

Level 1 WZW Superselection Sectors

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Received August 26, 1991

Abstract. The superselection structure of the Wess–Zumino–Witten theory based on the affine Lie algebra $\widehat{\mathfrak{so}}(N)$ at level one is investigated for arbitrary N . By making use of the free fermion representation of the affine algebra, the endomorphisms which represent the superselection sectors on the observable algebra can be constructed as endomorphisms of the underlying Majorana algebra. These endomorphisms do not close on the chiral algebra of the theory, but we are able to obtain a larger algebra on which the endomorphisms close. The composition of equivalence classes of the endomorphisms reproduces the WZW fusion rules.

1. Introduction

In quantum physics there has been a continuous effort to define the theory from first principles – the axiomatic approach. One of the main developments in this respect has been algebraic quantum field theory [1, 2] (for further references see [3]).

For many physicists a major drawback of the axiomatic approach to quantum field theory has been the scarcity of interesting examples. On the other hand two-dimensional conformal field theory (conformal field theory, for short) has produced a wealth of non-perturbatively solvable field theories. In fact rational conformal field theory has reached such a stage of maturity that there are attempts at a classification of all rational conformal field theories. Thus it is very natural to try to incorporate some of the conformal field theory examples into the framework of algebraic field theory. So far this has been done only for the simplest rational conformal field theory – that of the Ising model – by Mack and Schomerus [4].

Central to the description of conformal field theory is the infinite-dimensional symmetry algebra of the theory. Since conformal field theories are solvable theories it is no surprise that a large part of the information about the theory is contained in the symmetry algebra – the kinematics. Another specific feature of conformal field theory is holomorphic factorization. As a consequence one is led to consider a one-

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dimensional theory (the left or right chiral part) specified by the chiral algebra – the chiral half of the symmetry algebra. A careful analysis [5] shows that in fact the conformal field theories are living on a tube, and owing to the periodicity in the space direction the chiral parts live also on circles S^1 . To each highest weight module associated to a positive energy representation of the chiral algebra (the number of which is by definition finite for rational conformal field theories) corresponds a primary field which, when acting on the vacuum, produces the highest weight vector of the module. An important element of the theory are the chiral vertex operators. A chiral vertex operator at the point z can be thought of as an intertwiner of the representations provided by two primary fields at zero and at z and a third one at infinity. The data of the chiral vertex operators are equivalent to the three-point functions of the primaries, or to the knowledge of the operator product coefficients. The operator product coefficients are essentially the structure constants of the chiral part of the field algebra, which is obtained by adjoining the primary fields to the chiral algebra. Thus the knowledge of the representations of the chiral algebra and the operator product coefficients determines the theory completely. Somewhat coarser information is provided by the fusion rules. The fusion rule coefficients $N_{\pi_1 \pi_2}^{\pi_3}$ give the number of chiral vertex operators intertwining the representations $\pi_1 \times \pi_2$ and π_3 . The analogy with simple Lie algebras is that the chiral vertex operators correspond to the Clebsch–Gordan coefficients while the fusion rule coefficients correspond to the tensor product multiplicities. Even though the operators comprising the chiral algebra are mutually local (have trivial monodromy), in general the primary fields have non-trivial mutual monodromy and hence give rise to representations of the braid group.

Many of the structures encountered in conformal field theory are not due specifically to the conformal invariance, but are general to any two-dimensional quantum theory. In particular the appearance of braid group statistics is characteristic of any two-dimensional theory [6, 7]. Therefore trying to incorporate the different conformal field theory models (most important of all the Wess–Zumino–Witten (WZW) [8] models, believed to be the building blocks of rational conformal field theory) in algebraic field theory not only puts conformal field theory into a more general framework but also provides guidelines for the general theory.

In algebraic field theory the central role is played by the observable algebra [2, 6, 9]. For every double cone \mathcal{O} of the underlying Minkowski space-time there is a unital C^* -algebra $\mathcal{A}(\mathcal{O})$ of observables localized in this cone. The quasilocal algebra \mathcal{A} of observables is the completion of the union of the local algebras. The causal structure of space-time is implemented by the requirement that elements belonging to local algebras $\mathcal{A}(\mathcal{O}_1)$, $\mathcal{A}(\mathcal{O}_2)$ should commute if \mathcal{O}_1 and \mathcal{O}_2 are relatively space-like. The physical Hilbert space \mathcal{H} breaks up into a direct sum $\mathcal{H} = \bigoplus \mathcal{H}_i$ of orthogonal Hilbert spaces \mathcal{H}_i – the superselection sectors, carrying irreducible representations π_i of \mathcal{A} .

The guiding principle of algebraic field theory is that all the information be contained in the vacuum representation of the observable algebra. In particular, provided that $\pi_0(\mathcal{A})$ satisfies Haag duality, all the asymptotically vacuum-like representations π_i of \mathcal{A} are obtainable (up to unitary equivalence) by composing the vacuum representation π_0 with transportable, localized endomorphisms ρ_i of the observable algebra,

$$\pi_i \cong \pi_0 \circ \rho_i. \quad (1.1)$$

An endomorphism ρ is localized in \mathcal{O} iff ρ is the identity on $\mathcal{A}(\mathcal{O}^c)$, where \mathcal{O}^c is the causal complement of \mathcal{O} . It is transportable iff for any point x of Minkowski space-time there exists an endomorphism $\tilde{\rho}$ localized in $\tilde{\mathcal{O}} = \mathcal{O} + x$ which is equivalent to ρ . Two endomorphisms ρ and $\tilde{\rho}$ are defined to be (unitarily) equivalent if there exists a unitary U belonging to some local observable algebra such that

$$\tilde{\rho}(a) = U\rho(a)U^* \equiv \sigma_U(\rho(a)) \quad (1.2)$$

for all $a \in \mathcal{A}$. (In general, for any unitary $U \in \mathcal{A}$, one defines the corresponding inner automorphism σ_U of \mathcal{A} as

$$\sigma_U(a) = U \cdot a \cdot U^* \quad (1.3)$$

for all $a \in \mathcal{A}$.)

Endomorphisms in the same equivalence class $[\rho]$ lead to equivalent representations of the observable algebra. By locality we have $\pi_0(\mathcal{A}(\mathcal{O}^c)) \subseteq \pi_0(\mathcal{A}(\mathcal{O}))'$, where the prime denotes the commutant. The important assumption above, Haag duality, is the assertion that in fact this is an equality

$$\pi_0(\mathcal{A}(\mathcal{O}^c)) = \pi_0(\mathcal{A}(\mathcal{O}))'. \quad (1.4)$$

This means that, in a certain sense, the observable algebra \mathcal{A} is maximal. If Haag duality is satisfied, one usually identifies $\mathcal{A}(\mathcal{O})$ with its vacuum representation $\pi_0(\mathcal{A}(\mathcal{O}))$.

