Commun. Math. Phys. 147, 295–328 (1992)

Communications in
Mathematical
Physics

Characters and Fusion Rules for W-Algebras via Quantized Drinfeld–Sokolov Reduction

Edward Frenkel[1] *, Victor Kac[2] ** and Minoru Wakimoto[3] ***

[1] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
[2] Department of Mathematics, M.I.T., Cambridge, MA 02139, USA
[3] Department of Mathematics, Mie University, Tsu 514, Japan

Received November 8, 1991

Abstract. Using the cohomological approach to W-algebras, we calculate characters and fusion coefficients for their representations obtained from modular invariant representations of affine algebras by the quantized Drinfeld–Sokolov reduction.

0. Introduction

The study of extended conformal algebras has been playing an increasingly important role in the recent development of conformal field theory. Among them the W-algebras have attracted much attention in the past few years. The first example of a W-algebra was discovered by Zamolodchikov [37] in an attempt to classify extended conformal algebras with two generating fields. (Further classification of W-algebras generated by two or three fields may be found in [8, 9].) There have been developed several approaches since then to the construction of a general W-algebra.

In the series of papers [15–17, 31] Fateev, Zamolodchikov and Lukyanov defined W-algebras associated to simple finite-dimensional Lie algebras $\bar{\mathfrak{g}}$ of type A_ℓ and D_ℓ by explicitly quantizing the corresponding Miura transformations and derived some results on their "minimal" representations. They put results in a form suitable for an arbitrary simply laced $\bar{\mathfrak{g}}$. At the same time Bilal and Gervais studied W-algebras as the algebras of symmetries of Toda theories [7].

In [2, 9, 11] the W-algebras appeared as the chiral algebras in coset models. In [34, 1] they also appeared in an attempt to generalize the Sugawara construction to higher degree Casimirs. All these constructions are closely related to the invariants of the Weyl group $\bar{W}$ of $\bar{\mathfrak{g}}$, hence the name W-algebra.

We adopt the point of view of the paper [21] by Feigin and one of the authors of the present paper, where the W-algebra $W(\bar{\mathfrak{g}})$, associated to any simple

* Supported in part by Junior Fellowship from Harvard Society of Fellows
** Supported in part by NSF grants DMS-8802489 and DMS-9103792
*** Supported in part by RIMS-91 Project

finite-dimensional Lie algebra $\bar{\mathfrak{g}}$, naturally appears as a result of quantization of the (classical) Drinfeld–Sokolov reduction [14]. Namely, $W(\bar{\mathfrak{g}})$ is realized as the cohomology of a BRST complex involving the universal enveloping algebra $U(\mathfrak{g})$ of the affine algebra $\mathfrak{g}$ associated to $\bar{\mathfrak{g}}$ and the ghosts associated to the currents of a maximal nilpotent subalgebra $\bar{\mathfrak{n}}$ of $\bar{\mathfrak{g}}$. For (classical) simply laced algebras this construction gives the same result as in [16, 17].

This approach allows one not only to define the W-algebras, but also to construct a functor F from the category of positive energy representations of the affine algebra $\mathfrak{g}$ to the category of positive energy representations of the algebra $W(\bar{\mathfrak{g}})$. Namely, the $W(\bar{\mathfrak{g}})$-module corresponding to a $\mathfrak{g}$-module M is the cohomology of a BRST complex associated to M.

The most important representations of affine algebras to which we apply this functor are the admissible (conjecturally = all modular invariant) representations of $\mathfrak{g}$ of fractional levels k, discovered and classified in [26–28] by two of the authors of the present paper.

In the particular case $\mathfrak{g} = \widehat{sl}_2$ all modular invariant representations of level different from -2 have level $k = -2 + p/p'$ [26], and it was shown in [6, 19] that the functor F sends these representations either to zero or to the irreducible representations of the Virasoro algebra from the (p, p')-minimal model [3]. This makes us to believe that F sends a modular invariant representation of an arbitrary affine algebra $\mathfrak{g}$ either to zero or to an irreducible "minimal" representation of $W(\bar{\mathfrak{g}})$. This irreducibility is our basic conjecture.

We use the functor F to evaluate the characters of the $W(\bar{\mathfrak{g}})$-modules thus obtained as residues of affine characters. Our results completely agree with the results and conjectures of [7, 9, 16]. The fact that the characters of the minimal series of the Virasoro algebra are residues of affine $\widehat{sl}_2$-characters was first observed in [32]. The functor F gives a simple explanation of this phenomenon.

Our calculations give much information about the conformal field theory with $W(\bar{\mathfrak{g}})$-symmetry which as yet has not been rigorously defined. In particular, we can apply the Verlinde formula [35] to the modular transformation of $W(\bar{\mathfrak{g}})$-characters (found in [17, 28]) to describe the fusion algebra of the (conjectured) minimal $W(\bar{\mathfrak{g}})$-models (in the simply-laced case).

It is interesting that if we apply Verlinde's argument to the affine characters at a fractional level, then the fusion coefficients may be negative. However, the functor F corrects the situation. It sends some different $\mathfrak{g}$-modules to the same $W(\bar{\mathfrak{g}})$-modules and "erases" some of the $\mathfrak{g}$-modules, so that the resulting fusion coefficients for the W-algebra are positive integers.

Note that the characters of $W(\bar{\mathfrak{g}})$-modules computed by means of the quantum Drinfeld–Sokolov reduction coincide with those of the $\mathfrak{g}_1 \oplus \mathfrak{g}_{k'}/\mathfrak{g}_{1+k'}$ coset model in the case of a simply-laced $\bar{\mathfrak{g}}$. The connection between the quantum Drinfeld–Sokolov reduction and the coset models still remains a mystery.

Below we describe the contents of the paper. In Sects. 1.1 and 1.2 we recall the necessary information about an affine Kac–Moody algebra $\mathfrak{g}$, its affine Weyl group W and especially the "enlarged" affine Weyl group $\tilde{W} = \tilde{W}_+ \rtimes W$, where $\tilde{W}_+$ is a group of symmetries of the Dynkin diagram of $\mathfrak{g}$ isomorphic to the center of the simply connected group corresponding to $\bar{\mathfrak{g}}$. In Sects. 1.3–1.5 we recall the definition and properties of the principal admissible weights Pr^k [26, 27]. Here k stands for the level; it has the form $k = -h^\vee + p/p'$, where $h^\vee$ is the dual Coxeter number and p, p' are relatively prime positive integers such that $p \geqq h^\vee$. We also recall [27] the definition of the subset N_-^k of "$-$"-nondegenerate weights, and the map

$\varphi^- : Pr^k \to I_{p,p'} \cup \{0\}$ which is 0 outside N^k_- and $\varphi^- : N^k_- \to I_{p,p'}$ is $|\bar{W}|$ to 1. Here

$$I_{p,p'} = (P_+^{p-h^\vee} \times P_+^{\vee p'-h})/\tilde{W}_+ ,$$

where P^m_+ (respectively $P^{\vee m}_+$) is the set of dominant integral weights (resp. coweights) for $\mathfrak{g}$ of level m. The map φ^- corresponds to the first quantized Drinfeld–Sokolov reduction. Furthermore, defining a bijective map f of Pr^k onto itself, we introduce the map $\varphi^+ : Pr^k \to I_{p,p'} \cup \{0\}$ by $\varphi^+ = \varphi^- \circ f$ and characterize the set N^k_+ of " + "-nondegenerate weights. ($N^k_\pm \neq \emptyset$ only if $p' \geqq h$, the Coxeter number.) The map φ^+ corresponds to the second quantized Drinfeld–Sokolov reduction. In Sects. 1.6 and 1.7 we recall formulas for "normalized" characters of irreducible representations with principal admissible highest weights and their properties under modular transformations [27, 28]. In Sect. 1.8 we define the (multidimensional) residue and express the residues of normalized characters of admissible representations in terms of certain functions $\varphi_{\lambda,\mu}$ ($\lambda, \mu \in I_{p,p'}$) which in Sect. 3 will turn out to be the characters of W-algebras.

In Sect. 2.1 we describe two classical Drinfeld–Sokolov reductions, with respect to the set of positive roots (" + "-case) and negative roots (" − " case). In Sect. 2.2 we quantize these reductions by means of the semi-infinite cohomology. Though " + " and " − " reductions give isomorphic Poisson algebras and their quantizations give isomorphic W-algebras, it is important to consider both reductions. This is because the " + " reduction can be obtained using the techniques of conformal field theory. (It is much easier to perform calculations using operator product expansions [3, 9].) On the other hand, in the " − " reduction the picture is in many respects much simpler (for example, it is much easier to calculate the value of the highest weight vector). In Sect. 2.3 we construct, using the two quantum reductions, functors $F_\pm$ from the category of positive energy $\mathfrak{g}$-modules to the category of modules over the corresponding W-algebra $W_k^\pm(\bar{\mathfrak{g}})$. We prove here an important vanishing theorem (Theorem 2.3) which gives a sufficient condition of vanishing of the $W_k^\pm(\bar{\mathfrak{g}})$-module $F_-(M)$ for a $\mathfrak{g}$-module M.

In Sect. 3.1 we construct a Virasoro subalgebra of the algebra $W_k^+(\bar{\mathfrak{g}})$ (the energy momentum field $T(z)$). In Sect. 3.2 we calculate the Euler character of the $W_k^\pm(\bar{\mathfrak{g}})$-module $F_\pm(M)$ in terms of residues of affine characters (Proposition 3.2.3 and Theorem 3.2). The key fact here is Proposition 3.2.2 which asserts that functors $F_\pm$ map positive energy $\mathfrak{g}$-modules to positive energy $W_k^\pm(\bar{\mathfrak{g}})$-modules. In Sect. 3.3 we prove some properties of the W-algebras using representation theory. Using this and the fundamental irreducibility Conjecture $3.4_\pm$, we derive in Sect. 3.4 the character formula for $F_\pm(L(\Lambda))$, where $\Lambda \in N^k_\pm$, and show that these $W_k^\pm(\bar{\mathfrak{g}})$-modules are parameterized by the set $I_{p,p'}$. In Sect. 3.5 we state conjectures on resolutions, which, in particular, imply the irreducibility conjecture.

Finally, in Sect. 4 we use results of Sects. 1 and 3 to derive fusion rules for the W-algebra in the case of simply laced $\bar{\mathfrak{g}}$. It turns out, in particular, that if $|\tilde{W}_+|$ is relatively prime to p or to p' (which holds for all k in all cases except for $\bar{\mathfrak{g}}$ of type A_n, where n is not a power of a prime number) then the fusion algebra for $W_k^\pm(\bar{\mathfrak{g}})$ is isomorphic to $\mathscr{A}^{p-h} \otimes \mathscr{A}_1^{p'-h}$ or to $\mathscr{A}_1^{p-h} \otimes \mathscr{A}^{p'-h}$ respectively, where $\mathscr{A}^m$ denote the fusion algebra of level m for $\mathfrak{g}$ and $\mathscr{A}_1^m$ denote its subalgebra corresponding to radical weights (Theorem 4.3′).

The main results of the paper have been announced in [21 and 28].

1. Principal Admissible Highest Weight Representations of Affine Algebras

1.1. Preliminaries on $\bar{\mathfrak{g}}$. Let $\bar{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$ of rank ℓ. Choose a Cartan subalgebra $\bar{\mathfrak{h}}$ of $\bar{\mathfrak{g}}$ and let $\bar{\Delta}^\vee \subset \bar{\mathfrak{h}}$ and $\bar{\Delta} \subset \bar{\mathfrak{h}}^*$ be the sets of coroots and roots respectively. Let $\bar{Q}^\vee \subset \bar{\mathfrak{h}}$ be the coroot lattice and let $\bar{Q}^* \subset \bar{\mathfrak{h}}$ be the dual to the root lattice. One knows that $\bar{Q}^* \supset \bar{Q}^\vee$ and that the group $\bar{Q}^*/\bar{Q}^\vee$ is isomorphic to the center of the simply connected Lie group corresponding to $\bar{\mathfrak{g}}$.

Choose a subset of positive roots $\bar{\Delta}_+ \subset \bar{\Delta}$ and let $\bar{\Delta}_+^\vee$ be the corresponding subset of positive coroots in $\bar{\Delta}^\vee$. Let $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$ be the corresponding triangular decomposition of $\bar{\mathfrak{g}}$. Let $\bar{\Pi} = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_\ell\}$ and $\bar{\Pi}^\vee = \{\bar{\alpha}_1^\vee, \ldots, \bar{\alpha}_\ell^\vee\}$ be the sets of simple roots and simple coroots respectively. Let $-\bar{\alpha}_0 = \sum_{i=1}^\ell a_i\bar{\alpha}_i \in \bar{\Delta}_+$ and $-\bar{\alpha}_0^\vee = \sum_{i=1}^\ell a_i^\vee \bar{\alpha}_i^\vee \in \bar{\Delta}_+^\vee$ be the highest root and the corresponding coroot respectively, and let $a_0 = a_0^\vee = 1$. The numbers

$$h = \sum_{i=0}^{\ell} a_i \quad \text{and} \quad h^\vee = \sum_{i=0}^{\ell} a_i^\vee$$

are called the Coxeter number and the dual Coxeter number respectively. Let J be a subset of the set $\{0, 1, \ldots, \ell\}$ consisting of those i for which $a_i = 1$. One has: $|\bar{Q}^*/\bar{Q}^\vee| = |J|$.

Let $\bar{\Lambda}_i \in \bar{\mathfrak{h}}^*$ (resp. $\bar{\Lambda}_i^\vee \in \bar{\mathfrak{h}}$), $i = 1, \ldots, \ell$, be the fundamental weights (resp. coweights), i.e. $\langle \bar{\Lambda}_i, \bar{\alpha}_j^\vee \rangle = \delta_{ij}$ (resp. $\langle \bar{\Lambda}_i^\vee, \bar{\alpha}_j \rangle = \delta_{ij}$), and let $\bar{\Lambda}_0 = \bar{\Lambda}_0^\vee = 0$.

Let $(x|y) = \phi(x, y)/2h^\vee$ be the normalized invariant bilinear form on $\bar{\mathfrak{g}}$, where ϕ is the Killing form. We identify $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}^*$ using the form. Then we have:

$$\bar{\alpha}^\vee = 2\bar{\alpha}/(\bar{\alpha}|\bar{\alpha}) \in \bar{\Delta}^\vee \quad \text{for } \bar{\alpha} \in \bar{\Delta}, \tag{1.1.1}$$

$$(\bar{\alpha}_i|\bar{\alpha}_i) = 2a_i^\vee/a_i, \quad i = 0, \ldots, \ell\,. \tag{1.1.2}$$

It follows that $(\bar{\Lambda}_i|\bar{\alpha}_i) = a_i^\vee/a_i\, (i = 1, \ldots, l)$, hence $\bar{\Lambda}_i \in \bar{Q}^*$ if $i \in J$. Thus, we have

$$\{\bar{\Lambda}_i\}_{i\in J} \text{ is a set of representatives of } \bar{Q}^* \bmod \bar{Q}^\vee. \tag{1.1.3}$$

All possible values of the ratio $a_i/a_i^\vee$ are 1 if $\bar{\mathfrak{g}}$ is of type A_ℓ, D_ℓ or E_ℓ, 1 and 2 if $\bar{\mathfrak{g}}$ is of type B_ℓ, C_ℓ or F_4, and 1 and 3 if $\bar{\mathfrak{g}}$ is of type G_2. We let $r^\vee = \max_i(a_i/a_i^\vee)$. The case $r^\vee = 1$ (resp. $r^\vee > 1$) is called simply-laced (resp. non-simply-laced).

Let $\bar{W} \subset GL(\bar{\mathfrak{h}})$ be the Weyl group of $\bar{\mathfrak{g}}$. We denote by $\bar{W}_+$ the subgroup of elements of $\bar{W}$ that map the set $\{\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_\ell\}$ into itself. Note that the set $\{\bar{\alpha}_0, \ldots, \bar{\alpha}_\ell\}\backslash\{\bar{\alpha}_j\}$ is a root basis of $\bar{\Delta}$ if and only if $j \in J$. Since $\bar{W}$ acts simply transitively on root bases, we conclude that for each $j \in J$ there exists a unique $\bar{w}_j \in \bar{W}_+$ such that $\bar{\alpha}_j = \bar{w}_j\bar{\alpha}_0$, and that

$$\bar{W}_+ = \{\bar{w}_j\}_{j\in J}\,. \tag{1.1.4}$$

Proposition 1.1. $\varepsilon(\bar{w}_j) = (-1)^{2(\bar{\Lambda}_j|\bar{\rho})}$.

Proof. Let $\alpha = \sum_i m_i\bar{\alpha}_i \in \bar{\Delta}_+$. Then, for $j \in J$ we have: either $m_j = 0$, then $\bar{w}_j^{-1}\alpha \in \bar{\Delta}_+$ or $m_j = 1$, then $\bar{w}_j^{-1}\alpha \in -\bar{\Delta}_+$. Hence we have:

$$|\bar{w}_j^{-1}\bar{\Delta}_+ \cap -\bar{\Delta}_+| = \sum_{\alpha\in\bar{\Delta}_+} (\alpha|\bar{\Lambda}_j) = 2(\bar{\rho}|\bar{\Lambda}_j). \quad \square$$

1.2. Affine Algebra $\mathfrak{g}$ *and the Groups* W *and* $\tilde{W}$ (see [24] for details). Let $\mathfrak{g} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} + \mathbb{C}K$ be the affine algebra associated to $\mathfrak{g}$:

$$[a(m), b(n)] = [a, b](m + n) + m\delta_{m,-n}(a|b)K\ , \quad [K, \mathfrak{g}] = 0\ , \tag{1.2.1}$$

where $a, b \in \bar{\mathfrak{g}}$, $m, n \in \mathbb{Z}$ and $a(m)$ stands for $t^m \otimes a$. Let $\mathfrak{h} = \bar{\mathfrak{h}} + \mathbb{C}K$ be the Cartan subalgebra of $\mathfrak{g}$. The bilinear form $(.|.)$ extends trivially from $\bar{\mathfrak{h}}$ to $\mathfrak{h}$, so that $\mathbb{C}K$ is its kernel and further to the whole $\mathfrak{g}$ by $(a(m)|b(n)) = \delta_{m,-n}(a|b)$. Given $\lambda \in \mathfrak{h}^*$, we denote by $\bar{\lambda} \in \bar{\mathfrak{h}}^*$ its restriction to $\bar{\mathfrak{h}}$; the number $\langle \lambda, K \rangle$ is called the level of λ.

Let $\Delta_+^{\vee\,\mathrm{re}} = \bar{\Delta}_+^\vee \cup \{\alpha + nK | \alpha \in \bar{\Delta}^\vee_{\mathrm{short}}, n \in \mathbb{N}\} \cup \{\alpha + r^\vee nK | \alpha \in \bar{\Delta}^\vee_{\mathrm{long}}, n \in \mathbb{N}\}$ be the set of positive real coroots, $\Delta^{\vee\,\mathrm{re}} = \Delta_+^{\vee\,\mathrm{re}} \cup -\Delta_+^{\vee\,\mathrm{re}}$ the set of all real coroots, $\Pi^\vee = \{\alpha_i^\vee =: \delta_{i0}K + \bar{\alpha}_i^\vee \,|\, i = 0, \ldots, \ell\}$ the set of simple coroots. Note that $K = \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee$.

We have the following action $\alpha \mapsto t_\alpha$ of $\bar{\mathfrak{h}}^*$ on $\mathfrak{h}$:

$$t_\alpha(v) = v - \langle v, \alpha \rangle K, \quad v \in \mathfrak{h}$$

(the contragredient action on $\mathfrak{h}^*$ being $t_\alpha(\lambda) = \lambda + \langle \lambda, K \rangle \alpha$). For a subset $L \subset \bar{\mathfrak{h}}^*$ let $t_L = \{t_\alpha | \alpha \in L\}$.

