

# The Faddeev–Popov Procedure and Application to Bosonic Strings:

An Infinite Dimensional Point of View

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Received August 1, 1991; in revised form December 11, 1991

**Abstract.** A generalisation of the finite dimensional presentation of the Faddeev–Popov procedure is derived in an infinite dimensional framework for gauge theories with finite dimensional moduli space using heat-kernel regularised determinants. It is shown that the infinite dimensional Faddeev–Popov determinant is – up to a finite dimensional determinant determined by a choice of a slice – canonically determined by the geometrical data defining the gauge theory, namely a fibre bundle  $P \rightarrow P/G$  with structure group  $G$  and the invariance group of a metric structure given on the total space  $P$ . The case of (closed) bosonic string theory is discussed.

## 0. Introduction

The Faddeev–Popov procedure for gauge theories originally introduced by Faddeev and Popov in the context of Yang–Mills theories [1] has been discussed by many authors in the physics literature in the context of string theory (see e.g. [2]) from a topological point of view (see e.g. [3–5]) as well as from a geometrical stand-point (see e.g. [6–8]). It essentially yields a formal procedure to write a functional integral on the space  $P$  of paths arising from the functional quantisation of a classical action invariant under the action of the gauge group  $G$  as an integral on the quotient space  $P/G$  (or a submanifold  $\Sigma$  of  $P$  isomorphic to this quotient). If the quotient space is finite dimensional as in the case of bosonic string theory (it is given by the Teichmüller space of a Riemann surface), this procedure reduces a formal integration on an infinite dimensional space, the space of configurations to an integration on a finite dimensional manifold. “Factorising out” the gauge group in this way gives rise to a jacobian determinant, the formal Faddeev–Popov determinant. Some important clarifications were made as to the geometrical meaning behind this formal procedure [6, 7]. This geometrical interpretation was done in a finite dimensional setting with the implicit point of view that the infinite dimensional set up inherent to functional integration can be seen as a generalisation.

In this paper, we want to discuss how far this generalisation to an infinite dimensional framework can be made precise from a mathematical point of view.

We shall more specifically concentrate on the case of (closed) bosonic string theory.

In this infinite dimensional presentation of the Faddeev–Popov procedure, we stress the role of elliptic operators pointing out that the Faddeev–Popov operator is essentially built up from elliptic operators on compact surfaces. Using the heat kernel regularisation method for determinants of elliptic operators on compact surfaces, we extend the notion of regularised determinant to the class of operators of interest in the Faddeev Popov procedure.

Let us briefly describe how the Faddeev–Popov operator arises in this procedure. A natural way of parametrising the manifold  $P$  locally around  $p$  is to look for a local cross section  $\Sigma_p$  in  $P$  at point  $p$ . If the tangent map  $\tau_p$  at point  $p$  to the action of the group  $G$  is injective, the map  $G \times \Sigma_p \rightarrow P$  yields a one to one local parametrisation of  $P$  around  $p$  in terms of the gauge group  $G$  and the slice  $\Sigma_p$ . Changing from one local cross section to another gives rise to a change of parametrisation and hence to a jacobian operator tangent to the transformation going from one parametrisation to the other. In gauge field theories, when  $\tau_p$  has an injective symbol, starting from a given local cross section  $\Sigma_p$  at point  $p$  one can choose a local cross section *orthogonal* to the fibre at point  $p$  and the corresponding Jacobian map is called the *Faddeev Popov operator*. In Yang-Mills theory for example, a local cross section around  $p$  orthogonal to the fibre at point  $p$  is given by the affine space  $\{p + \text{Ker } \tau_p^*\}$ , where  $\tau_p^*$  is the adjoint of  $\tau_p$  given a riemannian structure on  $P$ . Note that this however does not a priori yield a global cross section since Gribov ambiguities can arise [3].

Inserting this new parametrisation into the formal functional integral on the space  $P$  gives rise to a formal jacobian determinant, the Faddeev–Popov determinant denoted by “ $\det F_p$ ” which coincides with “ $\det \tau_p$ ” up to a finite dimensional determinant. Up to this finite dimensional determinant which depends on the choice of the slice  $\Sigma_p$ , this Faddeev–Popov determinant is canonically determined by the geometric data  $P \rightarrow P/G$  through the operator  $\tau_p$ .

We shall give a detailed description of this Faddeev–Popov operator and show how in the infinite dimensional setting, one can compute a regularised version of this determinant (extending the notion of heat-kernel regularisation of determinants of elliptic operators to a class of operators of interest for this Faddeev–Popov operator). We prove it coincides (under an ellipticity assumption on  $\tau_p^* \tau_p$  and up to a finite dimensional determinant which depends on the choice of the slice  $\Sigma_p$ ) with a regularised version of the operator  $\tau_p$  so that the Faddeev–Popov determinant is canonically defined in terms of the geometrical data  $P \rightarrow P/G$  and a choice of the slice.

We consider two cases, namely first the case when the total space  $P$  is equipped with a metric structure invariant under the whole structure group  $G$  and then the general case when the metric structure on  $P$  is only invariant under a subgroup  $K$  of the structure group  $G$  which we then assume to be a semi-direct product  $G = H \rtimes K$ , where  $H$  is a group acting on  $K$  by  $k \rightarrow k^h$ , this action satisfying the condition  $(k^{h_1})^{h_2} = k^{h_1 h_2}$ . We shall illustrate this in the string theory where  $G$  is the invariance group of the classical string and  $H$  the invariance group remaining at the quantised level in non-critical dimension.

Ultimately, in the case of a gauge theory with finite dimensional moduli space  $P/G$ , the Faddeev–Popov procedure as described above enables us to define – under

precise conditions on the integrand – a renormalised path integral “ $\left(\int_P h(p)d[p]\right)_{\text{ren}}$ ” where  $h$  is a functional on  $P$  and  $d[p]$  a formal Lebesgue measure on path space as an integral on a finite dimensional manifold  $\Sigma \cong P/G$ . More precisely, for  $\varepsilon > 0$  let us denote by  $\det_\varepsilon(F_p)$  the  $\varepsilon$ - heat kernel regularised determinant of the Faddeev Popov operator (defined in Sect. II). If for an  $\varepsilon$ -renormalised version  $h_\varepsilon$  of  $h$  (which we shall describe in the particular case of string theory),  $h_\varepsilon(p) \det_\varepsilon(F_p)$  is gauge invariant and equivalent when  $\varepsilon$  goes to zero to a density  $\rho_\varepsilon(x)$  on  $\Sigma$ , then  $\left(\int_P h(p)d[p]\right)_{\text{ren}}$  is defined as the integral on the finite dimensional space  $\Sigma$ :

$$\lim_{\varepsilon \rightarrow 0} \int_\Sigma \rho_\varepsilon(x)d[x]$$

whenever this limit exists. This renormalised functional integral is the mathematical object implicitly referred to in the physics literature when writing  $\int_P h(p)d[p]$ . We shall illustrate this in more detail in the case of string theory.

The paper is organised as follows. In Sect. I, we briefly recall the geometrical setting for the Faddeev–Popov procedure and define the Faddeev–Popov operator which, in the context of string theory, is essentially built up from elliptic operators acting on functional spaces and as such, does not have a well defined determinant.

In Sect. II, we recall the heat kernel regularisation procedure for elliptic operators on compact surfaces (without boundary) and extend it to a more general class of operators, namely to certain triangular matrix operators on product bundles involving elliptic operators on a compact surface (without boundary).

In Sect. III, applying the results of the preceding section, we define a regularised Faddeev–Popov determinant which coincides with the usual Faddeev–Popov determinant encountered in the physics literature and define a renormalised functional integral on the bundle  $P$  when the moduli space  $P/G$  is finite dimensional.