The field algebra \mathcal{F} is obtained from \mathcal{A} by adjoining charged fields f to \mathcal{A} which realize the endomorphisms, i.e., $af = f\rho(a)$ for all $a \in \mathcal{A}$.

When one is looking at a conformal field theory from the point of view of superselection sectors, one must identify the various objects appearing in the conformal field theory with corresponding objects of the algebraic theory. In general these identifications have not yet been proven rigorously, but it is intuitively clear that the following “translation” of notions takes place. The chiral algebra plays the role of the observable algebra. Indeed, the fields contained in the chiral algebra are mutually local and moreover, as is usual in quantum theory, the generators of the symmetries are observable. The irreducible representations of the chiral algebra should be considered as the superselection sectors. Thus the primary fields correspond to the “charged” fields implementing the corresponding endomorphisms of the observable algebra. The fusion of two primaries corresponds to the composition of the respective transportable endomorphisms. If in the composition we are restricting ourselves only to equivalence classes of endomorphisms, then we get the fusion rules:

$$[\rho_{i_1} \circ \rho_{i_2}] = \sum_{i_3} N_{i_1 i_2}^{i_3} [\rho_{i_3}]. \quad (1.5)$$

When incorporating a conformal field theory model in the framework of algebraic field theory, one would like to use the knowledge one has about the positive energy representations of the chiral algebra. This means in particular that we will relax the canonical framework [2, 6, 9] and allow the observable algebra to contain unbounded operators – the generators of the chiral algebra. Another difference is that one allows global elements in \mathcal{A} ; this is because our “space-time” has the topology of the circle (for example, the central element Y which implements the translation by 2π is in \mathcal{A}). This means that the observable algebra cannot be identified with its

vacuum representation π_0 , since π_0 is not faithful. Having relaxed the quasilocality of \mathcal{A} it is natural to look for non-localized endomorphisms of the chiral algebra that lead to its inequivalent representations. One finds that in order to obtain these endomorphisms, it is necessary to extend the chiral algebra. This should in fact be expected because one does not have Haag duality for the chiral algebra. Let us give a pedestrian explanation of why e.g. the Virasoro algebra Vir has to be enlarged to obtain the observable algebra. Consider the algebra $\text{Vir}(I)$ of diffeomorphisms that are localized in an interval $I \subset S^1$, i.e., outside of this interval they act as the identity map. The double commutant $\text{Vir}(I)''$ of $\text{Vir}(I)$ in the space $\mathcal{B}(\mathcal{H}_0)$ of bounded operators in \mathcal{H}_0 can serve as a local observable algebra and satisfies Haag duality [10]. The algebra $\text{Vir}(I)''$ certainly contains not only the diffeomorphisms, but also bounded functions of operators that act by multiplication by functions that are constant on $I^c = S^1 \setminus I$. Therefore a necessary condition for being able to describe the observable algebra $\mathcal{A}(I)$ satisfying Haag duality as bounded functions of a Lie algebra of unbounded observables, is that we extend $\text{Vir}(I)$ by the Lie algebra of functions on the circle. Of course one must check that this extended observable algebra has the same superselection sectors as the chiral algebra we started with. In addition, having obtained the non-localized endomorphisms, one has to show that it is possible to find localized ones giving rise to the same representations.

This program was carried out in [4] for the Ising model. In this model one has three inequivalent physical representations of the chiral algebra (the $c = 1/2$ Virasoro algebra $\text{Vir}_{1/2}$); these are the representations corresponding to the identity, the energy operator, and the spin operator. The key point in [4] was the explicit construction of the non-local endomorphisms that correspond to these three representations.

Technically the endomorphisms were constructed by employing the free fermion realization of the Ising model. It is well known that a free Majorana fermion has an energy-momentum tensor which is bilinear in the fermions and has central charge $1/2$. Thus one is looking for endomorphisms of the Majorana algebra which, when composed with the vacuum representation and restricted to the Virasoro algebra, give the three inequivalent representations of the $c = 1/2$ Virasoro algebra. In fact it turns out that these endomorphisms do not leave the Virasoro algebra invariant; rather, the algebra of observables on which the constructed endomorphisms act is an extension of the algebra of vector fields (centerless Virasoro algebra) and functions on the circle by an infinite orthogonal Lie algebra O_∞ (consisting of finite linear combinations of fermion bilinears) and two central elements [4]. A question which was not investigated in [4] is “*why this algebra of observables.*” In this paper we also do not have much to say about this issue; instead we will try to enlarge the set of examples, hoping that this could provide a clue to the answer.

Since the fermionic realization of the Ising model has been the key to the explicit construction of the endomorphisms, we will consider here the next most simple generalization – WZW models based on level 1 affine orthogonal Lie algebras $\widehat{\mathfrak{so}}(N)_1$. For such models one can represent the Virasoro generators and the Kac-Moody currents in terms of bilinears of N fermions.

The representations of $\widehat{\mathfrak{so}}(N)_1$ are of course known [11]. For N even we have four physical representations – the basic representation, the vector, the spinor and the conjugate spinor; these will be denoted symbolically by $1, v, s,$ and $c,$ respectively. In this symbolic notation, the fusion rules read

$$v * v = s * s = c * c = 1, \quad s * c = v, \quad \text{for } N \in 4\mathbb{Z}, \quad (1.6)$$

respectively

$$v * v = s * c = 1, \quad s * s = c * c = v, \quad \text{for } N \in 4\mathbb{Z} + 2. \quad (1.7)$$

Recalling that primary fields for which the fusion product with any primary field produces exactly one primary field are called simple currents [12–14], we see that for even N all the representations $1, v, s, c$ correspond to primary fields that are simple currents. It is known that the simple currents of a conformal field theory are precisely those primaries which have quantum dimension equal to 1 [14]. In terms of algebraic field theory, the quantum dimension of a primary field corresponds to the statistical dimension of the associated endomorphism of the observable algebra, and it can be shown that the endomorphisms with statistical dimension 1 are precisely the automorphisms [15]. Thus simple currents correspond to endomorphisms which in fact are automorphisms, and hence for even N we will be looking for automorphisms leading to the fusion rules (1.6, 1.7). (Theories where all endomorphisms are automorphisms have also been analysed, in the context of the \hat{u}_1 Kac-Moody algebra, in [16].)