Recall that the Weyl group W of $\mathfrak{g}$ is a semidirect product:

$$W = t_{\bar{Q}^\vee} \rtimes \bar{W}\ . \tag{1.2.2}$$

It turns out that $\Delta^{\vee\,\mathrm{re}} = W(\Pi^\vee)$ is invariant with respect to a larger group $\tilde{W} := t_{\bar{Q}^*} \rtimes \bar{W}$, which will play an important role in our considerations. Let $\tilde{W}_+ = \{w \in \tilde{W} | w(\Pi^\vee) = \Pi^\vee\}$. Since W acts simply transitively on root bases, we have:

$$\tilde{W} = \tilde{W}_+ \ltimes W\ . \tag{1.2.3}$$

Using (1.1.3 and 4), the group $\tilde{W}_+$ can be described explicitly as follows:

$$\tilde{W}_+ = \{w_j\}_{j \in J}, \quad \text{where} \quad w_j = t_{\bar{\Lambda}_j} \bar{w}_j\ . \tag{1.2.4}$$

we have canonical isomorphisms

$$\bar{W}_+ \xleftarrow{\sim} \tilde{W}_+ \xrightarrow{\sim} \bar{Q}^*/\bar{Q}^\vee\ , \tag{1.2.5}$$

which are induced by the canonical homomorphism $\tilde{W} \to \bar{W}$ using (1.2.3) and the definition of $\tilde{W}$.

Proposition 1.2. (see e.g. [25]). *The group* $\tilde{W}_+$ *is a unique normal subgroup of the group* Aut $\Pi^\vee$ *that satisfies the following two properties*:

$$\mathrm{Aut}\, \Pi^\vee = \mathrm{Aut}\, \bar{\Pi}^\vee \ltimes \tilde{W}_+ \quad \textit{and} \quad \tilde{W}_+ \simeq \bar{Q}^*/\bar{Q}^\vee\ . \quad \square$$

Let $\Lambda_0 \in \mathfrak{h}^*$ be defined by $\Lambda_0|_{\bar{\mathfrak{h}}} = 0$, $\langle \Lambda_0, K \rangle = 1$. Define the fundamental weights (resp. coweights) by

$$\Lambda_i = \bar{\Lambda}_i + a_i^\vee \Lambda_0 \text{ (resp. } \Lambda_i^\vee = \bar{\Lambda}_i^\vee + a_i \Lambda_0), \quad i = 0, \ldots, \ell\ .$$

One has

$$w_j(\alpha_0^\vee) = \alpha_j^\vee \quad \text{and} \quad w_j(\Lambda_0) = \Lambda_j = \Lambda_j^\vee \quad \text{if } j \in J\ . \tag{1.2.6}$$

Let $\rho = \sum_{i=0}^{\ell} \Lambda_i$, $\rho^\vee = \sum_{i=0}^{\ell} \Lambda_i^\vee$. Then

$$\langle \rho, K \rangle = h^\vee, \quad \langle \rho^\vee, K \rangle = h\ . \tag{1.2.7}$$

Let $P = \sum_i \mathbb{Z}\Lambda_i$ and $P^\vee = \sum_i \mathbb{Z}\Lambda_i^\vee$ be the sets of integral weights and coweights and let $P_+ = \sum_i \mathbb{Z}_+\Lambda_i$ and $P_+^\vee = \sum_i \mathbb{Z}_+\Lambda_i^\vee$ be the sets of dominant integral weights and coweights. Let P^k, P_+^k etc. be the subsets of level k weights.

Note that the bilinear form $(.|.)$ on $\mathfrak{h}$ is W-invariant, but there is no non-trivial W-invariant bilinear form on $\mathfrak{h}^*$. The situation can be fixed, however, as follows. One enlarges $\mathfrak{h}$ by a one-dimensional space $\mathbb{C}d$ by letting $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}d$ and extends the bilinear form $(.|.)$ from $\mathfrak{h}$ to $\tilde{\mathfrak{h}}$ by letting $(d|\bar{\mathfrak{h}}) = 0$, $(d|d) = 0$, $(d|K) = 1$. Then there is a unique way to extend the action of W from $\mathfrak{h}$ to $\tilde{\mathfrak{h}}$ so that the form $(.|.)$ is W-invariant, namely:

$$w(d) = d \text{ if } w \in \bar{W}, \quad t_\alpha(d) = d + \alpha - \tfrac{1}{2}(\alpha|\alpha)K \ .$$

The resulting formula for the automorphism t_α of $\tilde{\mathfrak{h}}$ is:

$$t_\alpha(v) = v + (v|K)\alpha - ((v|\alpha) + \tfrac{1}{2}(\alpha|\alpha)(v|K))K, \quad v \in \tilde{\mathfrak{h}} \ . \tag{1.2.8}$$

We identify $\mathfrak{h}^*$ with the subspace of linear functions on $\tilde{\mathfrak{h}}$ which vanish on d. This subspace is not W-invariant, as we can see from

$$t_\alpha(\lambda) = \lambda + \langle\lambda, K\rangle\alpha - (\langle\lambda, \alpha\rangle + \tfrac{1}{2}(\alpha|\alpha)\langle\lambda, K\rangle)\delta, \quad \lambda \in \tilde{\mathfrak{h}}^* \ , \tag{1.2.9}$$

where δ is defined by

$$\delta|_{\mathfrak{h}} = 0, \quad \langle\delta, d\rangle = 1 \ .$$

As the bilinear form $(.|.)$ is non-degenerate on $\tilde{\mathfrak{h}}$, it induces one on $\tilde{\mathfrak{h}}^*$ which extends that on $\bar{\mathfrak{h}}^*$ by

$$(\bar{\mathfrak{h}}^*|\mathbb{C}\Lambda_0 + \mathbb{C}\delta) = 0, \quad (\delta|\delta) = (\Lambda_0|\Lambda_0) = 0, \quad (\Lambda_0|\delta) = 1 \ . \tag{1.2.10}$$

Let $\Delta_+^{\mathrm{re}} = \bar{\Delta}_+ \cup \{\alpha + n\delta | \alpha \in \bar{\Delta}, n \in \mathbb{N}\}$ be the set of positive real roots.

1.3. Principal Admissible Weights. Given $\Lambda \in \mathfrak{h}^*$, let $R^\Lambda = \{\alpha \in \Delta^{\vee\,\mathrm{re}} | \langle\Lambda + \rho, \alpha\rangle \in \mathbb{Z}\}$, $R_+^\Lambda = R^\Lambda \cap \Delta_+^{\vee\,\mathrm{re}}$. Recall that Λ is called a *principal admissible weight* [27] if it satisfies the following two properties:

$$\langle\Lambda + \rho, \alpha\rangle \notin -\mathbb{Z}_+ \quad \text{for all } \alpha \in \Delta_+^{\vee\,\mathrm{re}} \ , \tag{1.3.1}$$

$$R^\Lambda \text{ is isomorphic to } \Delta^{\vee\,\mathrm{re}} \ . \tag{1.3.2}$$

Note that all dominant integral weights are principal admissible. Recall the description of all principal admissible weights [27]. Let $u \in \mathbb{N}$ and let $R_{[u]} = \bar{\Delta}_+^\vee \cup \{\alpha + nuK | \alpha \in \bar{\Delta}^\vee, n \in \mathbb{N}\}$. One has (cf. (1.2.6)):

$$t_{u\bar{\Lambda}_j}\bar{w}_j R_{[u]} \subset R_{[u]} \quad \text{for } j \in J \ . \tag{1.3.3}$$

Given $y \in \tilde{W}$, denote by $P_{u,y}$ the set of all principal admissible Λ such that $R_+^\Lambda = y(R_{[u]})$, and by $P_{u,y}^k$ the subset of $P_{u,y}$ of weights of level k. Denote by Pr^k the set of all principal admissible weights of level k.

A rational number k with the denominator $u \in \mathbb{N}$ is called *principal admissible* if

$$u(k + h^\vee) \geqq h^\vee \quad \text{and} \quad (u, r^\vee) = 1 \ . \tag{1.3.4}$$

Letting

$$p = u(k + h^\vee) \ ,$$

conditions (1.3.4) can be rewritten as

$$p, u \in \mathbb{N}, \quad p \geqq h^\vee, \quad (p, u) = 1, \quad (u, r^\vee) = 1 \,. \tag{1.3.5}$$

Recall the shifted action of $\tilde{W}$: $w \,.\, \Lambda = w(\Lambda + \rho) - \rho$.

Theorem 1.3 [27]. (a) $P^k_{u,y}$ is *non-empty if and only if the following two conditions hold*:

$$k \text{ is a principal admissible number with the denominator } u \,, \tag{1.3.6}$$

$$y(R_{[u]}) \subset \Delta^{\vee\,\mathrm{re}}_+ \,. \tag{1.3.7}$$

(b) *If* (k, u, y) *and* (k, u, y_1) *are two triples satisfying* (1.3.6) *and* (1.3.7) *then the following statements are equivalent*:

(i) $P^k_{u,y}$ *and* P^k_{u,y_1} *have a non-empty intersection,*
(ii) $P^k_{u,y}$ *and* P^k_{u,y_1} *coincide,*
(iii) $y(R_{[u]}) = y_1(R_{[u]})$,
(iv) $y_1 = y t_{u\bar{\Lambda}_j} w_j$ *for some* $j \in J$.

(c) *If* (1.3.6) *and* (1.3.7) *hold, then*

$$P^k_{u,y} = \{ y \,.\, (\Lambda^0 - (u-1)(k + h^\vee)\Lambda_0) | \Lambda^0 \in P^{u(k+h^\vee)-h^\vee}_+ \} \,.$$

(d) Pr^k *is nonempty if and only if* k *is principal admissible.*

(e) $Pr^k = \bigcup_y P^k_{u,y}$, *where* $u \in \mathbb{N}$ *is the denominator of* k *and* $y \in \tilde{W}$ *satisfies* (1.3.7). □

Lemma 1.3. *Let* $y = t_\beta \bar{y}$. *Then condition* (1.3.7) *is equivalent to each of the following two conditions*:

$$(\bar{y}^{-1}\beta | \alpha_i^\vee) \leqq 0 \quad \text{for } i = 1, \dots, \ell; \quad (\bar{y}^{-1}\beta | \alpha_0^\vee) \leqq u \,. \tag{1.3.8}$$

$$0 \leqq -(\bar{y}^{-1}\beta | \alpha) \leqq u \quad \text{for all } \alpha \in \bar{\Delta}_+ \,. \tag{1.3.9}$$

Proof. is straightforward.

Remarks 1.3. (a) We have a bijective map $\Lambda \mapsto \Lambda^0$ between $P^k_{u,y}$ and $P^{u(k+h^\vee)-h^\vee}_+$ defined by $\Lambda = y \,.\, (\Lambda^0 - (u-1)(k+h^\vee)\Lambda_0)$.

(b) Note that $k \in \mathbb{Z}_+$ is principal admissible and in this case, $Pr^k = P^k_+$.

(c) If k is principal admissible, then $k\Lambda_0 \in P^k_{u,1} \subset Pr^k$. This is called the *vacuum weight* of Pr^k.

1.4. The Maps Transpose and f. In this section we consider some important maps on the set Pr^k of all principal admissible weights. The first map is the *transpose* $\Lambda \mapsto {}^t\Lambda$ defined in [28] as follows. Let $\bar{R}^\Lambda = R^\Lambda \cap \bar{\Delta}^\vee$, $\bar{R}^\Lambda_+ = \bar{R}^\Lambda \cap \bar{\Delta}^\vee_+$, and let $\bar{W}^\Lambda$ be the subgroup of $\bar{W}$ generated by reflections in the elements of $\bar{R}^\Lambda$. The group $\bar{W}^\Lambda$ contains a unique element, denoted by $\bar{w}^\Lambda$, such that $\bar{w}^\Lambda \bar{R}^\Lambda_+ = -\bar{R}^\Lambda_+$. In particular, $\bar{w}^0 \bar{\Delta}^\vee_+ = -\bar{\Delta}^\vee_+$. Note that $(\bar{w}^\Lambda)^2 = 1$. Define $w^\Lambda \in \mathrm{Aut}\,\mathfrak{h}$ by

$$w^\Lambda(v) = -\bar{w}^\Lambda(v) \quad \text{if } v \in \bar{\mathfrak{h}}, \quad w^\Lambda(K) = K \,. \tag{1.4.1}$$

Then ${}^t\Lambda$ is defined by

$$ {}^t\Lambda + \rho = w^\Lambda(\Lambda + \rho). \tag{1.4.2}$$

In particular, one has:

$$ {}^t\Lambda = w^0(\Lambda) \quad \text{if } \Lambda \in P_+ \, . \tag{1.4.3}$$

Furthermore, we have: ${}^tP^k_{u,y} = P^k_{u,{}^ty}$, where for $y = t_\beta \bar{y}$ we let

$$ {}^t\bar{y} = \overline{yw^\Lambda}\bar{w}^0, \quad {}^ty = t_{-\beta}{}^t\bar{y} \, . \tag{1.4.4}$$

Explicitly, for $\Lambda = (t_\beta \bar{y}).(\Lambda^0 - (u-1)(k+h^\vee)\Lambda_0) \in P^k_{u,y}$ we have

$$ {}^t\Lambda = (t_{-\beta}{}^t\bar{y}).({}^t(\Lambda^0) - (u-1)(k+h^\vee)\Lambda_0) \in P^k_{u,{}^ty} \, . \tag{1.4.5}$$

It is clear that the transpose is an involutive map of Pr^k into itself which fixes the vacuum weight $k\Lambda_0$. It will appear in the calculation of the fusion algebra.

We turn now to the definition of the second map $f: Pr^k \to Pr^k$, which will link two quantum reductions considered in the next sections. For this we need the following lemma.

Lemma 1.4. ([27], Lemma 3.4a). *Given $\beta \in \bar{Q}^*$, there exists a unique $\gamma \in \bar{Q}^\vee$ and a unique $\bar{y} \in \bar{W}$ such that $(t_{\beta + u\gamma}\bar{y})R_{[u]} \subset \Delta^{\vee\text{re}}_+$.* □

Given $u \in \mathbb{N}$, we first define a map $y \mapsto y'$ of $\tilde{W}$ into itself. Let $y = t_\beta \bar{y}$; by Lemma 1.4, there exists a unique $\gamma \in \bar{Q}^\vee$ and a unique $\bar{y}' \in \bar{W}$ such that $(t_{\beta - \bar{\rho}^\vee + u\gamma}\bar{y}')R_{[u]} \subset \Delta^{\vee\text{re}}_+$. We let

$$\beta' = \beta - \bar{\rho}^\vee + u\gamma, \quad y' = t_{\beta'}\overline{y'} \, .$$

Using (1.3.3), it is clear that if y is replaced by $y_1 = yt_{u\bar{\Lambda}_j}\bar{w}_j$, then its image y' gets replaced by $y'_1 = y't_{u\bar{\Lambda}_j}\bar{w}_j$. It follows that the element

$$\hat{y} := y'y^{-1} \in \tilde{W}$$

remains unchanged if y is replaced by $yt_{u\bar{\Lambda}_j}\bar{w}_j$.

Now, for $\Lambda \in Pr^k$ there exists $y \in \tilde{W}$ such that $\Lambda \in P^k_{u,y}$ and we let $f(\Lambda) = \hat{y}.\Lambda$. Due to the above argument and Theorem 1.3, f is a well-defined bijective map on Pr^k, such that $f(P^k_{u,y}) = P^k_{u,y'}$.

Proposition 1.4. (a) $(f \circ \textit{transpose})^2 = 1$.

(b) *If $u \geqq h$, then $f(k\Lambda_0) = k\Lambda_0 - (k+h^\vee)\bar{\rho}^\vee$.*

Proof. Let $\Lambda \in P^k_{u,y}$, $y = t_\beta \bar{y}$, so that $\Lambda = y.(\Lambda^0 - (u-1)(k+h^\vee)\Lambda_0)$ for some $\Lambda^0 \in P_+^{u(k+h^\vee)-h^\vee}$. By Lemma 1.4, there exists a unique $\gamma \in \bar{Q}^\vee$ and a unique $\bar{y}' \in \bar{W}$ such that

$$ t_{\beta'}\bar{y}'R_{[u]} \subset \Delta^{\vee\text{re}}_+, \quad \text{where} \quad \beta' = -\beta - \bar{\rho}^\vee + u\gamma \, . \tag{1.4.6}$$

Then, by definition, $\Lambda' := f({}^t\Lambda) = (t_{\beta'}\bar{y}').({}^t(\Lambda^0) - (u-1)(k+h^\vee)\Lambda_0)$.

We need to show that $f({}^t\Lambda') = \Lambda$. Applying the above argument to β' and $\bar{y}'$ in place of β and $\bar{y}$, there exists a unique $\gamma' \in \bar{Q}^\vee$ and a unique $\bar{y}'' \in \bar{W}$ such that

$$ t_{\beta''}\bar{y}''R_{[u]} \subset \Delta^{\vee\text{re}}_+, \quad \text{where} \quad \beta'' = -\beta' - \bar{\rho}^\vee + u\gamma' \, . \tag{1.4.7}$$

But $\beta'' = \beta + u(\gamma' - \gamma)$ (see (1.4.6)). Hence, comparing (1.4.7) with $t_\beta \bar{y}R_{[u]} \subset \Delta^{\vee\text{re}}_+$ and using uniqueness in Lemma 1.4, we conclude that $\bar{y}'' = \bar{y}$ and $\beta'' = \beta$. Hence, $f({}^t\Lambda') := (t_{\beta''}\bar{y}'').(\Lambda^0 - (u-1)(k+h^\vee)\Lambda_0) = \Lambda$, proving (a).

Noting that $t_{-\bar{\rho}^\vee} R_{[u]} \subset \Delta_+^{\vee\mathrm{re}}$ if $u \geqq h$, we have under the assumption that $u \geqq h$: $f(k\Lambda_0) = t_{-\bar{\rho}^\vee} . (k\Lambda_0) = k\Lambda_0 - (k + h^\vee)\bar{\rho}^\vee$, proving (b). $\square$

1.5. The Maps $\varphi^\pm$ and the Corresponding Sets of Non-Degenerate Weights $N_\pm^k$. The proof of the following lemma is straightforward (cf. Lemma 1.3):

Lemma 1.5. *Let Λ be a principal admissible weight, $\Lambda \in P_{u,y}^k$, where $y = t_\beta \bar{y}$. Then the following conditions are equivalent*:

$$\langle \Lambda, \alpha \rangle \notin \mathbb{Z} \quad \textit{for all } \alpha \in \bar{\Delta}^\vee , \tag{1.5.1}$$

$$y(R_{[u]}) \subset \Delta_+^\vee \backslash \bar{\Delta}_+^\vee , \tag{1.5.2}$$

$$(\bar{y}^{-1}\beta | \alpha_i^\vee) < 0 \quad \textit{for } i = 1, \ldots, \ell; \quad (\bar{y}^{-1}\beta | \alpha_0^\vee) < u , \tag{1.5.3}$$

$$0 < -(\bar{y}^{-1}\beta | \alpha) < u \quad \textit{for all } \alpha \in \bar{\Delta}_+ , \tag{1.5.4}$$

$$(\beta | \alpha) \not\equiv 0 \bmod u \quad \textit{for all } \alpha \in \bar{\Delta} . \quad \square \tag{1.5.5}$$

In particular, all elements of $P_{u,y}^k$ either satisfy (1.5.1–5) or all do not. Given $\bar{y} \in \bar{W}$, let

$$\tilde{M}_{u,\bar{y}} = \{\beta \in \bar{Q}^* | 0 < -(\bar{y}^{-1}\beta | \alpha) < u \text{ for all } \alpha \in \bar{\Delta}_+\} ,$$

$$P_{\bar{y}}^k = \bigcup_{\beta \in \tilde{M}_{u,\bar{y}}} P_{u, t_\beta \bar{y}}^k ,$$

$$N_-^k = \bigcup_{\bar{y} \in \bar{W}} P_{\bar{y}}^k .$$

(Elements of the last set are called non-degenerate weights in [28]).