Finally in Sect. IV, we apply this procedure to the case of closed bosonic strings and then define a renormalised Polyakov integral on Teichmüller space.

### I. The Faddeev–Popov Map

In this section, we recall the geometrical setting for the Faddeev–Popov procedure. In order to describe this procedure independently of any measure theoretic notion (infinite dimensional Lebesgue measures are not well defined), we shall define the notion of Faddeev–Popov operator given a riemannian structure on  $P$ .

$\pi: P \rightarrow V$  will denote a  $C^\infty$  trivial principal fibre bundle on a smooth base manifold  $V$  with structure group  $G$  and canonical projection  $\pi$ . (In some cases, such as Yang-Mills theory, the bundle  $P$  is not trivial but up to a restriction to an open subset of the base manifold, we can however recover the above set up.) Let  $\sigma: V \rightarrow P$  denote a  $C^\infty$  global cross section of  $P$ .  $\Sigma \equiv \sigma(V)$  can be seen as a  $C^\infty$  submanifold of  $P$ . We shall assume that the group  $G$  acts on  $P$  smoothly by a right-hand side action:

$$G \times P \rightarrow P$$

$$(a, p) \rightarrow R_a p \equiv p.a.$$

We introduce two spaces at a point  $p \in P$ , namely the space:

$$W_p \equiv T_p(R_a \Sigma), \quad (1.1)$$

where  $(x, a)$  is the unique element in  $\Sigma \times G$  such that  $p = R_a x$ , and the space:

$$V_p \equiv T_p(F_p) \quad (1.2)$$

tangent to the fibre  $F_p \equiv \pi^{-1}(\pi(p))$  at point  $p$ . Then  $V_p = \text{Im } \tau_p$ , where

$$\tau_p: T_e G \rightarrow T_p P \quad (1.3)$$

is the tangent map at point  $e \in G$  to

$$\begin{aligned} G &\rightarrow P \\ a &\rightarrow R_a \cdot p = p \cdot a. \end{aligned} \quad (1.4)$$

Since  $\Sigma$  is a cross section for the bundle  $P \rightarrow P/G$ , the tangent space to the bundle  $P$  at point  $p$  splits into a direct sum:

$$\begin{aligned} T_p P &= T_p(F_p) \oplus T_p(R_a \Sigma) \\ &= \text{Im } \tau_p \oplus W_p. \end{aligned} \quad (1.5)$$

**Lemma 1.1.** *The map*

$$\begin{aligned} G \times \Sigma &\rightarrow P \\ (a, x) &\rightarrow R_a x \end{aligned}$$

is a  $C^\infty$  diffeomorphism and  $\tau_p$  is injective.

*Proof.* It is a  $C^\infty$  diffeomorphism since  $\Sigma$  is a  $C^\infty$  cross section for the action of  $G$ . Hence,  $T_p(R_a \Sigma) \oplus \text{Im } \tau_p = T_p P$ , since  $\Sigma$  induces a local cross section and the tangent map:

$$\begin{aligned} T_e G \times T_x \Sigma &\rightarrow T_p P \\ (u, h) &\rightarrow \tau_p u + R_a h \end{aligned}$$

is one to one and onto. It is clearly injective if and only if  $\tau_p$  is injective which proves the lemma. ■

We now equip the bundle  $P$  with a smooth riemannian structure. It induces a scalar product  $\langle \cdot, \cdot \rangle_p$  on the tangent space  $T_p P$  at any point  $p$  in  $P$ . We shall first consider the case when this riemannian structure is invariant under the whole group  $G$ , i.e.

$$\langle R_a^* h, R_a^* k \rangle_{R_a p} = \langle h, k \rangle_p \forall h, k \in T_p P, \forall a \in G. \quad (1.6)$$

In gauge field theories, one wants a local parametrisation of  $P$  induced by a local cross section orthogonal to the fibre. A change of local parametrisation of  $P$  from one induced by a given cross section around a point  $p$  to one induced by a local cross section orthogonal to the fibre at point  $p$  gives rise to a jacobian operator tangent to the transformation map, the Faddeev–Popov operator.

We shall make the following assumption under with Faddeev–Popov operators can be naturally constructed in gauge theories.

**Hyp 1.** *The gauge group  $G$  is an infinite dimensional smooth Frechet manifold equipped with a smooth Riemannian structure, and  $T_e G$  is the space of smooth sections of a*

bundle  $E$  on a compact surface  $\Lambda$ . The operator  $\tau_p$  is a differential operator on  $E$  of order larger or equal to 1 with smooth coefficients and has injective symbol.

In the sequel, we shall denote by  $\langle \cdot, \cdot \rangle$  the scalar product on  $T_e G$ . We shall first assume it is invariant under the action of the whole group  $G$  so that we define on  $T_a G, a \in G$ :

$$\langle R_a^* u, R_a^* v \rangle_a \equiv \langle u, v \rangle \forall a \in G, \forall u, v \in T_e G. \tag{1.7}$$

If  $A$  is a differential operator with  $C^\infty$  coefficients from  $T_e G = C^\infty(E)$  to  $T_p P$  which is densely defined,  $A^*$  denotes the adjoint of  $A$  with respect to  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_p$ . It is uniquely defined since the domain of  $A$  contains the space  $C^\infty(E)$  of  $C^\infty$  sections dense in the closure  $L^2(E)$ . Moreover, the coefficients of  $A$  being  $C^\infty$ , and the Riemannian structure being smooth, the operator  $A^*$  is well defined on the image of  $C^\infty(E)$  through  $A$  so that the operator  $A^*A$  makes sense on  $C^\infty(E)$  and can be extended in a unique way to a self adjoint operator which we denote by the same symbol

*Remark.* Notice that the injectivity of the symbol of  $\tau_p$  yields the ellipticity of the operator  $\tau_p^* \tau_p$  (see e.g. [9]).

Since  $\tau_p$  has an injective symbol, the space  $\text{Im } \tau_p$  is closed in  $T_p P$  and the following orthogonal splitting holds (see [9], Corollary 6.9):

$$\text{Im } \tau_p \oplus \text{Ker } \tau_p^* = T_p P.$$

Since  $\text{Im } \tau_p$  is closed, the orthogonal projection onto  $\text{Im } \tau_p$  is well defined and we shall denote it by  $\pi_p$ . In gauge field theories, when the manifold  $P$  is a  $C^\infty$  Hilbert manifold, using the implicit function theorem, one constructs a local cross section  $S_p$  of  $P$  with tangent space  $\text{Ker } \tau_p^*$  which is then naturally orthogonal to the orbit of  $p$  since the splitting  $\text{Ker } \tau_p^* \oplus \text{Im } \tau_p = T_p P$  is orthogonal. This gives rise to the following jacobian operator:

**Lemma 1.2.** *The map*

$$\begin{aligned} F_p: T_e G \times W_p &\rightarrow \text{Im } \tau_p \times \text{Ker } \tau_p^* \\ (u, h) &\rightarrow (\tau_p u + \pi_p h, (1 - \pi_p)h) \end{aligned}$$

is one to one and onto.