The case of odd N is similar to the Ising model: we have three physical representations – the basic, vector and spinor representations, denoted symbolically by $1, v$, and σ . The fusion rules of the N odd case are the same as the ones of the Ising model,

$$v * v = 1, \quad \sigma * \sigma = 1 + v, \quad \sigma * v = \sigma; \quad (1.8)$$

thus the identity and vector are simple currents, while the spinor is not. This means that we should be looking for an automorphism to represent the vector, and a proper endomorphism to represent the spinor.

With this input from the conformal theory it is possible to construct explicit endomorphisms of the Majorana algebra which, when composed with the vacuum representation and when restricted to the chiral algebra (the semidirect sum of the Virasoro algebra $\text{Vir}_{N/2}$ and $\widehat{\mathfrak{so}}(N)_1$), give the physical representations of the chiral algebra. The Majorana algebra and the chiral algebra and their representations are presented in Sect. 2, and the endomorphisms are introduced and analysed in Sect. 3. Section 2 also contains a brief discussion of the various local and global observable algebras. The global Lie algebra of observables, which we denote by $\bar{\mathcal{L}}$, is analysed further in Sect. 4. As in the case of the Ising model, the endomorphisms again take us out of the chiral algebra. This time besides the algebra of functions on the circle and the infinite orthogonal Lie algebra generated by fermion bilinears, one has to introduce an algebra containing $\widehat{\mathfrak{so}}(N)_1$; up to finite sums of fermion bilinears, the commutation relations of this algebra which we call $\tilde{\mathfrak{gl}}(N)$ coincide with those of the $\widehat{\mathfrak{sl}}(N) \oplus \hat{u}(1)$ Kac-Moody algebra. Together, the fermion bilinears, the Virasoro generators and the generators of $\tilde{\mathfrak{gl}}(N)$ form a Lie algebra, and they are transformed among themselves by the endomorphisms. It is a non-trivial result that a closed Lie algebra is obtained after a finite number of steps. We also exhibit in Sect. 4 a maximal abelian subalgebra $\bar{\mathcal{L}}_0 \subset \bar{\mathcal{L}}$ of the observable Lie algebra; somewhat unexpectedly, $\bar{\mathcal{L}}_0$ is not closed under the action of the endomorphisms. In Sect. 5 we show that the composition of (equivalence classes of) endomorphisms reproduces the fusion rules of the WZW theory. In an Appendix we have collected various aspects of the zero mode subalgebra of the current algebra and of its representations.

The theories described in this paper are rather special conformal field theories as they can be expressed through free fermions. One can however hope to treat also

more complicated theories along similar lines. In particular it may be possible to use conformal embeddings [17] to reduce theories with higher level Kac-Moody algebras to the level one case treated here [18]. This and other open questions are discussed in Sect. 6.

2. Majorana Fields and Observables

The distinctive feature of the conformal field theory of the $\widehat{\mathfrak{so}}(N)_1$ WZW theories which makes the algebraic treatment of the theory manageable is the fact that the theory can be expressed in terms of free fermions, so that the observables may be constructed in terms of bilinears of the fermions, and that the relations of the observable algebra can be deduced from the canonical anti-commutation relations of the fermions. The canonical anti-commutation relations for real free fermions $\psi^i(z)$, $i = 1, 2, \dots, N$, on the circle S^1 are given by

$$\{\psi^i(z), \psi^j(w)\} = 2\pi i \delta^{ij} \delta(z - w) \mathbf{1} \tag{2.1}$$

(to be precise, only fields obtained by smearing the $\psi^i(z)$ with appropriate test functions are operators for which the canonical anti-commutation relations make sense), and the hermiticity condition is

$$(\psi^i(z))^* = z^{-1} \psi^i(z^{-1}). \tag{2.2}$$

There are in fact two types of such fermions, namely Neveu–Schwarz (NS) and Ramond (R) fermions which are characterized by the boundary conditions $\psi^i_{\text{NS}}(e^{2\pi i} z) = \psi^i_{\text{NS}}(z)$ (NS fermions) and $\psi^i_{\text{R}}(e^{2\pi i} z) = -\psi^i_{\text{R}}(z)$ (R fermions), so that they may be expanded into a Fourier–Laurent series as $\psi^i(z) = \sum_r b^i_r z^{-r-1/2}$ with $r \in \mathbb{Z} + \frac{1}{2}$ (NS)

and $r \in \mathbb{Z}$ (R), respectively. For the present purpose it will be convenient to treat NS and R fermions in a unified way. This is done by admitting test functions which are single-valued continuous functions on the double cover \widetilde{S}^1 of S^1 (the coordinate on \widetilde{S}^1 will also be denoted by z). The description of the boundary condition then requires the introduction of a non-trivial central element which will be denoted by Y :

$$\psi^i(e^{2\pi i} z) = -Y \psi^i(z), \quad Y^* = Y, \quad Y^2 = \mathbf{1}. \tag{2.3}$$

Thus we define: The *universal Majorana algebra* of N fermions is the associative $*$ -algebra with identity $\mathbf{1}$ which is generated by Y and fields ψ^i , $i = 1, 2, \dots, N$, on \widetilde{S}^1 (more precisely, smeared with test functions of the type described above), subject to the hermiticity and boundary conditions (2.2, 2.3), and to

$$\begin{aligned} \{\psi^i(z), \psi^j(w)\} &= \pi i \delta^{ij} [\delta(z - w) \mathbf{1} - \delta(z - e^{2\pi i} w) Y], \\ [\psi^i(z), Y] &= 0. \end{aligned} \tag{2.4}$$

The expansion of the universal fermion fields into a Fourier–Laurent series reads

$$\psi^i(z) = \sum_{q \in \mathbb{Z}/2} b^i_q z^{-q-1/2}. \tag{2.5}$$

The algebra generated by the modes b^i_q and by Y will be denoted by $\text{Maj} \equiv \text{Maj}(N)$, and its restriction to $q \in \mathbb{Z} + \frac{1}{2}$ and to $q \in \mathbb{Z}$ by Maj_{NS} and Maj_{R} , respectively. In terms of the modes, the hermiticity property reads $(b^j_p)^* = b^j_{-p}$, and the relations (2.4)

become

$$\begin{aligned} \{b_p^i, b_q^j\} &= \frac{1}{2} \delta^{ij} \delta_{p+q,0} [\mathbf{1} + (-1)^{2q} Y], \\ [b_p^i, Y] &= 0. \end{aligned} \quad (2.6)$$

Evaluating the first of these relations for $p = q = 0$, one sees that, for any i ,

$$Y = 4b_0^i b_0^i - \mathbf{1}, \quad (2.7)$$

so that

$$Y b_p^i = b_p^i Y = (-1)^{2p} b_p^i \quad (2.8)$$

(and hence in particular $b_p^i b_q^j = 0$ if $b_p^i \in \text{Maj}_{\text{NS}}$ and $b_q^j \in \text{Maj}_{\text{R}}$).