Each set $P_{\bar{y}}^k$ admits the following nice parametrisation:

Proposition 1.5.1. [28] (a) *Let $\Lambda \in P_{\bar{y}}^k$. Then there exists a unique $\beta (\in \tilde{M}_{u,\bar{y}})$ such that $\Lambda \in P_{u,t_\beta \bar{y}}^k$. We let*

$$\varphi_{\bar{y}}^-(\Lambda) = (\Lambda^0, u\Lambda_0 - \bar{y}^{-1}(\beta) - \rho^\vee) .$$

This is a bijective map (here, as before, $p = u(k + h^\vee)$):

$$\varphi_{\bar{y}}^- : P_{\bar{y}}^k \to P_+^{p-h^\vee} \times P_+^{\vee u-h} ,$$

the converse map being

$$\psi_{\bar{y}}^-(\lambda, \mu) = \bar{y} . \lambda - \frac{p}{u} \bar{y}(\mu + \rho^\vee) + \frac{p}{u} \Lambda_0 .$$

In particular, $P_{\bar{y}}^k \neq \emptyset$ if and only if

$$k \textit{ is principal admissible and } u \geqq h . \tag{1.5.6}$$

(b) *Let k satisfy* (1.5.6) *and let $\bar{y}, \bar{y}_1 \in \bar{W}$. Then $\psi_{\bar{y}}^-(\lambda, \mu) = \psi_{\bar{y}_1}^-(\lambda_1, \mu_1)$ if and only if $\bar{y}^{-1}\bar{y}_1 = \bar{w}_j$ for some $j \in J$ and $p^{-1}(\lambda - \bar{w}_j . \lambda_1) = u^{-1}(\mu - \bar{w}_j . \mu_1) = \bar{\Lambda}_j$.*

(c) *$P_{\bar{y}}^k$ and $P_{\bar{y}_1}^k$ are either disjoint or coincide and they coincide if and only if $\bar{y}^{-1}\bar{y}_1 \in \bar{W}_+$.* $\square$

Given relatively prime integers $p \geqq h^\vee$ and $p' \geqq h$, consider the set

$$I_{p,p'} = (P_+^{p-h^\vee} \times P_+^{\vee p'-h})/\tilde{W}_+$$

(where $w(\lambda, \mu) = (w\lambda, w\mu)$, $w \in \tilde{W}_+$). Let $\varphi^- : Pr^k \to I_{p,u}$ be the map defined as follows:

$$\varphi^-(\Lambda) = \varphi_{\bar{y}}^-(\Lambda) \text{ if } \Lambda \in P_{\bar{y}}^k, \quad \varphi^-(\Lambda) = 0 \text{ if } \Lambda \notin N_-^k .$$

It is easy to see, using Proposition 1.5.1b, that the map φ^- is well defined.

As we shall see, the map φ^- corresponds to the quantization of the first Drinfeld–Sokolov reduction. We turn now to the map $\varphi^+ := \varphi^- \circ f$ corresponding to the quantization of the second Drinfeld–Sokolov reduction.

Consider the set $N_+^k := f^{-1}(N_-^k) = \{\Lambda \in Pr^k | \varphi^+(\Lambda) \neq 0\}$. We give below a more explicit description of the set N_+^k and of the map φ^+. Proof is straightforward.

Proposition 1.5.2. (a) $N_+^k = \bigcup_{\beta,\bar{y}} P_{u,t_\beta \bar{y}}^k$, *where the union is taken over all* $\bar{y} \in \bar{W}$ *and all* $\beta \in \tilde{M}$ *such that*

$$(\beta - \rho^\vee | \alpha) \not\equiv 0 \bmod u \quad \textit{for all } \alpha \in \bar{\Delta} . \tag{1.5.7}$$

(b) *The map* φ^+ *is defined on* $\Lambda \in P_{u,t_\beta \bar{y}}^k \subset N_+^k$ *as follows. By Lemma* 1.4, *there exists a unique* $\bar{y}' \in \bar{W}$ *and a unique* $\gamma \in M$ *such that* $t_{\beta - \bar{\rho}^\vee + u\gamma} \bar{y}' R_{[u]} \subset \Delta_+^{\vee \mathrm{re}}$. *Then*

$$\varphi^+(\Lambda) = (\Lambda^0, u\Lambda_0 - \bar{y}'^{-1}(\beta - \bar{\rho}^\vee + u\gamma) - \rho^\vee) \bmod \tilde{W}_+ . \tag{1.5.8}$$

Equivalently, the weight $u\Lambda_0 - \bar{y}^{-1}(\beta - \bar{\rho}^\vee)$ *is cointegral regular, hence there exists a unique* $w_{\beta,\bar{y}} \in W$ *and a unique* $\mu \in P_+^{\vee u-h}$ *such that* $\mu + \rho^\vee = w(u\Lambda_0 - \bar{y}^{-1}(\beta - \bar{\rho}^\vee))$; *then* $\varphi^+(\Lambda) = (\Lambda^0, \mu) \bmod \tilde{W}_+$.

(c) *If* $\varphi^+(\Lambda) = (\lambda, \mu)$, then $\varphi^+(\sigma(\Lambda)) = ({}^t\lambda, {}^t\mu) \bmod \tilde{W}_+$, *where* $\sigma = f^{-1} \circ$ *transpose* $\circ f$.

(d) $\varphi^+(k\Lambda_0) = ((p - h^\vee)\Lambda_0, (u - h)\Lambda_0) \bmod \tilde{W}_+$. □

Proposition 1.5.3. (a) *Elements* $\Lambda, \Lambda' \in N_-^k$ *are such that* $\varphi^-(\Lambda) = \varphi^-(\Lambda')$ *if and only if* $\Lambda' = \bar{w} . \Lambda$ *for some* $\bar{w} \in \bar{W}$. *In particular, the map* $\varphi^- : N_-^k \to I_{p,u}$ *is* $|\bar{W}|$ *to* 1.

(b) *The same statement holds for* $\varphi^+ : N_+^k \to I_{p,u}$.

Proof. Let $\Lambda = y . (\lambda - (u - 1)(k + h^\vee)\Lambda_0) \in N_-^k$, where $y = t_\beta \bar{y} \in \tilde{W}$. Then, due to (1.5.2) we have:

$$\bar{w}y(R_{[u]}) \subset \Delta_+^\vee \backslash \bar{\Delta}_+^\vee \quad \text{for any } \bar{w} \in \bar{W} .$$

Hence by Theorem 1.3 and Lemma 1.5,

$$(\bar{w}y) . (\lambda - (u - 1)(k + h^\vee)\Lambda_0) \in N_-^k .$$

But $\bar{w}y = t_{\bar{w}\beta}(\overline{wy})$, hence by definition of φ_- (see Proposition 1.5.1), $\varphi_-(\bar{w} . \Lambda) = (\lambda, u\Lambda_0 - (\overline{wy})^{-1}(\bar{w}\beta) - \rho^\vee) = \varphi^-(\Lambda)$.

Conversely, suppose that $\varphi^-(\Lambda) = \varphi^-(\Lambda') = (\lambda, \mu) \in P_+^{p-h^\vee} \times P_+^{\vee p'-h}$, where $\Lambda' = y' . (\lambda' - (u - 1)(k + h^\vee)\Lambda_0) \in N_-^k$ and $y' = t_{\beta'} \bar{y}'$. By definition of φ^- we have: $\lambda = \lambda'$ and $\bar{y}^{-1}\beta = (\bar{y}')^{-1}\beta'$. Letting $\bar{w} = \bar{y}'\bar{y}^{-1}$, we thus have: $\bar{w}\beta = \beta'$. Hence $y' = t_{\beta'}\bar{y}' = t_{\bar{w}\beta}(\overline{wy}) = \bar{w}t_\beta \bar{y} = \bar{w}y$, and $\Lambda' = \bar{w} . \Lambda$. □

Here is a somewhat different description of the maps φ^+ and σ. Recall that by Lemma 1.4, given $u \in \mathbb{N}$, each $\beta \in \tilde{M}$ determines uniquely an element $y_\beta := t_{\beta + u\gamma}\bar{y} \in \tilde{W}$ by the condition $y_\beta(R_{[u]}) \subset \Delta_+^{\vee \text{re}}$. This gives us a map

$$\phi \colon P_+^{p - h^\vee} \times (\bar{Q}^*/u\bar{Q}^\vee) \to Pr^k$$

defined by $\phi(\Lambda^0, \beta) = y_\beta . (\Lambda^0 - (u-1)(k+h^\vee)\Lambda_0)$. The map ϕ induces a bijection:

$$(P_+^{p-h^\vee} \times (\bar{Q}^*/u\bar{Q}^\vee))/\tilde{W}_+ \xrightarrow{\sim} Pr^k ,$$

where $\tilde{W}_+$ acts on both factors as defined in 1.2.

Let now $\Lambda = \phi(\lambda, \beta) \in Pr^k$, where $(\lambda, \beta) \in (P_+^{p-h^\vee} \times (\bar{Q}^*/u\bar{Q}^\vee))$ mod $\tilde{W}_+$. Then $\Lambda \in N_+^k$ if and only if $\beta - \bar{\rho}^\vee$ is regular with respect to the group $\bar{W} \ltimes uM$. In this case we have:

$$\varphi^+(\Lambda) = (\lambda, (u-h)\Lambda_0 + \bar{\mu}) \bmod \tilde{W}_+ , \tag{1.5.9}$$

where $\bar{\mu} \in \bar{P}_+^{\vee u-h}$ is such that $\bar{\mu} + \bar{\rho}^\vee$ and $\beta - \bar{\rho}^\vee$ lie on the same $\bar{W} \ltimes u\bar{Q}^\vee$-orbit. Furthermore, we have:

$$\sigma(\Lambda) = \phi({}^t\lambda, -\beta + 2\bar{\rho}^\vee) . \tag{1.5.10}$$

1.6. Characters and Normalized Characters of an Affine Algebra. Let M be a $\mathfrak{g}$-module. It is called a *level* k module if $K = kI_M$. It is called a *restricted* module if for any $x \in \bar{\mathfrak{g}}$ and any $v \in M$ there exists n_0 such that $x(n)v = 0$ for $n > n_0$. If M is a restricted $\mathfrak{g}$-module of level $k \neq -h^\vee$ one defines the *Sugawara operators* on M by

$$S_n = \frac{1}{2(k+h^\vee)} \sum_{j \in \mathbb{Z}} \sum_i : u_i(-j)u^i(j+n): , \tag{1.6.1}$$

where $\{u_i\}$ and $\{u^i\}$ are bases of $\bar{\mathfrak{g}}$ such that $(u_i | u^j) = \delta_{ij}$. Recall that these operators define a representation of the Virasoro algebra with central charge

$$c_k = \frac{k(\dim \bar{\mathfrak{g}})}{k+h^\vee} . \tag{1.6.2}$$

A restricted $\mathfrak{g}$-module M of level $k \neq -h^\vee$ is called a *positive energy* module if S_0 is a diagonalizable operator on M with a discrete spectrum bounded below and each eigenspace of S_0 is a $\bar{\mathfrak{g}}$-module from the category $\mathcal{O}$. (Recall that a $\bar{\mathfrak{g}}$-module is said to be in the category $\mathcal{O}$ if it is finitely generated $\bar{\mathfrak{h}}$-diagonalizable and $\bar{\mathfrak{n}}_+$-finite.)

The most important examples of positive energy $\mathfrak{g}$-modules are irreducible highest weight modules $L(\Lambda)$, where $\Lambda \in \mathfrak{h}^*$ is of level $k \neq -h^\vee$, defined by the property that there exists a non-zero vector v_Λ such that [24]

$$\left(\bar{\mathfrak{n}}_+ + \sum_{k>0} t^k \bar{\mathfrak{g}}\right) v_\Lambda = 0, \quad hv_\Lambda = \langle \Lambda, h \rangle v_\Lambda \quad \text{for } h \in \mathfrak{h} .$$

Let $q = e^{2\pi i \tau}$, where $\tau \in \mathscr{H}_+$, the upper half-plane. Consider the domain $Y = \mathscr{H}_+ \times \bar{\mathfrak{h}} \times \mathbb{C}$. One defines the *character* ch_V of a positive energy $\mathfrak{g}$-module M of level k by the series:

$$\mathrm{ch}_M(\tau, x, t) = e^{2\pi i k t} \mathrm{tr}_M(q^{S_0} e^{2\pi i x}), \quad (\tau, x, t) \in Y .$$

This series converges to a holomorphic function in the following domain in Y [24, Chap. 11]:

$$\mathrm{Im}(x|\alpha) > 0 \text{ for } \alpha \in \bar{\Delta}_+^\vee \quad \text{and} \quad \mathrm{Im}(\bar{\alpha}_0^\vee|x) < \mathrm{Im}\,\tau\,, \tag{1.6.3}$$

and can be analytically extended to a meromorphic function in Y analytic outside the hyperplanes $(x|\alpha) = n$ for $\alpha \in \bar{\Delta}_+^\vee, n \in \mathbb{Z}$. If $\Lambda \in P_+^k$, then χ_Λ converges to a holomorphic function on the whole domain Y [24, Chap. 10].

Given $\Lambda \in \mathfrak{h}^*$ of level k one defines the normalized character χ_Λ of the $\mathfrak{g}$-module $L(\Lambda)$ by the formula:

$$\chi_\Lambda(\tau, x, t) = q^{-c_k/24}\mathrm{ch}_{L(\Lambda)}(\tau, x, t)\,.$$

We shall identify Y with a domain in $\tilde{\mathfrak{h}}$ by letting

$$(\tau, x, t) = 2\pi i(-\tau d + x + tK)\,.$$

In the case $\Lambda \in P^k_{u,t_\beta \bar{y}}$, the meromorphic function $\chi_\Lambda(\tau, x, t)$ in Y is given by the following formula [27, 28]:

$$\chi_\Lambda(\tau, x, t) = \frac{A_{\Lambda^0+\rho}(u\tau, \tau\bar{y}^{-1}(\beta) + \bar{y}^{-1}(x), u^{-1}(t + (x|\beta) + \frac{1}{2}\tau|\beta|^2))}{A_\rho(\tau, x, t)}\,. \tag{1.6.4}$$

Here, for $\lambda \in P_+^s$, $s \in \mathbb{N}$, we let

$$A_\lambda = q^{|\lambda|^2/2s} \sum_{w \in W} \varepsilon(w) e^{w(\lambda)}\,.$$

Recall the following simple (but useful) identity for $\lambda \in P_+^s$ and $\mu \in \mathfrak{h}^*$ such that $\langle \mu, K\rangle = m$ [28]:

$$A_\lambda\left(m\tau, -\tau\bar{\mu}, \tau\frac{|\bar{\mu}|^2}{2m}\right) = \sum_{w \in W} \varepsilon(w) q^{\frac{sm}{2}\left|\frac{w(\lambda)}{s} - \frac{\mu}{m}\right|^2}\,. \tag{1.6.5}$$

Recall also Macdonald's identity:

$$A_\rho = q^{|\bar{\rho}|^2/2h^\vee} e^{2\pi i((\bar{\rho}|x) + th^\vee)} \prod_{n \geqq 1}\left((1-q^n)^\ell \prod_{\gamma \in \bar{\Delta}_+} (1 - q^{n-1}e^\alpha)(1 - q^n e^{-\alpha})\right)\,. \tag{1.6.6}$$

1.7. Modular Transformations of Normalized Characters. The most interesting from the conformal field theory point of view are the modular invariant representations, which are defined as follows. Recall the action of the group $SL_2(\mathbb{Z})$ on Y:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, x, t) = \left(\frac{a\tau + b}{c\tau + d}, \frac{x}{c\tau + d}, t - \frac{c(x|x)}{2(c\tau + d)}\right)\,,$$

and its right action on functions on Y:

$$f(\tau, x, t)|_B = f(B \cdot (\tau, x, t)), \quad B \in SL_2(\mathbb{Z})\,.$$

The representation $L(\Lambda)$ is called modular invariant if χ_Λ is invariant with respect to a congruence subgroup of $SL_2(\mathbb{Z})$.

It was proved in [26] that $L(\Lambda)$ is modular invariant if Λ is admissible, i.e. (1.3.1) holds, and $\mathbb{Q}R^\Lambda = \mathbb{Q}\Delta^{\vee\mathrm{re}}$ (it was also conjectured there that there is no other modular invariant $L(\Lambda)$ of level $\neq -h^\vee$). According to the classification of admissible Λ given in [27], they can be described in terms of the principal admissible weights (in the type A case all admissible weights are principal admissible).

The normalized characters of the principal admissible representations have remarkable transformation properties. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Theorem 1.7. ([27, Theorem 3.6]). *Let $\Lambda \in P^k_{u,y}$, where $y = t_\beta \bar{y}$. Then*

$$\chi_\Lambda|_S = \sum_{\Lambda' \in Pr^k} S_{\Lambda\Lambda'} \chi_{\Lambda'},$$

where

$$S_{\Lambda\Lambda'} = i^{|\bar{\Delta}_+|} u^{-\ell} (k + h^\vee)^{-\ell/2} |M^*/M|^{-1/2} \varepsilon(\overline{y y'})$$
$$\times e^{-2\pi i((\Lambda^0 + \rho|\beta') + (\Lambda'^0 + \rho|\beta) + (k + h^\vee)(\beta|\beta'))} \sum_{w \in \bar{W}} \varepsilon(w) e^{-\frac{2\pi i}{k + h^\vee}(w(\Lambda^0 + \rho)|\Lambda'^0 + \rho)}. \quad (1.7.1)$$

(Here $\Lambda' \in P^k_{u,y'}$, $y' = t_{\beta'}\bar{y}'$.) In particular the space $\sum_{\Lambda \in Pr^k} \mathbb{C}\chi_\Lambda$ is $SL_2(\mathbb{Z})$-invariant. □

Recall also that the matrix $(S_{\Lambda\Lambda'})_{\Lambda, \Lambda' \in Pr^k}$ is a unitary symmetric matrix ([28, Proposition 4.3]).

1.8. The Residue of Affine Characters. Let $F(\tau, x, t)$ be a meromorphic function on Y. Define the *residue* of F by the following formula:

$$\mathrm{Res}_{x=0} F(\tau, x, t) = \lim_{\varepsilon \to 0} F(\tau, \varepsilon x, t) \prod_{\gamma \in \bar{\Delta}_+} (1 - e^{-2\pi i(\alpha|\varepsilon x)}).$$

This is a meromorphic function in $(\tau, t) \in \mathscr{H}_+ \times \mathbb{C}$.

Let p and p' be integers such that $p \geqq h^\vee$, $p' \geqq h$. For $\lambda \in P_+^{p - h^\vee}$, $\mu \in P_+^{\vee p' - h}$ let

$$\varphi_{\lambda,\mu}(\tau) = \eta(\tau)^{-\ell} \sum_{w \in W} \varepsilon(w) q^{\frac{pp'}{2}\left|\frac{w(\lambda + \rho)}{p} - \frac{\mu + \rho^\vee}{p'}\right|^2},$$

where $\eta(\tau) = q^{1/24} \prod_{n \in \mathbb{N}} (1 - q^n)$ is the Dedekind η-function. This is a holomorphic function in $\tau \in \mathscr{H}_+$.

