The Energy Operator for Infinite Statistics

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Abstract. We construct the energy operator for particles obeying infinite statistics defined by a q-deformation of the Heisenberg algebra.

The aim of this paper is to construct the energy operator for particles which obey the so-called infinite statistics defined by the q-deformation of the Heisenberg algebra. This topic was studied in the previous article [1], where a conjecture was formulated concerning the form of the energy operator. Our main result is a proof of this conjecture in a slightly modified form (cf. Remark 1).

We will essentially use the same notations as in [1]. Thus, T_{1k} will denote the particular elements of $\mathfrak{S}_n$ which send $[1, 2, \ldots, n]$ to $[k, 1, \ldots, k-1, k+1, \ldots, n]$, i.e.

$$T_{1k}(i) = \begin{cases} k, & \text{if } i = 1; \\ i-1, & \text{if } 1 < i \le k; \\ i, & \text{if } k < i \le n; \end{cases}$$

and $\mathfrak{S}_{n,p}$ will represent the following subsets of $\mathfrak{S}_n$:

$$\mathfrak{S}_{n,p} = \left\{ \sigma \in \mathfrak{S}_n, \text{ with } \sigma = T_{1k_1} T_{1k_2}, \ldots, T_{1k_p}, 1 < k_1 < \ldots < k_p \le n \right\},$$

for $1 \le p \le n - 1$ and $\mathfrak{S}_{n,0} = \{1\}$. (This differs from the definition of $\mathfrak{S}_{n,p}$ in [1].)

In [1] an $n! \times n!$ matrix $A_n(\pi, \sigma)$, $\pi, \sigma \in \mathfrak{S}_n$, with coefficients in $\mathbb{Z}[q]$ was studied and shown to be invertible for $|q| < 1$. As in [1], we will work with the group algebra $\mathbb{C}[\mathfrak{S}_n]$ rather than its matrix representation, so we have elements

$$\alpha_n = \sum_{\varrho \in \mathfrak{S}_n} A_n(\varrho, 1)\sigma = \sum_{\varrho \in \mathfrak{S}_n} q^{I(\varrho)}\varrho, \qquad \alpha_n^{-1} = \sum_{\varrho \in \mathfrak{S}_n} A_n^{-1}(\varrho, 1)\varrho. \tag{1}$$

Let $\mathscr{E}$ be the energy operator of particles obeying infinite statistics, defined by the commutation relation (1) in [1]. $\mathscr{E}$ acts on $\mathscr{H}(q)$ and each x_1 is an eigenvector of $\mathscr{E}$

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satisfying the eigenvalue equation

$$\mathscr{E}a^\dagger(l_n)\ldots a^\dagger(l_1)|0\rangle = \sum_{i=1}^{n} E(l_i)a^\dagger(l_n)\ldots a^\dagger(l_1)|0\rangle, \tag{2}$$

where $E(l_i)$ is the energy of a particle with momentum l_i.

Theorem. *The energy operator $\mathscr{E}$ has the form*

$$\mathscr{E} = \sum_{n\geq 1} \mathscr{E}_n,$$

with

$$\mathscr{E}_n = \sum_{k_1,\ldots,k_n} \sum_{\pi\in\mathfrak{S}_n} \sum_{i=1}^{n} c_i(q,\pi)E(k_{\pi(i)})a^\dagger(k_{\pi(n)})\ldots a^\dagger(k_{\pi(1)})a(k_1)\ldots a(k_n), \tag{3}$$

where the coefficients $c_i(q,\pi)$ are given by

$$\sum_{\pi\in\mathfrak{S}_n} \sum_{i=1}^{n} c_i(q,\pi)X^{i-1}\pi$$
$$= \alpha_n^{-1}(1-qXT_{12})(1-q^2XT_{13})\ldots(1-q^{n-1}XT_{1n}) \in \mathbb{C}[X][\mathfrak{S}_n]$$

or, explicitly,

$$c_i(q,\pi) = (-1)^{i-1}\sum_{\tau\in\mathfrak{S}_{n,i-1}} A_n^{-1}(q)(\pi,\tau)A_n(q)(\tau,1),$$

for all $\pi\in\mathfrak{S}_n$, $1\leq i\leq n$.