*Proof.* The map  $F_p$  is clearly injective if and only if  $\tau_p$  is injective. Let us check that it is onto. Take  $(k^1, k^2) \in \text{Im } \tau_p \times \text{Ker } \tau_p^*$ , the splitting (1.5) yields  $k^1 + k^2 = \tau_p u + h$ ,  $u \in T_e G, h \in W_p$ , so that  $k^1 = \pi_p(k^1 + k^2) = \tau_p u + \pi_p h, k^2 = (1 - \pi_p)h$  and  $F_p(u, h) = (k^1, k^2)$ . ■

**Definition.** *The map  $F_p$  of Lemma 1.2 is called the Faddeev–Popov operator associated to the group  $G$ .*

*Remark.* The construction of a Faddeev–Popov operator can be generalised to the case when the gauge group is a semi-direct product

$$G \equiv G' \rtimes G'' \tag{1.8}$$

of two groups as in the case of string theory, the group  $G'$  acting on  $G''$  by an action

satisfying the requirement:

$$(g''g')^{g^2} = g''g^1g^2 \forall g'_1, g'_2 \in G', \quad \forall g'' \in G''.$$

The underlying Riemannian structures are not invariant under the whole group  $G$  anymore but only under the group  $G'$ , the total group  $G$  being the structure group for the bundle. In the case of strings,  $G$  is the invariance group for the classical action describing the classical motion of the string whereas  $G'$  is the invariance group remaining at the quantised level in non-critical dimension, the group of invariance of the formal measures arising in the functional quantisation.

In this case, the operator  $\tau_p$  has the following shape:

$$\begin{aligned} \tau_p: T_e G' \times T_e G'' &\rightarrow T_p P \\ (u', u'') &\rightarrow \tau'_p(u') + \tau''_p(u''), \end{aligned}$$

where

$$\begin{aligned} \tau'_p: T_e G' &\rightarrow T_p P \\ u' &\rightarrow \tau'_p u' \equiv \tau_p(u') \end{aligned}$$

and

$$\begin{aligned} \tau''_p: T_e G'' &\rightarrow T_p P \\ u'' &\rightarrow \tau''_p u'' \equiv \tau_p(u''). \end{aligned}$$

Under the assumption Hyp. 1, the space  $\text{Im } \tau'_p$  is closed and we can define the orthogonal projection  $\pi''_p$  onto this space. Writing

$$\tau_p(u' + u'') = (\tau'_p - \pi''_p \tau'_p)u' + \pi''_p \tau'_p u' + \tau''_p u''$$

yields the following matrix representation for  $\tau_p$  as an operator from  $T_e G' \times T_e G''$  onto  $K_p^\perp \times K_p$  with  $K_p \equiv \text{Im } \tau_p$ :

$$\tau_p(u' + u'') = \begin{bmatrix} A_p & 0 \\ C_p & B_p \end{bmatrix} \begin{bmatrix} u' \\ u'' \end{bmatrix}, \quad (1.9)$$

where

$$A_p = (\mathbf{1} - \pi''_p) \tau'_p, \quad (1.9.a)$$

$$B_p = \tau''_p \quad (1.9.b)$$

and

$$C_p = \pi''_p \tau'_p. \quad (1.9.c)$$

The operator  $\tau_p$  then has injective symbol if and only if the operators  $A_p$  and  $B_p$  have injective symbol. In particular, both operators  $A_p^* A_p$  and  $B_p^* B_p$  are elliptic.

Lemma 1.2 clearly extends to this more general operator  $\tau_p$  and we can extend the notion of Faddeev–Popov operator to this case whenever  $\tau_p$  is injective. Notice that if  $G'' = \{e\}$ , we have  $\text{Im } \tau''_p = \{0\}$  and the matrix operator given in (1.9) reduces to the operator  $A_p = \tau_p$ .

In order to simplify the presentation of the Faddeev–Popov procedure, we shall assume that the operator  $B_p$  takes a simple form, this being the case in string theory. However, the setting could be generalised to any differential operator  $B_p$  with  $C^\infty$

coefficients and injective symbol as we shall point out throughout paper. We assume that  $T_e G''$  is canonically embedded by an isometry  $i_p$  into  $T_p P$  and that  $B_p u'' = i_p(u'')$ . We shall henceforth identify  $T_e G''$  with  $i_p(T_e G'')$  thus replacing the operator  $B_p$  in (1.9) by  $\mathbf{1}$ .

The Faddeev–Popov operator  $F_p$  associated to the gauge group  $G$  reads:

$$F_p = D_p + R_p \tag{1.10}$$

with

$$D_p = \begin{bmatrix} \tau_p & 0 \\ 0 & \mathbf{1} \end{bmatrix} \tag{1.11}$$

and

$$R_p = \begin{bmatrix} 0 & \pi_p/W_p \\ 0 & -\pi_p/W_p \end{bmatrix}, \tag{1.12}$$

where  $D_p$  and  $R_p$  are seen as operators from  $T_e G \times W_p$  to  $T_p P \cong \text{Im } \tau_p \times \text{Ker } \tau_p^*$ . Here  $\pi_p/W_p$  denotes the restriction of the projection  $\pi_p$  to the finite dimensional space  $W_p$ .

From now on, we shall assume that the moduli space is finite dimensional, i.e. that the following assumption is fulfilled

**Hyp 2.**  $\dim P/G < \infty$ .

Notice that the matrix  $R_p$  is then of finite rank and hence of finite trace. The operator  $D_p^* D_p$  on the other hand will essentially be an elliptic operator and to define its determinant requires a regularisation. The object of the following section is to define the Faddeev–Popov determinant, i.e. to give a meaning to the expression “ $\det F_p$ ” and to give a mathematical interpretation of the formal equality:

$$“\det(F_p) = (\det(\tau_p^* \tau_p))^{1/2} \det(\mathbf{1} - \pi_p)” \tag{1.13}$$

where  $\det(\mathbf{1} - \pi_p)$  is a finite dimensional determinant given by the determinant of the matrix of  $\mathbf{1} - \pi_p$  seen as operator from  $W_p$  onto  $\text{Ker } \tau_p^*$ .

## II. Regularised Determinants for Elliptic Operators

In this section, we shall extend the presentation of heat-kernel regularisation for determinants of elliptic operators which was done in [12 and 13] in the particular case of string theory to a more general setting so as to be able to define the regularised Faddeev–Popov determinant “ $\det F_p$ .” Let  $E$  be a smooth vector bundle with fibres of finite dimension based on a boundaryless  $C^\infty$  real compact manifold  $A$  of dimension  $k \geq 2$  and let  $C^\infty(E)$  be the vector space of smooth sections of  $E$ . Let  $H \equiv L^2(E)$  be the closure of  $C^\infty(E)$  with respect to the  $L^2$  scalar product induced by an  $L^2$ -scalar product  $\langle \cdot, \cdot \rangle$  on  $E$ . Let  $\text{Ell}^+(E)$  denote the space of positive elliptic self adjoint (with respect to  $\langle \cdot, \cdot \rangle$ ) operators on  $H$  of strictly positive order. We apply here classical results for elliptic operators on compact surfaces for which we refer the reader to [10]. Since  $A$  is compact, for  $A \in \text{Ell}^+(E)$ , the orthogonal space  $H_A \equiv (\text{Ker } A)^\perp$  to the kernel  $\text{Ker } A$  of  $A$  is invariant under  $A$  and we can define the restriction  $A/(\text{Ker } A)^\perp$

of  $A$  to this Hilbert space  $H_A$ . Let us set

$$A' \equiv A/H_A. \tag{2.1}$$

This defines a strictly positive self adjoint operator on  $H_A$ .

We now introduce a heat-kernel cut-off function which yields the usual heat-kernel regularisation for infinite dimensional determinants via the spectral theorem. For  $\varepsilon > 0, \lambda \in \mathbb{R}^+ \setminus \{0\}$  we set:

$$h_\varepsilon(\lambda) \equiv \exp\left(-\int_\varepsilon^{+\infty} t^{-1} e^{-t\lambda} dt\right) \tag{2.2}$$

and for  $A \in \text{Ell}^+(E)$ , we define  $h_\varepsilon(A')$  through the spectral theorem.