Proposition 1.8. *Let $\Lambda \in P^k_{u, t_\beta \bar{y}}$ be a principal admissible weight.*

(a) *There are two possibilities*:
(i) *$\Lambda \notin N^k_-$; in this case* $\mathrm{Res}_{x=0}(\chi_\Lambda(\tau, x, t)) = 0$;
(ii) *$\Lambda \in N^k_-$; in this case we let $(\lambda, \mu) = \varphi^-(\Lambda)$. Then*:

$$\eta(\tau)^{\dim \bar{\mathfrak{g}} - \ell} \mathrm{Res}_{x=0}\, \chi_\Lambda(\tau, x, 0) = \varphi_{\lambda,\mu}(\tau).$$

(b) *There are two possibilities*:
(i) *$\Lambda \notin N^k_+$; in this case* $\mathrm{Res}_{x=0}(\chi_\Lambda(\tau, -\tau\bar{\rho}^\vee + x, t)) = 0$;
(ii) *$\Lambda \in N^k_+$*; i.e. *$\Lambda \in P^k_{u, t_\beta \bar{y}}$, where $(\beta - \bar{\rho}^\vee|\alpha) \not\equiv 0 \bmod u$ for all $\alpha \in \bar{\Delta}$; in this case we let $(\lambda, \mu) = \varphi^+(\Lambda)$*, i.e. (*see Proposition* 1.5.2) *we let $\lambda = \Lambda^0$ and define* (*unique*) *$\mu \in P_+^{u-h}$ and $w \in W$ by*

$$\mu + \rho^\vee = w(u\Lambda_0 + \bar{y}^{-1}(\bar{\rho}^\vee - \beta)).$$

Then:

$$\eta^{\dim \bar{\mathfrak{g}} - \ell}(\tau)(-1)^{\ell(w) + 2(\bar{\rho}|\bar{\rho}^\vee) + |\bar{\Delta}_+|}\mathrm{Res}_{x=0}\,\chi_\Lambda(\tau, -\tau\bar{\rho}^\vee + x, \tau|\bar{\rho}^\vee|^2/2) = \varphi_{\lambda,\mu}(\tau)\,.$$

Proof. We use the explicit formula (1.6.4) for χ_Λ and identities (1.6.6) and (1.6.5). Since (a) was proved in [28, Proposition 4.2], we check here (b). The proof of (i) of (b) is the same as that of (a). In the case (ii) of (b) we have, using (1.6.4) and (1.6.6):

$$\mathrm{Res}_{x=0}\,\chi_\Lambda(\tau, -\tau\bar{\rho}^\vee + x, \tau|\bar{\rho}^\vee|^2/2) =$$

$$\frac{A_{\Lambda^0+\rho}\left(u\tau, \tau\bar{y}^{-1}(\beta - \bar{\rho}^\vee), \dfrac{\tau|\beta - \bar{\rho}^\vee|^2}{2u}\right)(-1)^{|\bar{\Delta}_+|}}{q^{|\bar{\rho}|^2/2h^\vee} q^{h^\vee|\bar{\rho}^\vee|^2/2 - \langle\bar{\rho},\bar{\rho}^\vee\rangle}\prod_{n\geqq 1}(1-q^n)^\ell \prod_{\alpha\in\bar{\Delta}_+}\left(\prod_{n\geqq 1}(1-q^{n-1+ht\alpha})\prod_{\substack{n\geqq 1\\ n\neq ht\alpha}}(1-q^{n-ht\alpha})\right)}\,.$$

Using (1.6.5), and the formulas

$$\prod_{n\geqq 1}(1-q^{n-1+a})\prod_{\substack{n\geqq 1\\ n\neq a}}(1-q^{n-a})$$

$$= (-1)^{a-1}q^{-a(a-1)/2}\prod_{n\geqq 1}(1-q^n)^2, \quad a\in\mathbb{N}\,, \tag{1.8.1}$$

$$|\bar{\rho}|^2/2h^\vee = \dim\bar{\mathfrak{g}}/24 \text{ ("strange formula")}\,, \tag{1.8.2}$$

$$\sum_{\alpha\in\bar{\Delta}_+} ht\alpha = 2(\bar{\rho}|\bar{\rho}^\vee)\,, \tag{1.8.3}$$

$$\sum_{\alpha\in\bar{\Delta}_+}(ht\alpha)^2 = h^\vee|\bar{\rho}^\vee|^2\,, \tag{1.8.4}$$

the previous formula gives the result. Note that (1.8.4) follows from

$$\sum_{\alpha\in\bar{\Delta}_+}(x|\alpha)(y|\alpha) = h^\vee(x|y). \quad \square \tag{1.8.5}$$

2. Two Quantum Reductions

2.1. Two Classical Drinfeld–Sokolov Reductions. Let $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$ be the triangular decomposition of the Lie algebra $\bar{\mathfrak{g}}$. Choose bases of $\bar{\mathfrak{n}}_+$ (resp. $\bar{\mathfrak{n}}_-$) consisting of root vectors $\{e_\alpha\}$ (resp. $\{e_{-\alpha}\}$), $\alpha\in\bar{\Delta}_+$, such that $(e_\alpha|e_{-\alpha}) = 1$. Consider the following two subalgebras of the affine algebra $\mathfrak{g}$:

$$\mathfrak{n}_\pm = \mathbb{C}[t, t^{-1}]\otimes\bar{\mathfrak{n}}_\pm\,.$$

Vectors $e_\alpha(m)$ (resp. $e_{-\alpha}(m)$), $\alpha\in\bar{\Delta}_+$, $m\in\mathbb{Z}$, form a basis of $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) and we let $e_\alpha(m)^*$ (resp. $e_{-\alpha}(m)^*$) be the dual basis of the space $\mathfrak{n}_+^*$ (resp. $\mathfrak{n}_-^*$) of linear functions on $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) which vanish on all but finitely many vectors of the basis. Using the bilinear form $(.|.)$ on $\mathfrak{g}$ we may identify $\mathfrak{n}_\pm^*$ with $\mathfrak{n}_\mp$ so that $e_{\pm\alpha}(m)^*$ gets identified with $e_{\mp\alpha}(-m)$. Set

$$p_+ = \sum_{\alpha\in\Pi} e_\alpha(-1)^*, \quad p_- = \sum_{\alpha\in\Pi} e_{-\alpha}(0)^*\,. \tag{2.1.1}$$

Note that p_+ (resp. p_-) is a character of $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$).

Let $\pi_\pm$ be the restriction map from the dual of $\mathfrak{g}$ to the dual of $\mathfrak{n}_\pm$. Denote by $\mathscr{F}_\pm(\bar{\mathfrak{g}})$ the algebra of local functionals on $\pi_\pm^{-1}(p_\pm)$ invariant under $\mathfrak{n}_\pm$. These two algebras have canonical structures of Poisson algebras induced from the canonical Poisson structure on the dual of $\mathfrak{g}$. These Poisson algebras are called the classical W-algebras. Drinfeld and Sokolov used them to write equations of KdV type as Hamiltonian systems [14]. For example, $\mathscr{F}_\pm(sl_n)$ is the Gelfand–Dikii algebra, which can be identified with the Poisson algebra of local functionals on the space of differential operators of the form $\partial^n + a_2(x)\partial^{n-2} + \ldots + a_n(x)$ on the circle.

According to [30], the Poisson algebra $\mathscr{F}_\pm(\bar{\mathfrak{g}})$ can be obtained as the cohomology of the complex which is the "local completion" of $(\mathbb{C}[\mathfrak{g}^*] \otimes \Lambda(\mathfrak{n} \oplus \mathfrak{n}^*), d)$, where $\mathbb{C}[\mathfrak{g}^*]$ is the Poisson algebra of polynomial functions on $\mathfrak{g}^*$ and $d = \{\varphi, \cdot\}$. Here

$$\varphi = \sum_\alpha e_\alpha \otimes \varphi_\alpha^* - \frac{1}{2} \sum_{\alpha,\beta,\gamma} c_{\alpha\beta}^\gamma \otimes \varphi_\gamma \varphi_\alpha^* \varphi_\beta^* + \sum_\alpha p(e_\alpha) \otimes \varphi_\alpha^* ,$$

where α, β, γ are roots of $\bar{\mathfrak{n}}_\pm$.

Indeed, the space $\mathscr{F}_\pm$ of functions on $\pi_\pm^{-1}(p_\pm)$ can be identified with the 0^{th} cohomology of the Koszul complex $\mathbb{C}[\mathfrak{g}^*] \otimes \Lambda(\mathfrak{n}_\pm)$ with respect to the differential

$$d_1 = \sum_\alpha e_\alpha \otimes \{\varphi_\alpha^*, \cdot\} + \sum_\alpha p(e_\alpha) \otimes \{\varphi_\alpha^*, \cdot\} .$$

The space of $\mathfrak{n}_\pm$-invariant functionals on $\pi_\pm^{-1}(p_\pm)$ is the 0^{th} cohomology of the standard cohomology complex $\mathscr{F}_\pm \otimes \Lambda^*(\mathfrak{n}_\pm^*)$ of the Lie algebra $\mathfrak{n}_\pm$. The differential of this complex is equal to

$$d_2 = \sum_\alpha \{e_\alpha, \cdot\} \otimes \varphi_\alpha^* - \frac{1}{2} \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta}^\gamma \otimes \{\varphi_\gamma \varphi_\alpha^* \varphi_\beta^*, \cdot\} .$$

Therefore, the space $\mathscr{F}_\pm(\bar{\mathfrak{g}})$ is the 0^{th} cohomology of the double complex $\mathbb{C}[\mathfrak{g}^*] \otimes \Lambda(\mathfrak{n}_\pm \oplus \mathfrak{n}_\pm^*)$ with respect to the differential $d_1 + d_2 = d$.

Remark 2.1. This construction can be generalized as follows. Let $\bar{\mathfrak{n}}_1$ be an ideal in $\bar{\mathfrak{n}} = \bar{\mathfrak{n}}_+$, and $p \in \bar{\mathfrak{n}}^*$ be such that its stabilizer in $\bar{\mathfrak{n}}$ is $\bar{\mathfrak{n}}_1$. Let $\mathfrak{n} = \mathbb{C}[t, t^{-1}] \otimes \bar{\mathfrak{n}}$, $\mathfrak{n}_1 = \mathbb{C}[t, t^{-1}] \otimes \bar{\mathfrak{n}}_1$. Let us define a linear functional $p \in \mathfrak{n}^*$ by $p(x(n)) = \delta_{n,-1}\bar{p}(x)$, so that p restricted to $\mathfrak{n}_1$ defines its one-dimensional representation $\mathbb{C}_p$. Let N and N_1 be Lie groups corresponding to Lie algebras $\mathfrak{n}$ and $\mathfrak{n}_1$. We can apply the Hamiltonian reduction to the orbit $\mathscr{O}_p$ of $p \in \mathfrak{n}^*$, which is isomorphic to $\tilde{N} = N/N_1$. Here $\tilde{N}$ is a Lie group, because N_1 is a normal subgroup of N. The Poisson algebra of local functionals on the reduced Hamiltonian space $\pi^{-1}(\mathscr{O}_p)/N$ coincides then with the 0^{th} cohomology of the local completion of the complex

$$\mathbb{C}[\mathfrak{g}^*] \otimes \Lambda^*(\mathfrak{n} \oplus \mathfrak{n}^*) \otimes \mathbb{C}[\mathscr{O}_p] ,$$

with respect to the differential

$$d_{st} = \left\{ \sum_\alpha e_\alpha \otimes \varphi_\alpha^* - \frac{1}{2} \sum_{\alpha,\beta,\gamma} c_{\alpha,\beta}^\gamma \otimes \varphi_\gamma \varphi_\alpha^* \varphi_\beta^*, \cdot \right\} \otimes 1 + \sum_\alpha 1 \otimes \{\varphi_\alpha^*, \cdot\} \otimes e_\alpha .$$

Here $\mathbb{C}[\mathscr{O}_p]$ is the space of polynomial functions on $\mathscr{O}_p$, and the Lie algebra $\mathfrak{n}$ acts on it as on the coinduced module $\mathrm{Hom}_{U(\mathfrak{n}_1)}(U(\mathfrak{n}), \mathbb{C}_p)$. If $\mathfrak{n}_1 = \mathfrak{n}$ and $p = p_+$, then

$\mathcal{O}_p = p_+$ and we have the Drinfeld–Sokolov reduction. Some other cases were considered in [29, 33, and 12].

2.2. Quantization of Drinfeld–Sokolov Reductions. As usual, the quantization procedure consists of replacing the Poisson algebra $\mathscr{L}(\mathfrak{g}^*)$ by the corresponding universal enveloping algebra and the Grassmann algebra by the Clifford algebra. This is explained below.

We shall view the space $\bar{\mathfrak{a}}^{\bar{1}}_{\pm} := \bar{\mathfrak{n}}_{\pm} \oplus \bar{\mathfrak{n}}^*_{\pm} (\equiv \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{n}}_-)$ as an odd commutative Lie superalgebra with the bilinear form $(.|.)$ restricted from $\bar{\mathfrak{g}}$. We define a Lie superalgebra $\bar{\mathfrak{a}}_{\pm}$ to be the orthogonal direct sum of the even Lie superalgebra $\bar{\mathfrak{a}}^{\bar{0}}_{\pm} := \bar{\mathfrak{g}}$ with the bilinear form $(.|.)$ and the odd Lie superalgebra $\bar{\mathfrak{a}}^{\bar{1}}$ with the bilinear form $(.|.)$. (Of course, the superalgebras $\bar{\mathfrak{a}}_+$ and $\bar{\mathfrak{a}}_-$ are naturally isomorphic.)

We consider the central extension

$$\mathfrak{a}_{\pm} = \mathbb{C}[t, t^{-1}] \otimes \bar{\mathfrak{a}}_{\pm} \oplus \mathbb{C}K \oplus \mathbb{C}K'$$

of the loop super-algebra $\mathbb{C}[t, t^{-1}] \otimes \bar{\mathfrak{a}}_{\pm}$ by letting the even and odd part commute, the bracket on the even part given by (1.2.1) and on the odd part by

$$[a(m), b(n)] = \delta_{m,-n}(a|b)K' , \tag{2.2.1}$$

where $a, b \in \bar{\mathfrak{a}}_{\pm}, m, n \in \mathbb{Z}$ and, as before, $a(m)$ stands for $t^m \otimes a$. Its even part $\mathfrak{a}^{\bar{0}}_{\pm} = \mathfrak{g} \oplus \mathbb{C}K'$ and its odd part $\mathfrak{a}^{\bar{1}}_{\pm} = \mathfrak{n}_{\pm} \oplus \mathfrak{n}^*_{\pm}$.

We have the following decomposition:

$$U'(\mathfrak{a}_{\pm}) := U(\mathfrak{a}_{\pm})/(K' - 1) = U(\mathfrak{g}) \otimes \mathscr{C}\ell(\mathfrak{a}^{\bar{1}}_{\pm}) .$$

Here $U(.)$ stands for the universal enveloping (super) algebra and $\mathscr{C}\ell(\mathfrak{a}^{\bar{1}}_{\pm})$ for the Clifford algebra on the space $\mathfrak{a}^{\bar{1}}_{\pm} (\equiv \mathfrak{n}_+ \oplus \mathfrak{n}_-)$ with the symmetric bilinear form induced from $\mathfrak{g}$.

Introduce a $\mathbb{Z}$-gradation of $U'(\mathfrak{a}_{\pm})$ by letting

$$\deg \mathfrak{g} = 0, \quad \deg \mathfrak{n}_{\pm} = -1, \quad \deg \mathfrak{n}^*_{\pm} = 1 . \tag{2.2.2}$$

Given $k \in \mathbb{C}$, let $U_k(\mathfrak{a}_{\pm}) = U'(\mathfrak{a}_{\pm})/(K - k)$ with the induced $\mathbb{Z}$-gradation (in other words, we fix the value of the affine central charge to be equal k).

Let z be an indeterminate. For an element $a \in \bar{\mathfrak{g}} + \bar{\mathfrak{n}}_{\pm}$ (resp. $\in \bar{\mathfrak{n}}^*_{\pm}$) of $\bar{\mathfrak{a}}_{\pm}$ we let

$$\Delta_a = 1 \,(\text{resp.} = 0) ,$$

and define the *elementary* field $a(z)$ of conformal dimension Δ_a as the series

$$a(z) = \sum_{m \in \mathbb{Z}} a(m) z^{-m-\Delta_a} .$$

Arbitrary fields are obtained from these by taking derivatives in z and normally ordered products a finite number of times. (Recall that one defines the derivative $\partial A(z)$ of a field $A(z)$ of conformal dimension Δ to be the field $\frac{d}{dz} A(z)$ of conformal dimension $\Delta + 1$, and the normally ordered product of fields $A(z)$ and $A_1(z)$ of conformal dimensions Δ and Δ_1 to be the field

$$:AA_1:(z) = A_-(z)A_1(z) \pm A_1(z)A_+(z)$$

of conformal dimension $\Delta + \Delta_1$. Here, as usual,

$$A_-(z) = \sum_{n \leqq -\Delta} A(n) z^{-n-\Delta}, \quad A_+(z) = \sum_{n > -\Delta} A(n) z^{-n-\Delta},$$

and the sign + (resp. −) is taken if at most one (resp. both) of the fields is odd.)

Let $U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}$ denote the $\mathbb{C}$-span of the coefficients of all fields (in a certain completion of $U_k(\mathfrak{a}_\pm)$). It is well-known that it is closed under the Lie (super) bracket. Also, it inherits $\mathbb{Z}$-gradation (2.2.2) from $U_k(\mathfrak{a}_\pm)$. ($U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}$ consists of the series whose symbols are local functionals on $\mathfrak{a}^*$ [20]. One also knows that the associative envelope of $U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}$ is its universal enveloping algebra [20].)

Denote by $\varphi_\alpha(m)$ and $\varphi_\alpha(m)^*$, $\alpha \in \bar{\Delta}_+$, $m \in \mathbb{Z}$, the generators of $\mathscr{C}\ell(\mathfrak{n}_+ \oplus \mathfrak{n}_+^*)$ (resp. $\mathscr{C}\ell(\mathfrak{n}_- \oplus \mathfrak{n}_-^*)$) which correspond to the elements $e_\alpha(m)$ and $e_\alpha(m)^*$ (resp. $e_{-\alpha}(m)$ and $e_{-\alpha}(m)^*$) of $\mathfrak{n}_+$ and $\mathfrak{n}_+^*$ (resp. $\mathfrak{n}_-$ and $\mathfrak{n}_-^*$). Consider the following fields (here normally ordered products are the usual products):

$$d_{\mathrm{st}}^{\pm}(z) = \sum_{\gamma \in \bar{\Delta}_+} e_{\pm\alpha}(z)\varphi_\alpha^*(z) - \frac{1}{2} \sum_{\gamma, \beta, \gamma \in \bar{\Delta}_+} c_{\alpha\beta}^{\gamma} \varphi_\gamma(z)\varphi_\alpha^*(z)\varphi_\beta^*(z),$$

where $[e_\alpha, e_\beta] = \sum_\gamma c_{\alpha\beta}^\gamma e_\gamma$.

Let $d_{\mathrm{st}}^{\pm} \in U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}$ be the coefficient of z^{-1} in $d_{\mathrm{st}}^{\pm}(z)$. It is easy to check that the singular part of the OPE $d_{\mathrm{st}}^{\pm}(z) d_{\mathrm{st}}^{\pm}(w)$ is 0 (see Sect. 3.1 for a digression on OPE). Hence $(d_{\mathrm{st}}^{\pm})^2 = 0$. The operator d_{st}^{+} (resp. d_{st}^{-}) is the standard differential of $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$)-cohomology. We let as before

$$p_+ = \sum_{\gamma \in \bar{\Pi}} \varphi_\alpha^*(1), \quad p_- = \sum_{\gamma \in \bar{\Pi}} \varphi_\alpha^*(0),$$

and define a new differential:

$$d_\pm = d_{\mathrm{st}}^{\pm} + p_\pm.$$

One easily checks that $d_{\mathrm{st}}^{\pm}$ and $p_\pm$ anticommute, hence $d_\pm^2 = 0$. Note that $d_\pm$ is an odd element of $U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}$ of degree 1. Hence the operator $D_\pm$ defined by

$$D_\pm(u) = [d_\pm, u], \quad u \in U_k(\mathfrak{a}_\pm)_{\mathrm{loc}},$$

equipes the $\mathbb{Z}$-graded Lie super-algebra $U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}$ with the structure of a differential graded Lie superalgebra. The corresponding cohomology is again a Lie superalgebra.

Note that the complexes $(U_k(\mathfrak{a}_+)_{\mathrm{loc}}, D_+)$ and $(U_k(\mathfrak{a}_-)_{\mathrm{loc}}, D_-)$ are naturally isomorphic. Indeed, let $\bar{w}^0$ be the involutive automorphism of $\bar{\mathfrak{g}}$ that maps $\bar{\mathfrak{n}}^+$ to $\bar{\mathfrak{n}}_-$ and induces the element $\bar{w}^0 \in \bar{W}$, and consider the element $\tilde{w} = \bar{w}^0 t_{-\bar{\rho}^\vee}$. We have:

$$\tilde{w}(\Lambda_0) = \Lambda_0 + \bar{\rho}^\vee - \tfrac{1}{2}|\bar{\rho}^\vee|^2\delta, \tag{2.2.3}$$

$$\tilde{w}(\alpha) = -{}^t\alpha + (ht\alpha)\delta \quad \text{if } \alpha \in \bar{\Delta}, \quad \text{where } {}^t\alpha = -\bar{w}^0(\alpha), \tag{2.2.4}$$

$$\tilde{w}(x) = \bar{w}^0(x) + (\bar{\rho}^\vee | x)K \quad \text{if } x \in \bar{\mathfrak{h}}, \tag{2.2.5}$$

$$\tilde{w}(e_\alpha(n)) = e_{-{}^t\alpha}(n + ht\alpha), \quad \tilde{w}(\varphi_\alpha(n)) = \varphi_{{}^t\alpha}(n + ht\alpha),$$

$$\tilde{w}(\varphi_\alpha^*(n)) = \varphi_{{}^t\alpha}^*(n - ht\alpha). \tag{2.2.6}$$

Using these formulas, we see that $\tilde{w}$ maps p_+ to p_-, $\mathfrak{n}_\pm$ to $\mathfrak{n}_\mp$ and induces an isomorphism of $U_k(\mathfrak{a}_+)$ and $U_k(\mathfrak{a}_-)$ which maps d_+ to d_-. Hence the cohomology of complexes $(U_k(\mathfrak{a}_+)_{\mathrm{loc}}, D_+)$ and $(U_k(\mathfrak{a}_-)_{\mathrm{loc}}, D_-)$ are isomorphic.