Remark 1. The theorem agrees with Zagier's conjecture in [1] except that he has $E(k_i)$ instead of $E(k_{\pi(i)})$. Thus the formulas agree if (and only if)

$$c_i(q,\pi) = c_{\pi(i)}(q,\pi),$$

for all $1\leq i\leq n$, $\pi\in\mathfrak{S}_n$. This is true for $n\leq 4$, but we do not know if it holds in general.

Remark 2. Although $\mathscr{E}$ contains an infinite sum, when applied on a given n-particle state, only the first n terms will give a nonzero contribution.

Remark 3. For $q=0$ this agrees with the results of Greenberg [2], who gave an expression for the energy operator of the form

$$E = \sum_{i} \mathscr{E}(i)n(i),$$

where the number operator $n(i)$ is given by

$$n(i) = \sum_{s\geq 0}\sum_{k_1,\ldots,k_s} a^\dagger(k_1)\ldots a^\dagger(k_s)a^\dagger(i)a(i)a(k_s)\ldots a(k_1),$$

with obvious notation.

   To prove this theorem we need some preparation. We know from [1] that the Hilbert space of states $\mathscr{H}(q)$ splits into an infinite direct sum of finite dimensional blocks. Each block is determined by the *unordered* n-tuple $\{k_1\ldots k_n\}$, whereas a

particular state in it is specified by an ordered version of that n-tuple. In other words, we identify the Fock space states with ordered sets $K = [k_1 \ldots k_n]$. For such a finite ordered set A we denote by $s(A)$ and $l(A)$ respectively the smallest and the largest element of A. Ordered sets can be concatenated, e.g., if we consider two disjoint ordered sets A_1 and A_2 we can form a new ordered set $A_1 \sqcup A_2$, such that if $a_i \in A_i$ then $a_1 < a_2$. Also, if B is a subset of an ordered set A, one can form the ordered set $A - B$. Moreover, we can invert the order of a given set, the new one being denoted by $\bar{A}$.

The permutation group $\mathfrak{S}_n$ acts naturally on the ordered sets of n elements; and this action extends to an action of the group algebra $\mathbb{C}[\mathfrak{S}_n]$ on the vector space $\mathscr{L}$ of formal linear combinations of such sets. If A is a given ordered set and $\sigma \in \mathfrak{S}_n$ we define $I_A(\sigma A) = I(\sigma)$.

We conclude these general considerations by introducing a linear evaluation map ξ acting on $\mathscr{L}[X]$ and defined by

$$\xi(\sigma A X^{i-1}) = E((\sigma A)(i))\sigma A.$$

In order to be able to determine the coefficients $c_i(q, \pi)$ in (3), we have to understand how the energy operator $\mathscr{E}$ and, in particular, each $\mathscr{E}_p$ acts on an arbitrary state. For that we will need two steps.

Proposition 1. *The action of the p-particle term of the energy operator on given n-particle state K is given by*

$$\mathscr{E}_p K = \xi\left(X^{n-p} \sum_{\substack{J \subset K \\ |J| = p}} q^{I_K((K-J) \sqcup J)}(K - J) \sqcup R_p(q, X) J \right), \tag{4}$$

where

$$R_p(q, X) = \alpha_p \sum_{\pi \in \mathfrak{S}_n} \sum_{i=1}^{p} c_i(q, \pi) X^{i-1} \pi, \tag{5}$$

for all $1 \leq p \leq n$.

Proof. To begin with, let us consider the case $p = n$. We have ([1], §2)

$$a(k_1) \ldots a(k_n) a^\dagger(l_n) \ldots a^\dagger(l_1)|0\rangle = \sum_{\sigma \in \mathfrak{S}_n} q^{I(\sigma)} \delta_{k_1 l_{\sigma(1)}} \ldots \delta_{k_n l_{\sigma(n)}}|0\rangle.$$

