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Complex Equivariant Intersection, Excess Normal Bundles and Bott-Chern Currents

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This paper is dedicated to Professor P. Malliavin

Abstract. The purpose of this paper is to establish an intersection formula in equivariant complex geometry, in the presence of an excess normal bundle. The contribution of the excess normal bundle to the formula appears through an additive genus ${}^K R$. In a forthcoming paper, an infinite dimensional analogue of this formula will be shown to be the result of Bismut-Lebeau on the behaviour of Quillen metrics under complex immersions.

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Introduction

This paper is the second of a series of three papers, which include [B1] and [B3], which are devoted to the role of excess normal bundles in complex intersection theory.

In [B1], we established a formula relating certain Bott-Chern currents associated to non-transversal complex submanifolds of a complex manifold. With respect to a similar formula which was established in [BGS5], the formula in [B1] contains a correcting term, which explicitly reflects the presence of an excess normal bundle $\tilde{N}$. In [BGS5], one could use the microlocal estimates of [B2] to take full advantage of the transversality of the considered manifolds. In [B1], one uses instead a method formally inspired by the proof by Bismut-Lebeau [BL1, 2] of a formula describing the behaviour of Quillen metrics on the determinant of the cohomology under complex embeddings.

In this paper, we solve a similar problem in complex equivariant intersection theory. In fact let LX be a compact complex Kähler manifold with Kähler form ω^{LX}, let K be a holomorphic Killing vector field on LX, let X be the zero set of K. Then a class of formulas, whose prototype is the Bott residue formula [Bo], relates integrals of certain forms over LX to integrals over X. These formulas have been made transparent in the context of equivariant cohomology by Berline-Vergne [BeV].

In [B5], inspired by our proof of such localization formulas [B4], and also by a loop space formulation of the construction of Quillen metrics, we constructed a K-invariant current ${}^K S_{\omega^{LX}}$ on LX, which solves the equation of currents

$$\frac{\bar{\partial}_K \partial_K}{2i\pi}\, {}^K S_{\omega^{LX}} = 1 - {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, \delta_X \,. \tag{0.1}$$

In (0.1), $\bar{\partial}_K \partial_K$ is a K-equivariant version of the operator $\bar{\partial}\partial$, and ${}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})$ is the equivariant maximal Chern class of the normal bundle $N_{X/LX}$ to X in LX in equivariant Chern-Weil theory. The construction of the current ${}^K S_{\omega^{LX}}$ refines the localization formulas of [Bo] and [BeV].

Let now (LE, g^{LE}) be a complex Hermitian equivariant vector bundle on LX, let σ be an equivariant holomorphic section of LE, which vanishes on a K-invariant complex submanifold LY of LX. By imitating a construction of Bismut-Gillet-Soulé [BGS5], we exhibit an explicit Euler-Green K-invariant current ${}^K\tilde{e}^{LX}(LE, g^{LE})$ on LX such that

$$\frac{\bar{\partial}_K \partial_K}{2i\pi}\, {}^K\tilde{e}^{LX}(LE, g^{LE}) = \delta_{LY} - {}^K c_{\max}(LE, g^{LE}) \,. \tag{0.2}$$

In (0.2), ${}^K c_{\max}(LE, g^{LE})$ is the equivariant maximal Chern class of (LE, g^{LE}) in equivariant Chern-Weil theory.

In general X and LY have a non-empty intersection Y. Let $\tilde{N}$ the corresponding equivariant excess normal bundle. Roughly speaking, we now want to give a formula for the height pairing of the K-invariant cycles X and LY. Still the fact that X and LY have a non-empty intersection and that the intersection is non-transversal makes such a formula highly non-trivial. Our main result, which is contained in Theorems 3.2

and 3.4, is a refinement of the previous described localization formulas in equivariant cohomology. It expresses a combination of integrals of currents over LX and LY in terms of integrals of other currents evaluated over X and Y. The presence of an excess normal bundle $\tilde{N}$ is reflected in the appearance of a mysterious genus ${}^K R(\tilde{N})$ in the final formula.

The formulas considered in Theorems 3.2 and 3.4 are of interest from several points of view. They could be the prototype of formulas in a still non-developed equivariant Arakelov intersection theory.

More surprising to us is the fact that as we will see in our next paper [B3], our main result has a well-defined formal extension in infinite dimensions, which, if properly interpreted, coincides with the main result of Bismut-Lebeau [BL1, 2] which concerns the behaviour of Quillen metrics under complex embeddings. In [B3], $i: Y \to X$ is an embedding of complex manifolds, and LX, LY are the loop spaces of X, Y. The analogy between the present paper and [BL2] is valid not only for the final result, but also for the intermediary steps. In fact, Sect. 3 of this paper has been written by strictly imitating the general organization of [BL2], so that a reader with a limited knowledge of both subjects can immediately perceive the analogy.

In particular, as we shall see in [B3], the infinite dimensional analogue of the mysterious genus ${}^K R$ is exactly the genus R introduced by Gillet and Soulé [GS1] in their conjectural formula of Riemann-Roch-Grothendieck in Arakelov theory. Using the main result of Bismut-Lebeau [BL1, 2], Gillet and Soulé [GS2, 3] have in fact proved the conjectured Riemann-Roch-Grothendieck formula.

This paper is organized as follows. In Sect. 1, we construct a form $\mathbb{B}(E, F, g^F)$ associated to an equivariant exact sequence of holomorphic Hermitian vector bundles

$$0 \to E \underset{i}{\to} F \underset{j}{\to} G \to 0\,, \tag{0.3}$$

and we calculate this form modulo ∂ and $\bar{\partial}$ coboundaries. This way, we produce a genus ${}^K D$ which is closely related to the genus ${}^K R$ of our final formula. In fact, Sect. 1 is the finite dimensional analogue of our paper [B6], where a genus D, closely related to the Gillet-Soulé genus R [GS1] was exhibited. This section relies on finite dimensional results of [B6, Sect. 9], where the analogy with previous infinite dimensional results of [B6] had been partly worked out. The organization of the proofs of the results of Sect. 1 is such as to allow the reader to grasp at least the formal resemblance with [B6].

In Sect. 2, we recall the construction in [B5] of the current ${}^K S_{\omega^{LX}}$. Also by imitating [BGS5], we construct the equivariant Euler-Green current ${}^K \tilde{e}^{LX}(LE, g^{LE})$.

Finally, in Sect. 3, we establish our intersection formula. As explained before, the general organization of this section is closely related to the organization of the paper of Bismut-Lebeau [BL2]. In particular a rectangular contour Γ in R_+^2, which played a crucial role in [BL2], reappears here, as it also did in the preceding paper of the series [B1]. The role of the contour Γ in [B1] and in the present paper is to overcome the presence of the excess normal bundle $\tilde{N}$.

I. Equivariant Short Exact Sequences and Bott-Chern Forms

The purpose of this section is to construct and to calculate certain Bott-Chern forms $\mathbb{B}(E, F, g^F)$ which are naturally associated to a short exact sequence

$$0 \to E \underset{i}{\to} F \underset{j}{\to} G \to 0$$

of holomorphic Hermitian vector bundles. Part of the technical machinery was already developed in [B6, Sect. 9], to which the reader is referred when necessary.

This section is organized as follows. In a), we introduce our main objects, and we describe the forms χ_T, $T > 0$ already introduced in [B6, Sect. 9]. In b), we recall the results of [B6] on the asymptotics of χ_T as $T \to 0$ or $T \to +\infty$. In c), we construct the form $\mathbb{B}(E, F, g^F)$ as the derivative at 0 of the Mellin transform of χ_T. In d), we calculate $\mathbb{B}(E, F, g^F)$ in terms of a Bott-Chern class in the sense of [BGS1], and of an additive genus ${}^K D$ evaluated on G. Finally in e), we introduce a closely related genus ${}^K R$.

This section should be considered as the continuation of [B6, Sect. 9]. Our formula for $\mathbb{B}(E, F, g^F)$ is in fact a finite dimensional analogue of the main result of [B6].

a) Equivariant Exact Sequences and Differential Forms

Let B a connected complex manifold.

Definition 1.1. Let P^B be the vector space of smooth forms which are sums of forms over B of type (p, p). Let $P^{B,0}$ be the set of $\omega \in P^B$ such that there exist smooth forms α, β for which $\omega = \partial\alpha + \bar{\partial}\beta$.

Let

$$0 \to E \underset{i}{\to} F \underset{j}{\to} G \to 0 \tag{1.1}$$

be a holomorphic acyclic complex of vector bundles over B. E will be considered as a holomorphic subbundle of F, and G is identified with F/E.

Let g^F be a Hermitian metric on F. g^F induces a Hermitian metric g^E on E. By identifying G to the orthogonal bundle to E in F, g^F also induces a Hermitian metric g^G on G. Let ∇^E, ∇^F, and ∇^G be the corresponding holomorphic Hermitian connections on E, F, and G, and let R^E, R^F, and R^G be their curvatures.

Let J^F be a holomorphic skew-adjoint section of $\operatorname{End} F$, which preserves E. Let J^E be the restriction of J^F to E and let J^G be the natural action of J^F on G. Then J^E and J^G are also holomorphic skew-adjoint sections of $\operatorname{End} E$ and $\operatorname{End} G$. Moreover J^E, J^F, and J^G are parallel with respect to the connections ∇^E, ∇^F, and ∇^G.

The connection ∇^F defines a natural splitting of TF into

$$TF = F \oplus T^H F, \tag{1.2}$$

where $T^H F$ is the horizontal subbundle of TF. We define $T^H E$ and $T^H G$ in a similar way.

If $z \in F$, we identify z with $Z = z + \bar{z} \in F_{\mathbb{R}}$. In particular $|Z|^2 = 2|z|^2$. Also $J^F Z \in F_{\mathbb{R}}$. Using (1.2), we consider $J^F Z = J^F z + J^F \bar{z}$ as a vertical holomorphic vector field on F. If TB is equipped with a Hermitian metric, we can lift the metric of TB to $T^H F$. We equip $TF = F \oplus T^H F$ with the orthogonal sum of the metrics on F and $T^H F$. Then $J^F Z$ is also a Killing vector field on F. Similarly, if $Z \in E$ or $Z \in G$, $J^E Z$ and $J^G Z$ will be considered as holomorphic Killing vector fields on E and G.

E is a complex submanifold of F. The vector field $J^F Z$ restricts to the vector field $J^E Y$ on E. Also, the projection map $j: F \to G$ maps the vector field $J^F Z$ into $J^G j(Z)$.

If $U \in T_{\mathbb{R}}F$, let $U^V \in F_{\mathbb{R}}$ be the projection of U on $F_{\mathbb{R}}$ with respect to the splitting (1.2) of $T_{\mathbb{R}}F$. Let $(J^F Z)'$ be the 1-form on $F, U \in T_{\mathbb{R}}F \to \langle J^F Z, U^V \rangle$.

Also, we identify J^F with the 2-form

$$U, U' \in T_{\mathbb{R}}F \to \langle U^V, J^F U'^V \rangle . \tag{1.3}$$

Let $i_{J^F Z}$ denote the interior multiplication by $J^F Z$ acting on $\Lambda(T^*_{\mathbb{R}}F)$ and let $L_{J^F Z}$ be the Lie derivative operator associated to the vector field $J^F Z$. Then the operators d, $i_{J^F Z}$ and $L_{J^F Z}$ act on the set of smooths sections of $\Lambda(T^*_{\mathbb{R}}F)$ over F. Moreover

$$L_{J^F Z} = (d + i_{J^F Z})^2 . \tag{1.4}$$

Let π be any of the projections E, F, $G \to B$. If ω is a form on B, we identify ω with the form $\pi^*\omega$ on E, F, or G.

The following result is proved in [B6, Proposition 9.1].

Proposition 1.2. *The following identities hold:*

$$\begin{gathered} -\frac{1}{2}(d + i_{J^F Z})(J^F Z) = -\frac{|J^F Z|^2}{2} + \frac{1}{2}\langle R^F J^F Z, Z\rangle + J^F , \\ L_{J^F Z}(J^F Z)' = 0 . \end{gathered} \tag{1.5}$$

Definition 1.3. Let ${}^K\alpha(F, g^F)$ be the smooth form on F

$${}^K\alpha(F, g^F) = \exp\left\{ -\frac{|J^F Z|^2}{2} + \frac{1}{2}\langle R^F J^F Z, Z\rangle + J^F \right\} . \tag{1.6}$$

The form ${}^K\alpha(F, g^F)$ was introduced in a different context in [B4, Proof of Theorem 1.3].

Theorem 1.4. *The form ${}^K\alpha(F, g^F)$ lies in P^F. Also*

$$(d + i_{J^F Z})\, {}^K\alpha(F, g^F) = 0 . \tag{1.7}$$

Proof. This result has been proved in [B6, Theorem 9.3]. It follows directly from Proposition 1.2. □

Observe that since J^F is a holomorphic skew-adjoint section of End(F), then

$$(\bar\partial + i_{J^F z})^2 = 0 , \qquad (\partial + i_{J^F \bar z})^2 = 0 . \tag{1.8}$$

Using (1.4) and (1.8), we find that

$$L_{J^F Z} = [\bar\partial + i_{J^F z}, \partial + i_{J^F \bar z}] . \tag{1.9}$$

Of course, similar identities hold for $L_{J^E Z}$ and $L_{J^G Z}$.

Definition 1.5. Set

$${}^K R^E = R^E + J^E , \qquad {}^K R^F = R^F + J^F , \qquad {}^K R^G = R^G + J^G . \tag{1.10}$$

Let $\mathbf{J}^G$ be the complex structure of $G_{\mathbb{R}}$.

We now use the formalism of Mathai-Quillen [MQ], which is briefly described in [B1, Sect. 3a)].

Definition 1.6. For $T \geq 0$, let ${}^K a_T(G, g^G)$, ${}^K c_T(G, g^G)$ be the forms on G

$$\begin{aligned}
{}^K a_T(G, g^G) &= \det\left(-\frac{{}^K R^G}{2i\pi}\right) \exp\left\{-T\left(\frac{|Z|^2}{2} + ({}^K R^G)^{-1}\right)\right\}, \\
{}^K c_T(G, g^G) &= \frac{\partial}{\partial b}\left[\det\left(-\frac{{}^K R^G}{2i\pi} - b\right)\right. \\
&\qquad \left.\exp\left\{-T\left(\frac{|Z|^2}{2} + ({}^K R^G + 2\pi b \mathbf{J}^G)^{-1}\right)\right\}\right]_{b=0}.
\end{aligned} \tag{1.11}$$

We recall a result of [B6, Theorem 9.5].

Theorem 1.7. *The forms ${}^K a_T(G, g^G)$, ${}^K c_T(G, g^G)$ lie in P^G. For any $T > 0$, the following identities hold:*

$$\begin{aligned}
(d + i_{J^G Z})^K a_T(G, g^G) &= 0, \\
\frac{\partial}{\partial T}\, {}^K a_T(G, g^G) &= \frac{1}{2i\pi}(\bar{\partial} + i_{J^G z})(\partial + i_{J^G \bar{z}})\, \frac{{}^K c_T}{T}(G, g^G).
\end{aligned} \tag{1.12}$$

Since J^F is a holomorphic skew-adjoint parallel section of $\operatorname{End} F$, one verifies easily that F splits holomorphically and orthogonally into

$$F = \bigoplus_{\lambda \in \Lambda} F^\lambda,$$

where Λ is a finite set of locally constant distinct purely imaginary numbers, and the F^λ's are nonzero holomorphic vector bundles. Moreover for any $\lambda \in \Lambda$, J^F acts on F^λ by multiplication by λ.

The acyclic complex (1.1) splits into a direct sum of holomorphic acyclic complexes

$$0 \to E^\lambda \underset{i}{\to} F^\lambda \underset{j}{\to} G^\lambda \to 0. \tag{1.13}$$

Again J^E and J^G act on E^λ and G^λ by multiplication by λ.

We now make the basic assumption that $0 \notin \Lambda$.

Therefore, for any $T > 0$, the forms

$${}^K\alpha(F, g^F) j^*({}^K a_T(G, g^G)), \qquad {}^K\alpha(F, g^F) j^*({}^K c_T(G, g^G))$$

on F are Gaussian shaped, i.e., exhibit a Gaussian like decay as $|Y| \to +\infty$.

In the sequel, $\int_F$ denotes integration along the fibre of $\pi : F \to B$.

Definition 1.8. For $T > 0$, let $\theta(T)$, $\chi(T)$ be the forms on B,

$$\begin{aligned}
\theta_T &= \int_F {}^K\alpha(F, g^F) j^*({}^K a_T(G, g^G)), \\
\chi_T &= \int_F {}^K\alpha(F, g^F) j^*({}^K c_T(G, g^G)).
\end{aligned} \tag{1.14}$$

The following result is proved in [B6, Theorem 9.7].

Theorem 1.9. *The forms θ_T, χ_T lie in P^B. The forms θ_T are closed and their cohomology class does not depend on T. More precisely, for $T > 0$,*

$$\frac{\partial}{\partial T}\theta_T = \frac{\bar{\partial}\partial}{2i\pi}\frac{\chi_T}{T}. \tag{1.15 }$$

Set

$$\begin{aligned}
{}^K c_{\max}(E, g^E) &= \det\left(-\frac{{}^K R^E}{2i\pi}\right) \\
{}^K c_{\max}^{-1}(E, g^E) &= \det{}^{-1}\left(-\frac{{}^K R^E}{2i\pi}\right), \\
{}^K c'_{\max}(E, g^E) &= \frac{\partial}{\partial b}\left[\det\left(-\frac{{}^K R^E}{2i\pi}+b\right)\right]_{b=0}, \\
({}^K c_{\max}^{-1})'(E, g^E) &= \frac{\partial}{\partial b}\left[\det\left(-\frac{{}^K R^E}{2i\pi}+b\right)\right]^{-1}_{b=0}.
\end{aligned} \tag{1.16}$$

We make the convention that if $E = \{0\}$, then

$${}^K c_{\max}(E, g^E) = 1\,, \qquad {}^K c'_{\max}(E, g^E) = 0\,, \qquad ({}^K c'^{-1}_{\max})(E, g^E) = 0\,. \tag{1.17}$$

Note that since J^E is parallel with respect to the connection ∇^E, the forms in (1.16) are closed. Of course they lie in P^B. Similar forms can be defined which are associated to (F, g^F), (G, g^G).

b) The Asymptotics of the Forms θ_T and χ_T

The following result is proved in [B6, Theorem 9.9].

Theorem 1.10. *As $T \to 0$,*

$$\begin{aligned}
\theta_T &= {}^K c_{\max}^{-1}(F, g^F)\,{}^K c_{\max}(G, g^G) + O(T)\,, \\
\chi_T &= -{}^K c_{\max}^{-1}(F, g^F)\,{}^K c'_{\max}(G, g^G) + O(T)\,.
\end{aligned} \tag{1.18}$$

As $T \to +\infty$,

$$\theta_T = {}^K c_{\max}^{-1}(E, g^E) + O\left(\frac{1}{T}\right), \qquad \chi_T = O\left(\frac{1}{T}\right). \tag{1.19}$$

Set

$$\chi_0 = \lim_{T\to 0}\chi_T\,. \tag{1.20}$$

The form χ_0 is calculated in the right-hand side of (1.18).

c) The Form $\mathbb{B}(E, F, g^F)$

Definition 1.11. For $s \in \mathbb{C}$, $0 < \mathrm{Re}(s) < 1$, set

$$A(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} T^{s-1}\chi_T\,dT\,. \tag{1.21}$$

Using Theorem 1.10, it is clear that $A(s)$ is a meromorphic function of s, which extends to a holomorphic function near $s = 0$.

Definition 1.12. Set

$$\mathbb{B}(E,F,g^F) = \frac{\partial}{\partial s} A(0)\,. \tag{1.22}$$

Then $\mathbb{B}(E,F,g^F)$ is a smooth form on B.

Proposition 1.13. *The following identity holds*

$$\mathbb{B}(E,F,g^F) = \int_0^1 (\chi_T - \chi_0)\,\frac{dT}{T} + \int_1^{+\infty} \chi_T\,\frac{dT}{T} - \Gamma'(1)\chi_0\,. \tag{1.23}$$

Proof. Equation (1.23) follows from Theorem 1.10. □

By [BGS5, Theorem 3.15] or by [B1, Remark 3.7], the current on G

$$-\frac{\partial}{\partial b}\left[\det\left(-\frac{{}^K R^G}{2\pi i} - b\right)\operatorname{Log}\left(\frac{|Z|^2}{2} + ({}^K R^G + 2\pi b\mathbf{J}^G)^{-1}\right)\right]_{b=0} \tag{1.24}$$

is locally integrable.

Theorem 1.14. *The following identity holds*

$$\begin{aligned}\mathbb{B}(E,F,g^F) = \int_F \exp\left\{-\frac{|J^F Z|^2}{2} + \frac{1}{2}\langle R^F J^F Z, Z\rangle + J^F\right\}\\ j^*\Bigg[-\frac{\partial}{\partial b}\Bigg[\det\left(-\frac{{}^K R^G}{2\pi i} - b\right)\\ \operatorname{Log}\left(\frac{|Z|^2}{2} + ({}^K R^G + 2\pi b\mathbf{J}^G)^{-1}\right)\Bigg]_{b=0}\Bigg]\,.\end{aligned} \tag{1.25}$$

Proof. Equation (1.25) follows from (1.11), (1.14), (1.21). □

Remark 1.15. Note that the local integrability of the current (1.24) on G plays a key role in making sense of the right-hand side (1.25).

Theorem 1.16. *The form $\mathbb{B}(E,F,g^F)$ lies in P^B. Moreover the following equation holds*

$$\frac{\bar{\partial}\partial}{2i\pi}\,\mathbb{B}(E,F,g^F) = {}^K c_{\max}^{-1}(E,g^E) - {}^K c_{\max}(G,g^G)\,{}^K c_{\max}^{-1}(F,g^F)\,. \tag{1.26}$$

Proof. Equation (1.26) follows from (1.15), (1.18), (1.19). □

Remark 1.17. Equation (1.26) can also be considered as a consequence of [BGS5, Theorems 3.14 and 3.15], of Proposition 1.2 and of Theorem 1.14.

d) Evaluation of $\mathbb{B}(E,F,g^F)$ in $P^B/P^{B,0}$

If $\lambda \in \Lambda$, let g^{E^λ} be the Hermitian metric induced by g^E on E^λ. Let ∇^{E^λ} be the holomorphic Hermitian connection on $(E^\lambda, g^{E^\lambda})$ and let R^{E^λ} be its curvature. Then

$${}^K c_{\max}^{-1}(E,g^E) = \prod_{\lambda\in\Lambda} \det{}^{-1}\left(\frac{-\lambda - R^{E^\lambda}}{2i\pi}\right)\,. \tag{1.27}$$

Therefore ${}^K c_{\max}^{-1}(E, g^E)$ is obtained by a usual construction in Chern-Weil theory. Of course similar considerations apply to ${}^K c_{\max}^{-1}(F, g^F)$ and ${}^K c_{\max}^{-1}(G, g^G)$.

