda + \rho, \alpha)}) . \tag{6.2}$$

This can be used to show that $L^1(\lambda | \Lambda)$ is an irreducible $W(g)$ module with central charge $c = \left. \frac{k \dim g}{k + \check{h}} \right|_{k=1} = r$. The field generators are given by

$$C^i(z) = \frac{1}{N v^i} d_{a_1 \ldots a_i} (x^{a_1} \ldots x^{a_i})(z) \tag{6.3}$$

with the d-symbols chosen symmetric and traceless and N is a normalization factor to be specified later. These fields are primary w.r.t. $C^2 = L[x]$ (cf. [4]) and close under operator product expansion. For the proof one has to show that the states of the form (2.1) built from the modes of $C^i(z)$ are independent. This can be done in the level 1 vertex operator realization of $\hat{g}$ ([32]; the argument is reproduced in [13]).

Now observe that (6.2) coincides with the character of the irrational $W(sl(r+1))$ modules (4.30), i.e.

$$e^{2\pi i\tau\cdot\frac{1}{2}(s_+^2-1)(\Lambda+\rho,\rho)}\operatorname{ch}\mathscr{L}(I(\Lambda,0)) = \operatorname{ch}L^1(\lambda|\Lambda)$$
$$= e^{2\pi i\tau\cdot\frac{1}{2}(s_-^2-1)(\Lambda+\rho,\rho)}\operatorname{ch}\mathscr{L}(I(0,\Lambda))\,, \qquad (6.4)$$

where $\Lambda\in P_+\cap(Q+\lambda)$ and $\mathscr{L}(I(\Lambda_+,\Lambda_-))$ is the irrational $W(sl(r+1))$ module of central charge $c=r-48s_0^2\rho^2$. This means that $L^1(\lambda|\Lambda)$ and $\mathscr{L}(I(\Lambda,0))$ are isomorphic as graded vector spaces. In fact, the isomorphism can be made explicit and is essentially given by the free field realization of $\hat{g}$ in terms of r free bose fields and $|\Delta_+|$ bosonic $\beta\gamma$ pairs. This can be regarded as an infinite dimensional analogue of the Harish–Chandra isomorphism.

Let g in the following be $sl(r+1)$ and $g=n_-\oplus h\oplus n_+$ a triangular decomposition. Let $E_{\pm\alpha}$ be Cartan step operators and H_i, $1\leqq i\leqq r$ be any basis of the Cartan subalgebra. The Poincaré-Birkhoff-Witt (PBW) theorem states that a basis of the universal enveloping algebra $\mathscr{U}(g)$ of g is then given by the lexicographically ordered monomials $E^{i_1}_{-\alpha_1}\dots E^{i_s}_{-\alpha_s}H_1^{j_1}\dots H_r^{j_r}E^{k_1}_{\alpha_1}\dots E^{k_s}_{\alpha_s}$, with the step operators written in an arbitrary but fixed order of the positive roots $\alpha_1>\dots>\alpha_s$. One has the following simple facts (see e.g. [29]).

a) $\mathscr{U}(g)=\mathscr{U}(h)\oplus(\mathscr{U}(g)n_+\oplus n_-\mathscr{U}(g))$.
b) For elements C in the center $\mathscr{Z}(g)$ of g, the projection onto the second factor in a) lies in $\mathscr{U}(g)n_+$.

Let $\gamma\colon\mathscr{U}(g)\to\mathscr{U}(h)$ denote the projection onto the first factor in a) s.t. $C-\gamma(C)$ lies in $\mathscr{U}(g)n_+$ for $C\in\mathscr{Z}(g)$. Set $\sigma(H)=H-(\rho,H)$ for $H\in h$.