Thus, applying $\mathscr{E}_n$ on an n-particle state we obtain

$$\mathscr{E}_n K = \sum_{\sigma, \pi \in \mathfrak{S}_n} \sum_{i=1}^{n} q^{I(\sigma)} c_i(q, \pi) E((\sigma \pi K)(i))\sigma \pi K$$

$$= \xi\left(\sum_{\sigma, \pi \in \mathfrak{S}_n} \sum_{i=1}^{n} q^{I(\sigma)} c_i(q, \pi) X^{i-1} \sigma \pi K \right)$$

$$= \xi(R_n(q, X) K).$$

We must now determine how a generic term $\mathscr{E}_p$ acts on the n-particle state. Its action can be described in the following way: it chooses a subset $J \subset K$, $|J| = p$, such that the p annihilation operators of $\mathscr{E}_p$ will "contract" with the p creation operators of J, leaving the remaining creation operators of K in unaltered order i.e., characterized by the set $(K - J)$. This yields a new n-particle state, characterized

by the permutation $(K - J) \sqcup J$ multiplied by the numerical coefficient incurred in by repeated application of the commutation relation (1) in [1] and which is given by $q^{I_K((K-J)\sqcup J)}$. Clearly, $R_p(q, X)$ acts now on J and, because the evaluation map ξ is defined on the whole n-particle state, we have to shift the polynomial in X by a common factor $X^{|K-J|} = X^{n-p}$ in order to obtain the correct energies. Hence, it follows that

$$\mathscr{E}_p K = \xi \left(X^{n-p} \sum_{\substack{J \subset K \\ |J|=p}} q^{I_K((K-J)\sqcup J)} (K - J) \sqcup R_p(q, X) J \right). \quad \square$$

Proposition 2. *The action of the group ring element $R_p(q, X)$ on the ordered set J is given by*

$$R_p(q, X) J = \sum_{\substack{L \subset J \\ s(J) \notin L}} q^{I_J(\bar{L} \sqcup (J-L))} (\bar{L} \sqcup (J - L))(-X)^{|L|}. \tag{6}$$

Proof. We shall essentially show that (6) yields the correct energy operator, i.e., that it satisfies the eigenvalue equation. Therefore, we insert $R_p(q, X)$ in the expression for $\mathscr{E}_p$ and we compute

$$\mathscr{E}_p K = \xi \left(X^{n-p} \sum_{\substack{J \subset K \\ |J|=p}} \sum_{\substack{L \subset J \\ s(J) \notin L}} q^{I_K((K-J)\sqcup J) + I_J(\bar{L}\sqcup(J-L))} ((K-J)\sqcup\bar{L}\sqcup(J-L))(-X)^{|L|} \right).$$

But, obviously,

$$I_K((K - J) \sqcup J) + I_J(\bar{L} \sqcup (J - L)) = I_K((K - J) \sqcup \bar{L} \sqcup (J - L)),$$

such that we obtain

$$\mathscr{E}_p K = \xi \left(\sum_{\substack{J \subset K \\ |J|=p}} \sum_{\substack{L \subset J \\ s(J) \notin L}} (-1)^{|L|} q^{I_K((K-J)\sqcup\bar{L}\sqcup(J-L))} ((K-J)\sqcup\bar{L}\sqcup(J-L)) X^{n-p+|L|} \right).$$

For given J and L, we consider those terms in the sum which are characterized by $l(K - J) > l(L)$. Then the corresponding set can be viewed in another way, namely,

$$(K - J) \sqcup \bar{L} \sqcup (J - L) = ((K - J) - \{l(K - J)\}) \sqcup (\{l(K - J)\} \sqcup \bar{L}) \sqcup (J - L),$$

having now $l((K - J) - \{l(K - J)\}) < l(\{(K - J)\} \sqcup \bar{L})$ and thus corresponding to another set which contributes as well to the sum. As one can easily see, these two terms will occur with identical coefficients but with opposite signs and will therefore cancel.

Thus, it only remains to discuss the case $L = \varnothing$. If $l(K - J) > s(J)$, then we can proceed analogously, writing

$$(K - J) \sqcup \varnothing \sqcup J = ((K - J) - \{l(K - J)\}) \sqcup \{l(K - J)\} \sqcup J,$$

such that we obtain the usual cancellation. But if $l(K-J) > s$, then $(K-J) \sqcup J = K$, and we finally obtain

$$\mathscr{E}K = \sum_{p=1}^{n} \mathscr{E}_p K$$

$$= \xi \left(\sum_{p=1}^{n} \sum_{\substack{J \subset K \\ |J|=p}} q^{I_K((K-J) \sqcup J)} (K-J) \sqcup J X^{n-p} \right)$$

$$= \sum_{i=1}^{n} E(K_i) K . \quad \square$$

Now we are ready to prove the theorem stated at the very beginning.