By the theory of Bott-Chern classes developed in [BGS1, Sect. 1f)], there exists a uniquely defined class $\widetilde{{}^K c_{\max}^{-1}}(E, F, g^F) \in P^B/P^{B,0}$ such that

$$\frac{\bar{\partial}\partial}{2i\pi} \widetilde{{}^K c_{\max}^{-1}}(E, F, g^F) = {}^K c_{\max}^{-1}(F, g^F) - {}^K c_{\max}^{-1}(E, g^E)\, {}^K c_{\max}^{-1}(G, g^G)\,. \tag{1.28}$$

The class $\widetilde{{}^K c_{\max}^{-1}}(E, F, g^F)$ is normalized by the two conditions:

- It is functorial with respect to pull-backs.
- $\widetilde{{}^K c_{\max}^{-1}}(E, F, g^F)$ vanishes when the equivariant exact sequence (1.1) splits holomorphically and metrically. This exactly means that for every $\lambda \in \Lambda$, we have an identification of holomorphic Hermitian vector bundles $F^\lambda = E^\lambda \oplus G^\lambda$, and i and j are the obvious injection and projection maps.

Definition 1.18. Let ${}^K D(G, g^G)$ be the smooth form on B

$${}^K D(G, g^G) = \mathrm{Tr}\left[-\left(\frac{J^G + R^G}{2i\pi}\right)^{-1} (\Gamma'(1) - \mathrm{Log}(-J^{G^2} - J^G R^G))\right]. \tag{1.29}$$

We can write ${}^K D(G, g^G)$ in the form

$${}^K D(G, g^G) = \sum_{\lambda \in \Lambda} \mathrm{Tr}\left[-\left(\frac{\lambda + R^{G^\lambda}}{2i\pi}\right)^{-1} (\Gamma'(1) - \mathrm{Log}(-\lambda^2 - \lambda R^{G^\lambda}))\right]. \tag{1.30}$$

It is clear that ${}^K D(G, g^G)$ is a closed form which lies in P^B. Also if g^G varies in the class of metrics on G such that the G^λ's remain mutually orthogonal in G, the class ${}^K D(G)$ of ${}^K D(G, g^G)$ in $P^B/P^{B,0}$ does not depend on g^G.

Similarly, we denote by ${}^K c_{\max}(E)$, ${}^K c_{\max}^{-1}(E)$ the classes of ${}^K c_{\max}(E, g^E)$, ${}^K c_{\max}^{-1}(E, g^E)$ in $P^B/P^{B,0}$. These classes do not depend on the metric g^E as g^E varies in the class of metrics preserving the mutual orthogonality of the E^λ's.

Theorem 1.19. *The following identity holds*

$$\begin{aligned} \mathbb{B}(E, F, g^F) = {} & -{}^K c_{\max}(G, g^G)\, \widetilde{{}^K c_{\max}^{-1}}(E, F, g^F) \\ & + {}^K c_{\max}^{-1}(E)\, {}^K D(G) \quad \text{in} \quad P^B/P^{B,0}\,. \end{aligned} \tag{1.31}$$

Proof. We proceed exactly as in [B6, Proof of Theorem 8.5].

Let $\mathbb{P}^1$ be the one-dimensional complex projective plane equipped with two distinguished points $\{0\}$ and $\{\infty\}$ and with the meromorphic coordinate z. By [BGS1, Theorem 1.29] or by the Grassmann graph construction of Baum-Fulton-MacPherson [BaFMa] which is explained in detail in [BGS5, Sect. 4], we can construct over $B \times \mathbb{P}^1$ an acyclic complex of holomorphic vector bundles

$$0 \to E' \underset{i'}{\to} F' \underset{j'}{\to} G' \to 0\,, \tag{1.32}$$

which is a direct sum in $\lambda \in \Lambda$ of the holomorphic complexes

$$0 \to E'^\lambda \underset{i'}{\to} F'^\lambda \underset{j'}{\to} G'^\lambda \to 0 \tag{1.33}$$

and which has the following two properties:

• The restriction of the complex (1.32) to $B \times \{0\}$ coincides with the complex (1.1).
• On $B\times\{\infty\}$, the complex (1.32) splits, i.e. for each $\lambda \in \Lambda$, we have an identification of holomorphic vector bundles

$$F'^{\lambda}_{|B\times\{\infty\}} = E'^{\lambda}_{|B\times\{\infty\}} \oplus G'^{\lambda}_{|B\times\{\infty\}}, \tag{1.34}$$

and i' and j' are the obvious injection and projection maps.

Let $g^{F'}$ be a Hermitian metric on F', which is such that the F'^{λ}'s are mutually orthogonal in F', and which has the following two properties:
• The restriction of $g^{F'}$ to $B \times \{0\}$ coincides with g^F.
• On $B \times \{\infty\}$, for any $\lambda \in \Lambda$, the splitting (1.34) is orthogonal.

As in Sect. 1a), from the metric $g^{F'}$, we construct metrics $g^{E'}$, $g^{G'}$ on E', G'. Clearly, on $B \times \{0\}$, $g^{E'}$, $g^{G'}$ coincide with g^E, g^G. Also for $T \geq 0$, let χ'_T be the associated form (1.14) on $B \times \mathbb{P}^1$ associated to the complex (1.32).

Over $\mathbb{P}^1$, we have the equation of currents

$$\frac{\bar\partial\partial}{2i\pi}(\operatorname{Log}|z|^2) = \delta_{\{0\}} - \delta_{\{\infty\}}. \tag{1.35}$$

Set

$$\beta(E',F',g^{F'}) = \mathbb{B}(E',F',g^{F'}) + {}^K c_{\max}(G',g^{G'})\widetilde{{}^K c^{-1}_{\max}}(E',F',g^{F'}). \tag{1.36}$$

By (1.26), (1.28), it is clear that

$$\frac{\bar\partial\partial}{2i\pi}\beta(E',F',g^{F'}) = 0. \tag{1.37}$$

Also

$$\begin{aligned}\frac{\bar\partial\partial}{2i\pi}&(\operatorname{Log}|z|^2)\beta(E',F',g^{F'}) - (\operatorname{Log}|z|^2)\frac{\bar\partial\partial}{2i\pi}\beta(E',F',g^{F'})\\ &= \frac{\bar\partial}{2i\pi}(\partial(\operatorname{Log}|z|^2)\beta(E',F',g^{F'})) + \frac{\partial}{2i\pi}((\operatorname{Log}|z|^2)\bar\partial\beta(E',F',g^{F'})).\end{aligned} \tag{1.38}$$

If κ is a smooth form on $B \times \mathbb{P}^1$, let κ_0, κ_∞ be the restrictions of κ to $B \times \{0\}$, $B \times \{\infty\}$ respectively. We will consider κ_0, κ_∞ as forms on B.

Using (1.38) and integrating along the fibre of the projection map $B \times \mathbb{P}^1 \to B$, we get

$$\beta(E',F',g^{F'})_0 - \beta(E',F',g^{F'})_\infty \in P^{B,0}. \tag{1.39}$$

Clearly

$$\begin{aligned}\beta(E',F',g^{F'})_0 &= \mathbb{B}(E,F,g^F)\\ &\quad + {}^K c_{\max}(G,g^G)\,\widetilde{{}^K c^{-1}_{\max}}(E,F,g^F) \quad \text{in} \quad P^B/P^{B,0}.\end{aligned} \tag{1.40}$$

Also by construction

$$\widetilde{{}^K c^{-1}_{\max}}(E',F',g^{F'})_\infty = 0 \quad \text{in} \quad P^B/P^{B,0}. \tag{1.41}$$

We now calculate $\mathbb{B}(E',F',g^{F'})_\infty$. In the sequel, all our constructions will be done on $B \times \{\infty\}$.

Since the complex (1.32) splits on $B \times \{\infty\}$, we see that if $Z \in F'_{|B\times\{\infty\}}$, $Z = Y + Y'$, $Y \in E'_{|B\times\{\infty\}}$, $Y' \in G'_{|B\times\{\infty\}}$, then over $B \times \{\infty\}$,

$$
\begin{aligned}
{}^K\alpha(F', g^{F'}) = \exp&\left\{ -\frac{|J^{E'}Y|^2}{2} + \frac{1}{2}\langle R^{E'}J^{E'}Y, Y\rangle + J^{E'} \right\} \\
&\exp\left\{ -\frac{|J^{G'}Y'|^2}{2} + \frac{1}{2}\langle R^{G'}J^{G'}Y', Y'\rangle + J^{G'} \right\}. \qquad (1.42)
\end{aligned}
$$

So, over $B \times \{\infty\}$, for $T > 0$, we get

$$
\begin{aligned}
\chi'_{T\infty} = &\int_{E'} \exp\left\{ -\frac{|J^{E'}Y|^2}{2} + \frac{1}{2}\langle R^{E'}J^{E'}Y, Y\rangle + J^{E'} \right\} \\
&\frac{\partial}{\partial b}\left[\int_{G'} \det\left(\frac{-R^{G'}}{2i\pi} - b \right) \exp\left\{ -\frac{|J^{G'}Y'|^2}{2} + \frac{1}{2}\langle R^{G'}J^{G'}Y', Y'\rangle \right.\right. \\
&\left.\left. -\frac{T|Y'|^2}{2} + J^{G'} - T(J^{G'} + R^{G'} + 2\pi b\mathbf{J}^{G'})^{-1} \right\} \right]_{b=0}. \qquad (1.43)
\end{aligned}
$$

By an easy calculation (which is done in [B4, Eqs. (1.21)–(1.23)]), we see that

$$
\int_{E'} \exp\left\{ -\frac{|J^{E'}Y|^2}{2} + \frac{1}{2}\langle J^{E'}R^{E'}Y, Y\rangle + J^{E'} \right\} = {}^K c_{\max}^{-1}(E', g^{E'}). \qquad (1.44)
$$

Also one immediately verifies that

$$
\begin{aligned}
&\int_{G'} \det\left(-\frac{{}^K R^{G'}}{2i\pi} - b \right) \exp\left\{ -\frac{|J^{G'}Y'|^2}{2} + \frac{1}{2}\langle J^{G'}R^{G'}Y', Y'\rangle \right. \\
&\qquad \left. -\frac{T|Y'|^2}{2} + J^{G'} - T(J^{G'} + R^{G'} + 2\pi b\mathbf{J}^{G'})^{-1} \right\} \\
&= (2\pi)^{\dim G'} \det\left(-\frac{J^{G'} + R^{G'}}{2i\pi} - b \right) \frac{\det\left(\dfrac{-J^{G'} + T(J^{G'} + R^{G'} + 2\pi bi)^{-1}}{i} \right)}{\det(-J^{G'2} + T - R^{G'}J^{G'})} \\
&= \frac{\det(-J^{G'2} + T - J^{G'}(R^{G'} + 2\pi bi))}{\det(-J^{G'2} + T - J^{G'}R^{G'})}. \qquad (1.45)
\end{aligned}
$$

From (1.45), we get

$$
\begin{aligned}
&\frac{\partial}{\partial b}\left[\int_{G'} \det\left(-\frac{{}^K R^{G'}}{2i\pi} - b \right) \exp\left\{ -\frac{|J^{G'}Y'|^2}{2} + \frac{1}{2}\langle R^{G'}J^{G'}Y', Y'\rangle \right.\right. \\
&\qquad \left.\left. -\frac{T|Y'|^2}{2} + J^{G'} - T(J^{G'} + R^{G'} + 2\pi b\mathbf{J}^{G'})^{-1} \right\} \right]_{b=0} \\
&\quad = -2\pi \operatorname{Tr}[iJ^{G'}(-J^{G'2} + T - J^{G'}R^{G'})^{-1}] \qquad (1.46)
\end{aligned}
$$

Now by an elementary calculation which is made in [B6, Appendix, Eq. (7)], we know that for $s \in \mathbb{C}$, $0 < \mathrm{Re}(s) < 1$, then

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} T^{s-1} \left[\frac{iJ^{G'}}{-J^{G'2} + T - J^{G'} R^{G'}} \right] dT$$
$$= \Gamma(1-s) \operatorname{Tr}[iJ^{G'}(-J^{G'2} - J^{G'} R^{G'})^{s-1}] . \tag{1.47}$$

From (1.43)–(1.47), we deduce that over $B \times \{\infty\}$, then

$$A(s)_\infty = (-2\pi^K c_{\max}^{-1}(E', g^{E'}) \Gamma(1-s) \operatorname{Tr}[iJ^{G'}(-J^{G'2} - J^{G'} R^{G'})^{s-1}])_\infty . \tag{1.48}$$

Using (1.48), we see that

$$\mathbb{B}(E', F', g^{F'})_\infty = {}^K c_{\max}^{-1}(E', g^{E'})_\infty \operatorname{Tr} \left[- \left(\frac{J^{G'} + R^{G'}}{2i\pi} \right)^{-1} \right.$$
$$\left. (\Gamma'(1) - \operatorname{Log}(-J^{G'2} - J^{G'} R^{G'})) \right]_\infty . \tag{1.49}$$

Equivalently,

$$\mathbb{B}(E', F', g^{F'})_\infty = {}^K c_{\max}^{-1}(E', g^{E'})_\infty {}^K D(G', g^{G'})_\infty . \tag{1.50}$$

Clearly

$$\frac{\bar{\partial}\partial}{2i\pi} (\operatorname{Log} |z|^2) {}^K c_{\max}^{-1}(E', g^{E'}) {}^K D(G', g^{G'})$$
$$= \frac{\bar{\partial}}{2i\pi} [\partial(\operatorname{Log} |z|^2) {}^K c_{\max}^{-1}(E', g^{E'}) {}^K D(G', g^{G'})] . \tag{1.51}$$

Using (1.35), (1.51) and integrating along the fibre of the projection map $B \times \mathbb{P}^1 \to B$, we get

$${}^K c_{\max}^{-1}(E, g^E) {}^K D(G, g^G) - {}^K c_{\max}^{-1}(E', g^{E'})_\infty {}^K D(G', g^{G'})_\infty \in P^{B,0} . \tag{1.52}$$

From (1.39)–(1.41), (1.50), (1.52), we get (1.31). □

e) The Genus ${}^K R$

Definition 1.20. Let ${}^K R(G, g^G) \in P^B$ be given by

$${}^K R(G, g^G) = \operatorname{Tr} \left[- \left(\frac{J^G + R^G}{2i\pi} \right)^{-1} (2\Gamma'(1) - \operatorname{Log}(-(J^G)^2 - J^G R^G)) \right] . \tag{1.53}$$

The considerations we made for ${}^K D(G, g^G)$ also apply to ${}^K R(G, g^G)$. In particular ${}^K R(G, g^G)$ is a closed form. We denote by ${}^K R(G)$ the class of ${}^K R(G, g^G)$ in $P^B / P^{B,0}$. Similarly $\dfrac{{}^K c'_{\max}(G)}{{}^K c_{\max}(G)}$ denotes the class of $\dfrac{{}^K c'_{\max}(G, g^G)}{{}^K c_{\max}(G, g^G)}$ in $P^B / P^{B,0}$.

Proposition 1.21. *The following identity holds*

$${}^K R(G) = {}^K D(G) + \Gamma'(1) \frac{{}^K c'_{\max}(G)}{{}^K c_{\max}(G)} . \tag{1.54}$$

Proof. Clearly,

$$\frac{{}^K c'_{\max}(G, g^G)}{{}^K c_{\max}(G, g^G)} = \mathrm{Tr}\left[- \left(\frac{J^G + R^G}{2i\pi} \right)^{-1} \right]. \tag{1.55}$$

Equation (1.54) follows from (1.29), (1.53), (1.55). □

Definition 1.22. If $\mu \in \mathbb{R}^*$, $x \in \mathbb{C}$, set

$$R^\mu(x) = (\mu + x)^{-1} \left(2\Gamma'(1) - 2\,\mathrm{Log}\,|2\pi\mu| - \mathrm{Log}\left(1 + \frac{x}{\mu}\right)\right). \tag{1.56}$$

In the sequel, we make $\sum_{k=1}^{j} \frac{1}{k} = 0$ if $j = 0$.

Proposition 1.23. *For $x \in \mathbb{C}$, $|x| \leq \mu$, the following identity holds*

$$R^\mu(x) = \frac{1}{\mu} \sum_{j=0}^{+\infty} \left(2\Gamma'(1) - 2\,\mathrm{Log}\,|2\pi\mu| + \sum_{k=1}^{j} \frac{1}{k} \right) \left(\frac{-x}{\mu} \right)^j. \tag{1.57}$$

Proof. Equation (1.57) follows from a trivial calculation which is left to the reader. □

We identify R^μ with the corresponding additive genus.

Let $M \subset \mathbb{R}^*$ be the spectrum of $\frac{-J^G}{2i\pi}$.

Proposition 1.24. *The following identity holds*

$${}^K R(G) = \sum_{\mu \in M} R^\mu(G^{-2i\pi\mu}). \tag{1.58}$$

Proof. Equation (1.58) follows from (1.53), (1.56). □

II. Equivariant Bott-Chern Currents

The purpose of this section is to recall the construction in [B5] of the equivariant Bott-Chern current ${}^K S_{\omega^{LX}}$, and also to construct the equivariant Euler-Green current ${}^K \tilde{e}^{LX}(LE, g^{LE})$ by extending results of [BGS5].

This section is organized as follows. In a), we recall results of [B4, 5]. In b), we construct the current ${}^K S_{\omega^{LX}}$ on LX. In c), we introduce the equivariant holomorphic Hermitian vector bundle (LE, g^{LE}). In d), using the formalism of Mathai-Quillen [MQ], we introduce equivariant Thom forms and we prove equivariant double transgression formulas. In e), we construct an equivariant Euler-Green current on a K-invariant complex submanifold Y', which is denoted ${}^K \tilde{e}^{Y'}(LE, g^{LE})$. Finally in f), we establish some properties of the current ${}^K \tilde{e}^{LX}(LE, g^{LE})$.

a) Holomorphic Killing Vector Fields and Localization

Let LX be a compact complex manifold. Let $\mathbf{J}^{TLX}$ be the complex structure on $T_{\mathbb{R}}LX$. Let g^{TLX} be a Hermitian metric on TX. Let ω^{LX} be the Kähler form of LX, i.e. if $U, V \in T_{\mathbb{R}}LX$, set

$$\omega^{LX}(U, V) = \langle U, \mathbf{J}^{TLX} V \rangle. \tag{2.1}$$

Then ω^{LX} is a $(1,1)$ form on LX.

In the sequel, we assume that the manifold (LX, g^{TLX}) is Kähler, or equivalently that the form ω^{LX} is closed.

Let ∇^{TLX} be the holomorphic Hermitian connection on (TLX, g^{TLX}). The connection ∇^{TLX} induces the Levi-Civita connection on $T_{\mathbb{R}}LX$. Let R^{TLX} be the curvature of ∇^{TLX}.

Let K be a holomorphic Killing vector field on LX. Then the form ω^{LX} is K-invariant. Set

$$X = \{u \in LX; K(u) = 0\}. \tag{2.2}$$

Then X is a complex totally geodesic submanifold of LX. Let f be the embedding $X \to LX$. Set

$$\omega^X = f^* \omega^{LX}. \tag{2.3}$$

The form ω^X is the Kähler form of the metric g^{TX} induced by g^{TLX} on TX, and ω^X is closed, i.e. the manifold (X, g^{TX}) is also Kähler.

Let $N_{X/LX}$ be the normal bundle to X in LX. We identify $N_{X/LX}$ to the orthogonal bundle to TX in $TLX_{|X}$. Then $TLX_{|X}$ splits holomorphically into

$$TLX_{|X} = TX \oplus N_{X/LX}. \tag{2.4}$$

Also TX and $N_{X/LX}$ are orthogonal subbundles of $TLX_{|X}$. Let $g^{N_{X/LX}}$ be the metric induced by the metric $g^{TLX_{|X}}$ on $N_{X/LX}$.

Let ∇^{TX}, $\nabla^{N_{X/LX}}$ be the holomorphic Hermitian connections on (TX, g^{TX}), $(N_{X/LX}, g^{N_{X/LX}})$. Then with respect to the splitting (2.4) of $TLX_{|X}$, we know that $\nabla^{TLX_{|X}} = \nabla^{TX} \oplus \nabla^{N_{X/LX}}$. Let R^{TX}, $R^{N_{X/LX}}$ be the curvatures of ∇^{TX}, $\nabla^{N_{X/LX}}$.

If $U \in N_{X/LX}$, set

$$J^{N_{X/LX}} U = \frac{\partial X}{\partial u}(U). \tag{2.5}$$

Then $J^{N_{X/LX}}$ is a skew-adjoint invertible endomorphism of $N_{X/LX}$, which is parallel with respect to the connection $\nabla^{N_{X/LX}}$. In particular over each connected component of X, the eigenvalues of $J^{N_{X/LX}}$ are nonzero and are constant. The vector bundle $N_{X/LX}$ then splits holomorphically and metrically as the direct sum of the various eigenbundles of $J^{N_{X/LX}}$. We will use freely the notation of Sect. 1 with respect to the couple $(N_{X/LX}, J^{N_{X/LX}})$.