Theorem 6.1. (*Harish–Chandra*). *The map $\sigma\circ\gamma\colon\mathscr{U}(g)\to\mathscr{U}(h)$ is an algebra isomorphism of $\mathscr{Z}(g)$ onto the algebra of Weyl invariant polynomials in $\mathscr{U}(h)$.*

A proof can be found in any textbook, for example [29]. In particular, the theorem allows to calculate the eigenvalues of the Casimir operators from the Weyl invariant polynomials by a simple shift σ^{-1} in the Cartan subalgebra generators. For comparison with the infinite dimensional case, consider $sl(3)$ as a nontrivial example. In terms of the Chevalley generators the Casimir operators read[3]

$$C^2=\delta_{ab}x^ax^b$$
$$=\frac{1}{3}[h_1^2+h_1h_2+h_2^2]+\frac{1}{2}[f_1e_1+f_2e_2+f_3e_3+(e\leftrightarrow f)] \qquad (6.5.a)$$
$$=-\hat{h}_1\hat{h}_2-\hat{h}_2\hat{h}_3-\hat{h}_1\hat{h}_3+\hat{h}_1-\hat{h}_2+f_1e_1+f_2e_2+f_3e_3, \qquad (6.5.b)$$

[3] We do not agree with the expression for C^3 given in Eq. (4.4) of [36]

$$
\begin{aligned}
C^3 &= \frac{2}{3} d_{abc} x^a x^b x^c \\
&= \frac{1}{27}[2h_1^3 + 3h_1^2 h_2^2 - 3h_1^2 h_2 - 2h_2^3] \\
&\quad + \frac{1}{6}[f(h_1 + 2h_2)e_1 - f_2(2h_1 + h_2)e_2 + f_3(h_1 - h_2)e_3 \\
&\quad + (f_1 f_2 + f_2 f_1)e_1 + f_2 e_3 f_1 + f_1 e_3 f_2 + e_3(f_1 f_2 + f_2 f_1) + (e \leftrightarrow f)] \qquad (6.6.a) \\
&= -\hat{h}_1 \hat{h}_2 \hat{h}_3 + \hat{h}_3(\hat{h}_1 - \hat{h}_2) + \hat{h}_3 \\
&\quad + f_1 \hat{h}_1 e_1 + f_2 \hat{h}_2 e_2 + f_3 \hat{h}_3 e_3 + f_3 e_1 e_2 + f_1 f_2 e_3 + f_1 e_1 + f_2 e_2 . \qquad (6.6.b)
\end{aligned}
$$

Here $e_3 = [e_1, e_2]$, $f_3 = [f_2, f_1]$ and our d-symbols are normalized s.t. $[t_a, t_b]_+ = \frac{3}{4}\delta_{ab} + 2d_{abc}t_c$, if t_a, $a = 1, \ldots, 8$ are the Gell–Mann matrices. The second form of C^2, C^3 is obtained respectively by rewriting all monomials in the PBW basis. For convenience we also switched to the usual overcomplete basis $\hat{h}_i$, $1 \leqq i \leqq r+1$ in the Cartan subalgebra. The relation to the Chevalley basis is $h_i = \hat{h}_{i+1} - \hat{h}_{i+2}(\hat{h}_{r+2} = \hat{h}_1)$. Besides $\sum_i \hat{h}_i = 0$ one has for this choice $[e_i, \hat{h}_i] = 0 = [f_i, \hat{h}_i]$. The first term in brackets is respectively the Weyl invariant polynomial and one can verify from the PBW forms (6.5.b), (6.6.b) that the eigenvalues of C^2, C^3 on some highest weight vector are in fact obtained by the shift σ^{-1} from them. Generally we normalize the Casimir operator C^k s.t. the leading term in $\mathscr{U}(h)$ is given by minus the symmetric polynomial of power k in $\hat{h}_i$.

Let now $f_i(z)$, $h_i(z)$, $e_i(z)$ be Chevalley field generators of $\widehat{sl}(r+1)$ at some level k and define overcomplete Cartan subalgebra fields by

$$
\hat{h}_i(z) = v(h_{i+1} - h_{i+2})(z), \quad v = \sqrt{k + \check{h}} \cdot \qquad (6.7)
$$