As is quite obvious, each of the subalgebras Maj_{NS} and Maj_{R} has only a single (inequivalent) faithful irreducible unitary *-representation; the corresponding irreducible unitary *-representations of the universal Majorana algebra Maj , denoted as π_{NS} and π_{R} , are then defined by

$$\begin{aligned} \pi_{\text{NS}}(b_p^i) &= 0 \quad \text{for } p \in \mathbb{Z}, & \pi_{\text{NS}}(Y) &= -1, \\ \pi_{\text{R}}(b_p^i) &= 0 \quad \text{for } p \in \mathbb{Z} + \frac{1}{2}, & \pi_{\text{R}}(Y) &= 1. \end{aligned} \quad (2.9)$$

They obey

$$\text{Maj}/\text{Ker}(\pi_{\text{NS}}) \cong \text{Maj}_{\text{NS}}, \quad \text{Maj}/\text{Ker}(\pi_{\text{R}}) \cong \text{Maj}_{\text{R}}. \quad (2.10)$$

The corresponding modules \mathcal{H}_{NS} and \mathcal{H}_{R} are highest weight modules, i.e. they contain highest weight vectors $|\text{NS}\rangle$ and $|\text{R}\rangle$ obeying

$$\begin{aligned} b_q^i |\text{NS}\rangle &= 0 \quad \text{for } b_q^i \in \text{Maj}_{\text{NS}}, \quad q > 0, \\ b_q^i |\text{R}\rangle &= 0 \quad \text{for } b_q^i \in \text{Maj}_{\text{R}}, \quad q > 0, \end{aligned} \quad (2.11)$$

and \mathcal{H}_{NS} and \mathcal{H}_{R} are spanned by the vectors

$$|(\vec{i}\vec{q})_m\rangle_{\text{NS}} \equiv |i_1 q_1, i_2 q_2, \dots, i_m q_m\rangle_{\text{NS}} := b_{-q_m}^{i_m} \cdots b_{-q_2}^{i_2} b_{-q_1}^{i_1} |\text{NS}\rangle \quad (2.12)$$

and

$$|(\vec{i}\vec{q})_m\rangle_{\text{R}} \equiv |i_1 q_1, i_2 q_2, \dots, i_m q_m\rangle_{\text{R}} := b_{-q_m}^{i_m} \cdots b_{-q_2}^{i_2} b_{-q_1}^{i_1} |\text{R}\rangle \quad (2.13)$$

with $b_{-q_l}^{i_l} \in \text{Maj}_{\text{NS}}$ and $b_{-q_l}^{i_l} \in \text{Maj}_{\text{R}}$, respectively, where $m \in \mathbb{Z}_{\geq 0}$ and $q_{l+1} \geq q_l \geq 0$ for $l = 1, 2, \dots, m-1$. For future convenience we will also write $\mathcal{H}_{\text{NS}} = \mathcal{H}_{\text{NS}}^{\text{even}} \oplus \mathcal{H}_{\text{NS}}^{\text{odd}}$, where $\mathcal{H}_{\text{NS}}^{\text{even}}$ and $\mathcal{H}_{\text{NS}}^{\text{odd}}$ are spanned by the vectors $|(\vec{i}\vec{q})_m\rangle_{\text{NS}}$ with m even and m odd, respectively. Also, we define the *grade* of a state in \mathcal{H}_{NS} or \mathcal{H}_{R} as the non-negative integer that one obtains by adding up minus the mode numbers of the fermion operators which create the state by acting on the highest weight state of the space, i.e. as $\sum_{s=1}^m q_s$ in the notation of (2.12) and (2.13).

The global observable algebra $\bar{\mathcal{A}}$ which is the algebra of our main interest will be a subalgebra of an appropriate completion by infinite power series of Maj . Moreover, as we will see, $\bar{\mathcal{A}}$ is in fact the enveloping algebra of a Lie algebra; we will use the notation $\bar{\mathcal{L}}$ for this underlying Lie algebra of global observables. The Lie algebra $\bar{\mathcal{L}}$ will be constructed by adjoining further generators to the infinite orthogonal Lie algebra $O_\infty(N)$ which is defined as

$$O_\infty(N) := \text{span}\{\mathbf{1}, b_p^i b_q^j | i, j \in \{1, 2, \dots, N\}, p, q \in \frac{1}{2}\mathbb{Z}, p - q \in \mathbb{Z}\} \quad (2.14)$$

(note that owing to (2.7) this includes also the central generator Y). The full set of infinite series of such bilinears that are needed for the completion $\widehat{\mathcal{L}}$ will be obtained in Sect. 5. For the moment we content ourselves with including in the observable algebra those operators which according to the introduction should certainly be present, namely the generators of the chiral symmetry algebra of the conformal field theory under consideration. For WZW theories, the chiral algebra is the semidirect sum of the Virasoro algebra Vir with an untwisted affine Kac-Moody algebra. Therefore we introduce the following infinite series of fermion bilinears:

$$J_m^a = \frac{1}{2} \sum_{i,j} (T^a)_{ij} \sum_{q \in \mathbb{Z}/2} :b_q^i b_{m-q}^j:, \tag{2.15}$$

$$L_m = -\frac{1}{2} \sum_i \sum_{q \in \mathbb{Z}/2} \binom{i-m}{2} :b_q^i b_{m-q}^i: + \frac{N}{32} (\mathbf{1} + Y) \delta_{m,0} \tag{2.16}$$

for $m \in \mathbb{Z}$. Here $T^a, a = 1, 2, \dots, N(N-1)/2$, are the matrix generators of the simple Lie algebra $\mathfrak{so}(N)$ in the defining (vector) representation (taken to be real and antisymmetric). The colons denote the normal ordering of the modes b_q^i , defined as

$$:b_p^i b_q^j: = \begin{cases} b_p^i b_q^j & \text{for } p < 0, \\ -b_q^j b_p^i & \text{for } p \geq 0. \end{cases} \tag{2.17}$$