Let $K^{(1,0)}$, $K^{(0,1)}$ be the components of K in $T^{(1,0)}X$, $T^{(0,1)}X$ respectively, so that $K = K^{(1,0)} + K^{(0,1)}$. Set

$$\partial_K = \partial + i_{K^{(0,1)}}, \qquad \bar{\partial}_K = \bar{\partial} + i_{K^{(1,0)}}. \tag{2.6}$$

Since K is holomorphic, then

$$\partial_K^2 = 0, \qquad \bar{\partial}_K^2 = 0. \tag{2.7}$$

Let L_K be the Lie derivative operator with respect to K. Then

$$L_K = (d + i_K)^2. \tag{2.8}$$

From (2.7), (2.8), we deduce that

$$L_K = \bar{\partial}_K \partial_K + \partial_K \bar{\partial}_K. \tag{2.9}$$

Definition 2.1. Let ${}^K P^{LX}$ be the set of smooth forms on LX which are K-invariant and are sums of forms ot type (p,p). Let ${}^K P^{LX,0}$ be the set of smooth forms $\alpha \in {}^K P^{LX}$ such that there exist K-invariant forms β, γ on LX for which $\alpha = \partial_K \beta + \bar{\partial}_K \gamma$.

Clearly $\omega^{LX} \in {}^K P^{LX}$. Observe that by (2.9), if α is a K-invariant form, then

$$\bar{\partial}_K \partial_K \alpha = -\partial_K \bar{\partial}_K \alpha . \tag{2.10}$$

Therefore, when acting on K-invariant forms, the operators ∂_K and $\bar{\partial}_K$ anticommute as the usual ∂ and $\bar{\partial}$ operators.

Let K' be the 1-form on LX which corresponds to K by the metric g^{TLX}. Clearly

$$L_K K' = 0 . \tag{2.11}$$

Since ω^{LX} is a closed form, $\partial \omega^{LX} = 0$, $\bar{\partial} \omega^{LX} = 0$. By [B5, Eq. (14)], we get

$$\bar{\partial}_K \partial_K \sqrt{-1}\, \omega^{LX} = \frac{-(d + i_K) K'}{2} . \tag{2.12}$$

Definition 2.2. For $t > 0$, set

$$\begin{aligned} {}^K\alpha_t &= \exp\left(\frac{\bar{\partial}_K \partial_K \sqrt{-1}\, \omega^{LX}}{t}\right), \\ {}^K\gamma_t &= \frac{2\pi \omega^{LX}}{t} \exp\left(\frac{\bar{\partial}_K \partial_K \sqrt{-1}\, \omega^{LX}}{t}\right). \end{aligned} \tag{2.13}$$

Theorem 2.3. *For any $t > 0$, the forms ${}^K\alpha_t$ and ${}^K\gamma_t$ lie in ${}^K P^{LX}$. Moreover the following identities hold*

$$\partial_K {}^K\alpha_t = 0 , \qquad \bar{\partial}_K {}^K\alpha_t = 0 , \qquad \frac{\partial}{\partial t}\, {}^K\alpha_t = \frac{1}{t} \frac{\bar{\partial}_K \partial_K}{2i\pi}\, {}^K\gamma_t . \tag{2.14}$$

Proof. These results were already proved in [B5, Proposition 5]. In fact using (2.7), (2.10), we obtain the first identities in (2.14). Also

$$\begin{aligned} \frac{\partial}{\partial t}\, {}^K\alpha_t &= -\frac{1}{t^2}\, \bar{\partial}_K \partial_K (\sqrt{-1}\, \omega^{LX})\, {}^K\alpha_t \\ &= -\frac{1}{t^2}\, \bar{\partial}_K \partial_K [\sqrt{-1}\, \omega^{LX}\, {}^K\alpha_t] . \end{aligned} \tag{2.15}$$

The third identity in (2.14) follows. □

Let $\mathscr{D}'(LX)$ be the set of currents on LX. Let $\mathscr{D}_{N^*_{X/LX,\mathbb{R}}}(LX)$ be the set of currents on LX whose wave front set is included in $N^*_{X/LX,\mathbb{R}}$.

Then by [H, p. 262], $\mathscr{D}'_{N^*_{X/LX,\mathbb{R}}}(LX)$ can be naturally equipped with a family of semi-norms. In fact, let V be an open set in LX which is holomorphically equivalent to an open ball in $\mathbb{C}^{\dim LX} \cong \mathbb{R}^{2\dim LX}$. Over V, we identify $T^*_{\mathbb{R}} LX$ with $V \times \mathbb{R}^{2\dim LX}$. Let Γ be a closed cone in $\mathbb{R}^{2\dim LX}$ such that on $V \cap X$, $\Gamma \cap N^*_{X/LX,\mathbb{R}} = \{0\}$. Let φ be a smooth current with support in V, and let m be an integer. Let $\wedge$ denote Fourier transform.

If $\alpha \in \mathscr{D}'_{N^*_{X/LX,\mathbb{R}}}(LX)$, set

$$p_{V,\Gamma,\varphi,m}(\alpha) = \sup_{\xi \in \Gamma} |\xi|^m \, |\widehat{\varphi\alpha}(\xi)| . \tag{2.16}$$

Take $\alpha \in \mathscr{D}'_{N^*_{X/LX,\mathbb{R}}}(LX)$. We will say that a sequence of currents $\alpha_n \in \mathscr{D}'_{N^*_{X/LX,\mathbb{R}}}(LX)$ converges to α in $\mathscr{D}'_{N^*_{X/LX,\mathbb{R}}}(LX)$ if:

- α_n converges to α in $\mathscr{D}'(LX)$.

• If V, Γ, φ, m are taken as in (2.16), then $p_{V,\Gamma,\varphi,m}(\alpha_n - \alpha) \to 0$.

Definition 2.4. Let ${}^K P_X^{LX}$ be the set of K-invariant currents on LX which are sums of currents of type (p,p), whose wave front set is included in $N^*_{X/LX,\mathbb{R}}$. Let ${}^K P_X^{LX,0}$ be the set of currents $a \in {}^K P_X^{LX}$ such that there exist K-invariant currents b, c, whose wave front sets are included in $N^*_{X/LX,\mathbb{R}}$, and for which $a = \partial_K b + \bar{\partial}_K c$.

We equip ${}^K P_X^{LX}$, ${}^K P_X^{LX,0}$ with the topology induced by $\mathscr{D}'_{N^*_{X/LX,\mathbb{R}}}(LX)$.

Let $\| \|_{C^1(LX)}$ be a norm on the Banach space of forms μ on LX which are continuous with continuous first derivative.

Theorem 2.5. *There exists a constant $C > 0$ such that for any $t \in]0,1]$, for any smooth differential form μ on LX, then*

$$\left| \int_{LX} \mu \{ {}^K\alpha_t - {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \delta_X \} \right| \leq C\sqrt{t} \|\mu\|_{C^1(LX)} . \tag{2.17}$$

If V, Γ, φ, m are taken as in (2.16), *there exists $C' > 0$ such that for any $t \in]0,1]$,*

$$p_{V,\Gamma,\varphi,m}(\alpha_t - {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \delta_X) \leq C' \sqrt{t} . \tag{2.18}$$

Proof. By (2.12), we know that

$$\int_{LX} \mu \exp\left(\frac{\bar{\partial}_K \partial_K \sqrt{-1}\, \omega^{LX}}{t} \right) = \int_{LX} \mu \exp\left(- \frac{(d + i_K) K'}{2t} \right) . \tag{2.19}$$

By [B4, second proof of Theorem 1.3] and [B5, Theorem 2], we know that as $t \to 0$,

$$\int_{LX} \mu \exp\left(\frac{\bar{\partial}_K \partial_K \sqrt{-1} \omega^{LX}}{t} \right) \to \int_X \frac{\mu}{\det\left(- \frac{J^{N_{X/LX}} + R^{N_{X/LX}}}{2i\pi} \right)} . \tag{2.20}$$

More precisely, the techniques of [B4] easily show that (2.17) holds. The proof of [B4, Theorem 1.3] is closely related to the proof of [B2, Theorem 3.2]. By proceeding as in [B2, Eqs. (3.121)–(3.127)], we obtain (2.18). □

Remark 2.6. From the proof of (2.17), one also easily finds that there exist currents $\theta_1, \ldots, \theta_k, \ldots$ on LX, which lie in ${}^K P_X^{LX}$ and are concentrated on X, such that as $t \to 0$, for any $k \in \mathbb{N}$,

$$\begin{aligned} \int_{LX} \mu \exp\left(\frac{\bar{\partial}_K \partial_K \sqrt{-1}\, \omega^{LX}}{t} \right) &= \int_X \frac{\mu}{\det\left(- \frac{J^{N_{X/LX}} + R^{N_{X/LX}}}{2i\pi} \right)} \\ &\quad + \sum_{j=1}^{k} \int_{LX} \mu \theta_j t^j + o(t^k) . \end{aligned} \tag{2.21}$$

The main point in (2.21) is that only integral (and no half-integrals) powers of t appear, essentially because the integral of an odd polynomial on $\mathbb{R}^n$ with respect to a Gaussian distribution vanishes.

b) An Equivariant Bott-Chern Current

We now recall the construction in [B5] of an equivariant Bott-Chern current on LX associated to the embedding $f: X \to LX$.

Theorem 2.7. *For any smooth form μ on LX, for any $k \in \mathbb{N}$, as $t \to 0$,*

$$\int_{LX} \mu^K \gamma_t = \int_X \mu 2\pi\omega^{LX} \, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \frac{1}{t} + \sum_{j=0}^{k} \int_{LX} \mu 2\pi\omega^{LX} \theta_{j+1} t^j + o(t^k). \tag{2.22}$$

In particular

$$2\pi\omega^{LX}\theta_1 = ({}^K c_{\max}^{-1})'(N_{X/LX}, g^{N_{X/LX}})\delta_X \quad \text{in} \quad {}^K P_X^{LX} / {}^K P_X^{LX,0}. \tag{2.23}$$

Proof. Equation (2.22) follows from (2.21), (2.23) was proved in [B5, Eqs. (40)–(49)]. □

Remark 2.8. From (2.23), we find that

$$\bar{\partial}_K \partial_K (2\pi\omega^{LX}\theta_1) = 0. \tag{2.24}$$

Observe that (2.24) also follows from (2.14) and (2.22).

Let μ be a smooth differential form on LX. By (2.22), the function of $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$,

$$F_\mu^1(s) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left\{ \int_{LX} \mu^K \gamma_t^{LX} \right\} dt \tag{2.25}$$

extends to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic near $s = 0$. Also for $s \in \mathbb{C}$, $\mathrm{Re}(s) < 1$, the function

$$F_\mu^2(s) = \frac{1}{\Gamma(s)} \int_1^{+\infty} t^{s-1} \left\{ \int_{LX} \mu^K \gamma_t^{LX} \right\} dt \tag{2.26}$$

is holomorphic.

Therefore the function $F_\mu^1(s) + F_\mu^2(s)$ is holomorphic near $s = 0$.

Definition 2.9. Let ${}^K S_{\omega^{LX}}$ be the current on LX such that if μ is a smooth differential form on LX, then

$$\int_{LX} \mu^K S_{\omega^{LX}} = \frac{\partial}{\partial s}(F_\mu^1 + F_\mu^2)(0). \tag{2.27}$$

Remark 2.10. The current ${}^K S_{\omega^{LX}}$ is exactly the current $2\pi i \zeta'_{\omega^{LX}/2}(0)$ which was constructed in [B5, Sect. C].

Proposition 2.11. *For any smooth form μ on LX, the following identity holds*

$$\begin{aligned}
&\int_{LX} \mu^K S_{\omega^{LX}} \\
&= \int_0^1 \left\{ \int_{LX} \mu({}^K\gamma_t - 2\pi\omega^{LX\,K} c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\,\delta_X \frac{1}{t} - 2\pi\omega^{LX}\theta_1) \right\} \frac{dt}{t} \\
&+ \int_1^{+\infty} \left\{ \int_{LX} \mu^K\gamma_t \right\} \frac{dt}{t} \\
&- \int_X \mu 2\pi\omega^{X\,K} c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \\
&- \Gamma'(1) \int_{LX} \mu 2\pi\omega^{LX}\theta_1 .
\end{aligned} \tag{2.28}$$

Proof. Using (2.22), we immediately obtain (2.28) □

We now state a result which was proved in [B5, Theorem 6].

Theorem 2.12. *The current ${}^K S_{\omega^{LX}}$ lies in ${}^K P_X^{LX}$. Also the following equation of currents on LX holds*

$$\frac{\bar{\partial}_K \partial_K}{2i\pi}\, {}^K S_{\omega^{LX}} = 1 - {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\,\delta_X . \tag{2.29}$$

Proof. Our theorem follows from Theorems 2.3 and 2.5, from (2.22), (2.24) and from Proposition 2.11. □

c) An Equivariant Vector Bundle on LX

Let (LE, g^{LE}) be a holomorphic Hermitian vector bundle on LX. Let ∇^{LE} be the holomorphic Hermitian connection on (LE, g^{LE}) and let $R^{LE} = (\nabla^{LE})^2$ be its curvature.

Let Q^{LE} be the $GL(\dim LE)$ bundle of frames in LE. Let α be the connection form on Q^{LE} associated to the connection ∇^{LE}.

We assume that the vector field K lifts to a $GL(\dim LE)$-invariant vector field K^{LE} on Q^{LE}, which is holomorphic, and preserves the metric g^{LE}. Then $\alpha(K^{LE})$ is the equivariant representation of a smooth skew-adjoint section of End(LE), which we denote J^{LE}.

Let σ be a holomorphic K^{LE}-invariant section of LE. Set

$$LY = \{u \in LX; \sigma(u) = 0\} . \tag{2.30}$$

We assume that if $u \in LY$, the rank of $d\sigma(u): TLX_u \to LE_u$ is equal to $\dim LE_u$.

Then LY is a compact K-invariant complex submanifold of LX. Let i be the embedding $LY \to LX$. Set

$$\omega^{LY} = i^* \omega^{LX} . \tag{2.31}$$

Then ω^{LY} is the Kähler form of the K-invariant metric g^{TLY} induced by g^{TLX} on TLY.

Let $N_{LY/LX}$ be the normal bundle to LY. By identifying $N_{LY/LX}$ with the orthogonal bundle $TLY^{\perp}$ to TLY in $TLX_{|LY}$ with respect to the metric $g^{TLX_{|LY}}$, we thus equip $N_{LY/LX}$ with a Hermitian metric $g^{N_{LY/LX}}$.

Also $d\sigma : N_{LY/LX} \to LE_{|LY}$ is an identification of holomorphic vector bundles. We assume that $d\sigma$ also identifies the metrics $g^{N_{LY/LX}}$ and $g^{LE_{|LY}}$. Observe that given the metric g^{TLX} on TLX, one can always find a K^{LE}-invariant metric g^{LE} on LE such that this is the case.

Let P^{TLY}, $P^{N_{LY/LX}}$ be the orthogonal projection operators from $TLX_{|LY}$ on TLY, $N_{LY/LX}$ respectively.

Clearly since K is holomorphic and Killing, (TLX, g^{TLX}) is an example of (LE, g^{LE}), to which the action of K on LX lifts. One easily verifies that

$$J^{TLX} = \nabla^{TLX} K . \tag{2.32}$$

Also the vector field $K^{TLX}_{|LY}$ preserves (TLY, g^{TLY}). Therefore $K^{TLX}_{|LY}$ also preserves $(N_{LY/LX}, g^{N_{LY/LX}})$.

Let J^{TLY}, $J^{N_{LY/LX}}$ be the obvious analogues of J^{LE}. One verifies that

$$\begin{aligned} J^{TLY} &= P^{TLY} J^{TLX}_{|LY} P^{TLY} , \\ J^{N_{LY/LX}} &= P^{N_{LY/LX}} J^{TLX}_{|LY} P^{N_{LY/LX}} . \end{aligned} \tag{2.33}$$

Also since σ is a K^{LE}-invariant section of E, one sees easily that under the identification of holomorphic Hermitian vector bundles $N_{LY/LX} = LE_{|LY}$, then

$$J^{N_{LY/LX}} = J^{LE}_{|LY} . \tag{2.34}$$

d) Equivariant Double Transgression Formulas

We now closely imitate [BGS5] to construct equivariant Thom forms and equivariant Euler-Green currents on the total space of LE. We still use the formalism of Mathai-Quillen [MQ] described in [B1, Sect. 3a)].

Set

$$\partial_{K^{LE}} = \partial + i_{K^{LE(0,1)}} , \qquad \bar\partial_{K^{LE}} = \bar\partial + i_{K^{LE(1,0)}} . \tag{2.35}$$

As in (2.7), (2.9), we find that

$$\begin{aligned} (\partial_{K^{LE}})^2 = 0 , \quad & (\bar\partial_{K^{LE}})^2 = 0 , \\ L_{K^{LE}} = \bar\partial_{K^{LE}} \partial_{K^{LE}} & + \partial_{K^{LE}} \bar\partial_{K^{LE}} . \end{aligned} \tag{2.36}$$

Since K^{LE} is a holomorphic vector field which preserves the metric g^{LE}, it also preserves the connection ∇^{LE}.

Definition 2.13. The equivariant curvature ${}^K R^{LE}$ is defined by

$$ {}^K R^{LE} = J^{LE} + R^{LE} . \tag{2.37}$$

${}^K R^{LE}$ is the sum of $(0,0)$ and of a $(1,1)$ form on LX taking values in skew-adjoint elements of $\mathrm{End}(LE)$.

Let $\mathbf{J}^{LE}$ be the complex structure on $LE_{\mathbb{R}}$.

Recall that if $z \in LE$, we identify z to $Z = z + \bar z \in LE_{\mathbb{R}}$, so that $|Z|^2 = 2|z|^2$.

Definition 2.14. For $u \geq 0$, let ${}^K a_u(LE, g^{LE})$, ${}^K c_u(LE, g^{LE})$ be the smooth forms on LE,

$$
\begin{aligned}
{}^K a_u(LE, g^{LE}) &= \det\left(-\frac{{}^K R^{LE}}{2i\pi}\right) \exp\left\{-u\left(\frac{|Z|^2}{2} + ({}^K R^{LE})^{-1}\right)\right\}, \\
{}^K c_u(LE, g^{LE}) &= \frac{\partial}{\partial b}\left[\det\left(-\frac{{}^K R^{LE}}{2i\pi} - b\right)\right. \\
&\qquad \left.\exp\left\{-u\left(\frac{|Z|^2}{2} + ({}^K R^{LE} + 2\pi b \mathbf{J}^{LE})^{-1}\right)\right\}\right]_{b=0}
\end{aligned}
\tag{2.38}
$$

The analogue of [B1, Theorem 3.2] is as follows.

Theorem 2.15. *The forms ${}^K a_u(LE, g^{LE})$ and ${}^K c_u(LE, g^{LE})$ lie in ${}^{K^{LE}}P^{LE}$. Also*

$$
\begin{aligned}
\partial_{K^{LE}} {}^K a_u(LE, g^{LE}) &= 0\,, \\
\bar{\partial}_{K^{LE}} {}^K a_u(LE, g^{LE}) &= 0\,.
\end{aligned}
\tag{2.39}
$$

For any $u > 0$, the following identity holds

$$
\begin{aligned}
\frac{\partial}{\partial u} {}^K a_u(LE, g^{LE}) &= \frac{\bar{\partial}_{K^{LE}} \partial_{K^{LE}}}{2i\pi} \frac{{}^K c_u}{u}(LE, g^{LE}) \\
&= -\frac{\partial_{K^{LE}} \bar{\partial}_{K^{LE}}}{2i\pi} \frac{{}^K c_u}{u}(LE, g^{LE})\,.
\end{aligned}
\tag{2.40}
$$

Proof. When $K = 0$, formulas (2.39), (2.40) were established in [BGS5, Theorem 3.10] and recalled in [B1, Theorem 3.2]. In general, the formalism of equivariant cohomology of [BeV] shows that formulas (2.39), (2.40) are consequences of [BGS5, Theorem 3.10], essentially because the connection ∇^{LE} is K^{LE}-invariant. The fact that the forms ${}^K a_u(LE, g^{LE})$ and ${}^K c_u(LE, g^{LE})$ are K^{LE}-invariant follows tautologically.

Here we will simply check directly that the forms ${}^K a_u(LE, g^{LE})$ and ${}^K c_u(LE, g^{LE})$ are K^{LE}-invariant. In fact since the connection ∇^{LE} is K^{LE}-invariant, one deduces from [BeV] that

$$
\nabla^{LE} J^{LE} + i_K R^{LE} = 0\,.
\tag{2.41}
$$

In particular from (2.41), we find that

$$
\nabla_K^{LE} J^{LE} = 0\,.
\tag{2.42}
$$

Since ∇^{LE} is K^{LE}-invariant, R^{LE} is K^{LE}-invariant, i.e.

$$
\nabla_K^{LE} R^{LE} - [J^{LE}, R^{LE}] = 0\,.
\tag{2.43}
$$

From (2.42), (2.43), we get

$$
[\nabla_K^{LE} - J^{LE}, {}^K R^{LE}] = 0\,.
\tag{2.44}
$$

Using (2.44), we easily deduce that the forms ${}^K a_u(LE, g^{LE})$ and ${}^K c_u(LE, g^{LE})$ are K^{LE}-invariant. $\square$

e) Equivariant Bott-Chern Currents on K-invariant Submanifolds of LX

Let Y' be a K-invariant complex submanifold of LX. Let $i'{:}Y' \to LX$ be the corresponding embedding. Set

$$Y'' = LY \cap Y' . \tag{2.45}$$

We assume that Y'' is a complex submanifold of LX and that

$$TY'' = TLY \cap TY' . \tag{2.46}$$

Then Y'' is also a K-invariant submanifold of LX.

Let $N_{Y''/Y'}$ be the normal bundle to Y'' in Y'. Then we have an exact sequence of holomorphic vector bundles over Y'',

$$0 \to N_{Y''/Y'} \to N_{LY/LX_{|Y''}} \to \tilde{N} \to 0 . \tag{2.47}$$

In (2.47), $\tilde{N}$ is the excess normal bundle to Y'' in LX.

The vector bundle $N_{LY/LX_{|Y''}}$ is equipped with the metric $g^{N_{LY/LX_{|Y''}}}$. Let $g^{N_{Y''/Y'}}$ be the metric induced by $g^{N_{LY/LX_{|Y''}}}$ on $N_{Y''/Y'}$. We identify $\tilde{N}$ with the orthogonal bundle to $N_{Y''/Y'}$ in $N_{LY/LX_{|Y''}}$. Let $g^{\tilde{N}}$ be the metric induced by $g^{N_{LY/LX_{|Y''}}}$ on $\tilde{N}$.

Let $\nabla^{\tilde{N}}$ be the holomorphic Hermitian connection on $(\tilde{N}, g^{\tilde{N}})$ and let $R^{\tilde{N}}$ be its curvature.