Let $\hat{g} = \hat{n}_+ \oplus \hat{h} \oplus \hat{n}_-$ be a triangular decomposition of $\widehat{sl}(r+1)$. The above form of the PBW basis is also valid on the enveloping algebra of normal ordered products of the affine field generators. We adopt a right nesting convention for repeated normal ordered products, i.e. $ABCD$ shall be shorthand for $(A(B(CD)))$ etc. Consider now the generalized Sugawara fields (6.3). They are not Casimir operators of the affine algebra. In fact, for $k \neq -\check{h}$ an affine algebra does not admit Casimir operators other than the quadratic [38]. (Instead theta functions separate the Weyl group orbits [39].) For $k = -\check{h}$, an infinite set of Casimir operators exists and is given by the modes of generalized Sugawara-type fields [40]. For level $k = 1$ one has the following weaker analogue. Let $\hat{\gamma}$ denote the projection onto the Cartan subalgebra piece in the PBW basis of $\widehat{sl}(r+1)$. Remarkably, as in the finite dimensional case, this Cartan subalgebra piece can be given in closed form.

Theorem 6.2. *The analogues of statements* a), b) *hold for* $\mathscr{U}(\hat{g})$ *at level* $k = 1$ *with* $\mathscr{L}(g)$ *replaced by the* g *singlets* $L_1(\lambda|\Lambda)$. *The Cartan subalgebra piece* $\hat{\gamma}C^k(z) =: W^k[\hat{h}]$ *of the Sugawara fields* $C^k(z) = \frac{1}{Nv^k} d_{a_1 \ldots a_k}(x^{a_1} \ldots x^{a_k})(z)$ *is given by*

$$
\tau\left[\frac{1}{v}\partial_z + \hat{h}_{r+1}\right] \ldots \left[\frac{1}{v}\partial_z + \hat{h}_1\right] = -\sum_{k=-1}^{r+1} W^k[\hat{h}](z)\left(\frac{1}{v}\partial_z\right)^{v-k} . \qquad (6.8)
$$

Remark. $\hat{h}_i(z)$ does not obey the same commutation relations as $\hat{h}_i \cdot \partial_z \phi$ in (4.4). Nevertheless (6.8) is form-identical to a Miura transformation with $2s_0 = \frac{1}{v} = (\check{h} + 1)^{-1/2}$.

Proof. The first part is shown as in the finite dimensional case. For the second part, note that, with the above normalization, the leading term of $\hat{\gamma} C^k(z)$ will be given by minus the (normal ordered) symmetric polynomial of power k in $\hat{h}_i(z)$. Rewriting unordered monomials in the PBW form, additional $\mathscr{U}(\hat{h})$ terms are generated with a derivative for each power less than k. The result follows by comparison of the zero mode pieces. The eigenvalue of the zero mode of $\hat{\gamma} C^k(z)$ coincides (up to a factor) with that of γC^k. In the basis $\hat{h}_i$ the shift factors $\hat{h}_i - \sigma^{-1} \hat{h}_{i+1}$ come in pairs $\rho \cdot \hat{h}_{i+1} = -\rho \cdot \hat{h}_{i+1}$, $i = 1, 3, \ldots, r$, r odd; $\rho \cdot \hat{h}_{i+1} = -\rho \cdot \hat{h}_{i+1}$, $i = 1, 3, \ldots, r-1$, r even. In the differential operator (6.8) this sign pattern arises from the τ symmetrization $\tau\left(\frac{1}{v}\partial_z \hat{h}_i\right) = -\frac{1}{v}\hat{h}_{i+1}$, with the r even/odd subcases as above. The absolute values of $\rho \cdot \hat{h}_i$ account for symmetry factors and, for example, by using the formulae of [35] one can check that they come out correctly. □

As an example, consider again $W(sl(3))$. Calculation gives

$$C^2 = -\hat{h}_1\hat{h}_2 - \hat{h}_2\hat{h}_3 - \hat{h}_1\hat{h}_3 - \frac{1}{v}\partial_z(\hat{h}_1 - \hat{h}_2) + f_1 e_1 + f_2 e_2 + f_3 e_3 ,$$

$$\begin{aligned} C^3 = & -\hat{h}_1\hat{h}_2\hat{h}_3 - \frac{1}{2v}(\partial_z(\hat{h}_3(\hat{h}_1 - \hat{h}_2)) + \hat{h}_1\partial_z\hat{h}_2 - \hat{h}_2\partial_z\hat{h}_1) + \frac{1}{2v^2}\partial_z{}^2\hat{h}_3 \\ & + f_1\hat{h}_1 e_1 + f_2\hat{h}_2 e_2 + f_3\hat{h}_3 e_3 + f_3 e_1 e_2 + f_1 f_2 e_3 \\ & + \frac{1}{2}[\partial_z(f_1 e_1) - \partial_z(f_2 e_2) + \partial_z f_3 e_3 - f_3 \partial_z e_3] , \end{aligned} \tag{6.9}$$

as it should.