It was conjectured in [20] (and proved in [23] for generic k) that the i^{th} cohomology of these complexes vanishes if $i \neq 0$. The 0^{th} cohomology is a Lie algebra which is called the *W-algebra associated to* $\bar{\mathfrak{g}}$ and is denoted by $W_k^\pm(\bar{\mathfrak{g}})$. This is a natural quantization of the classical W-algebra. Namely, it was shown in [23] that $W_k^+(\bar{\mathfrak{g}})$ is a quantum deformation of the classical W-algebra. Its properties are described in Sect. 3.3.

2.3. The Functors $F_\pm^i$. Let $\Lambda_\pm$ be the module of semi-infinite forms over the algebra $\mathscr{C}\ell(\bar{\mathfrak{a}}_\pm^1)$, i.e. the irreducible module with the cyclic vector $v_\pm$ satisfying the following conditions ($\alpha \in \bar{\Delta}_+$):

$$\varphi_\alpha^*(m)v_- = 0 \text{ if } m \geqq 0, \quad \varphi_\alpha(m)v_- = 0 \text{ if } m > 0\,,$$

$$\varphi_\alpha^*(m)v_+ = 0 \text{ if } m > 0, \quad \varphi_\alpha(m)v_+ = 0 \text{ if } m \geqq 0\,.$$

Letting $\deg v_\pm = 0$, $\Lambda_\pm$ inherits $\mathbb{Z}$-gradation from $\mathscr{C}\ell(\bar{\mathfrak{a}}_\pm^1)$ (given by (2.2.2)):

$$\Lambda_\pm = \sum_{m \in \mathbb{Z}} \Lambda_\pm^m\,. \tag{2.3.1}$$

Given a restricted $\mathfrak{g}$-module M, let

$$C_\pm(M) = M \otimes \Lambda_\pm = \sum_{j \in \mathbb{Z}} C_\pm^j(M), \quad \text{where} \quad C_\pm^j(M) = M \otimes \Lambda_\pm^j\,.$$

The Lie superalgebra $U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}$ acts on $C_\pm(M)$. In particular, the element $d_\pm$ acts on $C_\pm(M)$ shifting the $\mathbb{Z}$-degree by 1. Let $H_\pm(M) = \bigoplus_{j \in \mathbb{Z}} H_\pm^j(M)$ be the cohomology of the complex $(C_\pm(M), d_\pm)$. The representation of $U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}$ on $C_\pm(M)$ induces a representation of the Lie algebra $W_k^\pm(\bar{\mathfrak{g}})$ on each space $H_\pm^j(M)$.

Thus, we get functors, which we denote by $F_\pm^j$, from the category of positive energy $\mathfrak{g}$-modules to the category of $W_k^\pm(\mathfrak{g})$-modules, that send M to $H_\pm^j(M)$.

In order to prove a vanishing theorem, we need the following standard lemma.

Lemma 2.3. *Let (C, d) be a complex,* i.e. $d(C^j) \subset C^{j+1}$ *and* $d^2 = 0$. *Let $\delta\colon C \to C$ be such that $\delta(C^j) \subset C^{j-1}$ and $d\delta + \delta d = A$ is an invertible operator on C. Then the cohomology of the complex (C, d) is zero.*

Proof. First, note that $dA = d\delta d = Ad$. Given $\omega \in C$ such that $d\omega = 0$, let $\omega' = \delta(A^{-1}\omega)$. Then $d\omega' = d\delta(A^{-1}\omega) = \omega - \delta d(A^{-1}\omega) = \omega$ since A and d commute. $\square$

Theorem 2.3. *If M is a positive energy $\mathfrak{g}$-module such that $e_{-\alpha_i}$ is locally nilpotent for some $\alpha_i \in \bar{\Pi}$, then $H_-(M) = 0$.*

Proof. Let $\delta = \varphi_{\alpha_i}(0)$. Then we have:

$$d\delta + \delta d = 1 + n\,,$$

where $n = e_{-\alpha_i}(0) + \sum_{\gamma \in \bar{\Delta}_+} \sum_j c_{\alpha,\alpha_i}^{\alpha+\alpha_i} \varphi_{\alpha+\alpha_i}(-j)\varphi_\alpha^*(j)$. The operator n is locally nilpotent on $C(M)$ since $e_{-\alpha_i}(0)$ is, hence $1 + n$ is invertible and we apply Lemma 2.3. $\square$

Remark 2.3. We can also quantize the generalized Drinfeld–Sokolov reduction, described in Remark 2.1. The quantum BRST complex then is $U_k(\tilde{\mathfrak{a}})_{\mathrm{loc}}$, where $\tilde{\mathfrak{a}} = (\mathfrak{g} \oplus \tilde{\mathfrak{n}})_{\bar{0}} \oplus (\mathfrak{n} \oplus \mathfrak{n}^*)_{\bar{1}}$ where $\tilde{\mathfrak{n}} = \mathfrak{n}/\mathfrak{n}_1$. The differential $\tilde{D}$ is the supercommutator with d_{st}, and the action of $\mathfrak{n}$ on $U_k(\tilde{\mathfrak{a}})$ is twisted by p. Let us denote the 0^{th} cohomology of this algebra by $W_k(\bar{\mathfrak{g}}, \bar{\mathfrak{n}}_1, \bar{p})$. We also have functors from the category of positive-energy modules over $\mathfrak{g}$ to the category of $W_k(\bar{\mathfrak{g}}, \bar{\mathfrak{n}}_1, \bar{p})$-modules, sending a $\mathfrak{g}$-module M to $H^j(\mathfrak{n}, M \otimes N_p)$, where $N_p = \mathrm{Hom}_{U(\mathfrak{n}_1)}(U(\mathfrak{n}), \mathbb{C}_p)$. The algebra $W_k(sl_3, \bar{\mathfrak{n}}_1, \bar{p})$ where $\mathfrak{n}_1$ is the span of the generator of the maximal root, was considered in [5].

3. A Calculation of Characters of W-Algebras

3.1. The Virasoro Subalgebra. Let $k \neq -h^\vee$. Then the W-algebras $W_k^+(\bar{\mathfrak{g}})$ contains a subalgebra Vir isomorphic to the Virasoro algebra. In order to give a formula for its generating field $T(z) = \sum_{m \in \mathbb{Z}} L_m^+ z^{-m-2}$, we choose an orthonormal basis $\{u_i\}$ of $\bar{\mathfrak{h}}$. Then

$$T(z) = S(z) + \partial_z \bar{\rho}^\vee(z) + T_{\mathrm{gh}}(z) \,,$$

where $S(z)$ is the Sugawara field (cf. Sect. 1.6):

$$S(z) = \frac{1}{2(k+h^\vee)} \sum_{\alpha \in \bar{\Delta}} \left(:e_\alpha(z) e_{-\alpha}(z): + \sum_i :u_i(z)^2: \right) ,$$

and $T_{\mathrm{gh}}(z)$ is the ghost field:

$$T_{\mathrm{gh}}(z) = \sum_{\alpha \in \bar{\Delta}_+} (ht\alpha : \partial\varphi_\alpha(z)\varphi_\alpha^*(z): + (1 - ht\alpha) : \partial\varphi_\alpha^*(z)\varphi_\alpha(z):) \,.$$

We have to show that

$$[T(z), d_+] = 0 \,. \tag{3.1.1}$$

In order to avoid lengthy calculations, we use some well known field-theoretic techniques that we now recall. Given two fields $A(z)$ and $A_1(z)$ of conformal dimensions Δ and Δ_1, we may write their *operator product expansion* (OPE):

$$A(z)A_1(w) = \sum_{j \geqq -\Delta - \Delta_1} C_j(w)(z-w)^j \,,$$

where $C_j(z)$ are some fields. The sum of terms with $j < 0$, the singular part of the OPE, determines the (super) bracket of Fourier coefficients. One says that fields do not interact if the singular part of the OPE is 0 (in this case the Fourier coefficients (super) commute). The regular part of the OPE is unimportant for calculation of (super) commutators and is usually dropped.

One calls a field $t(z)$ of conformal dimension 2 an *energy-momentum field* with *central charge* c if

$$t(z)t(w) = \frac{c/2}{(z-w)^4} + \frac{2t(w)}{(z-w)^2} + \frac{\partial t(w)}{z-w}, \quad \text{where } c \in \mathbb{C} \,.$$

This OPE is equivalent to the property that $t(z) = \sum_{n \in \mathbb{Z}} t_n z^{-n-2}$ and the t_n obey the commutation relations of the Virasoro algebra with central charge c.

A field $A(z)$ is called *primary* of conformal dimension Δ with respect to $t(z)$ if

$$t(z)A(w) = \frac{\Delta A(w)}{(z-w)^2} + \frac{\partial A(w)}{z-w}.$$

In order to prove (3.1.1) we use the following simple lemma.

Lemma 3.1.1. *If $A(z) = \sum_n A_n z^{-n-1}$ is a primary field of conformal dimension 1 with respect to an energy-momentum field $t(z)$, then $[t(z), A_0] = 0$.* □

Let us recall some of the OPE's:

Lemma 3.1.2. (a) *$S(z)$ is an energy-momentum field with central charge c_k (given by* (1.6.2)*).*

(b) *The field $T_{gh}^{\lambda,\alpha} := \lambda : \partial\varphi_\alpha(z)\varphi_\alpha^*(z) : + (1-\lambda) : \partial\varphi_\alpha^*(z)\varphi_\alpha(z) :$ is an energy-momentum field with central charge*

$$c^\lambda = -12\lambda^2 + 12\lambda - 2.$$

(c) *Elementary fields $a(z)$, $a \in \bar{\mathfrak{g}}$, are primary with respect to $S(z)$ of conformal dimension* 1.

(d) *Elementary fields $\varphi_\alpha(z)$ and $\varphi_\alpha^*(z)$ are primary with respect to $T_{gh}^{\lambda,\alpha}$ of conformal dimension λ and $1-\lambda$ respectively.*

(e) *One has the following* OPE *between elementary fields*:

$$a(z)b(w) = \frac{[a,b](w)}{z-w} + \frac{(a|b)k}{(z-w)^2}, \quad a, b \in \bar{\mathfrak{g}},$$

$$\varphi_\alpha(z)\varphi_\alpha^*(w) = \frac{1}{z-w},$$

all other pairs of elementary fields do not interact. □

The following lemma is immediate by Lemma 3.1.2, using the usual Wick formula for free fermionic fields.

Lemma 3.1.3. (a) *$T_+(z)$ is an energy-momentum field with central charge*

$$c(k) = c_k - 12k|\bar{\rho}^\vee|^2 - 2\sum_{\gamma\in\bar{\Delta}_+}(6(ht\alpha)^2 - 6ht\alpha + 1), \tag{3.1.2}$$

where c_k is given by (1.6.2).

(b) *The field $d_+(z) = d_{st}^+(z) + \sum_{\alpha\in\bar{\Pi}}\varphi_\alpha^*(z)$ is primary with respect to $T_+(z)$ of conformal dimension* 1. □

Formula (3.1.1) is immediate by Lemma 3.1.1 and Lemma 3.1.3b.

Remark 3.1. Using (1.8.2–4) we obtain another formula for $c(k)$:

$$\begin{aligned} c(k) &= c_k - \dim\bar{\mathfrak{g}} + \ell + 24(\bar{\rho}|\bar{\rho}^\vee) - 12(k+h^\vee)|\bar{\rho}^\vee|^2 \\ &= \ell - \frac{12}{k+h^\vee}|\bar{\rho}|^2 + 24(\bar{\rho}|\bar{\rho}^\vee) - 12(k+h^\vee)|\bar{\rho}^\vee|^2. \end{aligned} \tag{3.1.3}$$

We have the following explicit formula for L_0^+:

$$L_0^+ = S_0 - \bar{\rho}^\vee + L_{0,\mathrm{gh}}^+, \tag{3.1.4}$$

where

$$L^{+}_{0,\mathrm{gh}} = \sum_{\alpha \in \bar{\Delta}_+} \left(\sum_{m>0} (m - ht\alpha)\varphi_\alpha(-m)\varphi^*_\alpha(m) + \sum_{m \geqq 0} (m + ht\alpha)\varphi^*_\alpha(-m)\varphi_\alpha(m) \right)$$

$$= \sum_{\alpha \in \bar{\Delta}_+} \left(\sum_{m>0} (m\delta - \alpha | \Lambda_0 + \bar{\rho}^\vee)\varphi_\alpha(-m)\varphi^*_\alpha(m) \right.$$

$$\left. + \sum_{m \geqq 0} (m\delta + \alpha | \Lambda_0 + \bar{\rho}^\vee)\varphi^*_\alpha(-m)\varphi_\alpha(m) \right). \tag{3.1.5}$$

Applying the automorphism $\tilde{w}$ (see formulas (2.2.3)–(2.2.6)) we get a formula for $L^-_0 = \tilde{w}(L^+_0)$:

$$L^-_0 = S_0 - \tfrac{1}{2}(k + h^\vee)|\bar{\rho}^\vee|^2 + (\bar{\rho}|\bar{\rho}^\vee) + L^-_{0,\mathrm{gh}}, \tag{3.1.6}$$

where

$$L^-_{0,\mathrm{gh}} = \sum_{\alpha \in \bar{\Delta}_+} \left(\sum_{m \geqq 0} m\varphi_\alpha(-m)\varphi^*_\alpha(m) + \sum_{m>0} m\varphi^*_\alpha(-m)\varphi_\alpha(m) \right). \tag{3.1.7}$$

(In this calculation we use (1.8.3 and 4) along with the formula

$$S_0 = \frac{\Omega}{2(k + h^\vee)} - \Lambda_0,$$

where Ω is the affine Casimir [24, Chapter 2].)

3.2. A Calculation of the Euler Character of the $W^\pm_k$-Module $H_\pm(M)$. Define the Euler character of a direct sum of $W^\pm_k(\mathfrak{g})$-modules $V = \bigoplus_{j \in \mathbb{Z}} V_j$ by the formula

$$\operatorname{ch} V = \sum_{j \in \mathbb{Z}} (-1)^j \operatorname{tr}_{V_j} q^{L^\pm_0}.$$

In this definition we assume that ch V converges, by which we mean that $L^\pm_0$ is diagonalizable on V with finite-dimensional eigenspaces.

Let now M be a positive energy $\mathfrak{g}$-module. By the Euler–Poincaré principle we have:

$$\operatorname{ch} H_\pm(M) = \operatorname{ch} C_\pm(M)$$

if the right-hand side converges. Unfortunately, it does not converge.

To get around this difficulty, introduce a $\mathbb{Z}^2$-gradation of $C_\pm(M)$ in such a way that $\deg d^\pm_{\mathrm{st}} = (1, 0)$ and $\deg p_\pm = (0, 1)$. It follows that the cohomology of the complex $(C_\pm(M), d_\pm)$ may be calculated as follows. First, we calculate the cohomology $E_\pm(M)$ of the complex $(C_\pm(M), d^\pm_{\mathrm{st}})$. Then we consider the spectral sequence $(E_\pm(M)_{(j)}, d^\pm_{(j)})$, $j \geqq 1$, where $E_\pm(M)_{(1)} = E_\pm(M)$ and $d_{(1)} = p_\pm$. This spectral sequence converges, which is ensured by the following facts. The complex $C_\pm(M)$ decomposes into a direct sum of subcomplexes $C_\pm(M)_\lambda$ which are λ-eigenspaces of $L^\pm_0$. Furthermore, $C^p_\pm(M)_\lambda = \oplus_{i+j=p} C^{i,j}_\pm(M)_\lambda$, where j is bounded from above for any λ and p. Thus we have:

$$H_\pm(M) = \lim_{j \to \infty} E_\pm(M)_{(j)}. \tag{3.2.1}$$

In order to proceed, we need the following

Lemma 3.2. *For* $x \in \bar{\mathfrak{h}}$, *let*

$$\tilde{x}(z) = x(z) + \sum_{\alpha \in \bar{\Delta}_+} (x|\alpha) : \varphi_\alpha(z)\varphi_\alpha^*(z): . \tag{3.2.2}$$

Then

(a) $[d_+^{\mathrm{st}}, \tilde{x}(z)] = 0$.

(b) $\tilde{x}(z)\tilde{y}(w) = \dfrac{(x|y)(k + h^\vee)}{(z-w)^2}$.

Proof. One checks (a) directly, using Wick's theorem and the fact that $c_{\alpha\beta}^\gamma \neq 0$ only, if $\gamma = \alpha + \beta$, that the singular part of the OPE $d_{st}^+(z)\tilde{x}(w)$ is zero. (b) follows from (1.8.5). □

By Lemma 3.2b, the Fourier coefficients of the fields $\tilde{x}(z)$, $x \in \bar{\mathfrak{h}}$, form an oscillator algebra of level $k + h^\vee$, which we denote by $\Gamma(\bar{\mathfrak{g}})$. By Lemma 3.2a, $\Gamma(\bar{\mathfrak{g}})$ acts on $E_+(M)$.

Proposition 3.2.1 [20]. *The action of the W-algebra* $W_k^+(\bar{\mathfrak{g}})$ *on* $E_+(M)$ *can be expressed via the action of* $U(\Gamma(\bar{\mathfrak{g}}))_{\mathrm{loc}}$. *In particular, one has*

$$T(z) = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\ell} :\tilde{u}_i^2(z): + \partial\tilde{\bar{\rho}}^\vee(z) - \frac{1}{k+h^\vee}\partial\tilde{\bar{\rho}}(z),$$

where u_i *is an orthonormal basis of* $\bar{\mathfrak{h}}$. □

Fix now $x \in \bar{\mathfrak{h}}$ such that $(\alpha|x) < 0$ in the " + " case (resp. > 0 in the " − " case) for all $\alpha \in \bar{\Pi}$, and fix $\varepsilon > 0$. Then all eigenspaces of the operator $L_0^+ + \varepsilon\tilde{x}$ in $C_+(M)$, (resp. $L_0^- + \varepsilon\tilde{w}(\tilde{x})$) are finite-dimensional. Here

$$\tilde{x} = x + \sum_{m \in \mathbb{Z}} \sum_{\alpha \in \bar{\Delta}_+} (x|\alpha) : \varphi_\alpha(-m)\varphi_\alpha^*(m):$$

is the coefficient of z^{-1} of $\tilde{x}(z)$. Thus, by the Euler–Poincaré principle we have a rigorous formula:

$$\sum_{j \in \mathbb{Z}} (-1)^j \mathrm{tr}_{C_+^j(M)} q^{L_0^+ + \varepsilon\tilde{x}} = \sum_{j \in \mathbb{Z}} (-1)^j \mathrm{tr}_{E_+^j(M)} q^{L_0^+ + \varepsilon\tilde{x}}. \tag{3.2.3$_+$}$$

Letting $d = -S_0$ we extend any positive energy $\mathfrak{g}$-module M to an $\tilde{\mathfrak{h}}$-module (cf. Sect. 1.2) which decomposes into a direct sum of finite-dimensional weight spaces: $M = \bigoplus_{\lambda \in P(M)} M_\lambda$, where $P(M) \subset \tilde{\mathfrak{h}}^*$ is the set of weights of M (note that weight spaces with respect to $\mathfrak{h}$ and to S_0 may be infinite-dimensional). A weight $\mu \in P(M)$ is called maximal if $(\mu|\rho^\vee)$ is maximal. Fixing a maximal weight μ, we may define the height of $\lambda \in P(M)$ by $ht_\mu(\lambda) = (\mu - \lambda|\rho^\vee) \in \mathbb{Z}_+$.