By the same arguments as in Sect. 2c), we see that K lifts to a holomorphic vector field on $N_{Y''/Y'}$ which preserves the metrics $g^{N_{Y''/Y'}}$.

Let $P^{N_{Y''/Y'}}$, $P^{\tilde{N}}$ be the orthogonal projection operators from $N_{LY/LX_{|Y''}}$ on $N_{Y''/Y'}$, $\tilde{N}$ respectively. Let $J^{N_{Y''/Y'}}$ be the obvious analogue of J^{LE}. By (2.33), we find that

$$J^{N_{Y''/Y'}} = P^{N_{Y''/Y'}} J^{N_{LY/LX_{|Y''}}} P^{N_{Y''/Y'}} . \tag{2.48}$$

Since K lifts to a holomorphic vector field on $N_{LY/LX_{|Y''}}$ which preserves the metric $g^{N_{LY/LX_{|Y''}}}$, it also acts holomorphically on $\tilde{N}$ and preserves the metric $g^{\tilde{N}}$. If $J^{\tilde{N}}$ is the analogue of J^{LE} for $\tilde{N}$, one then finds that

$$J^{\tilde{N}} = P^{\tilde{N}} J^{N_{LY/LX}}_{|Y''} P^{\tilde{N}} . \tag{2.49}$$

In the sense of Definition 2.13, the equivariant curvature ${}^K R^{\tilde{N}}_{Y/X}$ on $(\tilde{N}, g^{\tilde{N}})$ is given by

$${}^K R^{\tilde{N}} = J^{\tilde{N}} + R^{\tilde{N}} . \tag{2.50}$$

We identify $\sigma \in LE$ to $s = \sigma + \bar{\sigma} \in LE_{\mathbb{R}}$, so that $|s|^2 = 2|\sigma|^2$.

Theorem 2.16. *The forms $s^{*}\,{}^K a_u(LE, g^{LE})$ and $s^{*}\,{}^K c_u(LE, g^{LE})$ lie in ${}^K P^{LX}$. Also*

$$\partial_K s^{*}\,{}^K a_u(LE, g^{LE}) = 0 , \qquad \bar{\partial}_K s^{*}\,{}^K a_u(LE, g^{LE}) = 0 . \tag{2.51}$$

For any $u > 0$, the following identity holds

$$\frac{\partial}{\partial u} s^{*}\,{}^{K}a_u(LE, g^{LE}) = \frac{\bar{\partial}_K \partial_K}{2i\pi} \frac{s^{*}\,{}^{K}c_u}{u}(LE, g^{LE}) = -\frac{\partial_K \bar{\partial}_K}{2i\pi} \frac{s^{*}\,{}^{K}c_u}{u}(LE, g^{LE}). \tag{2.52}$$

Proof. Since σ is holomorphic and K^{LE}-invariant, one sees easily that

$$\partial_K s^* = s^* \partial_{K^{LE}}; \qquad \bar{\partial}_K s^* = s^* \bar{\partial}_{K^{LE}}. \tag{2.53}$$

Our theorem follows from Theorem 2.15 and from (2.53). □

Let k be the embedding $Y'' \to Y'$. We use the notation of Sect. 2a) for currents on Y' whose wave front set is included in $N^*_{Y''/Y',\mathbb{R}}$.

Definition 2.17. Set

$$\begin{aligned} {}^{K}c_{\max}(\tilde{N}, g^{\tilde{N}}) &= \det\left(-\frac{{}^{K}R^{\tilde{N}}}{2i\pi}\right), \\ {}^{K}c'_{\max}(\tilde{N}, g^{\tilde{N}}) &= \frac{\partial}{\partial b}\left[\det\left(-\frac{{}^{K}R^{\tilde{N}}}{2i\pi} + b\right)\right]_{b=0}. \end{aligned} \tag{2.54}$$

If $\tilde{N} = 0$, we make the convention

$${}^{K}c_{\max}(\tilde{N}, g^{\tilde{N}}) = 1, \qquad {}^{K}c'_{\max}(\tilde{N}, g^{\tilde{N}}) = 0. \tag{2.55}$$

Theorem 2.18. *There exists a constant $C > 0$ such that if μ is a smooth differential form on Y', then for $u \geq 1$,*

$$\begin{aligned} \left|\int_{Y'} \mu(i'^* s^*\,{}^{K}a_u(LE, g^{LE}) - {}^{K}c_{\max}(\tilde{N}, g^{\tilde{N}})\delta_{Y''})\right| &\leq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(Y')}, \\ \left|\int_{Y'} \mu(i'^* s^*\,{}^{K}c_u(LE, g^{LE}) + {}^{K}c'_{\max}(\tilde{N}, g^{\tilde{N}})\delta_{Y''})\right| &\leq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(Y')}. \end{aligned} \tag{2.56}$$

If V, Γ, φ, m are taken as in (2.16) *with respect to the embedding $k : Y'' \to Y'$, there exists $C' > 0$ such that for $u \geq 1$,*

$$\begin{aligned} p_{V,\Gamma,\varphi,m}(i'^* s^*\,{}^{K}a_u(LE, g^{LE}) - {}^{K}c_{\max}(\tilde{N}, g^{\tilde{N}})\delta_{Y''}) &\leq \frac{C'}{\sqrt{u}}, \\ p_{V,\Gamma,\varphi,m}(i'^* s^*\,{}^{K}c_u(LE, g^{LE}) + {}^{K}c'_{\max}(\tilde{N}, g^{\tilde{N}})\delta_{Y''}) &\leq \frac{C'}{\sqrt{u}}. \end{aligned} \tag{2.57}$$

Proof. The proof of our theorem is essentially the same as the proof of [B2, Theorems 5.1 and 5.4] and [BGS5, Theorem 3.12]. In particular the fact that J^{LE} which appears in ${}^{K}R^{LE}$ is ultimately replaced by $J^{\tilde{N}}$ follows directly from (2.49) and from [B2, Eq. (5.23)]. □

Definition 2.19. For $s \in \mathbb{C}$, $0 < \mathrm{Re}(s) < \frac{1}{2}$, let ${}^{K}H^{Y'}(LE, g^{LE})(s)$ be the current on Y',

$$\begin{aligned} {}^{K}H^{Y'}(LE, g^{LE})(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1}(&i'^* s^*\,{}^{K}c_u(LE, g^{LE}) \\ &+ {}^{K}c'_{\max}(\tilde{N}, g^{\tilde{N}})\delta_{Y''})\,du. \end{aligned} \tag{2.58}$$

By Theorem 2.18, (2.58) extends to a holomorphic function near 0.

Definition 2.20. Let ${}^K\tilde{e}^{Y'}(LE, g^{LE})$ be the current on Y',

$$ {}^K\tilde{e}^{Y'}(LE, g^{LE}) = \frac{\partial}{\partial s}\, {}^K H^{Y'}(LE, g^{LE})(0). \tag{2.59}$$

Proposition 2.21. *The following identity holds*

$$\begin{aligned} {}^K\tilde{e}^{Y'}(LE, g^{LE}) = & \int_0^1 i'^* s^* \left({}^K c_u(LE, g^{LE}) - {}^K c_0(LE, g^{LE})\right) \frac{du}{u} \\ & + \int_1^{+\infty} \left(i'^* s^*\, {}^K c_u(LE, g^{LE}) + {}^K c'_{\max}(\tilde{N}, g^{\tilde{N}})\delta_{Y''}\right) \frac{du}{u} \\ & + \Gamma'(1)\left[i'^*\, {}^K c'_{\max}(LE, g^{LE}) - {}^K c'_{\max}(\tilde{N}, g^{\tilde{N}})\delta_{Y''}\right]. \end{aligned} \tag{2.60}$$

Proof. Equation (2.60) follows from (2.58). □

By replacing in Definition 2.4 LX and X by Y' and Y'', we define the sets of currents ${}^K P^{Y'}_{Y''}$, ${}^K P^{Y',0}_{Y''}$ on Y'.

Theorem 2.22. *The current ${}^K\tilde{e}^{Y'}(LE, g^{LE})$ lies in ${}^K P^{Y'}_{Y''}$. Moreover it verifies the equation of currents*

$$\frac{\bar{\partial}_K \partial_K}{2i\pi}\, {}^K\tilde{e}^{Y'}(LE, g^{LE}) = {}^K c_{\max}(\tilde{N}, g^{\tilde{N}})\delta_{Y''} - i'^*\, {}^K c_{\max}(LE, g^{LE}). \tag{2.61}$$

Proof. Using Theorems 2.16 and 2.18, the proof of our theorem is the same as the proof of Theorem 2.12. □

Remark 2.23. By exactly proceeding as in [BGS5, Theorem 3.15], we know that the current ${}^K\tilde{e}^{LX}(LE, g^{LE})$ is locally integrable, and is given by the formula

$$\begin{aligned} {}^K\tilde{e}^{LX}(LE, g^{LE}) = -\frac{\partial}{\partial b}\Bigg[& \det\left(-\frac{{}^K R^{LE}}{2i\pi} - b\right) \\ & \operatorname{Log}\left(\frac{|s|^2}{2} + s^*({}^K R^{LE} + 2\pi b \mathbf{J}^{LE})^{-1}\right)\Bigg]_{b=0}. \end{aligned} \tag{2.62}$$

Equivalently

$$\begin{aligned} {}^K\tilde{e}^{LX}(LE, g^{LE}) = -\frac{\partial}{\partial b}\Bigg[& \det\left(-\frac{{}^K R^{LE}}{2i\pi} - b\right) \operatorname{Log}\left(\frac{|s|^2}{2}\right) \\ & - \sum_{k=1}^{\dim LE - 1} \frac{2^k}{k|s|^{2k}}\, s^*(-({}^K R^{LE} + 2\pi b \mathbf{J}_{LE})^{-1})^k\Bigg]_{b=0}. \end{aligned} \tag{2.63}$$

The singularity of ${}^K\tilde{e}^{LX}(LE, g^{LE})$ near Y is of the form $|s|^{-2(\dim E - 1)}$, which is indeed locally integrable.

If Y' is transversal to LY, the restriction $i'^*\, {}^K\tilde{e}^{LX}(LE, g^{LE})$ of the current ${}^K\tilde{e}^{LX}(LE, g^{LE})$ to Y' is well-defined. As in [B1, Eq. (3.17)], we find that

$$ {}^K\tilde{e}^{Y'}(LE, g^{LE}) = i'^*\, {}^K\tilde{e}^{LX}(LE, g^{LE}). \tag{2.64}$$

If Y' is not transversal to Y, i.e. if $\tilde{N} \neq 0$, then ${}^K\tilde{e}^{Y'}(LE, g^{LE})$ is not locally integrable on Y' near Y''. More precisely, by proceeding as in [BGS4, proof of Theorem 3.4], we see that ${}^K\tilde{e}^{Y'}(LE, g^{LE})$ is smooth on Y'/Y'', and has a singularity near Y'' of the form $\dfrac{1}{|z|^{2\dim N_{Y''/Y'}}}$. In fact the obvious analogue of the final part of [B1, Remark 3.7] still holds in this case.

f) The Current ${}^K\tilde{e}^X(LE, g^{LE})$

If B is a complex manifold and if B' is a complex submanifold, by replacing in Definition 2.4 LX by B, X by B' and K by the 0 vector field, we define the sets of currents $P^B_{B'}$, $P^{B,0}_{B'}$ over B. Of course the condition of K-invariance of the currents is now empty.

We now specialize the results of Sect. 2e) to the case where $Y' = X$. X is of course K-invariant. Set

$$Y = LY \cap X . \tag{2.65}$$

Since the restriction of K to LY is also a Killing vector field, then Y is a complex submanifold of LY. Moreover since TX is the kernel of $J^{TLX}_{|X}$ and TY is the kernel of $J^{TLY}_{|Y}$, we find that

$$TY = TLY \cap TX . \tag{2.66}$$

With the notation of Sect. 2e), $Y'' = Y$.

Since K vanishes on X, using (2.41), we get

$$\nabla^{LE} J^{LE}_{|X} = 0 . \tag{2.67}$$

The tensor $J^{LE}_{|X}$ being parallel with respect to the connection ∇^{LE}, the eigenvalues of $J^{LE}_{|X}$ are locally constant. Let Λ denote the finite set of distinct locally constant eigenvalues of $J^{LE}_{|X}$. If $\lambda \in \Lambda$, let $LE^{\lambda}_{|X}$ be the eigensubbundle of $LE_{|X}$ associated to the eigenvalue λ of $J^{LE}_{|X}$. Then $LE_{|X}$ splits holomorphically into

$$LE_{|X} = \bigoplus_{\lambda \in \Lambda} LE^{\lambda}_{|X} , \tag{2.68}$$

and the splitting (2.68) is orthogonal with respect to the metric $g^{LE_{|X}}$.

Let $LE^0_{|X}$ be the eigensubbundle of $LE_{|X}$ associated to the eigenvalue 0 ($LE^0_{|X}$ may be reduced to 0) and let $LE^{0,\perp}_{|X}$ be its orthogonal with respect to the metric $g^{LE_{|X}}$. Then by (2.68), we deduce in particular that $LE_{|X}$ splits holomorphically into

$$LE_{|X} = LE^0_{|X} \oplus LE^{0,\perp}_{|X} . \tag{2.69}$$

Let $g^{LE^0_{|X}}$, $g^{LE^{0,\perp}_{|X}}$ be the Hermitian metrics on $LE^0_{|X}$, $LE^{0,\perp}_{|X}$ induced by the metric $g^{LE_{|X}}$.

Since the section σ of LE is K^{LE}-invariant, then

$$\nabla^{LE}_K \sigma = J^{LE} \sigma . \tag{2.70}$$

From (2.70), we deduce that

$$\sigma_{|X} \in LE^0_{|X} . \tag{2.71}$$

In the case considered here, the exact sequence (2.47) can be written in the form

$$0 \to N_{Y/X} \to N_{LY/LX_{|Y}} \to \tilde{N} \to 0\,. \tag{2.72}$$

We know that if $h \in TLX_{|X}$, then $J^{TLX}h = 0$ if and only if $h \in TX$. Moreover $TLY_{|Y}$ is stable under J^{TLX}. Therefore using (2.33), over Y, we have the identity

$$N_{Y/X} = \operatorname{Ker} J_{|Y}^{N_{LY/LX}}\,. \tag{2.73}$$

From (2.72), (2.73) we see that if $N^{0,\perp}_{LY/LX_{|Y}}$ is the direct sum of the eigenspaces of $J^{N_{LY/LX_{|Y}}}$ associated to nonzero eigenvalues of $J^{N_{LY/LX_{|Y}}}$, then $N^{0,\perp}_{LY/LX_{|Y}}$ is a holomorphic vector subbundle of $N_{LY/LX_{|Y}}$. Moreover $N_{LY/LX_{|Y}}$ splits holomorphically into

$$N_{LY/LX_{|Y}} = N_{Y/X} \oplus N^{0,\perp}_{LY/LX_{|Y}}\,, \tag{2.74}$$

and so

$$\tilde{N} = N^{0,\perp}_{LY/LX_{|Y}}\,. \tag{2.75}$$

Recall that $d\sigma_{|LY}$ identifies $N_{LY/LX}$ with $LE_{|LY}$. Using (2.34), it is clear that under this identification, the splittings (2.69) and (2.75) correspond. Of course by construction, the metrics also correspond. In particular $d\sigma_{|Y}$ identifies $N_{Y/X}$ with $LE^0_{|X}$.

We identify X with the zero section of $LE^0_{|X}$. The Euler-Green current $\tilde{e}(LE^0_{|X}, g^{LE^0_{|X}})$ on the total space of the holomorphic Hermitian vector bundle $(LE^0_{|X}, g^{LE^0_{|X}})$ was constructed in [BGS5, Sect. 3f)]. Then $\tilde{e}(LE^0_{|X}, g^{LE^0_{|X}})$ is a locally integrable current lying in $P_X^{LE^0_{|X}}$, such that

$$\frac{\bar{\partial}\partial}{2i\pi}\,\tilde{e}(LE^0_{|X}, g^{LE_{|X}}) = \delta_X - c_{\max}(LE^0_{|X}, g^{LE^0_{|X}})\,. \tag{2.76}$$

In the sequel, $\sigma_{|X}$ will be considered as a section of $LE^0_{|X}$, which vanishes exactly on Y. Set $s_{|X} = \sigma_{|X} + \bar{\sigma}_{|X}$. Then $s_{|X}$ is a smooth section of $LE^0_{|X,\mathbb{R}}$ over X.

By [BGS5, Remark 3.16], the current on X, $s^*_{|X}\tilde{e}(LE^0_{|X}, g^{LE^0_{|X}})$, is well-defined, lies in P_Y^X, and moreover

$$\frac{\bar{\partial}\partial}{2i\pi}\,s^*_{|X}\tilde{e}(LE^0_{|X}, g^{LE_{|X}}) = \delta_Y - c_{\max}(LE^0_{|X}, g^{LE^0_{|X}})\,. \tag{2.77}$$

The current $s^*_{|X}\tilde{e}(LE^0_{|X}, g^{LE^0_{|X}})$ is a special case of the current ${}^K\tilde{e}^{LX}(LE, g^{LE})$, when $K = 0$, $K^{LE} = 0$.

Theorem 2.24. *The following identity holds*

$${}^K\tilde{e}^X(LE, g^{LE}) = {}^K c_{\max}(LE^{0,\perp}_{|X}, g^{LE^{0,\perp}_{|X}})\, s^*_{|X}\tilde{e}(LE^0_{|X}, g^{LE^0_{|X}}) \quad \text{in} \quad P_Y^X/P_Y^{X,0}\,. \tag{2.78}$$

Proof. By proceeding as in the proof of [B1, Theorem 3.8], the proof of Theorem 2.24 is identical to the proof of [BGS5, Theorem 3.17]. □

III. Localization Formulas for Bott-Chern Currents

The purpose of this section is to prove localization formulas for Bott-Chern currents. More precisely, we prove in Theorems 3.2 and 3.4 that certain combination of Bott-Chern currents on LX differ of currents localized on X by sums of ∂_K and $\bar{\partial}_K$ coboundaries.

As explained in the introduction, the organization of this section is closely related to the organization of the paper by Bismut-Lebeau [BL2]. The analogy will in fact be further explored in [B3].

This paper is organized as follows. In a), we state our localization formulas. The rest of the section is devoted to the proof of these formulas. In b), and in relation with [BL2] and also with [B1], we construct a closed form on $R_+^* \times R_+^*$. In c), by integrating this form over a closed rectangular contour, we obtain a fundamental identity. By deforming the rectangle, we will in fact ultimately prove our main formulas.

In d), we state three intermediary results, which are needed in the proof of Theorems 3.2 and 3.4. The proofs of these results are delayed to Sects. 3h)–3j).

In e), using these intermediary results, we calculate the asymptotics of the fundamental identity of Sect. 3c). Section 3d) is in fact in close resemblance with [BL2, Sect. 6].

In f), we verify the consistency of our calculations, by checking that certain diverging terms in e) effectively cancel each other.

In g), we prove Theorem 3.4.

In h), i), j), we prove the three intermediary results stated in d).

In k), we give another approach to the results established in e), by a direct manipulation of certain Euler-Green currents, which are shown to exhibit mysterious and hidden algebraic properties.

Finally in l), we briefly check the proof of Theorem 3.2.

The techniques used in this section are very similar to techniques we used in [B1] to deal with another problem also involving an excess normal bundle.

a) A Fundamental Result

We make the same assumptions and we use the same notation as in Sect. 2. If $x \in X \cup LY$, set

$$\begin{aligned} N^*_{X/LX,\mathbb{R}} + N^*_{LY/LX,\mathbb{R}} &= N^*_{X/LX,\mathbb{R}} \quad \text{if} \quad x \in X \backslash Y \\ &= N^*_{LY/LX,\mathbb{R}} \quad \text{if} \quad x \in LY \backslash Y \\ &= N^*_{Y/LX,\mathbb{R}} \quad \text{if} \quad x \in Y\,. \end{aligned} \tag{3.1}$$

Definition 3.1. Let $\mathscr{D}'_{N^*_{X/LX,\mathbb{R}}+N^*_{LY/LX,\mathbb{R}}}(LX)$ be the set of currents on LX whose wave front set is included in $N^*_{X/LX,\mathbb{R}} + N^*_{LY/LX,\mathbb{R}}$.

Let $P^{LX}_{X\cup LY}$ be the set of K-invariant currents in $\mathscr{D}'_{N^*_{X/LX,\mathbb{R}}+N^*_{LY/LX,\mathbb{R}}}(LX)$ which are sums of currents of type (p,p).

Let $P^{LX,0}_{X\cup LY}$ be the set of currents $\omega \in P^{LX}_{X\cup LY}$ such that there exist K-invariant currents $\alpha, \beta \in \mathscr{D}'_{N^*_{X/LX,\mathbb{R}}+N^*_{LY/LX,\mathbb{R}}}(LX)$ for which $\omega = \partial_K \alpha + \bar{\partial}_K \beta$.

We now use the notation of Sect. 2f). Consider the exact sequences

$$\begin{aligned} &0 \to N_{Y/X} \to N_{LY/LX_{|Y}} \to \tilde{N} \to 0\,, \\ &0 \to N_{Y/LY} \underset{v}{\to} N_{X/LX_{|Y}} \underset{w}{\to} \tilde{N} \to 0\,. \end{aligned} \tag{3.2}$$

As we saw in (2.72)–(2.75), the first exact sequence splits holomorphically, so that

$$N_{LY/LX_{|Y}} = N_{Y/X} \oplus \tilde{N}\,. \tag{3.3}$$

By identifying $N_{LY/LX_{|Y}}$ and $N_{X/LX}$ with the orthogonal bundles to TLY and TX in $TLX_{|LY}$ and $TLX_{|X}$, $N_{LY/LX}$ and $N_{X/LX}$ are respectively equipped with induced metrics $g^{N_{LY/LX}}$ and $g^{N_{X/LX}}$. Using (3.2), these metrics induce the obvious natural metrics $g^{N_{Y/X}}$ and $g^{N_{Y/LY}}$ on $N_{Y/X}$ and $N_{Y/LY}$.