If  $J_1(H)$  denotes the set of compact operators  $A$  on  $H$  such that  $|A|$  has finite trace (the trace is taken with respect to the scalar product  $\langle \cdot, \cdot \rangle$ ) then for  $A = \mathbf{1} + C, C \in J_1(H)$ , we can define the determinant  $\det A$  of  $A$  as in [11] and it coincides with the product of the eigenvalues of  $A$  (theorem of Lidskii). The asymptotic behaviour of the eigenvalues of a positive self adjoint elliptic operator of strictly positive order on a compact boundaryless finite dimensional manifold yields that

$$h_\varepsilon(A') \in \mathbf{1} + J_1(H_A) \tag{2.3}$$

for all  $A \in \text{Ell}^+(E), \varepsilon > 0$  and hence the finiteness of the “ $\varepsilon$ -cut-off” determinant:

$$\det'_\varepsilon(A) \equiv \det(h_\varepsilon(A')), \quad \forall A \in \text{Ell}^+(E). \tag{2.4}$$

The regularised determinant is then defined as

$$\det'(A) \equiv \lim_{\varepsilon \rightarrow 0} \exp[\log \det h_\varepsilon(A') - \text{divergent terms}]. \tag{2.4 bis}$$

Let now  $E, F$  be two smooth vector bundles based on  $A$  with fibres of finite dimension and  $C^\infty(E), C^\infty(F)$  be the corresponding vector spaces of smooth sections.

Let  $A: C^\infty(E) \rightarrow C^\infty(F)$  be an operator such that  $A^*A \in \text{Ell}^+(E)$ , then applying formula (2.4), we can extend the notion of regularised determinant setting:

$$\det'_\varepsilon(A) = (\det(h_\varepsilon(A^*A)))^{1/2} \tag{2.5}$$

and

$$\det'(A) = \det(A^*A)^{1/2}. \tag{2.5 bis}$$

When the operator is injective, we shall omit the prime.

We now generalise this heat-kernel regularisation for determinants of a class of triangular matrix operators.

**Proposition 2.1.** *Let  $E = E_1 \times E_2, F = F_1 \times F_2$  be products of  $C^\infty$  bundles equipped with scalar products and*

$$T: C^\infty(E_1) \times C^\infty(E_2) \rightarrow C^\infty(F_1) \times C^\infty(F_2)$$

*be an injective operator of the form*

$$T = \begin{bmatrix} A & 0 \\ C & \mathbf{1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ C & \mathbf{1} \end{bmatrix}$$



such that:

- a) the operator  $A^*A \in \text{Ell}^+(E_1)$ ,
- b)  $C$  is a differential operator with  $C^\infty$  coefficients of strictly positive order. Then for  $\varepsilon > 0$ , the operator

$$(T^*T)_\varepsilon = \begin{bmatrix} \mathbf{1} & (1 + \varepsilon C^*C)^{-k} C^* \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} h_\varepsilon(A^*A) & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ C(1 + \varepsilon C^*C)^{-k} & \mathbf{1} \end{bmatrix}$$

is of the form “ $\mathbf{1} +$  a traceclass operator” and we have

$$\det(T^*T)_\varepsilon = \det h_\varepsilon(A^*A) = (\det_\varepsilon(A))^2.$$

*Remark.* As pointed out above, this could easily extend to the case

$$T = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix},$$

where  $B$  is a differential operator with  $C^\infty$  coefficients such that  $B^*B \in \text{Ell}^+(E_2)$ . We would have

$$\det(T^*T)_\varepsilon = \det h_\varepsilon(A^*A) \det h_\varepsilon(B^*B).$$

**Definition.** For  $T$  as in Proposition 2.1, we define the  $\varepsilon$ -cut-off determinant  $\det_\varepsilon(T) \equiv (\det(T^*T)_\varepsilon)^{1/2}$  so that

$$\det_\varepsilon(T) = \det_\varepsilon(A). \tag{2.6}$$

Notice that  $A^*A = A^*A'$  since  $T$  is injective. Moreover, the equality of the determinants also holds in the limit  $\varepsilon \rightarrow 0$  after removal of the divergences (see (2.5 bis)):

$$\det(T) = \det(A). \tag{2.6 bis}$$

*Proof of Proposition 2.1.* Let us write

$$(T^*T)_\varepsilon = \begin{bmatrix} \mathbf{1} & (1 + \varepsilon C^*C)^{-k} C^* \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} h_\varepsilon(A^*A) & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ C(1 + \varepsilon C^*C)^{-k} & \mathbf{1} \end{bmatrix}. \tag{2.7}$$

Since  $h_\varepsilon(A^*A)$  is of the form “ $\mathbf{1} +$  a trace-class operator,” so is the operator described by the middle matrix in (2.7). As for the two extreme matrices in (2.7), we observe that the operator  $C(1 + \varepsilon C^*C)^{-k}$  and its adjoint are pseudo-differential operators of order  $-m$  with  $m > k$  since  $k > 1$  (where  $k$  is as before the dimension of  $A$ ) and hence trace class (see e.g. [15] p. 308) so that these matrices represent operators of the form “ $\mathbf{1} +$  trace class.” The product of operators of the form “ $\mathbf{1} +$  trace class” being “ $\mathbf{1} +$  trace class” (see e.g. [11]), the first assertion of the proposition follows.

As for the second assertion, we use the fact that the determinant of an operator of the form “ $\mathbf{1} +$  trace class” is well defined and finite and we apply the product formula for determinants of such operators (see [11]). The determinants of the two extreme operator matrices in (2.7) are equal to 1 since they are of the form “ $1 +$  a nilpotent operator”; the determinant of the middle operator is exactly  $\det h_\varepsilon(A^*A)$  as the product of the eigenvalues. ■

Let us now briefly comment on this regularisation procedure; it essentially amounts to a renormalisation by which one divides the determinant of an operator

by an infinite constant independent of the operator in a sense made clear by the following proposition.

**Proposition 2.2**

- 1) Let  $A \in \text{Ell}^+(E)$ , where  $E$  is a vector bundle as above. For  $\varepsilon > 0$  there is a constant  $K_\varepsilon \equiv h_\varepsilon(1)^{-1}$  independent of  $A$  such that  $K_\varepsilon h_\varepsilon(A)$  converges pointwise to  $A$  on a core of  $A$ .
- 2) Let  $A, B, C$  and  $T$  be as in Proposition 2.1 and let us set for  $\bar{\varepsilon} \equiv (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ ,

$$(T^*T)_{\bar{\varepsilon}} \equiv \begin{bmatrix} \mathbf{1} & (\mathbf{1} + \varepsilon_1(C^*C))^{-k} C^* \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} K_{\varepsilon_2} h_{\varepsilon_2}(A^*A) & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ C(1 + \varepsilon_3 C^*C)^{-k} & \mathbf{1} \end{bmatrix}$$

then there is a sequence  $\bar{\varepsilon}(n) \in \mathbb{R}^3$  converging to  $(0, 0, 0)$  as  $n \rightarrow +\infty$  such that the family of operators  $(T^*T)_{\bar{\varepsilon}(n)}$  converges point-wise to the operator  $T^*T$  on a core of  $T^*T$ .