Although the (modes of the) fields $\hat{h}_i(z)$ in (6.8) and $i\hat{h}_i \cdot \partial_z \phi$ in (4.4) do not obey the same commutation relations, both can be set in correspondence by means of a free field realization of $\widehat{sl}(r+1)$. This realization employs r free bose fields ϕ^a and $|\Delta_+|$ bosonic $\beta\gamma$ pairs [41, 5]. One associates to each positive root $\alpha \in \Delta_+$ a first order bosonic $\beta\gamma$ pair

$$\beta^\alpha(z) = \sum_{n \in \mathbb{Z}} \beta_n^\alpha z^{-n-1}, \quad \gamma^\alpha(z) = \sum_{n \in \mathbb{Z}} \gamma_n^\alpha z^{-n} ,$$

satisfying $[\gamma_n^\beta, \beta_n^\alpha] = \gamma^{\beta\alpha}\delta_{n+m,0}$. This realization has been used in [5, 6, 7] to derive Fock space resolutions for irreducible $\widehat{sl}(r+1)$ modules. There also the explicit expressions for the realizations πe_i, πh_i, πf_i of the Chevalley fields generators can be found. The fields $\hat{h}_i$ take the form

$$\pi\hat{h}_i(z) = \hat{h}_i \cdot \partial_z \phi + \frac{1}{v}\sum_\alpha (\hat{h}_i, \alpha)\beta^\alpha\gamma^\alpha . \tag{6.10}$$

From Theorem 6.2 one therefore expects the $\beta\gamma$-independent part of πC^k to be just given by the Miura fields $W^k = W^k[\phi]$ in (4.4), with $2s_0 = \frac{1}{v}$. This is not entirely trivial, because also the $\mathscr{U}(\hat{g})\hat{n}_+$ part in $C^k(z)$ will develop $\beta\gamma$ independent terms. The character identity (6.4), however, tells that these have to drop out, whenever $2s_0 = 1/v$ amounts to an irrational screening parameter s_+^2. For level $k = 1$ this is the case if and only if $r + 2 \neq (m + 2)(m + 3)$, $m \geqq 0$ (where equality would give $s_+^2 = (m + 2)/(m + 3)$). Together one arrives at

Corollary 6.3. *Let $r + 2 \neq (m + 2)(m + 3)$. The map*

$$\pi\colon L^1(\lambda|\Lambda) \to \mathscr{L}(I(\Lambda, 0))$$

$$(x_{n_1}^{a_1} \dots x_{n_k}^{a_k})|0\rangle \to (x_{n_1}^{a_1} \dots x_{n_k}^{a_k})[\phi\beta\gamma]|0\rangle|_{\beta\gamma=0} \tag{6.11}$$

defines an isomorphism between graded vector spaces.

The structure of the singular vectors in both modules, of course, will be entirely different. The identity (6.3), however, implies that the infinitely many singular submodules of $L^1(\lambda|\Lambda)$ are contained in the finitely many singular submodules whose images under π are already present for c generic. In terms of the commutation relations this implies that the point $c = r$ is a generic point for the structure constants, i.e. that none of the composite fields on the r.h.s. of $[W_m^i, W_n^j]$ will drop out, compared to the case $c \in C_r$. For example, this can be used to lift the existence of infinite dimensional abelian subalgebras from $\mathscr{L}(I(0, 0))$ to $L^1(\lambda|0)$ [37].