On the other hand, in the exact sequences (3.2), we can identify $\tilde{N}$ to the orthogonal bundle to $N_{Y/X}$ in $N_{LY/LX_{|Y}}$ and also to the orthogonal bundle to $N_{Y/LY}$ in $N_{X/LX_{|Y}}$. It is then easy to check that $\tilde{N}$ inherits a common metric $g^{\tilde{N}}$. This metric is exactly the one which is obtained by identifying $\tilde{N}$ with the orthogonal bundle to $TX + TLY_{|X}$ in $TLX_{|X}$.

Finally, observe that the exact sequence of holomorphic Hermitian vector bundles on Y

$$0 \to N_{Y/LY} \underset{v}{\to} N_{X/LX_{|Y}} \underset{w}{\to} \tilde{N} \to 0 \tag{3.4}$$

verifies the assumptions of Sect. 1a) with respect to the action of $J^{N_{X/LX_{|Y}}}$, which restricts to $J^{N_{Y/LY}}$ on $N_{Y/LY}$. The assumptions of Sect. 1a) are verified in particular because $\mathrm{Ker}(J^{N_{X/LX}}) = 0$.

We can then use the notation of Sect. 1. Note that the class $\widetilde{{}^K c_{\max}^{-1}}(N_{Y/LY}, N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}}) \in P^Y/P^{Y,0}$ was constructed in Sect. 1d).

We now state the main result of this paper.

Theorem 3.2. *The following identity of currents on LX holds*

$$\begin{aligned} &-{}^K\tilde{e}^{LX}(LE, g^{LE}) - {}^K S_{\omega LX}\, {}^K c_{\max}(LE, g^{LE}) + {}^K S_{\omega LY}\, \delta_{LY} \\ &= -{}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K\tilde{e}^X(LE, g^{LE})\,\delta_X \\ &\quad + {}^K c_{\max}(\tilde{N}, g^{\tilde{N}})\, \widetilde{{}^K c_{\max}^{-1}}(N_{Y/LY}, N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}})\,\delta_Y \\ &\quad - {}^K c_{\max}^{-1}(N_{Y/LY})\, {}^K R(\tilde{N})\,\delta_Y \quad \text{in} \quad {}^K P^{LX}_{X\cup LY}/{}^K P^{LX,0}_{X\cup LY}\,. \end{aligned} \tag{3.5}$$

Equivalently,

$$\begin{aligned} &-{}^K\tilde{e}^{LX}(LE, g^{LE}) - {}^K S_{\omega LX}\, {}^K c_{\max}(LE, g^{LE}) + {}^K S_{\omega LY}\, \delta_{LY} \\ &= -{}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K\tilde{e}^X(LE, g^{LE})\,\delta_X \\ &\quad + {}^K c_{\max}(\tilde{N}, g^{\tilde{N}})\, \widetilde{{}^K c_{\max}^{-1}}(N_{Y/LY}, N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}})\,\delta_Y \\ &\quad - {}^K c_{\max}^{-1}(N_{X/LX})\, {}^K R(N_{X/LX})\, {}^K c_{\max}(LE)\,\delta_X \\ &\quad + {}^K c_{\max}^{-1}(N_{Y/LY})\, {}^K R(N_{Y/LY})\,\delta_Y \quad \text{in} \quad {}^K P^{LX}_{X\cup LY}/{}^K P^{LX,0}_{X\cup LY}\,. \end{aligned} \tag{3.6}$$

Remark 3.3. By applying the operator $\bar{\partial}_K \partial_K$ on both sides of (3.5), by using Theorems 2.12 and 2.22, and the fact that

$$\frac{\bar{\partial}_K \partial_K}{2i\pi} \left({}^K c_{\max}^{-1}(N_{Y/LY})\, {}^K R(\tilde{N})\, \delta_Y\right) = 0\,, \tag{3.7}$$

one obtains an already known identity. Identity (3.5) must be thought of as a considerable refinement of this identity.

It is here important to observe why (3.5) or (3.6) cannot be entirely trivial. In fact assume temporarily that X and LY are transversal in LX, i.e. that $\tilde{N} = 0$. Recall that ${}^K S_{\omega^{LX}} \in {}^K P_X^{LX}$, ${}^K \tilde{e}^{LX}(LE, g^{LE}) \in {}^K P_{LY}^{LX}$. By [H, Theorem 8.2.10], we can then form the product of currents ${}^K S_{\omega^{LX}}\, {}^K \tilde{e}^{LX}(LE, g^{LE})$ and the ordinary rules of calculus apply to this product. In particular

$$\begin{aligned} &\bar{\partial}_K \partial_K [{}^K S_{\omega^{LX}}]\, {}^K \tilde{e}^{LX}(LE, g^{LE}) - {}^K S_{\omega^{LX}}\, \bar{\partial}_K \partial_K\, {}^K \tilde{e}^{LX}(LE, g^{LE}) \\ &\quad = \bar{\partial}_K [\partial_K\, {}^K S_{\omega^{LX}}\, {}^K \tilde{e}^{LX}(LE, g^{LE})] + \partial_K [{}^K S_{\omega^{LX}}\, \bar{\partial}_K\, {}^K \tilde{e}^{LX}(LE, g^{LE})]\,. \end{aligned} \tag{3.8}$$

From (3.8), we get

$$\begin{aligned} &{}^K \tilde{e}^{LX}(LE, g^{LE}) - {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K \tilde{e}^{LX}(LE, g^{LE})\, \delta_X \\ &\quad - {}^K S_{\omega^{LX}}(\delta_{LY} - {}^K c_{\max}(LE, g^{LE})) \in {}^K P^{LX,0}\,, \end{aligned} \tag{3.9}$$

which is equivalent to (3.5), (3.6).

It should be no surprise that as in [B1, Theorem 2.8], the extra terms in (3.5), (3.6) with respect to (3.9) come from the fact that X and LY **are not transversal**, so that we get a contribution from the excess normal bundle $\tilde{N}$.

An obvious corollary of Theorem 3.2 is the following result.

Theorem 3.4. *Let $\mu \in {}^K P^{LX}$, which is such that $\partial_K \mu = 0$, $\bar{\partial}_K \mu = 0$. Then the following identity holds:*

$$\begin{aligned} &-\int_{LX} \mu\, {}^K \tilde{e}^{LX}(LE, g^{LE}) - \int_{LX} \mu\, {}^K S_{\omega^{LX}}\, {}^K c_{\max}(LE, g^{LE}) + \int_{LY} \mu\, {}^K S_{\omega^{LY}} \\ &\quad = -\int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K \tilde{e}^X(LE, g^{LE}) \\ &\qquad + \int_Y \mu\, {}^K c_{\max}(\tilde{N}, g^{\tilde{N}})\, \widetilde{{}^K c_{\max}^{-1}}(N_{Y/LY}, N_{X/LX|Y}, g^{N_{X/LX|Y}}) \\ &\qquad - \int_Y \mu\, {}^K c_{\max}^{-1}(N_{Y/LY})\, {}^K R(\tilde{N})\,. \end{aligned} \tag{3.10}$$

Equivalently,

$$\begin{aligned}
&-\int_{LX} \mu^K \tilde{e}^{LX}(LE, g^{LE}) - \int_{LX} \mu^K S_{\omega^{LX}}{}^K c_{\max}(LE, g^{LE}) + \int_{LY} \mu^K S_{\omega^{LY}} \\
&= -\int_X \mu^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})^K \tilde{e}^X(LE, g^{LE}) \\
&\quad + \int_Y \mu^K c_{\max}(\tilde{N}, g^{\tilde{N}})^K \widetilde{c_{\max}^{-1}}(N_{Y/LY}, N_{X/LX|Y}, g^{N_{X/LX|Y}}) \\
&\quad - \int_X \mu^K c_{\max}^{-1}(N_{X/LX})^K R(N_{X/LX})^K c_{\max}(LE) \\
&\quad + \int_Y \mu^K c_{\max}^{-1}(N_{Y/LY})^K R(N_{Y/LY}) .
\end{aligned} \tag{3.11}$$

Remark 3.5. Theorem 3.4 will be proved in detail in Sects. 3b)–3j). The proof of the more refined Theorem 3.2 essentially follows the same lines and will be sketched in Sect. 3l).

From now on, the assumptions of Theorem 3.4 will be in force.

b) A Closed One-Form on $R_+^* \times R_+^*$

A first step in the proof of Theorem 3.4 is as follows:

Theorem 3.6. *Let $\eta_{u,T}$ be the 1-form on $R_+^* \times R_+^*$*

$$\begin{aligned}
\eta_{u,T} &= \frac{du}{u} \int_{LX} \mu^K \gamma_u s^{*\,K} a_T(LE, g^{LE}) \\
&\quad + \frac{dT}{T} \int_{LX} \mu^K \alpha_u s^{*\,K} c_T(LE, g^{LE}) .
\end{aligned} \tag{3.12}$$

Then $\eta_{u,T}$ is a closed form.

Proof. Using Theorems 2.3 and 2.16, and also the fact that $\partial_K \mu = 0$, $\bar{\partial}_K \mu = 0$, we find that

$$\begin{aligned}
&\frac{\partial}{\partial T} \frac{1}{u} \int_{LX} \mu^K \gamma_u s^{*\,K} a_T(LE, g^{LE}) \\
&= \frac{1}{uT} \int_{LX} \mu^K \gamma_u \frac{\bar{\partial}_K \partial_K}{2i\pi} s^{*\,K} c_T(LE, g^{LE}) \\
&= \frac{1}{uT} \int_{LX} \mu \frac{\bar{\partial}_K \partial_K}{2i\pi} {}^K \gamma_u s^{*\,K} c_T(LE, g^{LE}) , \\
&\frac{\partial}{\partial u} \frac{1}{T} \int_{LX} \mu^K \alpha_u s^{*\,K} c_T(LE, g^{LE}) = \frac{1}{uT} \int_{LX} \mu \frac{\bar{\partial}_K \partial_K}{2i\pi} {}^K \gamma_u s^{*\,K} c_T(LE, g^{LE}) .
\end{aligned} \tag{3.13}$$

From (3.13), we see that the form $\eta_{u,T}$ is closed. □

c) A Contour Integral

We now fix constants ε, A, t_0, T_0 such that $0 < \varepsilon < 1 \leq A < +\infty$, $0 < t_0 < 1 \leq T_0 < +\infty$.

Let $\Gamma = \Gamma_{\varepsilon,A,t_0,T_0}$ be the oriented contour in R_+^{*2}

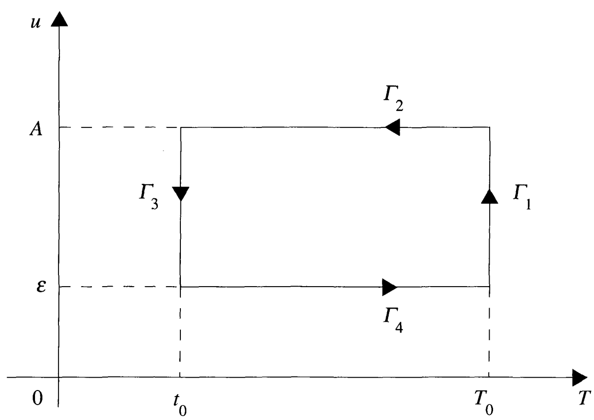

Fig. 1

As shown in Fig. 1, Γ is made of four oriented pieces:

$$\begin{aligned} &\Gamma_1 : T = T_0; \quad \varepsilon \leq u \leq A, \qquad &&\Gamma_2 : t_0 \leq T \leq T_0; \quad u = A, \\ &\Gamma_3 : T = t_0; \quad \varepsilon \leq u \leq A, \qquad &&\Gamma_4 : t_0 \leq T \leq T_0; \quad u = \varepsilon. \end{aligned}$$

The orientation of Γ_1, Γ_2, Γ_3, Γ_4 is indicated in Fig. 1.

For $1 \leq k \leq 4$,set

$$I_k^0 = \int_{\Gamma_k} \eta_{u,T} \,. \tag{3.14}$$

Theorem 3.7. *The following identity holds*

$$\sum_{k=1}^{4} I_k^0 = 0 \,. \tag{3.15}$$

Proof. Equation (3.15) is a trivial consequence of Theorem 3.6. □

Remark 3.8. We now will make $t_0 \to 0$, $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$ in this order in identity (3.15). Typically each term $I_k^0 (1 \leq k \leq 4)$ will diverge at one or several stages of this process. However because of the identity (3.15), the divergences will cancel, often for non-trivial reasons. Once the divergences will have been subtracted off, we will obtain an identity in Sect. 3f) which will lead us to the proof of Theorem 3.4.

d) Three Intermediary Results

We now state three intermediary essential results whose proof is delayed to Sects. 3h)–3j).

As explained in Sect. 3a), we apply the results of Sect. 1 to the exact sequence (3.4).

Theorem 3.8. *There exists $C>0$ such that for any $u\in]0,1]$, $T\in\left[0,\frac{1}{u}\right]$,*

$$\left|\int\limits_{LX}\mu^K\alpha_u s^{*\,K}c_T(LE,g^{LE})-\int\limits_X\mu^K c_{\max}^{-1}(N_{X/LX},g^{N_{X/LX}})\,s^{*\,K}c_T(LE,g^{LE})\right|$$
$$\leq C(u(1+T))^{1/2} \tag{3.16}$$

Theorem 3.9. *For any $T>0$, the following identity holds*

$$\lim_{u\to 0}\int\limits_{LX}\mu^K\alpha_u s^{*\,K}c_{T/u}(LE,g^{LE})$$
$$=\int\limits_Y\mu\int\limits_{N_{X/LX_{|Y}}}{}^K\alpha(N_{X/LX_{|Y}},g^{N_{X/LX_{|Y}}})\,w^{*\,K}c_T(\tilde{N},g^{\tilde{N}})\,. \tag{3.17}$$

Theorem 3.10. *There exists $C>0$ such that for any $u\in]0,1]$, and any $T\geq 1$,*

$$\left|\int\limits_{LX}\mu^K\alpha_u s^{*\,K}c_{T/u}(LE,g^{LE})\right|\leq\frac{C}{\sqrt{T}}\,. \tag{3.18}$$

Remark 3.11. Using Theorems 1.10 and 2.18, one can easily check that Theorems 3.8–3.10 are indeed compatible.

e) The Asymptotics of the I_k^0's

1. *The term I_1^0.* Clearly

$$I_1^0=\int\limits_\varepsilon^A\left\{\int\limits_{LX}\mu^K\gamma_u s^{*\,K}a_{T_0}(LE,g^{LE})\right\}\frac{du}{u}\,. \tag{3.19}$$

α) $t_0\to 0$. I_1^0 remains constant and equal to I_1^1.
β) $A\to+\infty$. We see that

$$I_1^1\to I_1^2=\int\limits_\varepsilon^{+\infty}\left\{\int\limits_{LX}\mu^K\gamma_u s^{*\,K}a_{T_0}(LE,g^{LE})\right\}\frac{du}{u}\,. \tag{3.20}$$

γ) $T_0\to+\infty$

The form $\int\limits_\varepsilon^{+\infty}{}^K\gamma_u\frac{du}{u}$ is smooth on LX. By using Theorem 2.18, we find that as $T_0\to+\infty$,

$$I_1^2\to I_1^3=\int\limits_\varepsilon^{+\infty}\left\{\int\limits_{LY}\mu^K\gamma_u\right\}\frac{du}{u}\,. \tag{3.21}$$

δ) $\varepsilon \to 0$. Since $\partial_K \mu = 0$, $\bar{\partial}_K \mu = 0$, using Theorem 2.7, we see that as $\varepsilon \to 0$,

$$I_1^3 + \int\limits_Y \mu 2\pi \omega^Y \, {}^K c_{\max}^{-1}(N_{Y/LY}, g^{N_{Y/LY}}) \left(1 - \frac{1}{\varepsilon}\right)$$
$$+ \int\limits_Y \mu({}^K c_{\max}^{-1})'(N_{Y/LY}, g^{N_{Y/LY}}) \operatorname{Log}(\varepsilon) \to I_1^4$$
$$= \int\limits_0^1 \Bigg\{ \int\limits_{LY} \mu({}^K \gamma_u - 2\pi \omega^{LY} \, {}^K c_{\max}^{-1}(N_{Y/LY}, g^{N_{Y/LY}}) \, \delta_Y \frac{1}{u}$$
$$- ({}^K c_{\max}^{-1})'(N_{Y/LY}, g^{N_{Y/LY}}) \, \delta_Y) \Bigg\} \frac{du}{u} + \int\limits_1^{+\infty} \Bigg\{ \int\limits_{LY} \mu^K \gamma_u \Bigg\} \frac{du}{u} \,. \tag{3.22}$$

ε) *Evaluation of* I_1^4.

Theorem 3.12. *The following identity holds*

$$I_1^4 = \int\limits_{LY} \mu^K S_{\omega^{LY}} + \int\limits_Y \mu(2\pi \omega^Y \, {}^K c_{\max}^{-1}(N_{Y/LY}, g^{N_{Y/LY}})$$
$$+ \Gamma'(1) ({}^K c_{\max}^{-1})'(N_{Y/LY}, g^{N_{Y/LY}})) \,. \tag{3.23}$$

Proof. Equation (3.23) follows from Theorem 2.7, from Proposition 2.11 and from (3.22). □

2. *The term* I_2^0. Clearly, I_2^0 is given by

$$I_2^0 = -\int\limits_{t_0}^{T_0} \Bigg\{ \int\limits_{LX} \mu^K \alpha_A s^* \, {}^K c_T(LE, g^{LE}) \Bigg\} \frac{dT}{T} \,. \tag{3.24}$$

α) $t_0 \to 0$. We have the obvious

$$s^* \, {}^K c_0(LE, g^{LE}) = -{}^K c'_{\max}(LE, g^{LE}) \,. \tag{3.25}$$

So, we find that as $t_0 \to 0$,

$$I_2^0 + \int\limits_{LX} \mu^K \alpha_A {}^K c'_{\max}(LE, g^{LE}) \operatorname{Log}(t_0) \to I_2^1$$
$$= -\int\limits_0^1 \Bigg\{ \int\limits_{LX} \mu^K \alpha_A (s^* \, {}^K c_T(LE, g^{LE}) - s^* \, {}^K c_0(LE, g^{LE})) \Bigg\} \frac{dT}{T}$$
$$- \int\limits_1^{T_0} \Bigg\{ \int\limits_{LX} \mu^K \alpha_A s^* \, {}^K c_T(LE, g^{LE}) \Bigg\} \frac{dT}{T} \,. \tag{3.26}$$

β) $A \to +\infty$. Obviously

$$I_2^1 \to I_2^2 = -\int_0^1 \left\{ \int_{LX} \mu(s^{*\,K} c_T(LE, g^{LE}) - s^{*\,K} c_0(LE, g^{LE})) \right\} \frac{dT}{T}$$
$$- \int_1^{T_0} \left\{ \int_{LX} \mu s^{*\,K} c_T(LE, g^{LE}) \right\} \frac{dT}{T} \,. \tag{3.27}$$

γ) $T_0 \to +\infty$. By Theorem 2.18, we know that as $T \to +\infty$,

$$\int_{LX} \mu s^{*\,K} c_T(LE, g^{LE}) = O\left(\frac{1}{\sqrt{T}}\right). \tag{3.28}$$

From (3.28), we find that as $T_0 \to +\infty$,

$$I_2^2 \to I_2^3 = -\int_0^1 \left\{ \int_{LX} \mu(s^{*\,K} c_T(LE, g^{LE}) - s^{*\,K} c_0(LE, g^{LE})) \right\} \frac{dT}{T}$$
$$- \int_1^{+\infty} \left\{ \int_{LX} \mu s^{*\,K} c_T(LE, g^{LE}) \right\} \frac{dT}{T} \,. \tag{3.29}$$

δ) $\varepsilon \to 0$. I_2^3 remains constant and equal to I_2^4.
ε) *Evaluation of* I_2^4.

Theorem 3.13. *The following identity holds*

$$I_2^4 = -\int_{LX} \mu^K \tilde{e}^{LX}(LE, g^{LE}) + \Gamma'(1) \int_{LX} \mu^K c'_{\max}(LE, g^{LE}) \,. \tag{3.30}$$

Proof. Equation (3.30) follows from Proposition 2.21 and from (3.29). □

3. *The term* I_3^0. We have the obvious

$$I_3^0 = -\int_\varepsilon^A \left\{ \int_{LX} \mu^K \gamma_u s^{*\,K} a_{t_0}(LE, g^{LE}) \right\} \frac{du}{u} \,. \tag{3.31}$$

α) $t_0 \to 0$. Clearly

$$I_3^0 \to I_3^1 = -\int_\varepsilon^A \left\{ \int_{LX} \mu^K \gamma_u{}^K c_{\max}(LE, g^{LE}) \right\} \frac{du}{u} \,. \tag{3.32}$$

β) $A \to +\infty$. As $A \to +\infty$, then

$$I_3^1 \to I_3^2 = -\int_\varepsilon^{+\infty} \left\{ \int_{LX} \mu^K \gamma_u{}^K c_{\max}(LE, g^{LE}) \right\} \frac{du}{u} \,. \tag{3.33}$$

γ) $T_0 \to +\infty$. I_3^2 remains constant and equal to I_3^3.

δ) $\varepsilon \to 0$. Using Theorem 2.7 and the fact that $\partial_K \mu = 0$, $\bar{\partial}_K \mu = 0$, we see that as $\varepsilon \to 0$,

$$
\begin{aligned}
I_3^3 &- \int_X \mu 2\pi\omega^X \, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \, {}^K c_{\max}(LE, g^{LE}) \left(1 - \frac{1}{\varepsilon}\right) \\
&- \int_X \mu ({}^K c_{\max}^{-1})' (N_{X/LX}, g^{N_{X/LX}}) \, {}^K c_{\max}(LE, g^{LE}) \operatorname{Log}(\varepsilon) \to I_3^4 \\
&= - \int_0^1 \Bigg\{ \int_{LX} \mu \Bigg({}^K \gamma_u - 2\pi\omega^{LX} \, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \, \delta_X \frac{1}{u} \\
&\quad - ({}^K c_{\max}^{-1})' (N_{X/LX}, g^{N_{X/LX}}) \, \delta_X \Bigg) {}^K c_{\max}(LE, g^{LE}) \Bigg\} \frac{du}{u} \\
&\quad - \int_1^{+\infty} \Bigg\{ \int_{LX} \mu \, {}^K \gamma_u \, {}^K c_{\max}(LE, g^{LE}) \Bigg\} \frac{du}{u} \,.
\end{aligned}
\tag{3.34}
$$

ε) *Evaluation of* I_3^4.