*Proof.* The first part of the proposition is a direct consequence of the definition of  $h_\varepsilon$ . Indeed, we have  $\exp - \int_\varepsilon^{\varepsilon/\lambda} t^{-1} e^{-t\lambda} dt \rightarrow \lambda$  when  $\varepsilon \rightarrow 0$  and hence

$$\left( \exp \int_{\varepsilon/\lambda}^\infty t^{-1} e^{-t\lambda} dt \right) h_\varepsilon(\lambda) \rightarrow \lambda$$

when  $\varepsilon \rightarrow 0$ , which is equivalent to  $h_\varepsilon(1)^{-1} h_\varepsilon(\lambda) \rightarrow \lambda$ .  $C^\infty(E)$  is a common core for a positive operator  $A$  and  $h_\varepsilon(A)$  and we have by the spectral theorem that  $K_\varepsilon h_\varepsilon(A)$  converges pointwise to  $A$  on this common core with  $K_\varepsilon = (h_\varepsilon(1))^{-1}$ .

The proof of part 2 essentially follows the lines of Prop. 6.7, p. 74 in [12]. We shall set the following notations

$$\Gamma_1^\varepsilon \equiv \begin{bmatrix} \mathbf{1} & 0 \\ C(1 + \varepsilon C^*C)^{-k} & \mathbf{1} \end{bmatrix},$$

$$\Gamma_2^\varepsilon \equiv \begin{bmatrix} K_\varepsilon h_\varepsilon(A^*A) & 0 \\ 0 & \mathbf{1} \end{bmatrix}$$

for  $\varepsilon > 0$ . The space  $L$  of finite linear combinations  $\sum_{k=0}^N a_k h_k$ ,  $N \in \mathbb{N}$ ,  $a_k \in \mathbb{Q}$ , where

$h_k, k \in \mathbb{N}$  is a fixed orthonormal basis in  $L^2(E)$  such that  $h_k \in C^\infty(E)$ , yields a common core for the operators  $\Gamma_i^\varepsilon$  and their adjoints since it is dense in  $C^\infty(E)$  and the operators are bounded.  $L$  is a countable space and we shall denote a generic element by  $l_i$ . The following inequalities hold by easy pointwise convergence considerations.

Setting  $\varepsilon(p) = \frac{1}{p}$ , from the pointwise convergence of  $\Gamma_1^{\varepsilon^*}$  to  $\Gamma_1^*$  when  $\varepsilon \rightarrow 0$  on the core, we have (denoting by  $\|\cdot\|$  the norm with respect to the  $L^2$  structure):

$$\forall N > 0, \forall i \in \mathbb{N}, \exists p_{i,N} > 0, \forall p > p_{i,N}$$

$$\|(\Gamma_1^{\varepsilon_1(p)})^* \Gamma_2 \Gamma_1 l_i - \Gamma_1^* \Gamma_2 \Gamma_1 l_i\| \leq \frac{1}{3N}$$

Furthermore, from the pointwise convergence of  $\Gamma_2^\varepsilon$  to  $\Gamma_2$  when  $\varepsilon \rightarrow 0$  on the core,

we can write:

$$\begin{aligned} \exists \varepsilon_2(p_{i,N}) > 0, \forall \varepsilon_2(p) \leq \varepsilon_2(p_{i,N}) \\ \| \Gamma_2^{\varepsilon_2(p)} \Gamma_1 l_i - \Gamma_2 \Gamma_1 l_i \| \leq \frac{1}{3N} \| (\Gamma_1^{\varepsilon_1(p)})^* \|^{-1}. \end{aligned}$$

Finally, the pointwise convergence of  $\Gamma_1^\varepsilon$  to  $\Gamma_1$  when  $\varepsilon \rightarrow 0$  on the core gives:

$$\begin{aligned} \exists \varepsilon_3(p_{i,N}) > 0, \forall \varepsilon_3(p) \leq \varepsilon_3(p_{i,N}) \\ \| \Gamma_1^{\varepsilon_3(p)} l_i - \Gamma_1 l_i \| \leq \frac{1}{3N} (\| (\Gamma_1^{\varepsilon_1(p)})^* \| \| \Gamma_2^{\varepsilon_2(p)} \|)^{-1}. \end{aligned}$$

Combining these three inequalities, the diagonal principle yields the following assertion:

$$\begin{aligned} \forall N > 0, \forall i \in \mathbb{N}, \exists n_{i,N} > 0, \forall n > n_{i,N} \\ \| (T^* T)_{\bar{\varepsilon}(n)} l_i - T^* T l_i \| \leq \frac{1}{N} \end{aligned}$$

using the formula

$$(T^* T)_{\bar{\varepsilon}} = \Gamma_1^{\varepsilon_1^*} \Gamma_2^{\varepsilon_2} \Gamma_1^{\varepsilon_3}.$$

This ends the proof of the proposition. ■

### III. The Regularised Faddeev–Popov Determinant

The aim of this section is to give meaning to the Faddeev–Popov determinant “ $\det F_p$ .” Recall that  $F_p = D_p + R_p$  with  $D_p$  and  $R_p$  given by (1.11) and (1.12). Since by definition of a Faddeev–Popov operator,  $\tau_p$  is invertible, so is  $D_p$  and we can write:

$$F_p = D_p(1 + D_p^{-1} R_p) \tag{3.1}$$

so that  $F_p^* F_p = (1 + D_p^{-1} R_p)^* D_p^* D_p (1 + D_p^{-1} R_p)$ .

Recall that  $D_p = \begin{bmatrix} \tau_p & 0 \\ 0 & \mathbf{1} \end{bmatrix}$  where  $\tau_p$  is a differential operator of the form (1.9). For  $\varepsilon > 0$ , we define the  $\varepsilon$ -cut-off Faddeev–Popov operator:

$$(F_p^* F_p)_\varepsilon \equiv (\mathbf{1} + D_p^{-1} R_p)^* (D_p^* D_p)_\varepsilon (\mathbf{1} + D_p^{-1} R_p), \tag{3.2}$$

where

$$(D_p^* D_p)_\varepsilon \equiv \begin{bmatrix} (\tau_p^* \tau_p)_\varepsilon & 0 \\ 0 & \mathbf{1} \end{bmatrix}$$

and with  $(\tau_p^* \tau_p)_\varepsilon$  defined as in Proposition 2.1 with  $T \equiv \tau_p$ ,  $A \equiv A_p$ ,  $B \equiv B_p = \mathbf{1}$ ,  $C \equiv C_p$ , defined in (1.9.a–c). This operator is also a determinant class operator as is shown in the following proposition:

**Proposition 3.1.** For  $\varepsilon > 0$ , the cut-off Faddeev–Popov operator  $(F_p^* F_p)_\varepsilon$  is an operator

of the form “ $\mathbf{1} + \text{trace class}$ ” and we can define the  $\varepsilon$ -cut-off Faddeev–Popov determinant:

$$\det_\varepsilon F_p \equiv (\det(F_p^* F_p)_\varepsilon)^{1/2}.$$

Then

$$\det_\varepsilon F_p = \det(\mathbf{1} - \pi_p) \det(\tau_p^* \tau_p)_\varepsilon^{1/2} = \det(\mathbf{1} - \pi_p) \det_\varepsilon(A_p),$$

where

$$\det(\mathbf{1} - \pi_p) = \frac{\det(\langle \Psi_p^i, \chi_p^j \rangle_p)}{\det(\langle \Psi_p^i, \Psi_p^j \rangle_p)^{1/2}}:$$

$\Psi_p^i, i = 1, \dots, \dim P/G$ , is a basis (non-necessarily orthonormal) of  $\text{Ker } \tau_p^*$  and  $\chi_p^i, i = 1, \dots, \dim P/G$  is an orthonormal basis of  $W_p$ .