By regularity in s_0, the fact that $\pi C^k|_{\beta\gamma=0} = W^k[\phi]$ can be extended to all values of s_+^2 (although, of course, for s_+^2 rational, the isomorphy 6.3 will in general cease to hold). Once again, consider $sl(3)$ for illustration. With $\beta^{\alpha_i} = \beta^i$, $\gamma^{\alpha^i} = \gamma^i$, $i = 1, 2, 3$, $\alpha_3 = \alpha_1 + \alpha_2$, one finds

$$\pi C^2 = -\frac{1}{2}\partial\phi\cdot\partial\phi - \frac{1}{v}\partial^2\phi - \sum_{k=1}^{3} \beta^k\partial\gamma^k ,$$

$$\pi C^3 = W^3[\phi] + C_{\mathrm{mix}}^3 + \frac{1}{v}C^3[\beta\gamma] , \tag{6.12}$$

where $W^3[\phi]$ is the τ-invariant Miura generator and

$$C_{\mathrm{mix}}^3 = -\sum_{k=1}^{3} i\hat{h}_k\cdot\partial\phi\beta^k\gamma^k - i(\hat{h}_3 - \hat{h}_1)\cdot\partial\phi\beta^3\partial\gamma^1\gamma^2$$

$$\begin{aligned} C^3[\beta\gamma] = {} & \beta^2\gamma^2(\beta^3\partial\gamma^1\gamma^2 + \beta^3\partial\gamma^3 - \beta^1\partial\gamma^1) \\ & + \frac{1}{2}(\partial\beta^1\partial\gamma^1 + \partial\beta^2\partial\gamma^2 - \partial\beta^3\partial\gamma^1\gamma^2 - 2\beta^1\beta^2\partial\gamma^3) \\ & + \frac{1}{2}(\beta^2\partial^2\gamma^3 + \beta^3\partial^2\gamma^3 + \beta^3\partial^2\gamma^1\gamma^2) . \end{aligned} \tag{6.13}$$

One can check that πC^2, πC^3 are $sl(3)$ singlets and that πC^3 is primary w.r.t. the bosonized Sugawara field πC^2.

7. Conclusion

The existence of Miura-type free field realizations of $W(sl(n))$ has been established. The problem of the closure of the algebra has been reduced to a finite dimensional quantum group problem, for q not a root of unity, which is solved by direct construction of the intertwiners. The extension at least to Casimir algebras $W(g)$ based on simply laced g is unproblematic. For q a root of unity additional intertwining operators are present. The representation theory of the W-algebra would then presumably be characterized by a suitable analogue of the affine Weyl group. The structure of the irreducible representations for irrational values of the screening parameter has been found to be form-identical to that of the underlying simple Lie algebra; paralleling the $\widehat{sl}(n)$ modules for irrational level k. As graded vector spaces these irrational $W(\widehat{sl}(n))$ modules are isomorphic to the space of $sl(n)$ singlets in integrable $sl(n)$ modules at level $k = 1$. The isomorphim is given by the $\phi\beta\gamma$ free field realizations of $sl(n)$. One might expect the pure $\beta\gamma$ pieces $C^k[\beta\gamma]$ of the images πC^k to form again a realization of $W(sl(n))$. A calculation (using (6.13), (6.14) and the closure of the C^2, C^3 algebra) shows, however, that this is not the case for $W(s\underset{\smile}{l}(3))$. The $\beta\gamma$ Fock space is closely related to the singular $\widehat{sl}(n)$ modules at $k = -\check{h}$, so that the fields $C^k[\beta\gamma]$ may have significance there [5, 42]. Theorem 6.2 should extend to all Casimir algebras. This would provide a very systematic way to define free field realizations as the images of the set of Sugawara operators under π in (6.12). These realizations would automatically possess the correct symmetries of the Dynkin diagram. Further, no fermions would be needed for non-simply laced algebras, reflecting the corresponding property of the $\phi\beta\gamma$ free field realization [22]. In extension to the $sl(n)$ situation the intrinsic significance of these bases should lie in the fact that they are members of the equivalence class of bases in which the structure constants are polynomial in c. The existence of a Cartan basis then provides a route to infinite dimensional abelian subalgebras.