Theorem 3.14. *The following identity holds*

$$
\begin{aligned}
I_3^4 = &- \int_{LX} \mu \, {}^K S_{\omega^{LX}} \, {}^K c_{\max}(LE, g^{LE}) \\
&- \int_X \mu (2\pi\omega^X \, {}^K c_{\max}^{-1}(N_{X/LX}) + \Gamma'(1) ({}^K c_{\max}^{-1})' (N_{X/LX})) \, {}^K c_{\max}(LE) \,.
\end{aligned}
\tag{3.35}
$$

Proof. Using Proposition 2.11 and (3.34), (3.35) follows. □

4. *The term* I_4^0. Clearly

$$
I_4^0 = \int_{t_0}^{T_0} \Bigg\{ \int_{LX} \mu \, {}^K \alpha_\varepsilon s^* \, {}^K c_T(LE, g^{LE}) \Bigg\} \frac{dT}{T} \,.
\tag{3.36}
$$

α) $t_0 \to 0$. Obviously,

$$
\begin{aligned}
I_4^0 &- \int_{LX} \mu \, {}^K \alpha_\varepsilon \, {}^K c'_{\max}(LE, g^{LE}) \operatorname{Log}(t_0) \to I_4^1 \\
&= \int_0^1 \Bigg\{ \int_{LX} \mu \, {}^K \alpha_\varepsilon s^* ({}^K c_T(LE, g^{LE}) - {}^K c_0(LE, g^{LE})) \Bigg\} \frac{dT}{T} \\
&\quad + \int_1^{T_0} \Bigg\{ \int_{LX} \mu \, {}^K \alpha_\varepsilon s^* \, {}^K c_T(LE, g^{LE}) \Bigg\} \frac{dT}{T} \,.
\end{aligned}
\tag{3.37}
$$

β) $A \to +\infty$. I_4^1 remains constant and equal to I_4^2.

γ) $T_0 \to +\infty$. Using Theorem 2.18, we find that

$$I_4^2 \to I_4^3 = \int_0^1 \left\{ \int_{LX} \mu^K \alpha_\varepsilon s^* ({}^K c_T(LE, g^{LE}) - {}^K c_0(LE, g^{LE})) \right\} \frac{dT}{T} + \int_1^{+\infty} \left\{ \int_{LX} \mu^K \alpha_\varepsilon s^* \, {}^K c_T(LE, g^{LE}) \right\} \frac{dT}{T} . \tag{3.38}$$

δ) $\varepsilon \to 0$. Set

$$\begin{aligned} J_1^0 &= \int_0^1 \left\{ \int_{LX} \mu^K \alpha_\varepsilon s^* ({}^K c_T(LE, g^{LE}) - {}^K c_0(LE, g^{LE})) \right\} \frac{dT}{T} , \\ J_2^0 &= \int_\varepsilon^1 \left\{ \int_{LX} \mu^K \alpha_\varepsilon s^{*K} c_{T/\varepsilon}(LE, g^{LE}) \right\} \frac{dT}{T} , \\ J_3^0 &= \int_1^{+\infty} \left\{ \int_{LX} \mu^K \alpha_\varepsilon s^{*K} c_{T/\varepsilon}(LE, g^{LE}) \right\} \frac{dT}{T} . \end{aligned} \tag{3.39}$$

Then

$$I_4^3 = J_1^0 + J_2^0 + J_3^0 . \tag{3.40}$$

1. *The term* J_1^0. Using Theorem 2.5, it is clear that as $\varepsilon \to 0$,

$$J_1^0 \to J_1^1 = \int_0^1 \left\{ \int_Y \mu^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \, s^* ({}^K c_T(LE, g^{LE}) - {}^K c_0(LE, g^{LE})) \right\} \frac{dT}{T} . \tag{3.41}$$

2. *The term* J_2^0. We make the crucial step of writing J_2^0 in the form

$$J_2^0 = \int_\varepsilon^1 \left\{ \int_{LX} \mu^K \alpha_\varepsilon s^* \, {}^K c_{T/\varepsilon}(LE, g^{LE}) - \int_X \mu^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \, s^* \, {}^K c_{T/\varepsilon}(LE, g^{LE}) \right\} \frac{dT}{T} + \int_1^{1/\varepsilon} \left\{ \int_X \mu^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \, s^* \, {}^K c_T(LE, g^{LE}) \right\} \frac{dT}{T} . \tag{3.42}$$

By Theorem 3.8, we know that for $\varepsilon \in]0,1]$, $T \in [\varepsilon, 1]$, then

$$\left| \int_{LX} \mu^K \alpha_\varepsilon s^* \, {}^K c_{T/\varepsilon}(LE, g^{LE}) - \int_X \mu^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \, s^* \, {}^K c_{T/\varepsilon}(LE, g^{LE}) \right| \leq C(\varepsilon + T)^{1/2} \leq C(2T)^{1/2} . \tag{3.43}$$

Using (3.43) and Theorems 2.18 and 3.9, we see that as $\varepsilon \to 0$,

$$\int_{\varepsilon}^{1} \left\{ \int_{LX} \mu\, {}^K\alpha_\varepsilon s^*\, {}^K c_{T/\varepsilon}(LE, g^{LE}) - \int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, s^*\, {}^K c_{T/\varepsilon}(LE, g^{LE}) \right\} \frac{dT}{T}$$
$$\to \int_0^1 \left\{ \int_Y \mu \left(\int_{N_{X/LX|Y}} {}^K\alpha(N_{X/LX|Y}, g^{N_{X/LX|Y}})\, w^*\, {}^K c_T(\tilde{N}, g^{\tilde{N}}) \right) + \int_Y \mu\, {}^K c_{\max}^{-1}(N_{X/LX|Y}, g^{N_{X/LX|Y}})\, {}^K c'_{\max}(\tilde{N}, g^{\tilde{N}}) \right\} \frac{dT}{T}\,. \tag{3.44}$$

Of course by Theorem 1.10, we know that as $T \to 0$,

$$\int_{N_{X/LX|Y}} {}^K\alpha(N_{X/LX|Y}, g^{N_{X/LX|Y}})\, w^*\, {}^K c_T(\tilde{N}, g^{\tilde{N}}) = -{}^K c_{\max}^{-1}(N_{X/LX|Y}, g^{N_{X/LX|Y}})\, {}^K c'_{\max}(\tilde{N}, g^{\tilde{N}}) + O(T)\,, \tag{3.45}$$

so that (3.44) makes sense.

By Theorem 2.18, we see that as $\varepsilon \to 0$,

$$\int_1^{1/\varepsilon} \left\{ \int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, s^*\, {}^K c_T(LE, g^{LE}) \right\} \frac{dT}{T} - \left(\int_Y \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K c'_{\max}(\tilde{N}, g^{\tilde{N}}) \right)$$
$$\operatorname{Log}(\varepsilon) \to \int_1^{+\infty} \left\{ \int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, s^*\, {}^K c_T(LE, g^{LE}) + \int_Y \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K c'_{\max}(\tilde{N}, g^{\tilde{N}}) \right\} \frac{dT}{T}\,. \tag{3.46}$$

We now use the notation of Sect. 1 with respect to the exact sequence (3.4). From (3.42)–(3.46), we see that as $\varepsilon \to 0$,

$$J_2^0 - \left(\int_Y \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K c'_{\max}(\tilde{N}, g^{\tilde{N}}) \right)$$
$$\operatorname{Log}(\varepsilon) \to J_2^1 = \int_0^1 \int_Y \mu(\chi_T - \chi_0) \frac{dT}{T}$$

$$+\int_1^{+\infty}\left\{\int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, s^*\, {}^K c_T(LE, g^{LE})\right.$$
$$\left.+\int_Y \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K c'_{\max}(\tilde N, g^{\tilde N})\right\}\frac{dT}{T}\,. \tag{3.47}$$

3. *The term* J_3^0. Using Theorems 3.9 and 3.10, we see that as $\varepsilon \to 0$,

$$J_3^0 \to J_3^1 = \int_1^{+\infty}\int_Y (\mu\chi_T)\frac{dT}{T}\,. \tag{3.48}$$

4. *The asymptotics of* I_4^3. Using (3.40), (3.41), (3.47), (3.48), we see that as $\varepsilon \to 0$,

$$I_4^3 - \left(\int_Y \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K c'_{\max}(\tilde N, g^{\tilde N})\right)$$
$$\mathrm{Log}(\varepsilon) \to I_4^4 = \int_0^1\left\{\int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\right.$$
$$\left. s^*({}^K c_T(LE, g^{LE}) - {}^K c_0(LE, g^{LE}))\right\}\frac{dT}{T}$$
$$+\int_1^{+\infty}\left\{\int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, s^*\, {}^K c_T(LE, g^{LE})\right.$$
$$\left.+\int_Y \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K c'_{\max}(\tilde N, g^{\tilde N})\right\}\frac{dT}{T}$$
$$+\int_0^1\int_Y \mu(\chi_T - \chi_0)\frac{dT}{T} + \int_1^{+\infty}\int_Y (\mu\chi_T)\frac{dT}{T}\,. \tag{3.49}$$

ε) *Evaluation of* I_4^4.

Theorem 3.15. *The following identity holds*

$$I_4^4 = \int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K \tilde e^X(LE, g^{LE})$$
$$+\int_Y \mu\mathbb{B}(N_{Y/LY}, N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}})$$
$$-\Gamma'(1)\int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K c'_{\max}(LE, g^{LE})\,. \tag{3.50}$$

Proof. Equation (3.50) follows from Propositions 1.13 and 2.21, and from (3.49). □

f) Matching the Divergences

We now establish an identity, which will lead us directly to the proof of Theorem 3.4.

Theorem 3.16. *The following identity holds*

$$\sum_{k=1}^{4} I_k^4 = 0 . \tag{3.51}$$

Proof. Recall that $\sum_{k=1}^{4} I_k^0 = 0$. The sum of the diverging terms at each of the four steps $t_0 \to 0$, $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$ is then tautologically zero. The identity (3.51) follows. We will here verify that the diverging terms effectively add up to zero. This will confirm that our previous calculations are correct. Also we will establish certain identities which will be useful when proving Theorem 3.4.

α) $t_0 \to 0$. By formulas (3.26) and (3.37), which concern the diverging terms I_2^0, I_4^0, we get

$$\left(\int_{LX} \mu\, {}^K\alpha_A {}^K c'_{\max}(LE, g^{LE}) - \int_{LX} \mu\, {}^K\alpha_\varepsilon {}^K c'_{\max}(LE, g^{LE}) \right) \operatorname{Log}(t_0) . \tag{3.52}$$

Since $\mu\, {}^K c'_{\max}(LE, g^{LE})$ is ∂_K and $\bar\partial_K$ closed, one concludes from Theorem 2.3 that (3.52) is effectively zero.

β) $A \to +\infty$. There is no divergence.

γ) $T_0 \to +\infty$. There is no divergence.

δ) $\varepsilon \to 0$. By formulas (3.22), (3.34), (3.49), which concern the terms I_1^3, I_3^3, I_4^3, we must calculate the expression

$$\begin{aligned}
&\left(\int_Y \mu 2\pi\omega^Y\, {}^K c_{\max}^{-1}(N_{Y/LY}, g^{N_{Y/LY}}) \right.\\
&\left. - \int_X \mu 2\pi\omega^X\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K c_{\max}(LE, g^{LE}) \right) \left(1 - \frac{1}{\varepsilon}\right)\\
&+ \left(\int_Y \mu ({}^K c_{\max}^{-1})' (N_{Y/LY}, g^{N_{Y/LY}}) \right.\\
&- \int_X \mu ({}^K c_{\max}^{-1})' (N_{X/LX}, g^{N_{X/LX}})\, {}^K c_{\max}(LE, g^{LE})\\
&\left. - \int_Y \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K c'_{\max}(\tilde N, g^{\tilde N}) \right) \operatorname{Log}(\varepsilon) .
\end{aligned} \tag{3.53}$$

Now by (2.69), we know that over X

$$f^*\, {}^K c_{\max}(LE, g^{LE}) = c_{\max}(LE_{|X}^{\{0\}}, g^{LE_{|X}^{\{0\}}})\, {}^K c_{\max}(LE_{|X}^{\{0\},\perp}, g^{LE_{|X}^{\{0\},\perp}}) . \tag{3.54}$$

The section $\sigma_{|X}$ of $LE^0_{|X}$ exactly vanishes on Y. Since the forms $f^*\mu$, $\omega^X \in P^X$ are closed, we deduce from Theorem 2.22, from (2.77) and (3.54),

$$\int_X \mu 2\pi\omega^X \, {}^K c^{-1}_{\max}(N_{X/LX}, g^{N_{X/LX}}) \, {}^K c_{\max}(LE, g^{LE})$$
$$= \int_Y \mu 2\pi\omega^Y \, {}^K c^{-1}_{\max}(N_{X/LX}, g^{N_{X/LX}}) \, {}^K c_{\max}(LE^{\{0\},\perp}_{|X}, g^{LE^{\{0\},\perp}_{|X}})$$
$$= \int_Y \mu 2\pi\omega^Y \, {}^K c^{-1}_{\max}(N_{X/LX}) \, {}^K c_{\max}(\tilde{N}) . \tag{3.55}$$

By (3.4), it is clear we have the identity

$${}^K c^{-1}_{\max}(N_{X/LX_{|Y}}) \, {}^K c_{\max}(\tilde{N}) = {}^K c^{-1}_{\max}(N_{Y/LY}) \quad \text{in} \quad P^Y/P^{Y,0} . \tag{3.56}$$

So from (3.55), (3.56), we see that

$$\int_X \mu 2\pi\omega^X \, {}^K c^{-1}_{\max}(N_{X/LX}, g^{N_{X/LX}}) \, {}^K c_{\max}(LE, g^{LE})$$
$$= \int_Y \mu 2\pi\omega^Y \, {}^K c^{-1}_{\max}(N_{Y/LY}) . \tag{3.57}$$

The same arguments as in (3.55) show that

$$\int_X \mu ({}^K c^{-1}_{\max})' (N_{X/LX}) \, {}^K c_{\max}(LE) = \int_Y \mu ({}^K c^{-1}_{\max})' (N_{X/LX}) \, {}^K c_{\max}(\tilde{N}) . \tag{3.58}$$

Clearly,

$$({}^K c^{-1}_{\max})' (N_{X/LX_{|Y}}) \, {}^K c_{\max}(\tilde{N})$$
$$= ({}^K c^{-1}_{\max})' (N_{Y/LY}) - {}^K c^{-1}_{\max}(N_{Y/LY}) \, {}^K c_{\max}(\tilde{N}) \, {}^K c'_{\max}(\tilde{N}) . \tag{3.59}$$

Equivalently,

$$({}^K c_{\max})' (N_{X/LX_{|Y}}) \, {}^K c_{\max}(\tilde{N})$$
$$= ({}^K c^{-1}_{\max})' (N_{Y/LY}) - {}^K c^{-1}_{\max}(N_{X/LX_{|Y}}) \, {}^K c'_{\max}(\tilde{N}) . \tag{3.60}$$

From (3.58), (3.60), we deduce that

$$\int_Y \mu ({}^K c^{-1}_{\max})' (N_{Y/LY}) - \int_X \mu ({}^K c^{-1}_{\max})' (N_{X/LX}) \, {}^K c_{\max}(LE)$$
$$- \int_Y \mu \, {}^K c^{-1}_{\max}(N_{X/LX}) \, {}^K c'_{\max}(\tilde{N}) = 0 . \tag{3.61}$$

Using (3.57), (3.61), we see that (3.53) is indeed equal to zero. The proof of Theorem 3.16 is completed. □

g) Proof of Theorem 3.4

Using Theorems 3.12–3.16, we get

$$
\begin{aligned}
&-\int_{LX} \mu\, {}^K S_{\omega^{LX}}\, {}^K c_{\max}(LE, g^{LE}) - \int_{LX} \mu\, {}^K \tilde{e}^{LX}(LE, g^{LE}) \\
&+\int_{LY} \mu\, {}^K S_{\omega^{LY}} + \int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K \tilde{e}^{LX}(LE, g^{LE}) \\
&+\int_Y \mu \mathbb{B}(N_{Y/LY}, N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}}) \\
&+\Gamma'(1) \Bigg[\int_Y \mu ({}^K c_{\max}^{-1})'(N_{Y/LY}) \\
&-\int_X \mu ({}^K c_{\max}^{-1})'(N_{X/LX})\, {}^K c_{\max}(LE) \\
&-\int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX})\, {}^K c'_{\max}(LE) + \int_{LX} \mu\, {}^K c'_{\max}(LE) \Bigg] \\
&+\int_Y \mu 2\pi \omega^Y\, {}^K c_{\max}^{-1}(N_{Y/LY}) \\
&-\int_X \mu 2\pi \omega^X\, {}^K c_{\max}^{-1}(N_{X/LX})\, {}^K c_{\max}(LE) = 0 .
\end{aligned}
\tag{3.62}
$$

Since $f^*\mu \in P^X$ and since $\partial f^*\mu = 0$, $\bar{\partial} f^*\mu = 0$, we deduce from Theorem 1.19 that

$$
\begin{aligned}
&\int_Y \mu \mathbb{B}(N_{Y/LY}, N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}}) \\
&\quad = -\int_Y \mu\, {}^K c_{\max}(\tilde{N}, g^{\tilde{N}})\, {}^K \widetilde{c_{\max}^{-1}}(N_{Y/LY}, N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}}) \\
&\qquad + \int_Y \mu\, {}^K c_{\max}^{-1}(N_{Y/LY})\, {}^K D(\tilde{N}) .
\end{aligned}
\tag{3.63}
$$

Also by (2.29) and (3.56), we obtain

$$
\begin{aligned}
\int_{LX} \mu\, {}^K c'_{\max}(LE) &= \int_X \mu\, {}^K c'_{\max}(LE)\, {}^K c_{\max}^{-1}(N_{X/LX}) , \\
\int_Y \mu\, {}^K c_{\max}^{-1}(N_{X/LX})\, {}^K c'_{\max}(\tilde{N}) &= \int_Y \mu\, {}^K c_{\max}^{-1}(N_{Y/LY}) \frac{{}^K c'_{\max}(\tilde{N})}{{}^K c_{\max}(\tilde{N})} .
\end{aligned}
\tag{3.64}
$$

From (3.57), (3.61)–(3.64), we get

$$\begin{aligned}
&-\int_{LX} \mu\, {}^K S_{\omega^{LX}}\, {}^K c_{\max}(LE, g^{LE}) - \int_{LX} \mu\, {}^K \tilde{e}^{LX}(LE, g^{LE}) \\
&+\int_{LY} \mu\, {}^K S_{\omega^{LY}} + \int_{X} \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K \tilde{e}^{X}(LE, g^{LE}) \\
&-\int_{Y} \mu\, {}^K c_{\max}(\tilde{N}, g^{\tilde{N}})\, {}^K \widetilde{c_{\max}^{-1}}(N_{Y/LY}, N_{X/LX|Y}, g^{N_{X/LX|Y}}) \\
&+\int_{Y} \mu\, {}^K c_{\max}^{-1}(N_{Y/LY}) \left({}^K D(\tilde{N}) + \Gamma'(1) \frac{{}^K c'_{\max}(\tilde{N})}{{}^K c_{\max}(\tilde{N})} \right) = 0\,. \qquad (3.65)
\end{aligned}$$

Using Proposition 1.21 and (3.65), we get (3.10). Also since the class ${}^K R$ is additive, we see that

$$\begin{aligned}
\int_Y \mu\, {}^K c_{\max}^{-1}(N_{Y/LY})\, {}^K R(\tilde{N}) = &\int_Y \mu\, {}^K c_{\max}^{-1}(N_{Y/LY})\, {}^K R(N_{X/LX}) \\
&- \int_Y \mu\, {}^K c_{\max}^{-1}(N_{Y/LY})\, {}^K R(N_{Y/LY})\,. \qquad (3.66)
\end{aligned}$$

By using (2.69), (2.74), (2.75), (3.56), (3.66), we thus find that

$$\begin{aligned}
\int_Y \mu\, {}^K c_{\max}^{-1}(N_{Y/LY})\, {}^K R(\tilde{N}) = &\int_X \mu\, {}^K c_{\max}^{-1}(N_{X/LX})\, {}^K R(N_{X/LX})\, {}^K c_{\max}(LE) \\
&- \int_Y \mu\, {}^K c_{\max}^{-1}(N_{Y/LY})\, {}^K R(N_{Y/LY})\,. \qquad (3.67)
\end{aligned}$$

Using (3.10), (3.67), we obtain (3.11). □

h) Proof of Theorem 3.8

By Theorem 2.5, it is clear that we may restrict ourselves to the case where $T \in \left[1, \frac{1}{u}\right]$.

If the support K of μ is included in $LX \backslash LY$, then the forms $\mu s^*\, {}^K c_T^*(LE, g^{LE})$ and their derivatives are uniformly bounded for $T \in [1, +\infty[$, and so, using again Theorem 2.5, (3.16) holds.

On the other hand as $T \to +\infty$, $|s^*\, {}^K c_T(LE, g^{LE})|$ grows at most like $T^{\dim LX}$. If the support of μ is included in $LX \backslash X$, there exists $c > 0$, $C > 0$ such that for $u \geq 1$, $T \leq \frac{1}{u}$,

$$|\mu\, {}^K \alpha_u s^*\, {}^K c_T(LE, g^{LE})| \leq c \exp\left(-\frac{C}{u}\right) \left(1 + \frac{1}{u^{\dim LX}}\right), \qquad (3.68)$$

and so (3.16) still holds.

So to prove Theorem 3.8, we may and we will assume that the support of μ is included in an arbitrarily small open neighborhood of Y in LX.

Take $y_0 \in Y$. Set

$$e = \dim N_{Y/X,y_0}\,, \qquad e' = \dim N_{Y/LY,y_0}\,, \qquad \tilde{e} = \dim \tilde{N}_{y_0}\,. \tag{3.69}$$

By (3.2), we find that

$$e + e' + \tilde{e} = \dim(N_{Y/LX})_{y_0}\,. \tag{3.70}$$

For $\eta > 0$, let $B_e(0,\eta)$, $B_{e'}(0,\eta)$, $B_{\tilde{e}}(0,\eta)$ be the open balls of center 0 and radius η in $\mathbb{C}^e = \mathbb{R}^{2e}$, $\mathbb{C}^{e'} = \mathbb{R}^{2e'}$, $\mathbb{C}^{\tilde{e}} = \mathbb{R}^{2\tilde{e}}$. Let $\mathscr{V}$ be an open neighborhood of y_0 in Y. If $\mathscr{V}$ and $\eta > 0$ are small enough, we can identify $\mathscr{V} \times B_e(0,\eta) \times B_{e'}(0,\eta) \times B_{\tilde{e}}(0,\eta)$ with an open neighborhood U_η of y_0 in LX.