Proof. Let us return now to the usual permutation language. One can easily see that the permutations of the form $\bar{L} \sqcup (J-L)$, with $|L| = s$ can be written as $T_{1m_1} \ldots T_{1m_s}$, with $1 < m_1 < \cdots < m_s \le n$, so that

$$R_n(q, X) = \sum_{s=0}^{n-1} \sum_{1 < m_1 < \cdots < m_s \le n} (-1)^s q^{(m_1-1)+(m_2-1)+\cdots+(m_s-1)} X^s T_{1m_1} \ldots T_{1m_s} ,$$

where we used the fact that $I(T_{1k}) = k - 1$.

Now we only have to identify this expression obtained for $R_n(q, X)$ with its definition (5), and we obtain the desired result; that is, the generating function for the coefficients $c_i(q, \pi)$ is given by

$$\sum_{\pi \in \mathfrak{S}_n} \sum_{i=1}^{n} c_i(q, \pi) X^{i-1} \pi = \alpha_n^{-1} (1 - qXT_{12})(1 - q^2 XT_{13}) \ldots (1 - q^{n-1} XT_{1n}),$$

with α_n given by (1).

The coefficients $c_i(q, \pi)$ can be also given in another equivalent form. Using (1), the right-hand side of the equation above can be written as

$$\left(\sum_{\varrho \in \mathfrak{S}_n} A_n^{-1}(\varrho, 1) \varrho \right) \left(\sum_{i=1}^{n} \sum_{\pi \in \mathfrak{S}_{n,i-1}} (-1)^{i-1} A_n(\pi, 1) X^{i-1} \pi \right)$$

$$= \sum_{i=1}^{n} (-1)^{i-1} X^{i-1} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \pi \in \mathfrak{S}_{n,i-1}}} A_n^{-1}(\sigma, \pi) A_n(\pi, 1) \sigma ,$$

where we made the substitution $\sigma = \varrho \pi \in \mathfrak{S}_n$ and we used the fact that $A_n^{-1}(\sigma, \pi) = A_n^{-1}(\sigma \pi^{-1}, 1)$. Thus, identifying with the left-hand side, we obtain

$$c_i(q, \sigma) = (-1)^{i-1} \sum_{\pi \in \mathfrak{S}_{n,i-1}} A_n^{-1}(\sigma, \pi) A_n(\pi, 1) .$$

It only remains to show that the solution obtained is unique. First of all it is obvious that the form (3) of the energy operator is the most general which can be assumed for such a system, so that we only need to consider the possibility of having another set of coefficients $c_i^*(q, \pi)$, such that the corresponding $\mathscr{E}^*$ yields the same

eigenvalue equation. Hence we must have $(\mathscr{E} - \mathscr{E}^*)K = 0$, $\forall K$. If we consider a 1-particle state, we immediately obtain $\Delta c_1(q, 1) \equiv c_1(q, 1) - c_1^*(q, 1) = 0$, for $n = 1$. Assume now $\Delta c_i(q, \pi) = 0$, for all $1 \leq i \leq p$, $\pi \in \mathfrak{S}_p$ in all orders $1 \leq p \leq n - 1$. Then for an n-particle state we will have $(\mathscr{E} - \mathscr{E}^*)K = (\mathscr{E}_n - \mathscr{E}_n^*)K = 0$. But, on the other hand

$$(\mathscr{E}_n - \mathscr{E}_n^*)K = \left(\sum_{\varrho, \pi \in \mathfrak{S}_n} \sum_{i=1}^{n} q^{I(\varrho\pi^{-1})} \Delta c_i(q, \pi) E(\varrho(i)) \varrho \right) K \,.$$

Taking into account the fact that ϱ and $E(\varrho(i))$ are linearly independent we get

$$\sum_{\pi \in \mathfrak{S}_n} A_n(q)(\varrho, \pi) \Delta c_i(q, \pi) = 0 \qquad \forall \varrho \in \mathfrak{S}_n, \, 1 \leq i \leq n \,.$$

Since $A_n(q)$ is invertible, it follows that $\Delta c_i(q, \pi) = 0$. Hence the energy operator is uniquely determined. □

References

1. Zagier, D.: Realizability of a Model in Infinite Statistics, 1991 Preprint MPI Bonn/University of Maryland. Commun. Math. Phys. **147**, 199–210 (1992)
2. Greenberg, O.W.: Example of infinite statistics. Phys. Rev. Lett. **64**, 705–708 (1990)

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