*Remark.* Since the divergences when  $\varepsilon \rightarrow 0$  of the  $\varepsilon$ -cut-off Faddeev–Popov determinant  $\det_\varepsilon(F_p)$  coincide with that of the  $\varepsilon$ -cut-off determinant  $\det_\varepsilon(A_p)$ , after removing the divergences, we have an equality of regularised determinants (see (2.5 bis)):

$$\det F_p = \det(\mathbf{1} - \pi_p) \det(A_p).$$

*Proof of Proposition 3.1.* As was pointed out earlier on under the assumption Hyp. 2,  $R_p$  being of finite rank is trace class, so both  $(\mathbf{1} + D_p^{-1} R_p)$  and  $(\mathbf{1} + D_p^{-1} R_p)^*$  are of the form “ $\mathbf{1} + \text{a traceclass operator}$ .” Thus the product  $(F_p^* F_p)_\varepsilon$  is also “ $\mathbf{1} + \text{traceclass}$ ” and has a well defined determinant which coincides with the product of the determinants of the three operators. By Proposition 2.1, we have  $\det(D_p^* D_p)_\varepsilon = \det_\varepsilon(A_p^* A_p)$ . On the other hand, by (1.11) and (1.12), we have

$$D_p^{-1} R_p = \begin{bmatrix} 0 & \tau_p^{-1} \pi_p \\ 0 & -\pi_p \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\pi_p \end{bmatrix} + \begin{bmatrix} 0 & \tau_p^{-1} \pi_p \\ 0 & 0 \end{bmatrix}.$$

But the last matrix in this sum is nilpotent and we have

$$\det(\mathbf{1} + D_p^{-1} R_p) = \det \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} - \pi_p \end{bmatrix} = \det((\mathbf{1} - \pi_p)/W_p).$$

Since  $\mathbf{1} - \pi_p$  is the orthogonal projection of  $W_p$  onto  $\text{Ker } \tau_p^*$ , the matrix  $\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} - \pi_p \end{bmatrix}$

can be written  $\begin{bmatrix} \mathbf{1} & 0 \\ 0 & (\langle \tilde{\Psi}_p^i, \chi_p^j \rangle_{ij}) \end{bmatrix}$ , where  $\tilde{\Psi}_p^i, i = 1, \dots, \dim P/G$  is an orthonormal basis of  $\text{Ker } \tau_p^*$  and  $\chi_p^i, i = 1, \dots, \dim P/G$  is an orthonormal basis of  $W_p$ . If we now express this matrix in terms of a non-necessarily orthonormal basis  $\Psi_p^i, i = 1, \dots, p$  of  $\text{Ker } \tau_p^*$ , and compute its determinant, we find the result announced in the proposition. A similar computation holds then for the adjoint expression, and we finally find

$$\det(F_p^* F_p)_\varepsilon = \det(\mathbf{1} - \pi_p)^2 \det(\tau_p^* \tau_p)_\varepsilon$$

with  $\det(\mathbf{1} - \pi_p)$  as in the proposition, which ends the proof since by Prop. 2.1 since we have  $\det(\tau_p^* \tau_p)_\varepsilon = \det h_\varepsilon(A_p^* A_p) = \det_\varepsilon(A_p)^2$ . ■

*Remark.* If we had  $\tau_p = \begin{bmatrix} A_p & 0 \\ C_p & B_p \end{bmatrix}$  with  $B_p^* B_p$  elliptic, then the  $\varepsilon$ -cut-off Faddeev–

Popov determinant would read:

$$\det_{\varepsilon}(F_p) = \det(\mathbf{1} - \pi_p)\det_{\varepsilon}(A_p)\det_{\varepsilon}(B_p).$$

#### IV. Application to Bosonic String Theory

In this section, we apply the framework set up above to the context of (closed) bosonic string theory. We shall use classical results in string theory and refer the reader to [12, 13] for further details. We first define the various objects introduced in the previous sections in this context.

Let  $A$  be a  $C^\infty$  real connected compact surface of genus  $p > 1$  (so that we take here  $k = \dim(A) = 2$ ) and let  $M(A)$  be the manifold of  $C^\infty$  Riemannian metrics on  $A$ . We shall denote by  $D_0(A)$  the group of  $C^\infty$  diffeomorphisms of  $A$  which are homotopic to identity. It coincides with the connected component of identity in the group  $D(A)$  of diffeomorphisms of  $A$  and acts smoothly on  $M(A)$  by pull-back. Let  $W(A)$  be the Weyl group  $\{e^\phi, \phi \in C^\infty(A, \mathbb{R})\}$ . It acts smoothly on  $M(A)$  by point-wise multiplication.

Let us define the bundle  $P$  of Sect. I. We first notice that setting  $G'' = W(A)$ ,  $G' = D_0(A)$ , the group  $G'$  acts on  $G''$  as follows:

$$\begin{aligned} W(A) &\rightarrow W(A) \\ \lambda &\rightarrow \lambda^f \equiv \lambda \circ f \end{aligned}$$

and the action obviously satisfies the relation:

$$(\lambda^{f_1})^{f_2} = \lambda^{f_1 \circ f_2}.$$

Hence we can define the semi-direct product  $G \equiv D_0(A) \rtimes W(A)$  with the product law:

$$[f_1, \lambda_1] \cdot [f_2, \lambda_2] = [f_1 \circ f_2, \lambda_1^{f_2} \lambda_2].$$

Notice that the group  $G$  has the structure of a  $C^\infty$  Fréchet manifold with tangent space  $C^\infty(TA) \times C^\infty(A, \mathbb{R})$ .

We now set

$$P: M(A) \rightarrow M(A)/D_0(A) \rtimes W(A). \tag{4.1}$$

$P$  indeed defines a  $C^\infty$  principal trivial fibre bundle (see e.g. [13] and references therein) with a  $C^\infty$  global section which we shall denote by  $\sigma$ .

Let us denote by  $\mathcal{F}(A)$  the Teichmüller space, i.e. the quotient space  $M(A)/W(A)/D_0(A)$ . It is a real smooth manifold of dimension  $6p - 6$ . Hence Hyp. 2 of Sect. 1 is fulfilled.

There is a global  $C^\infty$  cross-section:

$$\begin{aligned} \sigma: \mathcal{F}(A) &\rightarrow M(A) \\ t &\rightarrow g_t \end{aligned}$$

and any metric in  $M(A)$  can be written in a unique way as  $g = f^* e^\phi g_t$ ,  $f \in D_0(A)$ ,  $\phi \in C^\infty(A, \mathbb{R})$ ,  $t \in \mathcal{F}$ . For more details concerning this point, we refer the reader to [12] and [13].

Let us now introduce the space  $W_p$  of Sect. I. We have  $\Sigma = \{g_t, t \in \mathcal{T}(\Lambda)\}$ . For a metric  $g = f^*e^\phi g_t \in M(\Lambda)$ , we therefore set

$$W_g = T_g(f^*e^\phi \Sigma), \tag{4.2}$$

where  $C^\infty(S^2T^*)$  is the space of  $C^\infty$  sections of the bundle  $S^2T^*$  of symmetric 2 covariant tensors over  $\Lambda$ -notice that this space is just the tangent space to the  $C^\infty$  Frechet manifold  $M(\Lambda)$ .