By direct computation, one verifies that these combinations obey the commutation relations

$$[J_m^a, J_n^b] = m\kappa^{ab} \delta_{m+n,0} + f_c^{ab} J_{m+n}^c, \tag{2.18}$$

$$[L_m, L_n] = \frac{N}{24} (m^3 - m) \delta_{m+n,0} + (m-n) L_{m+n}, \tag{2.19}$$

$$[L_m, J_n^a] = -n J_{m+n}^a, \tag{2.20}$$

where f_c^{ab} and κ^{ab} are the structure constants and Cartan-Killing form of $\mathfrak{so}(N)$, respectively, i.e. $[T^a, T^b] = f_c^{ab} T^c, \text{tr}(T^a T^b) = \kappa^{ab}$, and the summation convention is used for the adjoint indices a, b, \dots . Thus the combinations (2.15, 2.16) generate the semidirect sum of the Virasoro algebra Vir (generated by the L_m) and the untwisted affine Lie algebra $\widehat{\mathfrak{so}}(N)$ (generated by the J_m^a), with the respective central generators fixed to the values $c = N/2$ and $k = 1$, respectively. In addition, of course the commutator of any element of $O_\infty(N)$ with an element of the Virasoro or the affine algebra is again in $O_\infty(N)$. Thus we see that the global observable algebra $\widehat{\mathcal{L}}$, we are finally interested in, contains the Lie algebra

$$\mathcal{L} = O_\infty(N) \oplus \text{Vir}_{N/2} \oplus \widehat{\mathfrak{so}}(N)_1 \tag{2.21}$$

(with “ \oplus ” denoting the semidirect sum) as a subalgebra.

The representation theory of the algebra $\widehat{\mathfrak{so}}(N)$ at level one (and of its semidirect sum with the Virasoro algebra) is well known [11]: up to equivalence, there are only three different integrable irreducible highest weight modules if N is odd, and four if N is even, namely the basic module, the vector module, and one or two inequivalent spinor modules. With respect to the $\mathfrak{so}(N)$ algebra generated by the zero modes J_0^a , they reduce to the trivial one-dimensional module, the N -dimensional vector module, and the $2^{\lfloor (N-1)/2 \rfloor}$ -dimensional¹ spinor module(s), respectively. Also, the

¹ Here and below we denote the integer part of a rational number x by $[x]$.

states spanning the highest weight modules of $\widehat{\mathfrak{so}}(N)$ can be chosen as eigenvectors of L_0 ; the L_0 -eigenvalues h are the sum of the grade of the state and the eigenvalue of the highest weight vector, which is $h_0 = 0$, $h_v = 1/2$, and $h_s = N/16$ for the basic, vector, and spinor modules, respectively.

Since the observables are made out of bilinears in the modes b_q^i , the representations π_{NS} and π_{R} of Maj restrict to representations of the algebra (2.21), which are however reducible. This is most easily seen by considering the action of the Virasoro generator L_0 . Because of (2.11) the highest weight vectors $|\text{NS}\rangle = |0\rangle$ of \mathcal{H}_{NS} and $|\text{R}\rangle$ of \mathcal{H}_{R} obey

$$\begin{aligned} L_0|0\rangle &= 0, \\ L_0|\text{R}\rangle &= \frac{N}{16}|\text{R}\rangle, \end{aligned} \tag{2.22}$$

while for the states

$$|i\rangle_{\text{NS}} := |i - \frac{1}{2}\rangle_{\text{NS}} = b_{-1/2}^i|0\rangle \tag{2.23}$$

one finds

$$L_0|i\rangle_{\text{NS}} = \frac{1}{2}|i\rangle_{\text{NS}}. \tag{2.24}$$

Using

$$[L_0, b_q^i] = -qb_q^i, \tag{2.25}$$

it then follows that

$$L_0|(\vec{i}\vec{q})_m\rangle_{\text{NS}} = h|(\vec{i}\vec{q})_m\rangle_{\text{NS}} \quad \text{with} \quad \begin{cases} h \in \mathbb{Z}_{\geq 0} & \text{for } |(\vec{i}\vec{q})_m\rangle_{\text{NS}} \in \mathcal{H}_{\text{NS}}^{\text{even}} \\ h \in \mathbb{Z}_{\geq 0} + \frac{1}{2} & \text{for } |(\vec{i}\vec{q})_m\rangle_{\text{NS}} \in \mathcal{H}_{\text{NS}}^{\text{odd}} \end{cases} \tag{2.26}$$

and

$$L_0|(\vec{i}\vec{q})_m\rangle_{\text{R}} = h|(\vec{i}\vec{q})_m\rangle_{\text{R}} \quad \text{with} \quad h \in \mathbb{Z}_{\geq 0} + \frac{N}{16}. \tag{2.27}$$

In particular, the lowest values $h = 0$ and $h = \frac{1}{2}$ in (2.26) are obtained iff $|(\vec{i}\vec{q})_m\rangle_{\text{NS}} = |0\rangle$ and $|(\vec{i}\vec{q})_m\rangle_{\text{NS}} = |i\rangle_{\text{NS}}$ (for some $i \in \{1, 2, \dots, N\}$), respectively, while the lowest value $h = N/16$ in (2.27) is the L_0 eigenvalue of each of the following 2^N independent states:

$$\begin{aligned} &|\text{R}\rangle \\ &b_0^i|\text{R}\rangle \quad 1 \leq i \leq N \\ &b_0^{i_2}b_0^{i_1}|\text{R}\rangle \quad 1 \leq i_1 < i_2 \leq N \\ &\vdots \\ &b_0^N b_0^{N-1} \dots b_0^1|\text{R}\rangle. \end{aligned} \tag{2.28}$$

It is also easily verified that the states $|0\rangle$, $|i\rangle_{\text{NS}}$ and the states (2.28) have the correct transformation properties under the horizontal subalgebra $\mathfrak{so}(N)$ of $\widehat{\mathfrak{so}}(N)$; $|0\rangle$ provides the singlet and the states $|i\rangle_{\text{NS}}$ span the vector module of $\mathfrak{so}(N)$, i.e.

$$J_0^a|0\rangle = 0, \quad J_0^a|i\rangle_{\text{NS}} = \sum_j (T^a)_{ji}|j\rangle_{\text{NS}}, \tag{2.29}$$

while the states (2.28) span $2^{[N/2]+1}$ irreducible spinor modules (more details are provided in the Appendix). Finally one can check that upon acting on the zero grade states with any fermion bilinear $b_p^i b_q^j$, the number of irreducible subspaces is not changed² and hence the multiplicities of the various irreducible $\widehat{\mathfrak{so}}(N)$ -modules in \mathcal{H}_{NS} and \mathcal{H}_{R} are the same as the multiplicities of the corresponding irreducible $\mathfrak{so}(N)$ -modules in the zero grade subspaces of \mathcal{H}_{NS} and \mathcal{H}_{R} . Putting this information together, we learn that the representations π_{NS} and π_{R} of the Majorana algebra restrict to representations of $\widehat{\mathfrak{so}}(N)_1$, and hence of (2.21), as follows: π_{NS} splits up into the direct sum of the basic and the vector representation, while π_{R} gives the direct sum of $2^{[N/2]+1}$ irreducible spinor representations.