Recall that the oscillator algebra $\Gamma(\bar{\mathfrak{g}})$ acts on $E_+(M)$, the semi-infinite cohomology of $\mathfrak{n}_+$, with coefficients in M. Since $k + h^\vee \neq 0$, it follows from representation theory of oscillator algebras [24, Chap. 9], that for each $\mu \in \mathfrak{h}^*$ of level k there exists a unique irreducible $\Gamma(\bar{\mathfrak{g}})$-module π_μ which admits a non-zero vector v_μ (vacuum vector) such that

$$\tilde{x}(m)v_\mu = 0 \text{ for } m > 0 \quad \text{and} \quad \tilde{x}(0)v_\mu = \mu(x)v_\mu \text{ for } x \in \bar{\mathfrak{h}},$$

and that $E_+(M)$ viewed as a $\Gamma(\bar{\mathfrak{g}})$-module decomposes into a direct sum of modules π_μ. Using the formula for $T(z)$ given by Proposition 3.2.1, we obtain

$$\mathrm{tr}_{\pi_\mu} q^{L_0^+} = q^{P(\mu)} \prod_{n=1}^{\infty} (1 - q^n)^{-\ell},$$

where

$$P(\mu)=\frac{|\bar{\mu}+\bar{\rho}|^2-|\bar{\rho}|^2}{2(k+h^\vee)}-(\mu|\bar{\rho}^\vee)\,.$$

In particular, all eigenspaces of L_0^+ in π_μ are finite-dimensional.

Now we are in a position to prove the following important proposition.

Proposition 3.2.2. *For any positive energy* $\mathfrak{g}$*-module* M *all eigenspaces of* $L_0^\pm$ *on* $E_\pm(M)$ *are finite-dimensional.*

Proof. We give the proof in the " + " case, the proof in the " − " case being similar. Let μ be a maximal weight of M. Denote by $\bar{\mathcal{O}}_\mu$ the category of positive energy modules M' such that all irreducible subquotients of M' are of the form $L(w\,.\,\mu)$, where $w\in W$, and $ht_\mu(\lambda)\in\mathbb{Z}_+$ if $\lambda\in P(M')$. We may assume using results of [13], that M is a module from the category $\bar{\mathcal{O}}_\mu$.

We shall prove by induction on n the following:

Claim (n): the multiplicity of π_λ in a $\mathfrak{g}$-module from the category $\bar{\mathcal{O}}_\mu$ is finite if $ht_\mu(\lambda)\leqq n$ and it is non-zero only if $\lambda=w\,.\,\mu$.

We shall repeatedly use the (obvious) fact that if $ht_\mu(\lambda)>0$ for all maximal weights λ of a $\mathfrak{g}$-module M' (in this case we write: $P(M')<\mu$), then π_μ does not occur in $E_+(M')$.

Let $B(\mu)$ be a Wakimoto module [18]; by its definition, $E_+^0(B(\mu))=\pi_\mu$ and $E_+^j(B(\mu))=0$ if $j\neq 0$. We have an exact sequence:

$$0\to U(\mu)\to B(\mu)\to\bar{B}(\mu)\to 0\,,$$

where $U(\mu)$ is a quotient of $M(\mu)$ and $P(\bar{B}(\mu))<\mu$. From the long exact sequence for semi-infinite cohomology, we conclude that the multiplicity of π_λ in $E_+(U(\mu))$ is equal to that in $E_+(\bar{B}(\mu))$. Hence, Claim (0) holds for $U(\mu)$ and, applying the inductive assumption for n to $\bar{B}(\mu)$, we derive Claim $(n+1)$ for $U(\mu)$. Thus, we have proved the claim for $U(\mu)$. The same argument applied to the exact sequence

$$0\to\bar{M}(\mu)\to M(\mu)\to U(\mu)\to 0$$

proves the claim for the Verma module $M(\mu)$. Similarly we prove the claim for any quotient of $M(\mu)$.

Finally, let $\mu_1,\ldots,\mu_n$ be all maximal weights of M (with their multiplicities). Consider the exact sequence

$$0\to\bigoplus_i\bar{M}(\mu_i)\to M\to\bar{M}\to 0\,,$$

where $\bar{M}(\mu_i)$ are some quotients of Verma modules. Applying the above argument to this exact sequence we prove the claim for M.

Proposition now follows since for any positive integer N there exists n such that $P(w\,.\,\mu)>N$ if $\ell(w)>n$. $\square$

Using Proposition 3.2.2, formulas (3.2.1) and $(3.2.3)_+$ now imply:

$$\operatorname{ch}H_+(M)=\lim_{\varepsilon\downarrow 0}\sum_{j\in\mathbb{Z}}(-1)^j\operatorname{tr}_{C_+^j(M)}q^{L_0^++\varepsilon\tilde{x}}\,.\tag{3.2.4$_+$}$$

Similarly, under the same assumption on M we have

$$\operatorname{ch}H_-(M)=\lim_{\varepsilon\downarrow 0}\sum_{j\in\mathbb{Z}}(-1)^j\operatorname{tr}_{C_-^j(M)}q^{L_0^-+\varepsilon\tilde{w}(\tilde{x})}\,.\tag{3.2.4$_-$}$$

We can prove now

Proposition 3.2.3. *Let M be a positive energy $\mathfrak{g}$-module of level k. Then*:

(a) $\mathrm{ch}\, H_-(M) = q^{\frac{1}{24}(c(k)-c_k)}\eta(\tau)^{\dim \bar{\mathfrak{g}} - \ell}\mathrm{Res}_{x=0}\mathrm{ch}_M(\tau, x, 0)$.

(b) $\mathrm{ch}\, H_+(M) = (-1)^{2(\bar\rho|\bar\rho^\vee)+|\bar\Delta_+|}q^{\frac{1}{24}(c(k)-c_k)}\eta(\tau)^{\dim \bar{\mathfrak{g}} - \ell}\mathrm{Res}_{x=0}\mathrm{ch}_M(\tau, -\tau\bar\rho^\vee + x, \tau|\bar\rho^\vee|^2/2)$.

Proof. Using (3.1.5) and (3.1.7), it is straightforward to derive the following formulas:

$$\sum_{j\in\mathbb{Z}}(-1)^j \mathrm{tr}_{\Lambda_+^j} q^{L^+_{0,\mathrm{gh}} + \varepsilon(\tilde{x}-x)} = \prod_{\alpha\in\Delta_+^{\mathrm{re}}}\left(1 - q^{(\Lambda_0 + \bar\rho^\vee - \varepsilon x|\alpha)}\right), \qquad (3.2.5_+)$$

$$\sum_{j\in\mathbb{Z}}(-1)^j \mathrm{tr}_{\Lambda_-^j} q^{L^-_{0,\mathrm{gh}} + \varepsilon\tilde{w}(\tilde{x}-x)} = \prod_{\alpha\in\Delta_+^{\mathrm{re}}}\left(1 - q^{(\Lambda_0 - \varepsilon\bar{w}^0(x)|\alpha)}\right). \qquad (3.2.5_-)$$

Since $C_\pm(M)$ is a tensor product of M and $\Lambda_\pm$, and since the operator $L_0^+ + \varepsilon\tilde{x}$ (resp. $L_0^- + \varepsilon\tilde{w}(\tilde{x})$) acts on M as $S_0 - \bar\rho^\vee + x$ (resp. $S_0 - \frac{1}{2}(k+h^\vee)|\bar\rho^\vee|^2 + (\bar\rho|\bar\rho^\vee) + \varepsilon\tilde{w}(x)$), and on Λ_+ (resp. Λ_-) as $L^+_{0,\mathrm{gh}} + \varepsilon(\tilde{x}-x)$ (resp. as $L^-_{0,\mathrm{gh}} + \varepsilon\tilde{w}(\tilde{x}-x)$), (a) and (b) follow from $(3.2.5_-)$ and $(3.2.5_+)$ respectively. $\square$

Comparing Proposition 3.2.3 with Proposition 1.8 (and its proof), we obtain

Theorem 3.2. *Let $\Lambda \in P^k_{u,t_\beta\bar{y}}$ be a principal admissible weight.*

(a) *If $\Lambda \in N^k_-$ and $\varphi^-(\Lambda) = (\lambda, \mu)$, then $q^{-c(k)/24}\mathrm{ch}\, H_-(L(\Lambda)) = \varphi_{\lambda,\mu}(\tau)$.*

(b) *If $\Lambda \in N^k_+$, $\varphi^+(\Lambda) = (\lambda, \mu)$ and w is the element of W defined in Proposition 1.8b(ii), then*

$$q^{-c(k)/24}\,\mathrm{ch}\, H_+(L(\Lambda)) = \varepsilon(w)\varphi_{\lambda,\mu}(\tau)\,. \qquad \square$$

3.3. Some Properties of W-Algebras. In order to state some properties of W-algebras define the Harish–Chandra homomorphism $\pi: U_k(\mathfrak{a}_\pm)_{\mathrm{loc}} \to U(\bar{\mathfrak{h}})$. For this note that $U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}$ is a direct sum of subspaces $U(\bar{\mathfrak{h}})$ and $\bar{\mathfrak{n}}_- U_k(\mathfrak{a}_\pm)_{\mathrm{loc}} + U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}\bar{\mathfrak{n}}_+ + \bar{\mathfrak{a}}^1_\pm U_k(\mathfrak{a}_\pm)_{\mathrm{loc}}$. We let π be the projection on the first summand. Let $d_1 = 2 < d_2 \leqq \ldots < d_\ell = h$ be the degrees of fundamental $\bar{W}$-invariants in $S(\bar{\mathfrak{h}})$.

It was shown in [21, 23] that the W-algebra $W_k^+(\bar{\mathfrak{g}})$ contains Fourier coefficients of ℓ fundamental fields $W_2^+(z) = T(z), W_{d_2}^+(z), \ldots, W_{d_\ell}^+(z)$, with the following properties:

(W1) The Lie algebra $W_k^+(\bar{\mathfrak{g}})$ is the linear span of the Fourier components of all fields obtained from the $W_j^+(z)$ by taking finitely many times normally ordered products and derivatives.

(W2) $W_j^+(z)$ has conformal dimension j:

$$W_j^+(z) = \sum_{m\in\mathbb{Z}} W_j^+(m)z^{-m-j}, \quad [W_j^+(m), L_0^+] = mW_j^+(m)\,.$$

(W3) The highest degree terms of $\pi(W_j^+(0))$, $j = d_1, \ldots, d_\ell$, generate the algebra of invariants $S(\bar{\mathfrak{h}})^{\bar{W}}$.

(W4) $[W_i^+(0), W_j^+(0)] = \sum_{s,m} c_{ij}(s,m)W_{s_1}^+(m_1)W_{s_2}^+(m_2)\ldots$, where $m_1 \leqq m_2 \leqq \ldots, \sum_r m_r = 0, \sum_r s_r < i+j$.

We let $W_j^-(m) = \tilde{w}^{-1}(W_j^+(m))$.

Proposition 3.3. (a) *Let* $P_j(\lambda) = \pi(W_j^-(0))(\lambda + \bar{\rho})$, $\lambda \in \bar{\mathfrak{h}}^*$. *Then the polynomials* P_j *generate the algebra* $S(\bar{\mathfrak{h}})^{\bar{W}}$.

(b) $c_{ij}(s, 0) = 0$ *for all* $i, j \in \{d_1, \ldots, d_\ell\}$, $s = (s_1, s_2, \ldots)$.

Proof. In view of (W3), in order to prove (a) it suffices to show that P_j is $\bar{W}$-invariant. Let r_i be the reflection with respect to α_i $(i = 1, \ldots, \ell)$. Let $n \in \mathbb{N}$ and let λ be a generic element in the hyperplane $\langle \lambda, \alpha_i^\pm \rangle = n$ in $\mathfrak{h}^*$ of level k. Consider the Wakimoto module $B(\lambda)$. Then we have either the exact sequence

$$0 \to L(\lambda) \to B(\lambda) \to L(r_i . \lambda) \to 0 \,,$$

or the same sequence with arrows reversed.

Recall that $H_-(L(\lambda)) = 0$ by Theorem 2.3. Note also that $E_-(B(\lambda)) = H_-(B(\lambda))$ and that $L(r_i . \lambda) = M(r_i . \lambda) = B(r_i . \lambda)$. From the d_--cohomology long exact sequence we now obtain that $H_-^0(L(r_i . \lambda)) = \pi_\lambda$. Since the vector of maximal weight of the $W_k^-(\bar{\mathfrak{g}})$-module $H_-^0(L(r_i . \lambda))$ is $v_{r_i . \lambda} \otimes v_-$, we deduce that

$$\pi(W_j^-(0))(r_i . \lambda) = \pi(W_j^-(0))(\lambda) \quad \text{for all } i \,.$$

This completes the proof of (a).

We prove (b) by induction i which we may assume to be $\leqq j$. For $i = 2$, (b) follows from (W2). By Proposition 3.2.3a, the vector $\tilde{v}_\lambda = v_\lambda \otimes v_-$ spans the eigenspace corresponding to the minimal eigenvalue of L_0^- in $H_-^0(M(\lambda))$. Since all $W_s^-(0)$ commute with L_0^-, we obtain that $W_s^-(0)\tilde{v}_\lambda = P_s(\bar{\lambda} - \bar{\rho})\tilde{v}_\lambda$ for all $\lambda \in \mathfrak{h}^*$. In particular, $[W_i^-(0), W_j^-(0)]\tilde{v}_\lambda = 0$, hence

$$\left(\sum c_{ij}(s, 0)\, W_{s_1}^-(0)\, W_{s_2}^-(0) \ldots\right)\tilde{v}_\lambda = 0 \qquad \text{or all } \lambda \,.$$

Since, by the inductive assumption all factors in the above sum commute (see (W4)), we deduce that $c_{ij}(s, 0) = 0$. $\square$

3.4. Positive Energy Modules Over W-Algebras. A $W_k^\pm(\bar{\mathfrak{g}})$-module is called a positive energy module if $L_0^\pm (= W_2^\pm(0))$ is diagonalizable with finite-dimensional eigenspaces and has a real discrete spectrum bounded below. Note that by Proposition 3.2.2, $F_\pm^j$ are functors from the category of positive energy $\mathfrak{g}$-modules to the category of positive energy $W_k^\pm(\bar{\mathfrak{g}})$-modules (see Sect. 2.3).

Lemma 3.4. *Let* M (resp. M') *be a quotient of a Verma module* $M(\lambda)$ (resp. $M(w . \lambda)$ *for some* $w \in \bar{W}$) *over* $\mathfrak{g}$ *and let* v (resp. v') *denote its highest weight vector. Then the eigenspace corresponding to the minimal eigenvalue of* $L_0^\pm$ *in* $F_\pm^0(M)$ (resp. $F_\pm^0(M')$) *is* 1*-dimensional and* $W_j^\pm(0)$*-invariant. Furthermore the eigenvalues of* $W_j^\pm(0)$ *in these eigenspaces are equal.*

Proof. The first claim of the lemma is clear and the second one follows from Proposition 3.3a. $\square$

Let now V be an irreducible positive energy $W_k^\pm(\bar{\mathfrak{g}})$-module. It is clear by Proposition 3.3b that the eigenspace of $L_0^\pm$ corresponding to the minimal eigenvalue is 1-dimensional. The ℓ-tuple of eigenvalues $(c_1, \ldots, c_\ell)$ of $W_{d_1}^\pm(0), \ldots, W_{d_\ell}^\pm(0)$ in this space is called the highest weight of V. Clearly, it determines V uniquely; moreover, there exists a unique irreducible positive energy module over $W_k^\pm(\bar{\mathfrak{g}})$ with a given highest weight.

Conjecture 3.4$_-$. Let $\Lambda \in N^k$ be a non-degenerate principal admissible weight. Then

(a) $F^0_-(L(\Lambda))$ is an irreducible $W_k^-(\bar{\mathfrak{g}})$-module.

(b) $F^j_-(L(\Lambda)) = 0$ if $j \neq 0$.

Conjecture 3.4$_-$ a together with Propositions 1.5.3 and 3.3a and Theorem 3.2a imply

Proposition 3.4. (a) *Λ and $\Lambda' \in N^k_-$ are such that $\varphi^-(\Lambda) = \varphi^-(\Lambda')$ if and only if $F^0_-(L(\Lambda))$ and $F^0_-(L(\Lambda'))$ are equivalent $W_k^-(\bar{\mathfrak{g}})$-modules. Thus, $W_k^-(\bar{\mathfrak{g}})$-modules obtained by applying the functor F^0_- to the set of $\mathfrak{g}$-modules $\{L(\Lambda) | \Lambda \in N^k_-\}$ is parametrized by the set $I_{p,p'}$ via the map φ^-, where p' is the denominator of k and $p = p'(k + h^\vee)$.*

(b) *Let $\Lambda \in N^k_-$ and let $(\lambda, \mu) = \varphi^-(\Lambda)$. Then:*

$$\mathrm{tr}_{F^0_-(L(\Lambda))} q^{L_0^- - c(k)/24} = \varphi_{\lambda, \mu}(\tau) \, .$$

Remark 3.4. (a) Formulas (3.1.3) for $k = -h^\vee + p/p'$ can be rewritten as follows:

$$c(k) = \ell - \frac{12|p'\bar{\rho} - p\bar{\rho}^\vee|^2}{pp'}, \quad \text{where } (p, p') = 1,\ p \geqq h^\vee, p' \geqq h \, ,$$

which in the simply laced case becomes (cf. [7, 9, 17]):

$$c(k) = \ell\left(1 - \frac{h(h+1)(p-p')^2}{pp'}\right), \quad \text{where } (p, p') = 1,\ p \geqq h, p' \geqq h \, .$$

(b) Let $h_{\lambda\mu}$ denote the lowest eigenvalue of $L_0^\pm$ in a $W_k^\pm(\bar{g})$-module labeled by $(\lambda, \mu) \in I_{p,p'}$. Then $h_{\lambda\mu} = \tilde{h}_{\lambda\mu} + c(k)$, where $\tilde{h}_{\lambda\mu}$ is the exponent of the leading term of $\varphi_{\lambda\mu}$. Using this one easily derives the formula (cf. [9]):

$$h_{\lambda\mu} = \frac{1}{2pp'}\left(|p'(\lambda + \rho) - p(\mu + \rho^\vee)|^2 - |p'\rho - p\rho^\vee|^2\right) .$$

(c) Let $\bar{\mathfrak{g}}$ be a simply laced simple Lie algebra of type different from A_ℓ. Then for each s such that $a_s > 1$ one has a family of non-principal admissible $\mathfrak{g}$-modules $L(\Lambda)$ parameterized by a finite set of Λ's which we denote by P^k_s. The set of all non-principal admissible $\mathfrak{g}$-modules is a union of these sets [27]. The sets P^k_s are explicitly described in [27, Theorem 2.3]. Here we only recall that $P^k_s \neq \emptyset$ if and only if $k + h = p/p'$, where p and p' are relatively prime positive integers such that

$$p \geqq \max(\dot{h}, \ddot{h}, \ldots) \, , \tag{3.4.1}$$

where $\dot{h}, \ddot{h}, \ldots$ are Coxeter numbers of Lie algebras $\dot{\bar{g}}, \ddot{\bar{\mathfrak{g}}}, \ldots$ whose Dynkin diagrams are connected components of the Dynkin diagram of $\mathfrak{g}$ with the s^{th} node deleted. In some cases (described in [27]), (3.4.1) should be a strict inequality. Furthermore, consider the set of coroots $\sigma_s(\Pi^\vee)$ (see [27]); this set of roots decomposes into an orthogonal disjoint union $\dot{\Pi}^\vee \cup \ddot{\Pi}^\vee \cup \ldots$. Define $\dot{k}, \ddot{k}, \ldots$ by $\dot{k} + \dot{h} = \ddot{k} + \ddot{h} = \ldots = k + h$. Let $\dot{\lambda}, \ddot{\lambda}, \ldots$ be dominant integral weights for $\dot{\Pi}^\vee, \ddot{\Pi}^\vee, \ldots$ of levels $p'(\dot{k} + \dot{h}) - \dot{h}$, $p'(\ddot{k} + \ddot{h}) - \ddot{h}, \ldots$. Then there exists a unique element $\lambda \in \mathfrak{h}^*$ such that $\lambda + \rho|_{\dot{\Pi}^\vee} = \dot{\lambda} + \dot{\rho}$, $\lambda + \rho | \ddot{\Pi}^\vee = \ddot{\lambda} + \ddot{\rho}, \ldots$. There exists a unique integer j_λ such that

$$a_s \bar{\lambda} \equiv j_\lambda \bar{\Lambda}_s \bmod \bar{P} \, .$$

Then all possible $(\lambda, \mu) = \varphi^{-}(\Lambda)$, that may occur (where $\Lambda \in P_s^k$, $\lambda \in P_+^{p-h}$, $\mu \in P_+^{p'-h}$) should satisfy the matching condition [27]:

$$(j_\lambda - j_\mu, a_s) = 1 \ . \tag{3.4.2}$$

Let now $N_s^k = \{\Lambda \in P_s^k | \langle \Lambda, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in \bar{\Delta}^\vee\}$ be the set of non-degenerate weights from P_s^k. Due to Theorem 2.3, given $\Lambda \in P_s^k$, we have $H_-(L(\Lambda)) = 0$ unless $\Lambda \in N_s^k$. The set N_s^k is non-empty if p' satisfies the same inequality (3.4.1) as p. In the case $\Lambda \in N_s^k$ we have, using Theorem 3.2a and [27, (3.4)]:

$$q^{-c(k)/24} \operatorname{ch} H_-(L(\Lambda)) = \varphi_{\dot{\lambda}\dot{\mu}}(\tau) \varphi_{\ddot{\lambda}\ddot{\mu}} \ldots \ .$$

In other words (provided that Conjecture 3.4$_-$ holds in this case) characters of corresponding representations of $W_k^-(\bar{\mathfrak{g}})$ are products of characters of $W_{\dot{k}}^-(\bar{\mathfrak{g}})$, $W_{\ddot{k}}^-(\bar{\mathfrak{g}}), \ldots$.