We now define for a point  $p$  in the bundle  $P$  (i.e. here for a metric  $g$  in the space  $M(\Lambda)$ ), the map  $\tau_p$  of Sect. I. Since diffeomorphisms act by pull-back and since the tangent space to  $D_0(\Lambda)$  is the space of  $C^\infty$  sections  $C^\infty(T\Lambda)$  of the tangent bundle  $T\Lambda$  over  $\Lambda$ , it is easy to see that the tangent map to the pull-back map is given at a point  $g \in M(\Lambda)$  by:

$$\begin{aligned} \tau_g: C^\infty(T\Lambda) \times C^\infty(\Lambda, \mathbb{R}) &\rightarrow C^\infty(S^2T^*) \\ (u, \lambda) &\rightarrow \nabla_g u + \lambda g. \end{aligned} \tag{4.3}$$

Here  $\nabla_g$  denotes the Lie derivative with respect to the metric  $g$ ,  $(\nabla_g u)_{ab} = \nabla_a u_b + \nabla_b u_a$ . Let us equip  $T_g M(\Lambda)$  with a scalar product:

$$\langle h, k \rangle_g \equiv \int_{\Lambda} d\xi \sqrt{g(\xi)} [G^{abcd}(\xi) + K g^{ab}(\xi) g^{cd}(\xi)] h_{ab}(\xi) k_{cd}(\xi) d\xi, \tag{4.4}$$

where  $K$  is a positive constant, (which we set equal to  $\frac{1}{4}$ )

$$G_{ab}^{cd} \equiv \frac{1}{2}(\delta_a^c \delta_b^d + \delta_b^c \delta_a^d - g_{ab} g^{cd})$$

and

$$G^{abcd} = g^{ae} g^{bf} G_{ef}^{cd}.$$

When restricted to tensors of the form  $h = \lambda g$  (pure trace tensors), this scalar product coincides with the scalar product on functions:

$$(\lambda, \mu)_g \equiv \int_{\Lambda} \sqrt{g(\xi)} \lambda(\xi) \mu(\xi) d\xi$$

so that the embedding:

$$\begin{aligned} i_g: C^\infty(\Lambda, \mathbb{R}) &\rightarrow T_g M(\Lambda) \\ \lambda &\rightarrow \lambda g \end{aligned}$$

is an isometry. As we did in the general setting, we shall henceforth identify  $C^\infty(\Lambda, \mathbb{R})$  with  $C^\infty(\Lambda, \mathbb{R})_g$ . Seen as a matrix operator as in (1.9), the operator  $\tau_g$  thus reads:

$$\tau_g = \begin{bmatrix} A_g & 0 \\ C_g & \mathbf{1} \end{bmatrix}$$

with

$$C_g = \tau_g - A_g,$$

and where  $A_g$  is defined as follows.

Let us first note that the natural scalar product  $\langle \cdot, \cdot \rangle_g$  for two tensors defined on  $T_g M(\Lambda)$  is not invariant under the whole gauge group  $G$  but only under the group  $D_0(\Lambda)$  of  $C^\infty$  diffeomorphisms homotopic to identity and that there is a

natural orthogonal splitting of  $T_g M(\Lambda)$  for the scalar product  $\langle \cdot, \cdot \rangle_g$  into traceless two tensors and pure trace tensors, namely:

$$T_g M(\Lambda) = C^\infty(S_0^2 T^*) \oplus C^\infty(\Lambda, \mathbb{R})g,$$

where  $C^\infty(S_0^2 T^*)$  is the space of smooth sections of the bundle of two symmetric traceless covariant tensors (the trace is taken with respect to  $g$ ). Since the image by  $\tau_g$  of the tangent space to the Weyl group  $W(\Lambda)$  is  $C^\infty(\Lambda, \mathbb{R})g$ , the operator  $A_g$  is obtained by projecting  $\tau_g$  onto the traceless tensors. We hence define (adopting the usual notations namely setting  $A_g \equiv P_g$ )

$$P_g : C^\infty(T\Lambda) \rightarrow C^\infty(S^2 T^*)$$

$$u \rightarrow \nabla_g u - \frac{1}{2} \text{tr}_g(\nabla_g u)g. \tag{4.5}$$

In matrix representation, the operator  $\tau_g$  reads:

$$\tau_g = \begin{bmatrix} P_g & 0 \\ \frac{1}{2} \text{tr}_g(\nabla_g(\cdot))g & \mathbf{1} \end{bmatrix}. \tag{4.6}$$

The operator  $P_g$  being injective for genus  $p > 1$ , so is the operator  $\tau_g$ . We can define for each  $g \in M(\Lambda)$ , the Faddeev–Popov operator  $F_g$  and write it  $F_g = D_g + R_g$  according to (1.10), (1.11) and (1.12) so that  $D_g = \begin{bmatrix} \tau_g & 0 \\ 0 & \mathbf{1} \end{bmatrix}$  and  $R_g = \begin{bmatrix} 0 & \pi_g \\ 0 & -\pi_g \end{bmatrix}$  where  $\pi_g$  is the orthogonal projection onto  $\text{Im } \tau_g$ .

Let us now check the remaining assumptions made in Sect. I for this data. Hyp. 1 is fulfilled since  $T_e G = C^\infty(T\Lambda \times (\Lambda \times \mathbb{R}))$  and  $\tau_g$  is a differential operator of order larger or equal to 1 since the metric  $g$  is  $C^\infty$ . Moreover it has injective symbol [14]. In particular, the operator  $P_g$  has injective symbol and  $P_g^* P_g$  is an elliptic operator on the compact surface  $\Lambda$  (see also [12, 13]). We pointed out above that Hyp. 2 is satisfied, since  $\dim M(\Lambda)/D_0(\Lambda) \times W(\Lambda) = 6p - 6$ , where  $p$  is the genus of  $\Lambda$ .

Hence we can apply the results of Propositions 2.1 and 3.1 to compute the  $\varepsilon$  regularised Faddeev–Popov determinant. We set  $T = \tau_g$ ,  $A = P_g$ ,  $C = \frac{1}{2} \text{tr}_g(\nabla_g(\cdot))g$  and define the  $\varepsilon$ -regularised operator  $(\tau_g^* \tau_g)_\varepsilon$  according to Proposition 2.1. As in (3.2) we can then define the  $\varepsilon$  cut-off Faddeev–Popov operator  $(F_g^* F_g)_\varepsilon$  and the  $\varepsilon$  cut-off Faddeev–Popov determinant is then given by Proposition 3.1,

$$\det_\varepsilon(F_g) = \det(\mathbf{1} - \pi_g) \det(\tau_g^* \tau_g)_\varepsilon^{1/2} = \det(\mathbf{1} - \pi_g) \det_\varepsilon(P_g),$$

where  $\mathbf{1} - \pi_g$  is as before the orthogonal projection from  $W_g$  onto  $\text{Ker } \tau_g^*$ . By (4.6), we see that the adjoint operator  $\tau_g^*$  is of the form  $\begin{bmatrix} P_g^* & x \\ 0 & \mathbf{1} \end{bmatrix}$ , so that

$$\text{Ker } \tau_g^* = \text{Ker } P_g^* \times \{0\} \cong \text{Ker } P_g^*$$

and  $\det(\mathbf{1} - \pi_g)$  coincides with this same determinant where now  $\pi_g$  is the orthogonal projection of  $W_g$  onto  $\text{Ker } P_g^*$ . Hence we find the usual expression of the Faddeev–Popov determinant for bosonic strings namely:

$$\det_\varepsilon(F_g) = \frac{\det(\langle \Psi_g^i, \chi_g^j \rangle_g)}{\det(\langle \Psi_g^i, \Psi_g^j \rangle_g)^{1/2}} (\det_\varepsilon(P_g^* P_g))^{1/2}. \tag{4.7}$$

$\Psi_g^i, i = 1, \dots, 6p - 6$  is a (non necessarily orthonormal) basis of  $\text{Ker } P_g^*$  and  $\chi_g^i, i = 1, \dots, 6p - 6$  is an orthonormal basis of  $W_g$ .