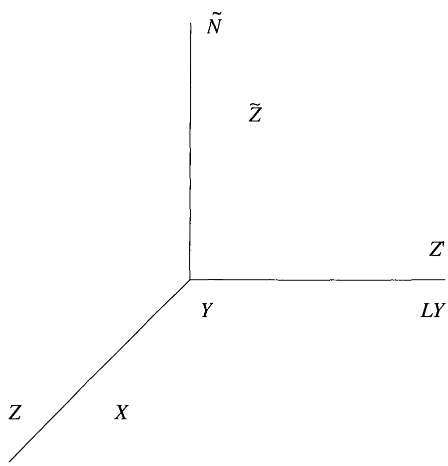

Fig. 2

Let $\sigma_{u,T}$ be the map from $\mathscr{V} \times \mathbb{R}^{2e} \times \mathbb{R}^{2e'} \times \mathbb{R}^{2\tilde{e}}$ into itself

$$\sigma_{u,T} : (y, Z, Z', \tilde{Z}) \to \left(y, \frac{Z}{\sqrt{T}}, \sqrt{u}\,Z', \sqrt{u}\,\tilde{Z}\right). \tag{3.71}$$

Assume that the support of μ is included in U_η. Clearly

$$\begin{aligned}
&\int_{LX} \mu\, {}^K\alpha_u s^{*\,K} c_T(LE, g^{LE}) \\
&\quad = \int_{\substack{y\in\mathscr{V} \\ |Z|\le\eta\sqrt{T} \\ |Z|,|Z'|\le\frac{\eta}{\sqrt{u}}}} (\sigma^*_{u,T}\mu)\,(\sigma^{*\,K}_{u,T}\alpha_u)\,\sigma^*_{u,T}(s^{*\,K}c_T(LE, g^{LE}))\,, \\
&\int_X \mu\, {}^K c^{-1}_{\max}(N_{X/LX}, g^{N_{X/LX}})\, s^{*\,K} c_T(LE, g^{LE}) \\
&\quad = \int_{\substack{y\in\mathscr{V} \\ |Z|\le\eta\sqrt{T}}} (\sigma^*_{0,T}\mu)\,(\sigma^{*\,K}_{0,T} c^{-1}_{\max}(N_{X/LX}, g^{N_{X/LX}}))\,\sigma^*_{0,T}(s^{*\,K}c_T(LE, g^{LE}))\,.
\end{aligned} \tag{3.72}$$

Set

$$\tau_T(y,Z,Z',\tilde{Z}) = \left(y,\ \frac{Z}{\sqrt{T}},\ Z',\tilde{Z}\right). \tag{3.73}$$

Recall that $N_{X/LX}$ is identified to the orthogonal bundle to TX in $TLX_{|X}$. As in (1.2), the connection $\nabla^{N_{X/LX}}$ induces a splitting $TN_{X/LX} = N_{X/LX} \oplus T^H N_{X/LX}$. If $U \in T_{\mathbb{R}} N_{X/LX}$, let U^V be the component of U in $T^H_{\mathbb{R}} N_{X/LX}$ with respect to this splitting. As in (1.3), we identify $J^{N_{X/LX}}$ with the 2-form $U, U' \in T_{\mathbb{R}} N_{X/LX} \to \langle U^V, J^{N_{X/LX}} U'^V\rangle$.

We consider $((y,Z),Z'+\tilde{Z})$ as lying in the total space of $N_{X/LX,\mathbb{R}}$. Therefore $J^{N_{X/LX}}$ is now a 2-form in the coordinates $(y,Z,Z',\tilde{Z})$. Similarly $R^{N_{X/LX}}_{(y,Z)}$ is a 2-form in the variables (y,Z) which lifts naturally to a 2-form in the variables $(y,Z,Z',\tilde{Z})$.

By proceeding as in [B2, proof of Theorem 3.2] and in [B4, proof of Theorem 1.3], we find that there exist $c>0$, $C>0$ such that if $0<u\le 1$, $T\ge 1$, $|Z|\le \eta\sqrt{T}$, $|Z'|\le \dfrac{\eta}{\sqrt{u}}$, $|\tilde{Z}|\le \dfrac{\eta}{\sqrt{u}}$, then

$$\begin{aligned}
&|\sigma^*_{u,T}{}^K\alpha_u| \le c\exp(-C(|Z'|^2+|\tilde{Z}|^2)),\\
&|\sigma^*_{u,T}{}^K\alpha_u(y,Z,Z',\tilde{Z}) - \tau^*_T(\exp\{-\tfrac{1}{2}\,|J^{N_{X/LX}}_{(y,Z)}(Z'+\tilde{Z})|^2\\
&\qquad +\tfrac{1}{2}\,\langle J^{N_{X/LX}}_{(y,Z)} R^{N_{X/LX}}_{(y,Z)}(Z'+\tilde{Z}),(Z'+\tilde{Z})\rangle + J^{N_{X/LX}}_{(y,Z)}\})|\\
&\quad \le c\sqrt{u}\exp(-C(|Z'|^2+|\tilde{Z}|^2)).
\end{aligned} \tag{3.74}$$

Set

$$\begin{aligned}
h_T(y,Z,Z',\tilde{Z}) &= \left(y,\ \frac{Z}{\sqrt{T}},\ Z',\ \frac{\tilde{Z}}{\sqrt{T}}\right),\\
j_{u,T}(y,Z,Z',\tilde{Z}) &= (y,Z,\sqrt{u}\,Z',\sqrt{uT}\,\tilde{Z}).
\end{aligned} \tag{3.75}$$

Clearly

$$\sigma_{u,T} = h_T j_{u,T}. \tag{3.76}$$

Therefore

$$\sigma^*_{u,T}(s^{*\,K}c_T(LE,g^{LE})) = j^*_{u,T}h^*_T(s^{*K}c_T(LE,g^{LE})), \tag{3.77}$$

and so

$$\begin{aligned}
&\sigma^*_{u,T}(s^{*\,K}c_T(LE,g^{LE})) - \sigma^*_{0,T}(s^{*\,K}c_T(LE,g^{LE}))\\
&\quad = (j^*_{u,T} - j^*_{0,T})\,h^*_T(s^{*\,K}c_T(LE,g^{LE})).
\end{aligned} \tag{3.78}$$

Now by proceeding as in [BGS5, proof of Theorem 3.12] and using the fact that $d\sigma : N_{LY/LX} \to LE_{|Y}$ is an isometry, it is clear that as $T\to+\infty$,

$$\begin{aligned}
h^*_T(s^{*\,K}c_T(LE,g^{LE}))\,(y,Z,Z',\tilde{Z}) \to \frac{\partial}{\partial b}\Bigg[-\det\left(\frac{{}^K R^{LE}}{2\pi i}(y,Z')-b\right)\\
\exp\left\{-\left(\frac{|Z+\tilde{Z}|^2}{2} + ({}^K R^{LE}(y,Z') + 2\pi b\mathbf{J}^{LE})^{-1}\right)\right\}\Bigg]_{b=0}.
\end{aligned} \tag{3.79}$$

Also, we find easily that for any differential operator P with constant coefficients in the variables $y, Z, Z', \tilde{Z}$, there exist $c_P > 0$, $C_P > 0$ such that for any $T \geq 1$, $|Z| \leq \eta\sqrt{T}$, $|Z'| \leq \sqrt{\eta}$, $|\tilde{Z}| \leq \eta\sqrt{T}$, then

$$|Ph_T^* s^{* \, K} c_T(LE, g^{LE})(y, Z, Z', \tilde{Z})| \leq c_P \exp(-C_P(|Z|^2 + |\tilde{Z}|^2)) . \tag{3.80}$$

Using (3.80), we see that there exist $c' > 0$, $C' > 0$ such that if $0 < u \leq 1$, $1 \leq T \leq \frac{1}{u}$, $|Z| \leq \eta\sqrt{T}$, $|Z'| \leq \frac{\eta}{\sqrt{u}}$, $|\tilde{Z}| \leq \frac{\eta}{\sqrt{u}}$, then

$$\begin{aligned} &|(\sigma_{u,T}^* s^{* \, K} c_T(LE, g^{LE}) - \sigma_{0,T}^* s^{* \, K} c_T(LE, g^{LE}))(y, Z, Z', \tilde{Z})| \\ &\quad \leq c'(\sqrt{u} + \sqrt{u}|Z'| + \sqrt{uT} + \sqrt{uT}|\tilde{Z}|) \exp(-C'|Z|^2) . \end{aligned} \tag{3.81}$$

From (3.74), (3.81), we find that there exist $c'' > 0$, $C'' > 0$ such that if $0 < u \leq 1$, $1 \leq T \leq \frac{1}{u}$, $|Z| \leq \eta\sqrt{T}$, $|Z'| \leq \frac{\eta}{\sqrt{u}}$. $|\tilde{Z}| \leq \frac{\eta}{\sqrt{u}}$, then

$$\begin{aligned} &|(\sigma_{u,T}^* \mu)(\sigma_{u,T}^{* \, K} \alpha_u)(\sigma_{u,T}^* s^{* \, K} c_T(LE, g^{LE}))(y, Z, Z', \tilde{Z}) \\ &\quad - (\sigma_{0,T}^* \mu)\tau_T^*(\exp\{\tfrac{1}{2}\langle J_{(y,Z)}^{N_{X/LX}} R_{(y,Z)}^{N_{X/LX}}(Z' + \tilde{Z}), (Z' + \tilde{Z})\rangle \\ &\quad - \tfrac{1}{2}|J_{(y,Z)}^{N_{X/LX}}(Z' + \tilde{Z})|^2 + J_{(y,Z)}^{N_{X/LX}}\})(\sigma_{0,T}^* s^{* \, K} c_T(LE, g^{LE}))(y, Z, Z', \tilde{Z})| \\ &\leq c''(\sqrt{u} + \sqrt{u}|Z'| + \sqrt{uT} + \sqrt{uT}|\tilde{Z}|) \exp(-C''(|Z|^2 + |Z'|^2 + |\tilde{Z}|^2)) . \end{aligned} \tag{3.82}$$

As in (1.44), we get

$$\begin{aligned} &\int_{N_{X/LX}} \exp\{\tfrac{1}{2}\langle J^{N_{X/LX}} R^{N_{X/LX}} U, U\rangle - \tfrac{1}{2}|J^{N_{X/LX}} U|^2 + J^{N_{X/LX}}\} \\ &\quad = {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) . \end{aligned} \tag{3.83}$$

Using (3.82), (3.83) we easily obtain the inequality (3.16) in the special case considered above.

By partition of unity, we get (3.16) in full generality. □

i) Proof of Theorem 3.9

It is clear that since on $LX \backslash Y$, either K or σ are nonzero, if $\mathscr{U}$ is an arbitrary open neighborhood of Y in LX, as $u \to 0$,

$$\int_{LX \backslash \mathscr{U}} \mu \, {}^K \alpha_u s^{* \, K} c_{T/u}(LE, g^{LE}) \to 0 . \tag{3.84}$$

We now take $y_0 \in Y$ and we use the notation in Sect. 3h). In particular, we choose $\mathscr{V}$ and $\eta > 0$ as in the proof of Theorem 3.8. Let σ_u be the map from $\mathscr{V} \times \mathbb{R}^{2e} \times \mathbb{R}^{2e'} \times \mathbb{R}^{2\tilde{e}}$ into itself

$$\sigma_u(y, Z, Z', \tilde{Z}) \to (y, \sqrt{u}Z, \sqrt{u}Z', \sqrt{u}\tilde{Z}) . \tag{3.85}$$

Then

$$\int_{\mathscr{U}_\eta} \mu\, {}^K\alpha_u s^*\, {}^K c_{T/u}(LE, g^{LE}) = \int_{\substack{y\in\mathscr{Y} \\ |Z|,|Z'|,|\tilde{Z}|\leq\eta/\sqrt{u}}} (\sigma_u^*\mu)(\sigma_u^*\, {}^K\alpha_u)\,\sigma_u^*(s^*\, {}^K c_{T/u}(LE, g^{LE}))\,. \tag{3.86}$$

Let ϱ, τ be the maps

$$\begin{aligned} &\varrho:(y, Z, Z', \tilde{Z}) \to (y, Z', \tilde{Z}) \in N_{X/LX_{|Y}}\,, \\ &\tau:(y, Z, Z', \tilde{Z}) \to (y, Z, \tilde{Z})\,. \end{aligned} \tag{3.87}$$

As explained after Eq. (3.73), ${}^K\alpha(N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}})$ can be considered as a form in the variables $(y, Z', \tilde{Z})$. By (3.74) and by [BGS4, proof of Theorem 3.12] (see also Eq. (3.79)), we know that

$$\begin{aligned} &\lim_{u\to 0} \sigma_u^{*K}\alpha_u = \varrho^*\, {}^K\alpha(N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}})\,, \\ &\lim_{u\to 0} \sigma_u^* s^*\, {}^K c_{T/u} = \tau^*\, {}^K c_T(N_{LY/LX_{|Y}}, g^{N_{LY/LX_{|Y}}})\,. \end{aligned} \tag{3.88}$$

By (3.86), (3.88) and by an easy application of the dominated convergence theorem, we find that

$$\lim_{u\to 0}\int_{LX} \mu\, {}^K\alpha_u s^*\, {}^K c_{T/u}(LE, g^{LE}) = \int_Y \mu \left[\int_{N_{Y/LX}} \varrho^*\, {}^K\alpha(N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}}) \right. \\ \left. \tau^*\, {}^K c_T(N_{LY/LX_{|Y}}, g^{N_{LY/LX_{|Y}}})\right]\,. \tag{3.89}$$

Recall that forms $a_T(N_{Y/X}, g^{N_{Y/X}})$ and $c_T(N_{Y/X}, g^{N_{Y/X}})$ were defined in [B1, Definition 3.1]. These forms are exactly the forms ${}^K a_T(N_{Y/X}, g^{N_{Y/X}})$ and ${}^K c_T(N_{Y/X}, g^{N_{Y/X}})$, with $J^{N_{Y/X}} = 0$.

Let p, p' be the projection maps $p: N_{LY/LX_{|Y}} \to N_{Y/X}, p': N_{LY/LX_{|Y}} \to \tilde{N}$. By (2.38), (2.74), (2.75), we get

$$\begin{aligned} {}^K c_T(N_{LY/LX_{|Y}}, g^{N_{LY/LX_{|Y}}}) &= p^* c_T(N_{Y/X}, g^{N_{Y/X}})\, p'^*\, {}^K a_T(\tilde{N}, g^{\tilde{N}}) \\ &\quad + p^* a_T(N_{Y/X}, g^{N_{Y/X}})\, p'^*\, {}^K c_T(\tilde{N}, g^{\tilde{N}})\,. \end{aligned} \tag{3.90}$$

By [MQ, Theorem 4.10], [BGS5, Eq. (3.58)], or by an easy direct calculation, we obtain

$$\int_{N_{Y/X}} a_T(N_{Y/X}, g^{N_{Y/X}}) = 1\,, \qquad \int_{N_{Y/X}} c_T(N_{Y/X}, g^{N_{Y/X}}) = 0\,. \tag{3.91}$$

Recall that w is the projection $N_{X/LX_{|Y}} \to \tilde{N}$. From (3.89)–(3.91), we deduce that

$$\lim_{u\to 0} \int_{LX} \mu\, {}^K\alpha_u s^*\, {}^K c_{T/u}(LE, g^{LE})$$
$$= \int_Y \mu \left[\int_{N_{X/LX_{|Y}}} {}^K\alpha(N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}})\, w^*\, {}^K c_T(\tilde{N}, g^{\tilde{N}}) \right]. \tag{3.92}$$

Using (3.92) and partition of unity, Theorem 3.9 holds in full generality. □

j) Proof of Theorem 3.10

We use the same notation as in the proofs of Theorems 3.8 and 3.9. If the support to μ is included in $LX \backslash LY$, there exist $c > 0$, $C > 0$ such that

$$|\mu s^*\, {}^K c_{T/u}(LE, g^{LE})| \leq c \exp\left(- \frac{CT}{u} \right). \tag{3.93}$$

The estimate (3.18) is then trivial.

If the support of μ is included in $LX \backslash X$, for $0 < u \leq 1$, the forms $\mu\, {}^K\alpha_u$ and their derivatives are uniformly bounded. Equation (3.18) then follows from (2.56). So as in the proof of Theorem 3.8, we may and we will assume that the support of μ is included in an arbitrary small open neighborhood of Y in LX.

We assume that y_0, $\mathscr{V}$, $\eta > 0$, U_η, are chosen as in the proof of Theorem 3.8, and also that the support of μ is included in U_η. Set

$$k_{u,T}(y, Z, Z', \tilde{Z}) = \left(y, \sqrt{\frac{u}{T}} Z, \sqrt{u} Z', \sqrt{\frac{u}{T}} \tilde{Z} \right). \tag{3.94}$$

Then

$$\int_{LX} \mu\, {}^K\alpha_u s^*\, {}^K c_{T/u}(LE, g^{LE})$$
$$= \int_{\substack{y \in \mathscr{V} \\ |Z|, |\tilde{Z}| \leq \sqrt{T/u}\,\eta \\ |Z'| \leq \frac{1}{\sqrt{u}} \eta}} (k_{u,T}^* \mu)\, (k_{u,T}^*\, {}^K\alpha_u)\, (k_{u,T}^* s^*\, {}^K c_{T/u}(LE, g^{LE})). \tag{3.95}$$

Set

$$\tau_u(y, Z, Z', \tilde{Z}) = (y, Z, \sqrt{u} Z', \tilde{Z}). \tag{3.96}$$

By proceeding as in [B2, proof of Theorem 3.2] and in [BGS5, proof of Theorem 3.12], we find that there exist $c \geq 0$, $C \geq 0$, such that if $|Z|, |\tilde{Z}| \leq \sqrt{\frac{T}{u}}\, \eta$, $|Z'| \leq \frac{1}{\sqrt{u}} \eta$, then

$$|k_{u,T}^* s^*\, {}^K c_{T/u}(LE, g^{LE}) - \tau_u^*\, {}^K c_{T/u}(N_{LY/LX}, g^{N_{LY/LX}})|$$
$$\leq c \sqrt{\frac{u}{T}} \exp(-C(|Z|^2 + |\tilde{Z}|^2)). \tag{3.97}$$

Set

$$\ell_u(y, Z, Z', \tilde{Z}) = (y, Z, \sqrt{u}\,Z', \sqrt{u}\,\tilde{Z}),$$
$$m_{u,T}(y, Z, Z', \tilde{Z}) = \left(y, \sqrt{\frac{u}{T}}\,Z, Z', \frac{\tilde{Z}}{\sqrt{T}}\right). \tag{3.98}$$

Then

$$k_{u,T} = \ell_u m_{u,T}\,. \tag{3.99}$$

Therefore

$$k^*_{u,T}(\mu^K \alpha_u) = m^*_{u,T}\ell^*_u(\mu^K \alpha_u)\,. \tag{3.100}$$

By proceeding as in [B2, proof of Theorem 3.2] and in [B4, proof of Theorem 1.3], we know that if P is any differential operator with constant coefficients, there exist $c > 0$, $C > 0$, such that if $u \in]0, 1]$, $|Z| \leq \eta$, $|Z'| \leq \eta/\sqrt{u}$, $|\tilde{Z}| \leq \eta/\sqrt{u}$, then

$$|P\ell^*_u(\mu^K \alpha_u)| \leq c\exp(-C(|Z'|^2 + |\tilde{Z}|^2))\,. \tag{3.101}$$

Forms on $\mathscr{Y} \times \mathbb{R}^{2e} \times \mathbb{R}^{2e'} \times \mathbb{R}^{2\tilde{e}}$ can be decomposed according to their partial degree in the Grassmann variables associated to the variables $(Z, \tilde{Z})$. If ω is a form, let ω^0 be the piece of ω of degree 0 in these Grassmann variables, and let $\omega^{>0}$ be the piece of ω which has nonzero degree in these variables, so that

$$\omega = \omega^0 + \omega^{>0}\,. \tag{3.102}$$

In particular

$$\ell^*_u(\mu^K \alpha_u) = (\ell^*_u(\mu^K \alpha_u))^0 + (\ell^*_u(\mu^K \alpha_u))^{>0}\,. \tag{3.103}$$

From (3.100), (3.101), we deduce that if $|Z| \leq \sqrt{T/u}\,\eta$, $|Z'| \leq \eta/\sqrt{u}$, $|\tilde{Z}| \leq \sqrt{T/u}\,\eta$, then

$$|(k^*_{u,T}\mu^K \alpha_u)^{>0}\,(y, Z, Z', \tilde{Z})| \leq \frac{c}{\sqrt{T}} \exp\left(- C\left(|Z'|^2 + \frac{|\tilde{Z}|^2}{T}\right)\right). \tag{3.104}$$

Also by (3.100), (3.101), under the same conditions on $Z, Z', \tilde{Z}$, we get

$$|(k^*_{u,T}(\mu^K \alpha_u))^0\,(y, Z, Z', \tilde{Z}) - (k^*_{u,T}(\mu^K \alpha_u))^0\,(y, 0, Z', 0)|$$
$$\leq c\left(\sqrt{\frac{u}{T}}\,|Z| + \frac{1}{\sqrt{T}}\,|\tilde{Z}|\right) \exp\left(- C\left(|Z'|^2 + \frac{|\tilde{Z}|^2}{T}\right)\right). \tag{3.105}$$

Finally by the obvious analogue of (3.91), or by a trivial calculation, we see that

$$\int_{N_{LY/LX}} {}^K c_{T/u}(N_{LY/LX}, g^{N_{LY/LX}}) = 0\,. \tag{3.106}$$

By combining (3.97), (3.104), (3.106), we get (3.18). Using partition of unity, we obtain (3.18) in full generality. □

k) Some Remarks on the Behaviour of the Term I_4^3 as $\varepsilon \to 0$

By (2.60), (3.38), it is clear that

$$I_4^3 = \int_{LX} \mu^K \alpha_\varepsilon {}^K\tilde{e}^{LX}(LE, g^{LE}) - \Gamma'(1) \int_{LX} \mu^K \alpha_\varepsilon {}^K c'_{\max}(LE, g^{LE}). \tag{3.107}$$

By Theorems 2.3 and 2.5, we know that

$$\int_{LX} \mu^K \alpha_\varepsilon {}^K c'_{\max}(LE, g^{LE}) = \int_X \mu^K c_{\max}^{-1}(N_{X/LX})^K c'_{\max}(LE). \tag{3.108}$$

Set

$$I_4'^3 = \int_{LX} \mu^K \alpha_\varepsilon {}^K\tilde{e}^{LX}(LE, g^{LE}). \tag{3.109}$$

Using (3.107)–(3.109), it is clear that, to calculate the asymptotics of I_4^3 as $\varepsilon \to 0$, we may instead replace I_4^3 by $I_4'^3$.