Theorem 3.2b leads us to a conjecture in the " + " case similar to Conjecture 3.4$_-$:

Conjecture 3.4$_+$. Let $\Lambda \in P_{u,t_\beta \bar{y}}^k \cap N_+^k$ (i.e. β satisfies (1.5.7)) and let $(\lambda, \mu) = \varphi^+(\Lambda)$ and $w \in W$ be the element defined in Proposition 1.8b(ii). Then:

(a) $F_+^{\ell t(w)}(L(\Lambda))$ is an irreducible (or zero) $W_k^+(\bar{\mathfrak{g}})$-module, where $\ell t(w) \in \mathbb{Z}$ is defined in [18].

(b) $F_+^j(L(\Lambda)) = 0$ if $j \neq \ell t(w)$. If $\Lambda \in Pr^k \backslash N_+^k$, then $F_+^j(L(\Lambda)) = 0$ for all j. (In the " $-$ " case this follows from Theorem 2.3.)

Of course, we have a corollary similar to Proposition 3.4 in this case as well.

3.5. Conjectures on Resolutions. One of the possible ways to prove Conjectures 3.4$_\pm$ is to use resolutions by Wakimoto modules. We will explain it for the " + " case.

Conjecture 3.5.1. Let $\Lambda \in P_{u,t_\beta \bar{y}}^k$ be a principal admissible weight and let $L(\Lambda)$ be the corresponding representation. Then there exists a complex (a two-sided BGG resolution) R_Λ, such that

$$R_\Lambda^i = \bigoplus_{\substack{s \in W^\Lambda \\ \ell t_\Lambda(s) = i}} B_{s.\Lambda} \ ,$$

and that all of its cohomologies but 0^{th} vanish, and the 0^{th} cohomology is isomorphic to $L(\Lambda)$. Here $\ell t_\Lambda(s)$ denotes $\ell t(\tilde{s})$, where $\tilde{s}$ is the image of $s \in W^\Lambda$ in W under their isomorphism.

The existence of this resolution has been proved for any integrable representation over arbitrary $\mathfrak{g}$ [18] and for any modular invariant representation over $A_1^{(1)}$ [4, 18]. Resolutions of this kind were extensively studied in [10].

If we apply our functor F_+^0 to this resolution, then we get the complex of modules $\tilde{R}_\Lambda$ over the W-algebra, such that

$$\tilde{R}_\Lambda^i = \bigoplus_{\substack{s \in W^\Lambda \\ \ell t_\Lambda(s) = i}} \pi_{s.\Lambda} \ ,$$

because $F_+^i(B_\mu) = 0$, $i \neq 0$, $F_+^0(B_\mu) = \pi_\mu$. By definition, the i^{th} cohomology of the complex $\tilde{R}_\Lambda$ coincides with $F_+^i(L(\Lambda))$.

Conjecture 3.5.2. $\tilde{R}_\Lambda$ is a resolution of an irreducible module $F_+^0(L(\Lambda))$ over the W-algebra: all of its cohomologies but $\ell t(w)^{\text{th}}$ (where w is the element, defined in

Proposition 1.8b(ii)) vanish, and the $\ell t(w)^{\text{th}}$ cohomology is isomorphic to an irreducible module (with character $\varphi_{\lambda,\mu}(\tau)$).

This conjecture has been proved for $\mathfrak{g} = sl_2$ in [22].

If Conjecture 3.5.2 is true in general, then we may calculate the character of the module $F_+^0(L(\Lambda))$ as

$$(-1)^{\ell t(w)} \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{tr}_{\tilde{R}_\Lambda^i} q^{L_0^+},$$

by the Euler–Poincaré principle. This is equal to

$$(-1)^{\ell t(w)} \sum_{s \in W^\Lambda} (-1)^{\ell t_\Lambda(s)} \operatorname{tr}_{\pi_{s.\Lambda}} q^{L_0^+} = (-1)^{\ell t(w)} \sum_{s \in W^\Lambda} (-1)^{\ell t_\Lambda(s)} q^{P(s.\Lambda)} \eta(\tau)^{-\ell}.$$

We get the same formula as the one obtained by means of the residue calculation.

Now Conjecture 3.4_+ follows from Conjectures 3.5.1 and 3.5.2. Note that the lowest eigenvalue of L_0^+ on $\tilde{R}_\Lambda$ is equal to $P(\tilde{w}.\Lambda)$, where $\tilde{w}$ is the element of W^Λ, which corresponds to the element w under the isomorphism $W \sim W^\Lambda$, and that the corresponding eigenspace is the span of the vacuum vector of $\pi_{\tilde{w}.\Lambda}$. Therefore this vector represents a cohomology class in $F_+^{\ell t(w)}(L(\Lambda))$, which is the highest weight vector of $F_+(L(\Lambda))$.

It is natural to assume that the $W_k(\mathfrak{g})$-modules which are the images of $L(\Lambda)$, $\Lambda \in N_+^k$ under the functor F_+ form a minimal model of the corresponding conformal field theory in the sense of [3]. In the next section we will prove that the linear span of characters of these modules form a representation of $SL_2(\mathbb{Z})$ and we will use the information on the action of this group to deduce the fusion algebra of this theory by means of the Verlinde argument.

4. Fusion Rules for W-Algebras in the Simply Laced Case

Throughout this section we will assume that $\bar{\mathfrak{g}}$ is simply laced (i.e. of type A_ℓ, D_ℓ or E_ℓ); equivalently: $a_i = a_i^\vee$ for all i (i.e. $r^\vee = 1$).

4.1. Some Properties of the Group $\tilde{W}_+$. We use here results and notation of Sect. 1.1. Let $\bar{Q}$ be the root lattice and $\bar{P}$ the weight lattice of $\bar{\mathfrak{g}}$. Note that

$$\bar{Q}^* = \bar{P}, \quad \bar{Q}^\vee = \bar{Q}. \tag{4.1.1}$$

Hence by (1.1.3) we have for any $k \in \mathbb{Z}$ relatively prime to $|J|$:

$$\{k\bar{\Lambda}_i\}_{i \in J} \text{ is a set of representatives of } \bar{P} \bmod \bar{Q}. \tag{4.1.2}$$

Lemma 4.1.1. *Let $k \in \mathbb{N}$ be relatively prime to $|J|$. Then for any $\lambda \in P^k$ there exists a unique $w \in \tilde{W}_+$ such that $\overline{w(\lambda)} \in \bar{Q}$.*

Proof. By (4.1.2), there exists a unique $j \in J$ such that $\bar{\lambda} \equiv k\bar{\Lambda}_j \bmod \bar{Q}$. Since (for any $k \in \mathbb{Z}$)

$$\overline{w_j(\lambda)} = \bar{w}_j(\bar{\lambda}) + k\bar{\Lambda}_j \tag{4.1.3}$$

and $w(\bar{\lambda}) \equiv \bar{\lambda} \bmod \bar{Q}$ if $w \in \bar{W}$, the lemma follows. $\square$

By (4.1.2), there exists a unique $s \in J$ such that

$$\bar{\Lambda}_s - \bar{\rho} \in \bar{Q} .$$

Of course, $s = 0$ if $\bar{\rho} \in \bar{Q}$. In all other cases s is listed below:

$$\begin{aligned} &s = \tfrac{1}{2}(l+1) && \text{if } \bar{\mathfrak{g}} \text{ is of type } A_\ell, \ell \text{ odd,} \\ &s = 1 && \text{if } \bar{\mathfrak{g}} \text{ is of type } D_\ell, \ell \equiv 2 \text{ or } 3 \bmod 4, \\ &s = 6 && \text{if } \bar{\mathfrak{g}} \text{ is of type } E_7. \end{aligned}$$

Note the following properties of s:

$$2\bar{\Lambda}_s \in \bar{Q}, \quad w_s^2 = 1 . \tag{4.1.4}$$

Lemma 4.1.2. *Given $j \in J$, let $i \in J$ be defined by $w_i = w_j^{-1}$* (i.e. $w_i(\Lambda_j) = \Lambda_0$).
(a) $\bar{w}_i(\bar{\Lambda}_j) + \bar{\Lambda}_i = 0$, *hence* $\bar{\Lambda}_i + \bar{\Lambda}_j \equiv 0 \bmod \bar{Q}$.
(b) *If $\lambda \in P_+^k$, then*

$$\bar{\lambda} + \bar{\rho} - (k + h^\vee)\bar{\Lambda}_j = \bar{w}_j(\overline{w_i(\lambda)} + \bar{\rho}) .$$

Proof. (a) (resp. (b)) follows from (4.1.3) where we let $j = i$ and replace λ by Λ_j (resp. by $\lambda + \rho - (k + h^\vee)\Lambda_j$, which has level 0). $\square$

Lemma 4.12a immediately implies

Lemma 4.1.3. *Let $j_1, j_2, j_3 \in J$ and let $\sigma_{j_t}^{-1} = \sigma_{i_t}$, $t = 1, 2, 3$. Then*

(a) $\bar{\Lambda}_{j_1} + \bar{\Lambda}_{j_2} + \bar{\Lambda}_{j_3} \in \bar{Q}$ iff $\bar{\Lambda}_{i_1} + \bar{\Lambda}_{i_2} + \bar{\Lambda}_{i_3} \in \bar{Q}$.
(b) $\bar{\Lambda}_{j_1} + \bar{\Lambda}_{j_2} + \bar{\Lambda}_{j_3} + \bar{\rho} \in \bar{Q}$ iff $\bar{\Lambda}_{i_1} + \bar{\Lambda}_{i_2} + \bar{\Lambda}_{i_3} + \bar{\rho} \in \bar{Q}$.

4.2. A Transformation Formula for Functions $\varphi_{\lambda,\mu}(\tau)$. Recall that

$$\varphi_{w(\lambda,\mu)} = \varphi_{\lambda,\mu} \text{ if } w \in \tilde{W}_+ , \tag{4.2.1}$$

where $w(\lambda, \mu) = (w\lambda, w\mu)$. The following transformation formula follows from Theorem 1.7 and Proposition 1.8a (see [28, proof of Theorem 4.4]): Let p and p' be relatively prime integers greater than or equal to h and let $(\lambda, \lambda') \in I_{p,p'}$. Then

$$\varphi_{\lambda,\lambda'}\left(-\frac{1}{\tau}\right) = \sum_{(\mu,\mu') \in I_{p,p}} S_{(\lambda,\lambda'),(\mu,\mu')}\varphi_{\mu,\mu'}(\tau) , \tag{4.2.2}$$

where

$$\begin{aligned} S_{(\lambda,\lambda')(\mu,\mu')} = {} & (pp')^{-\ell/2}|J|^{-1/2} e^{2\pi i((\bar{\lambda} + \bar{\rho}|\bar{\mu}' + \bar{\rho}) + (\bar{\lambda}' + \bar{\rho}|\bar{\mu} + \bar{\rho}))} \\ & \times \sum_{y \in \bar{W}} \varepsilon(y) e^{-\frac{2\pi i p}{p'}(\bar{\lambda}' + \bar{\rho}|y(\bar{\mu}' + \bar{\rho}))} \sum_{w \in \bar{W}} \varepsilon(w) e^{-\frac{2\pi i p'}{p}(\bar{\lambda} + \bar{\rho}|w(\bar{\mu} + \bar{\rho}))} . \end{aligned} \tag{4.2.3}$$

Note also a special case of Theorem 1.7 when $\lambda \in P_+^k$, $k \in \mathbb{Z}_+$ (and $\bar{\mathfrak{g}}$ is simply laced):

$$\chi_\lambda|_S = \sum_{\mu \in P_+^k} S_{\lambda,\mu}\chi_\mu , \tag{4.2.4}$$

where

$$S_{\lambda,\mu} = i^{|\bar{\Delta}_+|}(k+h)^{-\ell/2}|J|^{-1/2} \sum_{w \in \bar{W}} \varepsilon(w) e^{-\frac{2\pi i}{k+h}(w(\bar{\lambda}+\bar{\rho})|\bar{\mu}+\bar{\rho})} . \tag{4.2.5}$$

Recall also the following important fact [25]:

$$S_{w\lambda,\mu} = e^{-2\pi i(w\Lambda_0|\bar{\mu})} S_{\lambda,\mu} \quad \text{if } w \in \tilde{W}_+ \tag{4.2.6}$$

Our objective is to express the $S_{(\lambda,\lambda'),(\mu,\mu')}$ in terms of the $S_{\lambda,\mu}$. We shall assume that p' and $|J|$ are relatively prime. Then we may choose (and fix) positive integers a and a' such that

$$ap' \equiv 1 \bmod p|J|, \quad a'p \equiv 1 \bmod p', \quad a' \equiv 1 \bmod |J| . \tag{4.2.7}$$

Note that a' and $p'|J|$ are relatively prime, hence we can find a positive integer b' such that:

$$a'b' \equiv 1 \bmod p'|J| . \tag{4.2.8}$$

Lemma 4.2.1. *Suppose that $\bar{\lambda}' \in \bar{Q}$. Then*

(a) $X := a(p'(\bar{\lambda}+\bar{\rho}) - p(\bar{\lambda}'+\bar{\rho})) \equiv \bar{\lambda}+\bar{\rho} - p\bar{\Lambda}_s \bmod p\bar{Q}$.

(b) $Y := a'(p(\bar{\lambda}'+\bar{\rho}) - p'(\bar{\lambda}+\bar{\rho}))$

$$\equiv \begin{cases} \bar{\lambda}'+\bar{\rho}-p'\bar{\lambda} \bmod p'\bar{Q} & \textit{if } \bar{\rho}\in\bar{Q} \textit{ or } p \textit{ is even} \\ \bar{\lambda}'+\bar{\rho}-p'(\bar{\lambda}+\bar{\rho}) \bmod p'\bar{Q} & \textit{if } \bar{\rho}\notin\bar{Q} \textit{ and } p \textit{ is odd} . \end{cases}$$

Proof. By (4.2.7), $ap'(\bar{\lambda}+\bar{\rho}) \equiv \bar{\lambda}+\bar{\rho} \bmod \bar{Q}$, hence $X \equiv \bar{\lambda}+\bar{\rho} - ap\bar{\rho} \bmod p\bar{Q}$. If $\bar{\rho}\in\bar{Q}$, (a) follows. In the case $\bar{\rho}\notin\bar{Q}$, p' is odd since $|J|$ is even (see Sect. 4.1), hence a is odd. Hence $ap\bar{\rho} \equiv \bar{\rho} \bmod p\bar{Q}$ since $2\bar{\rho}\in\bar{Q}$ and (a) follows from the definition of Λ_s in Sect. 4.1. The proof of (b) is similar. □

Lemma 4.2.2. *Let $b\in\mathbb{N}$ be relatively prime to $p|J|$* (resp. $p'|J|$). *Let $\lambda\in P_+^{p-h}$* (resp. $\in P_+^{p'-h}$). *Define the map ϕ_b of P_+^{p-h} into itself and ε: $P_+^{p-h}\to\{\pm 1\}$* (resp. *ϕ_b' and ε_b' replacing p by p'*) *by*:

$$b(\lambda+\rho)-(b-1)h\Lambda_0 = w(\phi_b(\lambda)+\rho), \quad w\in W ;$$

$$\varepsilon_b(\lambda) = \varepsilon(w) .$$

Then the map ϕ_b (resp. ϕ_b') *is bijective.*

Proof. We have to show that if $\lambda, \lambda_1 \in P_+^{p-h}$ and $y\in W$ are such that

$$b(\lambda+\rho)-(b-1)p\Lambda_0 = y(b(\lambda_1+\rho)-(b-1)p\Lambda_0) , \tag{4.2.9}$$

then $y=1$. Note that $k := p/b - h$ is a principal admissible rational number with denominator b. Dividing both sides of (4.2.9) by b we get:

$$\lambda+\rho-(b-1)(k+h^\vee)\Lambda_0 = y(\lambda_1+\rho-(b-1)(k+h^\vee)\Lambda_0) .$$

So $\Lambda := \lambda+\rho-(b-1)(k+h^\vee)\Lambda_0$ and $\Lambda_1 := \lambda_1+\rho-(b-1)(k+h^\vee)\Lambda_0$ are principal admissible weights such that $R^\Lambda = R^{\Lambda_1} = R_{[b]}$ and $\Lambda = y\Lambda_1$. Hence $yR_{[b]} = R_{[b]}$, therefore there exists $t_\alpha\bar{w}\in\tilde{W}_+$ such that $y = t_{b\alpha}\bar{w}$. Since b and $|J|$ are relatively prime and $b\alpha\in M$, we deduce that $\alpha\in M$, hence $\alpha=0$ and $\bar{w}=1$ (see Sect. 1.2). Thus, $y=1$. □

Given $\lambda\in P_+^{p-h}$ (resp. $\lambda'\in P_+^{p'-h}$), we let $\lambda_b = \phi_b(\lambda)$, (resp. $\lambda_b' = \phi_b'(\lambda)$).