After removing the divergences when taking the limit  $\varepsilon \rightarrow 0$ , we have:

$$\det(F_g) = \frac{\det(\langle \Psi_g^i, \chi_g^j \rangle_g)}{\det(\langle \Psi_g^i, \Psi_g^j \rangle_g)^{1/2}} \det(P_g).$$

Let us now apply these results to define a renormalised ‘‘Polyakov path integral’’ on the space of configurations of a closed bosonic string evolving in space-time.

We consider as before a smooth compact boundaryless surface  $\Lambda$  embedded in  $\mathbb{R}^d$  modelling the evolution of a closed bosonic string in space time  $\mathbb{R}^d$ . The space of configurations is  $C^\infty(\Lambda, \mathbb{R}^d) \times M(\Lambda)$  and the quantification of the string model amounts to defining functional integrals of the form

$$“ \int_{C^\infty(\Lambda, \mathbb{R}^d) \times M(\Lambda)} f(x, g) e^{-(1/2)A(x, g)} dx dg ”$$

where  $dx dg$  is a formal Lebesgue integration on the product space  $C^\infty(\Lambda, \mathbb{R}^d) \times M(\Lambda)$ ,  $f(x, g)$  is a functional on this same product space, the space of configurations and  $A(x, g) = \int \sqrt{\det g} g^{ab} \partial_a x^\mu \partial_b x^\mu$  is the classical action corresponding to a metric  $g = g_{ab} d\eta_a^\Lambda d\eta_b^\Lambda$  on  $\Lambda$ , the determinant of which is denoted by  $\det g$  and the inverse matrix of which is written  $(g^{ab})$ .

After a gaussian integration in the variable  $x$  (corresponding to the embedding of the surface  $\Lambda$  in  $\mathbb{R}^d$ ), this boils down to defining formal integrals of the form

$$“ \int_{M(\Lambda)} H(g) \left( \frac{\det(-\Delta_g)}{\int_\Lambda \sqrt{\det g(\eta)}} \right)^{-d/2} dg ” = “ \int_{M(\Lambda)} h(g) dg, ” \tag{4.8}$$

where  $H$  is a functional on  $M(\Lambda)$ ,  $\Delta_g = \frac{1}{\sqrt{\det g}} \partial_a \sqrt{\det g} g^{ab} \partial_b$  is the Laplace Beltrami operator,  $\det(-\Delta_g)$  its formal determinant and

$$\Downarrow h(g) = H(g) \left( \frac{\det(-\Delta_g)}{\int_\Lambda \sqrt{\det g(\eta)}} \right)^{-d/2} \Downarrow .$$

Using the results of the preceding sections, we introduce a renormalised ‘‘Polyakov path integral’’ for closed bosonic strings which is the one implicitly referred to in the physics literature on functional integration of strings.

Since  $-\Delta_g$  is a positive self adjoint elliptic operator on the  $L^2$  closure  $L^2(\Lambda, \mathbb{R}^d)$  of  $C^\infty(\Lambda, \mathbb{R}^d)$  (with respect to the scalar product  $\langle \cdot, \cdot \rangle_g$  induced on scalar functions on  $\Lambda$  by the metric  $g$ ), we can define for  $\varepsilon > 0$ , the  $\varepsilon$ -cut-off determinant  $\det'_\varepsilon(-\Delta_g)^{-d/2}$  as in (2.4). It is by now a classical result (see references in [2]) that for dimension  $d = 26$  (critical dimension), and  $g = f^* e^\phi g_t$  (as described above), the expression

$$\varepsilon^{\alpha(\Lambda)} \exp \left\{ -\mu_\varepsilon \int_\Lambda \sqrt{\det g(\eta)} d\eta \right\} \left( \frac{\det'_\varepsilon(-\Delta_g)}{\int_\Lambda \sqrt{\det g(\eta)} d\eta} \right)^{-d/2} \det_\varepsilon(F_g)$$

is equivalent to a gauge invariant function  $\gamma_\varepsilon(g)$  when  $\varepsilon \rightarrow 0$  (and hence to a function



$\gamma_\varepsilon(t)$  of  $t \in \mathcal{T}(\Lambda)$ ). Here  $\exp\{-\mu_\varepsilon \int \sqrt{\det g(\eta)} d\eta\}$  is the cosmological constant term,  $\mu_\varepsilon$  an  $\varepsilon$  dependent positive constant tending to  $+\infty$  when  $\varepsilon \rightarrow 0$ ,  $\alpha(\Lambda)$  is a topological constant expressed in terms of the Euler characteristic  $\chi(\Lambda)$ .

Let now  $H(g)$ ,  $g = f^* e^\phi g_t$  be a gauge invariant functional on  $M(\Lambda)$ , which we can see as a function  $\bar{H}(t) = H(g_t)$  on  $\mathcal{T}(\Lambda)$ . Setting

$$h_\varepsilon(g) \equiv H(g) \varepsilon^{\alpha(\Lambda)} \exp\left\{-\mu_\varepsilon \int_\Lambda \sqrt{\det g(\eta)} d\eta\right\} \left(\frac{\det'_\varepsilon(-\Delta_g)}{\int_\Lambda \sqrt{\det g(\eta)} d\eta}\right)^{-d/2},$$

we see that  $h_\varepsilon(g) \det_\varepsilon(F_g)$  is equivalent when  $\varepsilon \rightarrow 0$  to a function  $\rho_\varepsilon(t) \equiv \bar{H}(t) \gamma_\varepsilon(t)$  on  $\mathcal{T}(\Lambda)$ . If  $\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{T}(\Lambda)} \rho_\varepsilon(t) dt$  exists, then the renormalised integral

$$\langle H \rangle_{\text{ren}} \equiv \left[ \int_{M(\Lambda)} h(g) dg \right]_{\text{ren}} \equiv \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{T}(\Lambda)} \rho_\varepsilon(t) dt$$

is finite and we have

$$\begin{aligned} \langle H \rangle_{\text{ren}} &= \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{T}(\Lambda)} \bar{H}(t) \gamma_\varepsilon(t) dt \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{T}(\Lambda)} \bar{H}(t) \varepsilon^{\alpha(\Lambda)} \exp\left\{-\mu_\varepsilon \int_\Lambda \sqrt{\det g(\eta)} d\eta\right\} \left(\frac{\det'_\varepsilon(-\Delta_g)}{\int_\Lambda \sqrt{\det g(\eta)} d\eta}\right)^{-d/2} \det_\varepsilon(F_g) dt, \end{aligned} \tag{4.9}$$

thus giving a meaning to the formal path integral (4.8).

### V. Summary

We have given a careful treatment of the Faddeev–Popov procedure applied to the functional quantisation of gauge theories with finite dimensional moduli space like string theory and shown how far the finite dimensional treatment of the procedure can be generalised to infinite dimensions. After having defined a notion of the Faddeev–Popov operator independent of measure theoretic considerations in terms of the geometrical data for the gauge theory, we extended the heat-kernel regularisation methods to classes of operator matrices to which belongs the Faddeev–Popov operator in order to define the Faddeev–Popov determinant. We prove this infinite dimensional determinant is canonically determined by the geometrical data – a fibre bundle  $P \rightarrow P/G$ , where  $P$  is equipped with a metric – defining the theory and a choice of a slice. This then enables us to define a priori a renormalised path integral in terms of a finite dimensional integral on moduli space.

*Acknowledgements.* I am very grateful to S. Albeverio for numerous invitations to the Ruhr University in Bochum where a great part of this paper was written and for his very helpful comments and to S. Scarlatti for many fruitful discussions related to this topic and for his constant encouragements. I am also indebted to Z. Jaskolski who kindly invited me to the I.C.T.P in Trieste, and I would like to thank him for important remarks he made to help me clarify some aspects of this paper.

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Communicated by H. Araki