By (2.63), we find that

$$\begin{aligned} I_4'^3 = \int_{LX} \mu^K \alpha_\varepsilon \Bigg\{ - \frac{\partial}{\partial b} \Bigg[\det\left(- \frac{{}^K R^{LE}}{2i\pi} - b\right) \Bigg(\operatorname{Log}\left(\frac{|s|^2}{2}\right) \\ - \sum_{k=1}^{\dim LE-1} \frac{2^k s^*}{k|s|^{2k}} ((-({}^K R^{LE} + 2\pi b \mathbf{J}^{LE})^{-1})^k) \Bigg) \Bigg] \Bigg\}_{b=0}. \end{aligned} \tag{3.110}$$

Observe that by (2.69),

$$\begin{aligned} &\det\left(- \frac{{}^K R^{LE_{|X}}}{2i\pi} - b\right) \\ &= \det\left(- \frac{R^{LE^0_{|X}}}{2i\pi} - b\right) \det\left(- \frac{{}^K R^{LE^{0,\perp}_{|X}}}{2i\pi} - b\right). \end{aligned} \tag{3.111}$$

Using (3.111) and the Mathai-Quillen formalism [MQ], we see that expressions over X like

$$\det\left(- \frac{{}^K R^{LE_{|X}}}{2i\pi} - b\right) (-(R^{LE^0_{|X}} + 2\pi b \mathbf{J}^{LE^0_{|X}})^{-1})^k \quad (0 \leq k \leq \dim E)$$

make sense.

In the sequel, we assume that X is connected, so that $\dim LE^0_X$ is a well-defined constant.

Theorem 3.17. *The following identities hold*

$$\lim_{\varepsilon\to 0}\int_{LX} \mu^K\alpha_\varepsilon \frac{\partial}{\partial b}\Bigg[-\det\left(-\frac{{}^K R^{LE}}{2i\pi}-b\right)\left(\mathrm{Log}\left(\frac{|s|^2}{2}\right)\right.$$

$$\left.\left. - \sum_1^{\dim LE^0_{|X}-1} \frac{2^k s^*}{k|s|^{2k}}\left(\left(-({}^K R^{LE}+2\pi b \mathbf{J}^{LE})^{-1}\right)^k\right)\right)\right]_{b=0}$$

$$= \int_X \mu^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \frac{\partial}{\partial b}\Bigg[-\det\left(-\frac{{}^K R^{LE}}{2i\pi}-b\right)\left(\mathrm{Log}\left(\frac{|s|^2}{2}\right)\right.$$

$$\left.\left. - \sum_1^{\dim LE^0_{|X}-1} \frac{2^k s^*}{k|s|^{2k}}\left(\left(-(R^{LE^0_{|X}}+2\pi b \mathbf{J}^{LE^0_{|X}})^{-1}\right)^k\right)\right)\right]_{b=0},$$

$$\lim_{\varepsilon\to 0}\int_{LX} \mu\alpha_\varepsilon \frac{\partial}{\partial b}\Bigg[-\det\left(-\frac{{}^K R^{LE}}{2i\pi}-b\right)$$

$$\left.\sum_{\dim LE^0_{|X}+1}^{\dim LE-1} \frac{2^k}{k|s|^{2k}} s^*\left(\left(-({}^K R^{LE}+2\pi b \mathbf{J}^{LE})^{-1}\right)^k\right)\right)\Bigg]_{b=0}$$

$$= \int_Y \mu\Bigg\{\int_{N_{X/LX_{|Y}}} {}^K\alpha(N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}}) \frac{\partial}{\partial b}\Bigg[\det\left(-\frac{{}^K R^{LE}}{2i\pi}-b\right)$$

$$w^*\left(\sum_{k=1}^{\dim \tilde{N}-1} \frac{2^k}{k|\tilde{Z}|^{2k}}\left(-({}^K R^{\tilde{N}}+2\pi b \mathbf{J}^{\tilde{N}})^{-1}\right)^k\right)\Bigg]_{b=0}\Bigg\}. \tag{3.112}$$

Proof. Observe that on X, the function $\mathrm{Log}\left(\frac{|s|^2}{2}\right)$ is integrable. Also for $1 \leq j <. \dim E = \dim N_{Y/X}$, one verifies that $\frac{1}{|s|^{2j}}$ is integrable over X. Over X, σ is a section of $LE^0_{|X}$. Some easy analysis, which essentially involves dominated convergence, then shows that the first identity in (3.112) holds.

If $\mathscr{U}$ is an open neighborhood of X in LX, it is clear that for any k, $1 \leq k \leq \dim LE - 1$, then

$$\lim_{\varepsilon\to 0}\int_{LX\setminus\mathscr{U}} \mu^K\alpha^\varepsilon \frac{\partial}{\partial b}\Bigg[-\det\left(-\frac{{}^K R^{LE}}{2i\pi}-b\right)\frac{2^k}{k|s|^{2k}}$$

$$s^*\left(\left(-({}^K R^{LE}+2\pi b \mathbf{J}^{LE})^{-1}\right)^k\right)\Bigg]_{b=0} = 0. \tag{3.113}$$

Recall that over X, σ is a section of $LE^0_{|X}$. It follows that for $k > \dim LE^0_{|X}$, over X

$$\det\left(-\frac{{}^K R^{LE}}{2i\pi}-b\right) s^*\left(-({}^K R^{LE}+2\pi b \mathbf{J}^{LE})^{-1}\right)^k = 0. \tag{3.114}$$

Let $\mathscr{U}'$ be an open set in LX which is at a positive distance from Y. From (3.114), we easily deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathscr{U}'} \mu\, {}^K\alpha_\varepsilon \frac{\partial}{\partial b}\Bigg[- \det\left(- \frac{{}^K R^{LE}}{2i\pi} - b\right)\left(\operatorname{Log}\left(\frac{|s|^2}{2}\right)\right. \\ \left.\left. - \sum_{k=1}^{\dim LE-1} \frac{2^k}{k|s|^{2k}} s^*((-({}^K R^{LE} + 2\pi b \mathbf{J}^{LE})^{-1})^k)\right)\Bigg]_{b=0} \\ = \int_{X\cap\mathscr{U}'} \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}}) \frac{\partial}{\partial b}\Bigg[- \det\left(- \frac{{}^K R^{LE}}{2i\pi} - b\right)\left(\operatorname{Log}\left(\frac{|s|^2}{2}\right)\right. \\ \left.\left. - \sum_{j=1}^{\dim LE^0_{|X}} \frac{2^k}{k|s|^{2k}} s^*((-({}^K R^{LE} + 2\pi b \mathbf{J}^{LE})^{-1})^k)\right)\Bigg]_{b=0}. \end{aligned} \tag{3.115}$$

Equivalently, using Remark 2.23, we find that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathscr{U}'} \mu\, {}^K\alpha_\varepsilon\, {}^K\tilde{e}^{LX}(LE, g^{LE}) \\ = \int_{X\cap\mathscr{U}'} \mu\, {}^K c_{\max}^{-1}(N_{X/LX}, g^{N_{X/LX}})\, {}^K\tilde{e}^X(LE, g^{LE}). \end{aligned} \tag{3.116}$$

Now the restriction of the current ${}^K\tilde{e}^X(LE, g^{LE})$ to $X\backslash Y$ is generally not locally integrable near Y because of the singular term $\dfrac{1}{|s|^{2\dim LE^0_{|X}}} = \dfrac{1}{|s|^{2\dim N_{Y/X}}}$ which appears in the analogue of (2.63). It follows in particular from (3.115) that for $k > \dim LE^0_{|X}$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathscr{U}'} \mu\, {}^K\alpha_\varepsilon \frac{\partial}{\partial b}\Bigg[- \det\left(- \frac{{}^K R^{LE}}{2i\pi} - b\right) \frac{2^k}{k|s|^{2k}} \\ s^*((-({}^K R^{LE} + 2\pi b \mathbf{J}^{LE})^{-1})^k)\Bigg]_{b=0} = 0. \end{aligned} \tag{3.117}$$

For $k > \dim LE^0_{|X}$, we now will study

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{LX} \mu\, {}^K\alpha_\varepsilon \frac{\partial}{\partial b}\Bigg[- \det\left(- \frac{{}^K R^{LE}}{2i\pi} - b\right) \frac{2^k}{k|s|^{2k}} \\ s^*((-({}^K R^{LE} + 2\pi b \mathbf{J}^{LE})^{-1})^k)\Bigg]_{b=0}. \end{aligned} \tag{3.118}$$

Let $\mathscr{U}$ be an arbitrary small open neighborhood of Y in LX. In view of (3.117), it is equivalent to study

$$\lim_{\varepsilon\to 0}\int_{\mathscr{U}} \mu^K\alpha_\varepsilon \frac{\partial}{\partial b}\Big[-\det\Big(-\frac{{}^K R^{LE}}{2i\pi}-b\Big)\frac{2^k}{k|s|^{2k}}$$
$$s^*((-({}^K R^{LE}+2\pi b\mathbf{J}^{LE})^{-1})^k)\Big]_{b=0}. \tag{3.119}$$

We now take $y_0\in Y$, $\eta>0$ as in the proof of Theorem 3.8, and we use the same notation. Set

$$\sigma_\varepsilon(y,Z,Z',\tilde{Z})=(y,\sqrt{\varepsilon}Z,\sqrt{\varepsilon}Z',\sqrt{\varepsilon}\tilde{Z}). \tag{3.120}$$

Clearly

$$\int_{\mathscr{U}_\eta} \mu^K\alpha_\varepsilon \frac{\partial}{\partial b}\Big[-\det\Big(-\frac{{}^K R^{LE}}{2i\pi}-b\Big)\frac{2^k}{k|s|^{2k}}$$
$$s^*((-({}^K R^{LE}+2\pi b\mathbf{J}^{LE})^{-1})^k)\Big]_{b=0}$$
$$=\int_{\substack{y\in\mathscr{Y}\\ |Z|,|Z'|,|\tilde{Z}|\le\eta/\sqrt{\varepsilon}}} (\sigma_\varepsilon^*\mu)\,\sigma_\varepsilon^*\Big({}^K\alpha_\varepsilon\frac{\partial}{\partial b}\Big[-\det\Big(-\frac{{}^K R^{LE}}{2i\pi}-b\Big)\Big)$$
$$\frac{2^k}{k|s|^{2k}}s^*((-({}^K R^{LE}+2\pi b\mathbf{J}^{LE})^{-1})^k)\Big]_{b=0}. \tag{3.121}$$

Let p,q be the obvious linear maps $N_{Y/LX}\to N_{X/LX_{|Y}}$, $N_{Y/LX}\to N_{LY/LX_{|Y}}=LE_{|Y}$. Using (3.74) and an easy argument on the asymptotic behaviour of $\sigma_\varepsilon^*\dfrac{1}{|s|^{2k}}$, we find that as $\varepsilon\to 0$,

$$\sigma_\varepsilon^*\Big({}^K\alpha_\varepsilon\frac{\partial}{\partial b}\Big[-\det\Big(-\frac{{}^K R^{LE}}{2i\pi}-b\Big)\frac{2^k}{k|s|^{2k}}s^*((-({}^K R^{LE}+2\pi b\mathbf{J}^{LE})^{-1})^k)\Big]_{b=0}\Big)$$
$$\to p^*\,{}^K\alpha(N_{X/LX_{|Y}},g^{N_{X/LX_{|Y}}})\frac{\partial}{\partial b}\Big[-\det\Big(-\frac{{}^K R^{LE}}{2i\pi}-b\Big)\frac{2^k}{k(|Z|^2+|\tilde{Z}|^2)^k}$$
$$q^*((-({}^K R^{LE}+2\pi b\mathbf{J}^{LE})^{-1})^k)\Big]_{b=0}. \tag{3.122}$$

Set $E=LE^0_{|Y}$. Equivalently $E=N_{Y/X}$. If $r,\tilde{r}$ are the projections $LE_{|Y}\to E$, $LE_{|Y}\to LE^{0,\perp}_{|Y}=\tilde{N}$, then

$$(-({}^K R^{LE}+2\pi b\mathbf{J}^{LE})^{-1})^k=\sum_{\ell=0}^{k}C_k^\ell r^*((-(R^E+2\pi b\mathbf{J}^E)^{-1})^\ell)$$
$$\tilde{r}^*((-({}^K R^{\tilde{N}}+2\pi b\mathbf{J}^{\tilde{N}})^{-1})^{k-\ell}). \tag{3.123}$$

Let dv_E be the oriented volume form on E. Then one has the identities

$$\begin{gathered}\det\left(-\frac{{}^K R^{LE}}{2i\pi}-b\right)=\det\left(-\frac{R^E}{2i\pi}-b\right)\det\left(-\frac{{}^K R^{\tilde N}}{2i\pi}-b\right),\\ \det\left(-\frac{R^E}{2i\pi}-b\right)(-(R^E+2\pi b\mathbf{J}^E)^{-1})^{\dim E}=\left(\frac{1}{2\pi}\right)^{\dim E}(\dim E)!\,dv_E\,.\end{gathered}\tag{3.124}$$

Moreover a trivial calculation shows that if $k>\dim E$, then for $\tilde Z\in\tilde N\backslash\{0\}$,

$$\int_{\mathbb{R}^{2\dim E}}\frac{dv_E(Z)}{(|Z|^2+|\tilde Z|^2)^k}=\frac{\Gamma(k-\dim E)\,\pi^{\dim E}}{\Gamma(k)\,|\tilde Z|^{2(k-\dim E)}}\,.\tag{3.125}$$

From (3.121)–(3.125) and from some non-entirely trivial algebra, we deduce that if $k>\dim E$

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathscr{U}_\eta}\mu\,{}^K\alpha_\varepsilon\frac{\partial}{\partial b}\Bigg[-\det\left(-\frac{{}^K R^{LE}}{2i\pi}-b\right)\frac{2^k}{k|s|^{2k}}\\&\qquad s^*((-({}^K R^{LE}+2\pi b\mathbf{J}^{LE})^{-1})^k)\Bigg]_{b=0}\\&=\int_Y\mu\int_{N_{X/LX_{|Y}}}{}^K\alpha(N_{X/LX_{|Y}},g^{N_{X/LX_{|Y}}})\\&\qquad\frac{\partial}{\partial b}\Bigg(-\det\left(-\frac{{}^K R^{\tilde N}}{2i\pi}-b\right)\left(\frac{1}{2\pi}\right)^{\dim E}(\dim E)!\,\frac{\Gamma(k-\dim E)}{\Gamma(k)}\,\pi^{\dim E}\\&\qquad C_k^{\dim E}\,\frac{2^k}{k}\,w^*\left(\frac{1}{|\tilde Z|^{2(k-\dim E)}}(-({}^K R^{\tilde N}+2\pi b\mathbf{J}^{\tilde N})^{-1})^{(k-\dim E)}\right)\Bigg]_{b=0}.\end{aligned}\tag{3.126}$$

Now,

$$\left(\frac{1}{2\pi}\right)^{\dim E}(\dim E)!\,\frac{\Gamma(k-\dim E)}{\Gamma(k)}\,\pi^{\dim E}C_k^{\dim E}\,\frac{2^k}{k}=\frac{2^{k-\dim E}}{k-\dim E}\,.\tag{3.127}$$

Using (3.117), (3.126), (3.127) and partition of unity, we see that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{LX}\mu\,{}^K\alpha_\varepsilon\frac{\partial}{\partial b}\Bigg[-\det\left(-\frac{{}^K R^{LE}}{2i\pi}-b\right)\frac{2^k}{k|s|^{2k}}\\&\qquad s^*((-({}^K R^{LE}+2\pi b\mathbf{J}^{LE})^{-1})^k)\Bigg]_{b=0}\\&=\int_Y\mu\Bigg\{\int_{N_{X/LX_{|Y}}}{}^K\alpha(N_{X/LX_{|Y}},g^{N_{X/LX_{|Y}}})w^*\frac{\partial}{\partial b}\Bigg[-\det\left(-\frac{{}^K R^{\tilde N}}{2i\pi}-b\right)\\&\qquad\frac{2^{k-\dim E}}{(k-\dim E)\,|\tilde Z|^{2(k-\dim E)}}(-({}^K R^{\tilde N}+2b\pi\mathbf{J}^{\tilde N})^{-1})^{k-\dim E}\Bigg]_{b=0}\Bigg\}.\end{aligned}\tag{3.128}$$

From (3.128), we obtain the second identity in (3.112). □

Remark 3.18. In view of (3.110), (3.112), we see that to calculate the asymptotics as $\varepsilon \to 0$ of $I_4'^3 = \int_{LX} \mu\, {}^K\alpha_\varepsilon\, {}^K\tilde{e}^{LX}(LE, g^{LE})$, the only quantity which is left to study is

$$\int_{LX} {}^K\alpha_\varepsilon \frac{\partial}{\partial b}\Bigg[-\det\Big(-\frac{{}^KR^{LE}}{2i\pi}-b\Big) \frac{2^{\dim LE^0_{|X}}}{\dim E|s|^{2\dim LE^0_{|X}}}\, s^*((-({}^KR^{LE}+2b\pi\mathbf{J}^{LE})^{-1})^{\dim LE^0_{|X}})\Bigg]_{b=0}. \quad (3.129)$$

It follows from (3.49), (3.107)–(3.109), (3.112), that, as $\varepsilon \to 0$, (3.129) diverges like $\mathrm{Log}(\varepsilon)$. In view of Theorem 3.15, once the logarithmic divergence is subtracted off, the remainder splits into two pieces, one which makes a missing locally non-integrable piece of the current $\tilde{e}^X(LE, g^{LE})$ appear in the expression $\int_X \mu\, {}^Kc^{-1}_{\max}(N_{X/LX},\, g^{N_{X/LX}})\, {}^K\tilde{e}^X(LE, g^{LE})$, the other piece which in particular makes the missing term appear in the expression for $\int_Y \mu\mathbb{B}(N_{Y/LY}, N_{X/LX_{|Y}}, g^{N_{X/LX_{|Y}}})$ given in (1.25) with respect to the sum of the right-hand sides of (3.112).

The main purpose of Theorems 3.8–3.10 has been to deal indirectly with these difficulties when studying the asymptotics of I_4^3 as $\varepsilon \to 0$.

The term (3.129) could be directly studied as $\varepsilon \to 0$ by the techniques of [BGS4, Sect. 3b)]. The description of the current $\tilde{e}^X(LE, g^{LE})$ as a principal part of its restriction to X/Y should then be used.

Another manifestation of the difficulty in studying (3.129) directly is made obvious by the fact that the integral

$$\int_X \frac{\partial}{\partial b}\Bigg[-\det\Big(-\frac{{}^KR^{LE}}{2i\pi}-b\Big)\frac{2^{\dim LE^0_{|X}}}{\dim E|s|^{2\dim LE^0_{|X}}}\, s^*((-({}^KR^{LE}+2\pi b\mathbf{J}^{LE})^{-1})^{\dim LE^0_{|X}})\Bigg]_{b=0}. \quad (3.130)$$

diverges.

As we shall see in [B3], the main point of the proof of Theorem 3.4 given in Sects. 3b)–3j) is that it has a formal extremely interesting analogue in infinite dimensions.

l) Proof of Theorem 3.2

We now briefly sketch the principle of the proof of the stronger Theorem 3.2. Let $\eta_{u,T}$ be the form on $R_+^* \times R_+^* \times LX$,

$$\eta_{u,T} = \frac{du}{u}\, {}^K\gamma_u s^*\, {}^Ka_T(LE, g^{LE}) + \frac{dT}{T}\, {}^K\alpha_u s^*\, {}^Kc_T(LE, g^{LE}).$$

$d_{u,T}$ denotes the exterior differentiation operator with respect to the variables u, T.

Theorem 3.19. *The following identity holds*

$$d_{u,T}\eta_{u,T} = \frac{du dT}{uT}\Bigg\{\bar\partial_K\left(\left(\frac{\bar\partial_K}{2i\pi}\, {}^K\gamma_u\right) s^*\, {}^Kc_T(LE, g^{LE})\right) + \partial_K\left({}^K\gamma_u \frac{\bar\partial_K}{2i\pi}\, s^*\, {}^Kc_T(LE, g^{LE})\right)\Bigg\}. \quad (3.131)$$

Proof. By proceeding as in the proof of Theorem 3.6, we immediately obtain (3.131). □

Let Γ be the oriented contour considered in Sect. 3c), and let Δ be its interior.

Theorem 3.20. *The following identity holds*

$$\int_{\Gamma} \eta_{u,T} = \bar{\partial}_K \int_{\Delta} \left(\frac{\partial_K}{2i\pi} {}^K\gamma_u \right) s^* \, {}^K c_T(LE, g^{LE}) \frac{dudT}{uT} + \partial_K \int_{\Delta} {}^K\gamma_u \frac{\bar{\partial}_K}{2i\pi} (s^* \, {}^K c_T(LE, g^{LE})) \frac{dudT}{uT} \,. \tag{3.132}$$

In particular

$$\int_{\Gamma} \eta_{u,T} \in {}^K P^{LX,0} \,. \tag{3.133}$$

The idea will be then to take the limit in (3.132) as $t_0 \to 0$, $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$. The intermediary steps are essentially the same as in the proof of Theorem 3.4, except that now one has to study carefully the right-hand side of (3.132). A similar difficulty in fact already appeared in the proof of [B1, Theorem 2.8].

Details of the proof of Theorem 3.2 are left to the reader.

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