The Lie algebras of local observables corresponding to the global algebras $O_\infty(N)$, $\text{Vir}_{N/2}$ and $\widehat{\mathfrak{so}}(N)_1$ are obtained by smearing the bilinears of fields $\psi_{\text{NS}}^i(z)$ and

$$\begin{aligned} T(z) &:= \sum_{m \in \mathbb{Z}} L_m z^{-m-2}, \\ J^a(z) &:= \sum_{m \in \mathbb{Z}} J_m^a z^{-m-1}, \end{aligned} \tag{2.30}$$

with appropriate test functions. Thus to $O_\infty(N)$ there corresponds, for any open interval $I \subset S^1$ whose closure is not all of S^1 , an algebra $(O_\infty(N))(I)$ of observables localized in I ; $(O_\infty(N))(I)$ is spanned by the identity $\mathbf{1}$ and the generators

$$\int_{S^1 \times S^1} dz dw (zw)^{-1/2} f_{ij}(z, w) \psi_{\text{NS}}^i(z) \psi_{\text{NS}}^j(w), \tag{2.31}$$

where f_{ij} is a real \mathbf{C}^∞ function with support on $I \times I \subset S^1 \times S^1$. Similarly, the algebra $\text{Vir}_{N/2}(I)$ is the real Lie algebra spanned by $\mathbf{1}$ and

$$\int_{S^1} dz z f(z) T(z), \tag{2.32}$$

with f a real \mathbf{C}^∞ function with support on I , and analogously for $(\widehat{\mathfrak{so}}(N)_1)(I)$. In the notation introduced here, the global algebras are presented as follows. $O_\infty(N)$ is the algebra spanned by $\mathbf{1}$ and the generators

$$\int_{\tilde{S}^1 \times \tilde{S}^1} dz dw (zw)^{-1/2} f_{ij}(z, w) \frac{1}{2} [\psi^i(z) \psi^j(w) + \psi^i(e^{2\pi i} z) \psi^j(e^{2\pi i} w)], \tag{2.33}$$

where f_{ij} is now a real \mathbf{C}^∞ function on $\tilde{S}^1 \times \tilde{S}^1$, and similarly for $\text{Vir}_{N/2}$ and $\widehat{\mathfrak{so}}(N)_1$. This definition of the global algebras illustrates the two main differences between the present treatment of these algebras and the canonical [2] one: first, we do not restrict to bounded operators (by using $f \cdot a$ rather than $\exp(f \cdot a)$ in the integrands), and second, by considering the double cover \tilde{S}^1 rather than S^1 .

One can also show (in full analogy with the treatment in [4]) that there exists an injective homomorphism from $(O_\infty(N))(I)$ to $O_\infty(N)$ for which the mapping prescription is independent of the interval I . This homomorphism acts as $\mathbf{1} \mapsto \mathbf{1}$ and

$$\begin{aligned} &\int_{I \times I} dz dw (zw)^{-1/2} f_{ij}(z, w) \psi_{\text{NS}}^i(z) \psi_{\text{NS}}^j(w) \\ &\mapsto \int_{\tilde{I} \times \tilde{I}} dz dw (zw)^{-1/2} f_{ij}(z, w) \frac{1}{2} [\psi^i(z) \psi^j(w) + \psi^i(e^{2\pi i} z) \psi^j(e^{2\pi i} w)]. \end{aligned} \tag{2.34}$$

² This is non-trivial in the case of spinor modules; some relevant formulae are again provided in the Appendix

Here the function f_{ij} on the left-hand side is regarded as a function on $\tilde{S}^1 \times \tilde{S}^1$ with support in $\tilde{I} \times \tilde{I}$, where \tilde{I} is one of the two disjoint intervals which form the pre-image of I under the projection from \tilde{S}^1 to S^1 .

3. Endomorphisms of the Majorana Algebra

3.1. *The Vector Endomorphism.* For arbitrary N , define the *vector endomorphism* ρ_v of Maj by

$$\rho_v(b_q^i) := \rho_{U_v}(b_q^i) = U_v \cdot b_q^i \cdot U_v^* \tag{3.1}$$

with the unitary operator

$$U_v = U_v^{-1} = \begin{cases} \frac{1}{\sqrt{N}} \sum_i (b_{1/2}^i + b_{-1/2}^i) & \text{on Maj}_{\text{NS}}, \\ \sqrt{\frac{2}{N}} \sum_i b_0^i & \text{on Maj}_{\text{R}}. \end{cases} \tag{3.2}$$

More explicitly, this transformation is given by

$$\rho_v(b_q^i) = -b_q^i \quad \text{for } q \neq 0, \pm \frac{1}{2} \tag{3.3}$$

and

$$\begin{aligned} \rho_v(b_{\pm 1/2}^i) &= -b_{\pm 1/2}^i + \frac{1}{N} \sum_j (b_{1/2}^j + b_{-1/2}^j), \\ \rho_v(b_0^i) &= -b_0^i + \frac{2}{N} \sum_j b_0^j. \end{aligned} \tag{3.4}$$

Inspection shows that ρ_v is a *-automorphism of Maj that projects to inner automorphisms of Maj_{NS} and Maj_{R} . Furthermore, owing to the fact that U_v is self-adjoint as well, one has

$$\rho_v \circ \rho_v = \text{id}. \tag{3.5}$$

(More explicitly, this can be seen as follows. Introducing the matrix I which has all entries equal to $\frac{1}{N}$, $I_{ij} = \frac{1}{N}$, and the vectors $b_0 = (b_0^1, b_0^2, \dots, b_0^N)^t$, $b_{\pm 1/2} = (b_{\pm 1/2}^1, b_{\pm 1/2}^2, \dots, b_{\pm 1/2}^N)^t$, (3.4) can be rewritten as $\rho_v(b_{\pm 1/2}) = -b_{\pm 1/2} + I \cdot (b_{1/2} + b_{-1/2})$, $\rho_v(b_0) = (2I - \mathbf{1}) \cdot b_0$. The result (3.5) then follows from the identity $I^2 = I$.)