Appendix

Here we summarize some facts related to the Bruhat ordering on the Weyl group of a simple Lie algebra. Let α_i, $1 \leqq i \leqq r$ be a system of simple roots and r_i the associated fundamental reflections that generate W. For $w \in W$ set

$$\Delta_+^w = \{\alpha \in \Delta_+ \,|\, w(\alpha) < 0\} = \Delta_+ \cap w^{-1}\Delta_-$$

with $\Delta_\pm$ being the positive/negative roots. The following facts can, for example, be found in [18, 28, 29]

A.1. For $\alpha \in \Delta_+$, r_α the reflection in α, $w \in W$: $l(r_\alpha w) > l(w)$ iff $w^{-1}(\alpha) > 0$, i.e. $\alpha \in \Delta_+^{w^{-1}r_\alpha}$.

Further, the length of a Weyl group element equals the order of Δ_+^w [29, Prop. 3.18]. This implies [28, Lemma 3.11.b]

$$\Delta_+^w = \{\alpha_{i_l}, r_{i_l}\alpha_{i_{l-1}}, \ldots, r_{i_l} \ldots r_{i_2}\alpha_{i_1}\},$$

where $w = r_{i_1} \dots r_{i_l}$ is a reduced expression. In particular, $\Delta_+ = \Delta_+^{w_0}$, if w_0 is the element of maximal length in W. One sees that for $\alpha \in \Delta_+^w$, the sequence $\alpha, r_{i_l}\alpha, \dots r_{i_1} \dots, r_{i_l}\alpha$ contains a unique simple root. Denoting it by $\alpha_{j(\alpha)}$, one has

A.2. $\lambda - w^{-1}\lambda = \sum_{\alpha \in \Delta_+^{w^{-1}}} (\lambda, \alpha_{j(\alpha)})\alpha,\ \lambda \in h^*$.

[Use the identity $\lambda - wr_i\lambda = \lambda - w\lambda + w(\lambda - r_i\lambda)$ and induction on $l(w)$.]
Directly from the definitions follows,

A.3. Each $1 \neq w \in W$ lies in the image of at least fundamental reflection w.r.t. the relation '$\leftarrow$', i.e. $w = r_i\tilde{w}$ for some $i \in \{1, \dots, r\}$, $\tilde{w} \in W$, $l(w) = l(\tilde{w}) + 1$.

[Otherwise $l(r_i w) > l(w)$ for all i so that by (A.1), $w(\alpha_i) > 0$ for all i, which forces w to be 1.] By induction it follows that for each $w \in W$, there exist fundamental reflections $r_{i_1}, \dots r_{i_l}$ s.t. $w = r_{i_l} \dots r_{i_1} \leftarrow \dots \leftarrow r_{i_2}r_{i_1} \leftarrow r_{i_1} \leftarrow 1$, where $w = r_{i_l} \dots r_{i_1}$ then is a reduced expression.

A.4. If $\alpha \in \Delta_+$, $l(r_\alpha w) = l(w) + k$, there exists a sequence $\beta_1, \dots \beta_k \in \Delta_+$ s.t.

$$r_\alpha w = r_{\beta_k} \dots r_{\beta_1} w \leftarrow r_{\beta_{k-1}} \dots r_{\beta_1} w \leftarrow r_{\beta_2} r_{\beta_1} w \leftarrow r_{\beta_1} w \leftarrow w .$$

[This can be extracted from the proof of Theorem 2 in [18].]

A.5. $\max(w\rho - r_\alpha w\rho, \rho) = r$, where the maximum is to be taken over all $w \in W$, $\alpha \in \Delta_+$, s.t. $r_\alpha w \leftarrow w$.

[Let $\alpha \in \Delta_+$ be a positive root for which $(w\rho - r_\alpha w\rho, \rho)$ takes its maximal value. As any two positive root systems are related by a unique Weyl group element, one may -w.l.o.g. take $\alpha = \alpha_i$ to be a simple root. Then $(w\rho - r_i w\rho, \rho) = (w\rho, \alpha_i)$. Let π be the permutation corresponding to w, in the basis $\hat{h}_1, \dots, \hat{h}_{r+1}$. Then, $\rho = \sum_{i=2}^{r+1}(r + 2 - i)\hat{h}_i$ and $w\rho = \sum_{i=2}^{r+1}(r + 2 - \pi(i))\hat{h}_i$, so that with $(\hat{h}_j, \alpha_k) = \delta_{j,k} - \delta_{j-1,k}$ one finds $\max(w\rho, \alpha_i) = (r + 1) - 1 = r$.]