Proposition 4.2. *Let (λ, λ') and (μ, μ') be elements of $P_+^{p-h} \times P_+^{p'-h}$ such that $\bar{\lambda}' \in \bar{Q}$. We define $j \in J$ by letting*

$$\bar{\Lambda}_j = \begin{cases} \bar{\lambda} \bmod \bar{Q} & \text{if } \bar{\rho} \in \bar{Q} \text{ or } p \text{ is even (case 1)} \\ \bar{\lambda} + \bar{\rho} \bmod \bar{Q} & \text{otherwise (case 2)}, \end{cases}$$

and let $w_i = w_j^{-1}$. Then

$$S_{(\lambda, \lambda'), (\mu, \mu')} = (-1)^{|\bar{\Delta}_+|} |J|^{1/2} \varepsilon(\bar{w}_s) \varepsilon(\bar{w}_j) \varepsilon_{p'}(\mu) \varepsilon'_{b'}(\mu') \times S_{w_s(\lambda), \mu_{p'}} S_{w_i(\lambda'), \mu'_{b'}} .$$

Proof. We rewrite (4.2.3) using the following calculations:

$$e^{2\pi i(\bar{\lambda} + \bar{\rho}|\bar{\mu}' + \bar{\rho})} \sum_{y \in \bar{W}} \varepsilon(y) e^{-\frac{2\pi i p}{p'}(\bar{\lambda}' + \bar{\rho}|y(\bar{\mu}' + \bar{\rho}))} = \sum_{y \in \bar{W}} \varepsilon(y) e^{-\frac{2\pi i}{p'}(p(\bar{\lambda}' + \bar{\rho}) - p'(\bar{\lambda} + \bar{\rho})|y(\bar{\mu}' + \bar{\rho}))}$$

$$= \sum_{y \in \bar{W}} \varepsilon(y) e^{-\frac{2\pi i}{p'}(a'(p(\bar{\lambda}' + \bar{\rho}) - p'(\bar{\lambda} + \bar{\rho}))|y(b'(\bar{\mu}' + \bar{\rho})))} \quad \text{(by (4.28))}$$

$$= \varepsilon'_{b'}(\mu') \sum_{y \in \bar{W}} \varepsilon(y) e^{-\frac{2\pi i}{p'}(a'(p(\bar{\lambda}' + \bar{\rho}) - p'(\bar{\lambda} + \bar{\rho})|y(\bar{\mu}'_{b'} + \bar{\rho})))}$$

$$= \varepsilon'_{b'}(\mu') \varepsilon(\bar{w}_j) \sum_{y \in \bar{W}} \varepsilon(y) e^{-\frac{2\pi i}{p'}(w_i(\bar{\lambda}') + \bar{\rho}|y(\bar{\mu}'_{b'} + \bar{\rho}))}$$

(by Lemmas 4.2.1b and 4.1.2b).

Similarly, we have:

$$e^{2\pi i(\bar{\lambda}' + \bar{\rho}|\bar{\mu} + \bar{\rho})} \sum_{w \in \bar{W}} \varepsilon(w) e^{-\frac{2\pi i p'}{p}(\bar{\lambda} + \bar{\rho}|w(\bar{\mu} + \bar{\rho}))}$$

$$= \sum_{w \in \bar{W}} \varepsilon(w) e^{-\frac{2\pi i}{p}(a(p'(\bar{\lambda} + \bar{\rho}) - p(\bar{\lambda}' + \bar{\rho})|w(p'(\bar{\mu} + \bar{\rho})))}$$

$$= \varepsilon_{p'}(\mu) \varepsilon(\bar{w}_s) \sum_{w \in \bar{W}} \varepsilon(w) e^{-\frac{2\pi i}{p}(w_s(\bar{\lambda}) + \bar{\rho}|w(\bar{\mu}'_{p'} + \bar{\rho}))}$$

by Lemma 4.1.1a.

Substituting in (4.2.3) and using (4.2.5) gives the result. □

4.3. A Calculation of the Fusion Rules. First, recall Verlinde's formula for fusion rules [35]. Suppose that we have a finite set I of representations of a "chiral algebra" such that

(i) the vacuum representation, labeled by 0, lies in I;

(ii) the linear span of normalized characters $\{\chi_\lambda\}_{\lambda \in I}$ is $SL_2(\mathbb{Z})$-invariant and the action of $S \in SL_2(\mathbb{Z})$ is given by a matrix $(S_{\lambda\mu})_{\lambda, \mu \in I}$.

Then the *fusion coefficients* $N_{\lambda\mu\nu}$ are given by the following formula:

$$N_{\lambda\mu\nu} = \sum_{\sigma \in I} \frac{S_{\lambda\sigma} S_{\mu\sigma} S_{\nu\sigma}}{S_{0\sigma}} . \tag{4.3.1}$$

If, in addition, an involutive map $\lambda \mapsto {}^t\lambda$ of I into itself is given, one defines the *fusion algebra* as an algebra over $\mathbb{C}$ with basis $\{\chi_\lambda\}_{\lambda \in I}$ and the following multiplication:

$$\chi_\lambda * \chi_\mu = \sum_{\nu \in I} N_{\lambda\mu {}^t\nu} \chi_\nu .$$

In the case when the chiral algebra is the affine algebra $\mathfrak{g}$ and the set of its representations is $L(\lambda)$, $\lambda \in P^m_+$, the vacuum representation is $L(m\Lambda_0)$ and one knows an explicit expression for the fusion coefficients $N_{\lambda,\mu,\nu}(\lambda, \mu, \nu \in P^m_+)$ [24, Exercise 13.35].

Let now k be a principal admissible rational number with the denominator p' and let $p = p'(k+h)$. Note that $p \geqq h$ and $(p, p') = 1$. Recall that $N^k_\pm \neq \emptyset$ if and only if $p' \geqq h$ (Proposition 1.5a). We consider the set of representations of the W-algebra $W_k(\bar{\mathfrak{g}})$ obtained from the principal admissible representations of $\mathfrak{g}$ by a quantum Drinfeld–Sokolov reduction. Recall that (by Proposition 3.4) these representations are parametrized by the set $I_{p,p'} = (P^{p-h}_+ \times P^{p'-h}_+)/\tilde{W}_+$, that the vacuum representation is labeled by $((p-h)\Lambda_0, (p'-h)\Lambda_0)$ and that a representation labeled by the pair (λ, μ) has normalized character $\varphi_{\lambda,\mu}(\tau)$.

Theorem 4.3. *Let p and p′ be integers such that* $p, p' \geqq h$ and $(p, p') = (p', |J|) = 1$. *In* $\Lambda_{(i)} \in I_{p,p'}$ $(i = 1, 2, 3)$ *choose a representative* (λ_i, λ'_i) *such that* $\bar{\lambda}'_i \in \bar{Q}$ *(see Lemma 4.1.1). Then*

$$N_{\Lambda_{(1)}\Lambda_{(2)}\Lambda_{(3)}} = N_{\lambda_1\lambda_2\lambda_3} N_{\lambda'_1\lambda'_2\lambda'_3}, \tag{4.3.2}$$

where $N_{\lambda_1\lambda_2\lambda_3}$ *and* $N_{\lambda'_1\lambda'_2\lambda'_3}$ *are fusion coefficients for* $\mathfrak{g}$. *(Similar result holds in the case when* $(p, |J|) = 1$.)

First, note the following lemma, which follows immediately from (4.2.6) and (4.3.1):

Lemma 4.3.1. *Let* $i_1, i_2, i_3 \in J$ *and* $\lambda_1, \lambda_2, \lambda_3 \in P^k_+$, $k \in \mathbb{N}$. *Then*

(a) *If* $\bar{\Lambda}_{i_1} + \bar{\Lambda}_{i_2} + \bar{\Lambda}_{i_3} \in \bar{Q}$, *then* $N_{w_{i_1}(\lambda_1), w_{i_2}(\lambda_2), w_{i_3}(\lambda_3)} = N_{\lambda_1\lambda_2\lambda_3}$.

(b) *If* $\bar{\Lambda}_{i_1} + \bar{\Lambda}_{i_2} + \bar{\Lambda}_{i_3} + \bar{\rho} \in \bar{Q}$, *then* $N_{w_s w_{i_1}(\lambda_1), w_{i_2}(\lambda_2), w_{i_3}(\lambda_3)} = N_{\lambda_1\lambda_2\lambda_3}$. □

A special case of Lemma 4.3.1a is

Lemma 4.3.2. *Let* $\lambda_1, \lambda_2, \lambda_3 \in P^k_+$. *Then* $N_{\lambda_1, w_s(\lambda_2), w_s(\lambda_3)} = N_{\lambda_1,\lambda_2,\lambda_3}$.

Proof of Theorem 4.3. Let $\lambda_0 = (p-h)\Lambda_0$, $\lambda'_0 = (p'-h)\Lambda_0$. For each $t = 0, 1, 2$ or 3, define $j(t) \in J$ by letting (cf. Proposition 4.2):

$$\bar{\Lambda}_{j(t)} \equiv \begin{cases} \bar{\lambda}_t \bmod \bar{Q} & \text{in case 1} \\ \bar{\lambda}_t + \bar{\rho} \bmod \bar{Q} & \text{in case 2} \end{cases}.$$

and define $i(t)$ by $w_{i(t)} = w_{j(t)}^{-1}$. In particular, $w_{i(0)} = 1$ in case 1, and $= w_s$ in case 2.

Now, using Verlinde's formula (4.3.1), formula (4.2.6), Proposition 4.2 and Lemma 4.2.2 we obtain the following formula:

$$\varepsilon N_{\Lambda_{(1)}\Lambda_{(2)}\Lambda_{(3)}} = N_{\lambda_1,\lambda_2,\lambda_3} N_{w_{i(1)}(\lambda'_1), w_{i(2)}(\lambda'_2), w_{i(3)}(\lambda'_3)}, \tag{4.3.3}$$

where $\varepsilon = \varepsilon(\bar{w}_{j(1)})\varepsilon(\bar{w}_{j(2)})\varepsilon(\bar{w}_{j(3)})\varepsilon_0$, and $\varepsilon_0 = 1$ in case 1, $\varepsilon_0 = \varepsilon(\bar{w}_s)$ in case 2.

Furthermore, it follows, for example, from the explicit formula in the affine case [24] that

$$N_{\lambda_1\lambda_2\lambda_3} \neq 0 \text{ implies } \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 \in \bar{Q}. \tag{4.3.4}$$

In case 1, we may assume, due to (4.3.4), that $\bar{\Lambda}_{j(1)} + \bar{\Lambda}_{j(2)} + \bar{\Lambda}_{j(3)} \in \bar{Q}$, hence $\bar{\Lambda}_{i(1)} + \bar{\Lambda}_{i(2)} + \bar{\Lambda}_{i(3)} \in \bar{Q}$ by Lemma 4.1.3a. It follows that $\varepsilon = 1$ and, by Lemma 4.3.1a, that $N_{w_{i(1)}(\lambda'_1), w_{i(2)}(\lambda'_2), w_{i(3)}(\lambda'_3)} = N_{\lambda'_1\lambda'_2\lambda'_3}$, proving the theorem.

In case 2 we similarly have $\bar{\Lambda}_{j(1)} + \bar{\Lambda}_{j(2)} + \bar{\Lambda}_{j(3)} + \bar{\rho} \in \bar{Q}$, hence $\bar{\Lambda}_{i(1)} + \bar{\Lambda}_{i(2)} + \bar{\Lambda}_{i(3)} + \bar{\rho} \in \bar{Q}$ by Lemma 4.1.3b. It follows that $\varepsilon = 1$ and, by Lemma 4.3.1b, that $N_{w_{i(1)}(\lambda'_1), w_{i(2)}(\lambda'_2), w_{i(3)}(\lambda'_3)} = N_{\lambda'_1,\lambda'_2,\lambda'_3}$. □

One has the following involutive automorphism of the set $I_{p,p'}$:

$$^t((\lambda, \lambda') \bmod \tilde{W}_+) = ({}^t\lambda, {}^t\lambda') \bmod \tilde{W}_+ ,$$

where ${}^t\lambda$ is defined by (1.4.3). Since $\bar{w}^0$ is the unique longest element in $\bar{W}$, w^0 commutes with $\tilde{W}_+$, hence this map is well-defined. Theorem 4.3 may be reformulated as follows.

Theorem 4.3′. *Let p and p' be relatively prime integers such that $p, p' \geqq h$, and assume that $(p, |J|) = 1$ (resp. $(p', |J|) = 1$). Let $\mathscr{A}^{p,p'}$ be the fusion algebra for the W-algebra $W_k(\bar{\mathfrak{g}})$, where $k = p/p' - h$. Given $m \in \mathbb{Z}_+$ let $\mathscr{A}^m$ denote the fusion algebra for the affine algebra $\mathfrak{g}$ with $K = m$ (and the index set P_+^m) and let $\mathscr{A}_1^m$ denote its subalgebra spanned by the χ_λ with $\bar{\lambda} \in \bar{Q}$. Then:*

$$\mathscr{A}^{p,p'} = \mathscr{A}_1^{p-h} \otimes \mathscr{A}^{p'-h} (\text{resp.} = \mathscr{A}^{p-h} \otimes \mathscr{A}_1^{p'-h}) .$$

Remark 4.3. If $|J|$ is a power of a prime number, then either p or p' is relatively prime to $|J|$. Thus, Theorem 4.3 describes fusion rules completely in all (simply laced) cases except for $\bar{\mathfrak{g}}$ of type A_n, n not a power or a prime.

The following result takes care of all cases (but it is not as nice as Theorem 4.3). Its proof is the same as that of Theorem 4.3.

Proposition 4.3. *Let p and p' be relatively prime integers greater than or equal to h. Choose integers b and b' such that $(p - b'p', |J|) = 1$ and $(p' - bp, |J|) = 1$. Let $\sigma = w_s$* (resp. $= 1$) *if b is even (resp. odd) and $\sigma' = w_s$* (resp. $= 1$) *if b' is even* (resp. odd). *Given $\Lambda^{(i)} \in I_{p,p'}$, $i = 1, 2, 3$, choose their representatives (λ_i, λ_i')* (resp. $(\lambda_i^*, \lambda_i^{*\prime})$) *such that $b(\bar{\lambda}_i + \bar{\rho}) - (\bar{\lambda}_i' + \bar{\rho}) \in \bar{Q}$* (resp. $b'(\bar{\lambda}_i^{*\prime} + \bar{\rho}) - (\bar{\lambda}_i^* + \bar{\rho}) \in \bar{Q}$). *Then*

$$N_{\Lambda^{(1)}\Lambda^{(2)}\Lambda^{(3)}} = N_{\sigma(\lambda_1)\lambda_2\lambda_3} N_{\sigma(\lambda_1^{*\prime})\lambda_2^{*\prime}\lambda_3^{*\prime}} . \quad \square$$

References

1. Bais, F.A., Bouwknegt, P., Schoutens, K., Surridge, M.: Extensions of the Virasoro algebra constructed from Kač–Moody algebras using higher order Casimir invariants. Nucl. Phys. **B304**, 348–370 (1988)
2. Bais, F.A., Bouwknegt, P., Schoutens, K., Surridge, M.: Coset construction for extended Virasoro algebras. Nucl. Phys. **B304**, 371–391 (1988)
3. Belavin, A., Polyakov, A., Zamolodchikov, A.B.: Infinite conformal symmetry in two dimensional quantum field theory. Nucl. Phys. **B241**, 333–380 (1984)
4. Bernard, D., Felder, G.: Fock representations and BRST cohomology in SL(2) current algebra. Commun. Math. Phys. **127**, 145–168 (1990)
5. Bershadsky, M.: Conformal field theory via Hamiltonian reduction. Preprint IASSNS-HEP-90144, 1990
6. Bershadsky, M., Ooguri, H.: Hidden SL(n)-symmetry in conformal field theory. Commun. Math. Phys. **126**, 49–84 (1989)
7. Bilal, A., Gervais, J.-L.: Conformal theories with non-linearly extended Virasoro symmetries and Lie algebra classification. In: Adv. Ser. Math. Phys. vol. **7**, pp. 483–526. Singapore: World Scientific 1989. See also Phys. Lett. **B206**, 412 (1988); Nucl. Phys. **B314**, 646 (1989); Nucl. Phys. **B318**, 579 (1989)
8. Blumenhagen, R., Flohr, M., Kleim, A., Nahm, W., Rechnagel, A., Varnhagen, R.: W-algebras with two and three generators. Nucl. Phys. **B354**, 255–289 (1991)
9. Bouwknegt, P.: Extended conformal algebras from Kac–Moody algebras. In: Advanced Ser. Math. Phys. vol. 7, pp. 527–555. Singapore: World Scientific 1989
10. Bouwknegt, P., McCarthy, J., Pilch, K.: Free field approach to 2-dimensional conformal field theory. Progr. Theor. Phys. Suppl. **102**, 67 (1990)
11. Bowcock, P., Goddard, P.: Coset constructions and extended conformal algebras. Nucl. Phys. **B305**, 685 (1988)

12. Burroughs, N.J., de Groot, M.F., Hollowood, T.J., Miramontes, J.L.: Generalized Drinfeld–Sokolov hierarchies II: The Hamiltonian structures. Preprint IASSNS-HEP-91142, 1991
13. Deodhar, V.V., Gabber, O., Kac, V.G.: Structure of some categories of representations of infinite-dimensional Lie algebras. Adv. Math. **45**, 92–116 (1982)
14. Drinfeld, V.G., Sokolov, V.V.: Lie algebra and the KdV type equations. Soviet J. Math. **30**, 1975–2036 (1985)
15. Fateev, V.A., Zamolodchikov, A.B.: Conformal quantum field theory models in two dimensions having Z_3 symmetry. Nucl. Phys. **B280** [FS18], 644–660 (1987)
16. Fateev, V.A., Lukyanov, S.L.: The models of two-dimensional conformal quantum field theory with $\mathbb{Z}_n$ symmetry. Int. J. Mod. Phys. **A3**, 507 (1988)
17. Fateev, V.A., Lukyanov, S.L.: Additional symmetries and exactly soluble models of two-dimensional conformal field theory. Kiev preprints, ITF-87-74, 75, 76 (1988)
18. Feigin, B., Frenkel, E.: Affine Kac–Moody algebras and semi-infinite flag manifolds. Commun. Math. Phys. **128**, 161–189 (1990)
19. Feigin, B., Frenkel, E.: Affine Kac–Moody algebras, bosonization and resolutions. Phys. Lett. **19**, 307–317 (1990)
20. Feigin, B., Frenkel, E.: Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras. preprint RIMS 796, 1991 (to appear in Proceedings of RIMS-91 Program)
21. Feigin, B., Frenkel, E.: Quantization of the Drinfeld–Sokolov reduction. Phys. Letts. B **246**, 75–81 (1990)
22. Felder, G.: BRST approach to minimal models. Nucl. Phys. **B317**, 215–236 (1989)
23. Frenkel, E.: W-algebras and Langlands–Drinfeld correspondence, to appear in Proceedings of Cargese Summer School, 1991
24. Kac, V.G.: Infinite-dimensional Lie algebras. 3-d edition, Cambridge: Cambridge University Press 1990
25. Kac, V.G., Wakimoto, M.: Modular and conformal invariance constraints in representation theory of affine algebras. Adv. Math. **70**, 156–236 (1988)
26. Kac, V.G., Wakimoto, M.: Modular invariant representations of infinite-dimensional Lie algebras and superalgebras. Proc. Natl. Sci. USA **85**, 4956–4960 (1988)
27. Kac, V.G., Wakimoto, M.: Classification of modular invariant representations of affine algebras. in: Adv. Ser. Math. Phys. vol. **7**, pp. 138–177. Singapore: World Scientific 1989
28. Kac, V.G., Wakimoto, M.: Branching functions for winding subalgebras and tensor products. Acta Applicandae Math. **21**, 3–39 (1990)
29. Khovanova, T.: Super KdV equation related to the Neveu–Schwartz-2 Lie superalgebra of string theory. Teor. Mat. Phys. **72**, 306–312 (1987)
30. Kostant, B., Sternberg, S.: Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. Ann. Phys. **176**, 49–113 (1987)
31. Lukyanov, S.L.: Quantization of the Gelfand–Dikii algebra. Funct. Anal. Appl. **22**:4, 1–10 (1988)
32. Mukhi, S., Panda, S.: Fractional level current algebras and the classification of characters. TIFR preprint, 1989
33. Polyakov, A.: Gauge transformations and diffeomorphisms. Int. J. Mod. Phys. **A5**, 833 (1990)
34. Thierry-Mieg, J.: Generalizations of the Sugawara construction. Proceedings of Cargese Summer School, 1987
35. Verlinde, E.: Fusion rules and modular transformations in 2D conformal field theory. Nucl. Phys. **B300**, 360–375 (1988)
36. Watts, G.T.M.: W-algebras and coset models. Phys. Lett. **B245**, 65–71 (1990)
37. Zamolodchikov, A.B.: Infinite additional symmetries in 2-dimensional conformal quantum field theory. Teor. Mat. Phys. **65**, 1205–1213 (1985)

Communicated by A. Jaffe

Note added in proof. Two of the authors of the present paper have shown recently that the quantum reduction applied to modular invariant representations of affine superalgebras $\mathrm{osp}(1,2n)^{(1)}$ (resp. $\mathrm{sl}(1,2n)^{(1)}$) gives the "minimal" series of representation of certain W-superalgebras. In particular, for $n = 1$ one recovers all "minimal" representations of $N = 1$ (resp. $N = 2$) superconformal algebras.