3.2. *The Spinor Endomorphisms.* We also want to identify further endomorphisms of Maj corresponding to the spinor representation(s) of $\text{so}(N)$. In order to simplify the expressions, let us first establish some additional notation. We denote by σ_i , $i = 1, 2, 3$, the Pauli matrices, and define the 2×2 -matrices

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \tag{3.6}$$

and the 4×4 -matrices

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} R & R \\ R & -R \end{pmatrix}, \quad P = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} \mathbf{1} - \sigma_2 & \mathbf{1} + \sigma_2 \\ \mathbf{1} + \sigma_2 & \mathbf{1} - \sigma_2 \end{pmatrix} \quad (3.7)$$

(so that e.g. $\sigma_1 = SR, \sigma_2 = -RS$ and $QM = MP$). We also define the vectors

$$\begin{aligned} B_0^i &= (b_0^{i-3}, b_0^{i-2}, b_0^{i-1}, b_0^i)^t && \text{for } i \in 4\mathbb{Z}, 4 \leq i \leq N, \\ B_{1/2}^i &= (b_{1/2}^{i-2}, b_{1/2}^{i-1}, b_{1/2}^i, b_{1/2}^{i+1})^t && \text{for } i \in 4\mathbb{Z}, 4 \leq i \leq N, \\ B_0^N &= (b_0^{N-1}, b_0^N)^t, \quad B_{1/2}^N = (b_{1/2}^N, b_{1/2}^{N+1})^t && \text{for } N \in 4\mathbb{Z} + 2. \end{aligned} \quad (3.8)$$

Finally, set

$$\text{Maj}^\circ = \begin{cases} \{b_0^{N-3}, b_0^{N-2}, b_0^{N-1}, b_0^N, b_{1/2}^{N-2}, b_{1/2}^{N-1}, b_{1/2}^N, b_{1/2}^{N+1}\} & \text{for } N \in 4\mathbb{Z}, \\ \{b_0^{N-1}, b_0^N, b_{1/2}^N, b_{1/2}^{N+1}\} & \text{for } N \in 4\mathbb{Z} + 2 \end{cases} \quad (3.9)$$

and denote

$$\eta \equiv \eta(j) = (-1)^j \quad (3.10)$$

for $j \in \{1, 2, \dots, N\}$.

Now for $N \in 2\mathbb{Z}$, we define the following *spinor endomorphisms* $\rho_{s,c}$ of Maj: For $N \in 4\mathbb{Z}$,

$$\rho_s(b_q^i) = \begin{cases} b_{q \mp \eta/2}^i & \text{for } q \in \mathbb{Z}_\pm \pm \frac{1}{2}\eta, \\ b_{q \pm \eta/2}^i & \text{for } q \in \mathbb{Z}_\pm, \end{cases} \quad (3.11)$$

except for b_q^i an entry of B_0^i or $B_{1/2}^i$, where

$$\begin{aligned} \rho_s(B_0^i) &= M^{-1} \cdot B_{1/2}^i, \\ \rho_s(B_{1/2}^i) &= M \cdot B_0^i, \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} \rho_c(b_q^i) &= -\rho_s(b_q^i) \text{ for } b_q^i \notin \text{Maj}^\circ, \\ \rho_c(B_0^N) &= -P \cdot M^{-1} \cdot B_{1/2}^N, \\ \rho_c(B_{1/2}^N) &= -M \cdot P \cdot B_0^N. \end{aligned} \quad (3.13)$$

For $N \in 4\mathbb{Z} + 2$,

$$\rho_s(b_q^i) = \begin{cases} \pm i b_{q \mp \eta/2}^i & \text{for } q \in \mathbb{Z}_\pm \pm \frac{1}{2}\eta, \\ \pm i b_{q \pm \eta/2}^i & \text{for } q \in \mathbb{Z}_\pm, \end{cases} \quad (3.14)$$

again except for entries of B_0^i or $B_{1/2}^i$, where

$$\begin{aligned} \left. \begin{aligned} \rho_s(B_0^i) &= M^{-1} \cdot B_{1/2}^i, \\ \rho_s(B_{1/2}^i) &= -M \cdot B_0^i, \end{aligned} \right\} && \text{for } i \in 4\mathbb{Z}, \\ \rho_s(B_0^N) &= R^{-1} \cdot B_{1/2}^N, \\ \rho_s(B_{1/2}^N) &= S^{-1} \cdot B_0^N, \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} \rho_c(b_q^i) &= -\rho_s(b_q^i) \quad \text{for } b_q^i \notin \text{Maj}^\circ, \\ \rho_c(B_0^N) &= S \cdot B_{1/2}^N, \\ \rho_c(B_{1/2}^N) &= R \cdot B_0^N. \end{aligned} \tag{3.16}$$

By direct computation, one verifies that $\rho_{s,c}$ are *-automorphisms of Maj.

Next we come to the case $N \in 2\mathbb{Z} + 1$, in which there is only one spinor endomorphism which we denote by ρ_σ . It is defined by

$$\rho_\sigma(b_q^j) = \tilde{\rho}_s(b_q^j) \quad \text{for } j < N \tag{3.17}$$

and

$$\rho_\sigma(b_q^N) = \begin{cases} b_{q \pm 1/2}^N & \text{for } q \in \frac{1}{2}\mathbb{Z}_\pm \text{ and } N \in 4\mathbb{Z} + 1 \\ \pm i b_{q \pm 1/2}^N & \text{for } q \in \frac{1}{2}\mathbb{Z}_\pm \text{ and } N \in 4\mathbb{Z} + 3 \\ \frac{i}{\sqrt{2}}(b_{1/2}^N - b_{-1/2}^N) & \text{for } q = 0. \end{cases} \tag{3.18}$$

In (3.17), $\tilde{\rho}_s$ means the spinor endomorphism ρ_s at $\tilde{N} = N - 1 \in 2\mathbb{Z}$. Clearly, ρ_σ is a *-endomorphism.

3.3. Composition of Endomorphisms. The fusion rules of the conformal field theory correspond to the natural composition of the representations π_u (with $u = v, s, c$ (N even), respectively $u = v, \sigma$ (N odd)) which is defined by

$$\pi_{u_1} \times \pi_{u_2} \cong \pi_0 \circ \rho_{u_1} \circ \rho_{u_2}, \tag{3.19}$$

(If ρ_{u_i} are localized endomorphisms, then this defines an operator product on the field algebra \mathcal{F} .) This definition is possible because the endomorphisms ρ_u obey

$$\pi_u \cong \pi_0 \circ \rho_u \tag{3.20}$$

for all bilinears in the fermion modes b_ρ^i , and hence for all (finite or infinite) sums thereof, in particular for the Kac-Moody and Virasoro generators J_m^a and T_m . (It is straightforward to check that the endomorphisms ρ_u constructed above indeed fulfill (3.20); for some details, see the appendix.)