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References

1. Zamolodchikov, A.B.: Infinite additional symmetries in two dimensional quantum field theory. Theor. Math. Phys. **65**, 1205–1213 (1986)
2. Fateev, V.A., Zamolodchikov, A.B.: Conformal quantum field theory models in two dimensions having Z_3 symmetry. Nucl. Phys. **B 280** [FS18], 644–660 (1987)
3. Fateev, V.A., Lukyanov, S.L.: The models of two-dimensional conformal quantum field theory with Z_n symmetry. Int. J. Mod. Phys. **A3**, 507–520 (1988)
4. Bouwknegt, P.: Extended conformal algebras from Kac–Moody algebras. Proc. of the meeting on 'Infinite dimensional Lie Algebras,' C.I.R.M. Luminy 1988
5. Feigin, B.L., Frenkel, E.V.: Representations of affine Kac Moody algebras and bosonization, In: V. Knizhnik, Memorial Volume, 'Physics and mathematics of strings,' Brink, L., Friedan, D., Polyakov, A.M. (eds.). Singapore: World Scientific 1990
6. Feigin, B.L., Frenkel, E.V.: The family of representations of affine Lie algebras., Usp. Mat. Nauk. **43**, 227–228 (1988). (Russ. Math. Surv. **43**, 221–222 (1989))
7. Bouwknegt, P., McCarthy, J., Pilch, K.: Quantum group structure in the Fock space resolutions of $\hat{sl}(n)$ representations. Commun. Math. Phys. **131**, 125–156 (1990)
8. Gomez, C., Sierra, G.: The quantum symmetry of rational conformal field theory. Nucl. Phys. **B352**, 791–828 (1991)

9. Niedermaier, M.: $W(sl(n))$: Existence, Cartan basis and infinite abelian subalgebras. To appear in: Proceedings of the 1991 Cargese Summer School on New Symmetry Principles in QFT
10. Jimbo, M.: A q-difference analogue of $\mathcal{U}(g)$ and the Yang–Baxter equation. Lett. Math. Phys. **10**, 63–69 (1985)
11. Drinfel'd, V.G.: Hopf algebras and the quantum Yang–Baxter equation. Sov. Math. Dokl. **32**, 254–258 (1985); Drinfel'd, V.G.: Quantum groups. Proc. of the International Congress of Math., Berkeley 1986
12. Lukyanov, S.L.: Quantization of the Gelfan'd Dikii brackets. Funct. Anal. Appl. **22**, 1–10 (1990)
13. Watts, G.M.T.: W-algebras and coset models. Phys. Lett. **245B**, 65–71 (1990)
14. Goddard, P.: Meromorphic conformal field theory. Proc. of the meeting on 'Infinite dimensional Lie algebras,' C.I.R.M. Luminy, 1988; Frenkel, I., Lepowsky, J., Meurman, A.: Vertex operator algebras and the monster. New York: Academicd Press 1988
15. Nahm, W.: Chiral algebras in 2-dim CFT. In: Proceedings 'Recent developments in CFT,' Trieste, 1989
16. Humphreys, J.E.: Introduction to Lie algebras and representation theory. Berlin, Heidelberg, New York: Springer 1972
17. Verma, D.N.: Structure of certain induced representations of complex semi-simple Lie algebras. Bull. Am. Math. Soc. **74**, 160–166 (1968)
18. Bernstein, I.N., Gelfan'd, I.M., Gelfan'd, S.I.: Structure of representations generated by vectors of highest weight. Funct. Anal. Appl. **5**, 1–8 (1971)
19. Bernstein, I.N., Gelfan'd I.M., Gelfan'd, S.I.: Differential operators on the base affine space and a study of g modules. Summer School of the Bolyai Jánas Math. Soc., I.M. Gelfan'd, New York (1975)
20. Feigin, B.L., Fuks, D.B.: Invariant skew-symmetric differential operators on the line and Verma module over the Virasoro algebra. Funct. Anal. App. **16**, 114–126 (1983)
21. Felder, G.: BRST approach to minimal models. Nucl. Phys. **B317**, 215–236 (1989), erratum, Nucl. Phys. **B324**, 548 (1989)
22. Bouwknegt, P., McCarthy, J., Pilch, K.: Free field approach to 2-dim CFTs. In: Proc. of the 1990 Yukawa Int. Seminar, Kyoto
23. Lusztig, G.: Quantum deformations of certain simple modules over enveloping algebras. Adv. Math. **70**, 237–249 (1988)
24. Kirilov, A.N., Reshetikhin, N.: q-Weyl group and a multiplicative formula for univeral R-matrices. Commun. Math. Phys. **134**, 421–432 (1990)
25. Rosso, M.: An analogue of P.B.W. theorem and the universal R-matrix for $\mathcal{U}_q(sl(N+1))$. Commun. Math. Phys. **124**, 307–318 (1989)
26. Rosso, M.: Finite dimensional representations of the quantum analogue of the enveloping algebra of a complex simple Lie algebra. Commun. Math. Phys. **117**, 581–593 (1988)
27. Kempf, G.: Cousin–Grothendieck complex of induced representations. Adv. Math. **29**, 310–396 (1978)
28. Kac, V.: Infinite dimensional Lie algebras. Cambridge: Cambridge University Press 1985
29. Knapp, A.W.: Lie groups, Lie algebras and Cohomology. Mathematical Notes Vol. **34**, Princeton, NJ: Princeton University Press 1988
30. Gervais, J.L., Bilal, A.: Conformal field theories with nonlinearly-extended Virasoro symmetries and Lie algebra classification. Proc. of the 'Meeting on Infinite Dimensional Lie algebras,' C.I.R.M. Luminy, 1988
31. Watts, G.M.T.: Determinant formulae for extended conformal algebras in two dimensional conformal field theory. Nucl. Phys. **B326**, 648–662 (1989); Mizoguchi, S.: Nonunitarity theorem for the A-type W_n algebra. Phys. Lett. **231B**, 112–118 (1989)
32. Nahm, W.: unpublished
33. Blumenhagen, R., Flohr, M., Kliem, A., Nahm, W., Recknagel, A., Varnhagen, R.: W algebras with two and three generators. Nucl. Phys. **B361**, 255–289 (1991)
34. Kausch, H.G., Watts, G.M.T.: A study of W algebras using Jacobi identities. Nucl. Phys. **B354**, 740–768 (1991)
35. Bilal, A.: A remark on the $N \to \infty$ limit of W_N algebras. Phys. Lett. **B227**, 406–410 (1989)
36. Bais, F.A., Bouwknegt, P., Surridge, M., Schoutens, K.: Extensions of the Virasoro algebra constructed from Kac Moody algebras using higher order Casimir invariants. Nucl. Phys. **B304**, 348–371 (1988)

37. Niedermaier, M.R.: Infinite abelian subalgebra of $W(sl(n))$. Nucl. Phys. **364**, 165 (1991)
38. Chari, V., Hangovan, S.: On the Harish–Chandra homomorphism for infinite dimensional Lie algebras. J. Algebra **90**, 476–490 (1984)
39. Kac, V.: Laplace operators of infinite dimensional Lie algebras and theta functions. Proc. Natl. Acad. Sci. USA **81**, 645–647 (1984)
40. Hayashi, T.: Sugawara operators and Kac–Kazdhan Conjecture. Invent. Math. **94**, 13–52 (1988); R. Goodman, N. Wallach,: Higher order Sugawara operators for affine Lie algebras Trans. AMS **315**, 1–56 (1989)
41. Wakimoto, M.: Fock representations of the affine Lie algebra $A_1^{(1)}$. Commun. Math. Phys. **104**, 605–609 (1986)
42. Feigin, B., Frenkel, E.: Bosonic ghost system and the Virasoro algebra. Phys. Lett. **B246**, 71–74 (1990)
43. Dobrev, V.K.: Singular vectors of quantum group representations for straight Lie algebra roots. Preprint Göttingen 1991

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