As a consequence of the definition (3.19) the fusion rules are isomorphic to the composition of equivalence classes of the endomorphisms ρ_u . Therefore we will now compute the various compositions of the endomorphisms ρ_u introduced above. These results will be the basis of the calculation of the fusion rules which will be performed in Sect. 5.

The result of the composition of two vector endomorphisms was already obtained in (3.5) above. The formulæ for all other possible compositions read as follows.

For $N \in 4\mathbb{Z}$,

$$\rho_s \circ \rho_s = \text{id} = \rho_c \circ \rho_c, \tag{3.21}$$

and

$$\begin{aligned} \rho_s \circ \rho_c &= \rho_c \circ \rho_s \\ \rho_c \circ \rho_s &= -\text{id} \text{ on } \text{Maj} \setminus \text{Maj}^\circ, \\ \rho_c \circ \rho_s(B_0^N) &= -P \cdot B_0^N, \quad \rho_c \circ \rho_s(B_{1/2}^N) = -Q \cdot B_{1/2}^N. \end{aligned} \tag{3.22}$$

For $N \in 4\mathbb{Z} + 2$,

$$\rho_s \circ \rho_c = \text{id} = \rho_c \circ \rho_s, \tag{3.23}$$

and

$$\begin{aligned} \rho_s \circ \rho_s &= \rho_c \circ \rho_c, \\ \rho_s \circ \rho_s &= -\text{id} \text{ on } \text{Maj} \setminus \text{Maj}^\circ, \\ \rho_s \circ \rho_s(B_0^N) &= \sigma_1 \cdot B_0^N, \quad \rho_s \circ \rho_s(B_{1/2}^N) = -\sigma_2 \cdot B_{1/2}^N. \end{aligned} \tag{3.24}$$

In particular, the compositions leading to non-trivial endomorphisms project onto inner automorphisms of Maj_{NS} and Maj_{R} :

$$\rho_s \circ \rho_c(b_q^i) = \rho_c \circ \rho_s(b_q^i) = U_{cs} \cdot b_q^i \cdot U_{cs}^* \text{ for } N \in 4\mathbb{Z} \tag{3.25}$$

and

$$\rho_s \circ \rho_s(b_q^i) = \rho_c \circ \rho_c(b_q^i) = U_{ss} \cdot b_q^i \cdot U_{ss}^* \text{ for } N \in 4\mathbb{Z} + 2, \tag{3.26}$$

with

$$\begin{aligned} U_{cs} &= \begin{cases} \frac{i}{\sqrt{2}}(b_{1/2}^{N-2} - ib_{-1/2}^{N-2} - b_{1/2}^N + ib_{-1/2}^N) & \text{on } \text{Maj}_{\text{NS}}, \\ (b_0^{N-1} + b_0^N) & \text{on } \text{Maj}_{\text{R}}, \end{cases} \\ U_{ss} &= \begin{cases} b_{1/2}^N + ib_{-1/2}^N & \text{on } \text{Maj}_{\text{NS}}, \\ (b_0^{N-1} + b_0^N) & \text{on } \text{Maj}_{\text{R}}. \end{cases} \end{aligned} \tag{3.27}$$

For odd N , one obtains the following results. From (3.18) we deduce that

$$\rho_\sigma \circ \rho_\sigma(b_q^j) = b_q^j \text{ for } j < N, N \in 4\mathbb{Z} + 1, \tag{3.28}$$

while

$$\rho_\sigma \circ \rho_\sigma(b_q^j) = -b_q^j \text{ for } j < N - 1, N \in 4\mathbb{Z} + 3, \tag{3.29}$$

and also for $j = N - 1$ except for

$$\begin{aligned} \rho_\sigma \circ \rho_\sigma(B_0^{N-1}) &= \sigma_1 B_0^{N-1}, \\ \rho_\sigma \circ \rho_\sigma(B_{1/2}^{N-1}) &= -\sigma_2 B_{1/2}^{N-1}. \end{aligned} \tag{3.30}$$

Here we used the corresponding formulæ (3.21, 3.24) for $\tilde{\rho}_s$. Finally, for $j = N$ it follows from (3.18) that

$$\rho_\sigma \circ \rho_\sigma(b_q^N) = \begin{cases} \left. \begin{aligned} &b_{q \pm 1}^N && \text{for } q \neq 0 \\ &\frac{i}{\sqrt{2}}(b_1^N - b_{-1}^N) && \text{for } q = 0 \end{aligned} \right\} & \text{for } N \in 4\mathbb{Z} + 1, \\ \left. \begin{aligned} &-b_{q \pm 1}^N && \text{for } q \neq 0 \\ &-\frac{1}{\sqrt{2}}(b_1^N + b_{-1}^N) && \text{for } q = 0 \end{aligned} \right\} & \text{for } N \in 4\mathbb{Z} + 3. \end{cases} \tag{3.31}$$

To determine $\rho_\sigma \circ \rho_\nu$ and $\rho_\nu \circ \rho_\sigma$, it is convenient to define the endomorphism $\tilde{\rho}_\nu = \sigma \tilde{U}$ with

$$\tilde{U} = \begin{cases} b_{1/2}^N + b_{-1/2}^N & \text{on Maj}_{NS}, \\ \sqrt{2}b_0^N & \text{on Maj}_R. \end{cases} \quad (3.32)$$

It is readily checked that this is a vector automorphism, i.e. $\tilde{\rho}_\nu \in [\rho_\nu]$; explicitly, its action is given by

$$\tilde{\rho}_\nu(b_q^j) = \begin{cases} -b_q^j & \text{for } j \neq N, \\ -b_q^N & \text{for } j = N, q \neq 0, \pm \frac{1}{2}, \\ b_{-q}^N & \text{for } j = N, q = 0, \pm \frac{1}{2}. \end{cases} \quad (3.33)$$

One verifies that $\tilde{\rho}_\nu \circ \rho_\sigma = -\rho_\sigma$. Similarly, $\rho_\sigma \circ \tilde{\rho}_\nu = -\rho_\sigma$ except when acting on b_0^N or on $b_{\pm 1/2}^N$, where in the first case $\rho_\sigma \circ \tilde{\rho}_\nu(b_0^N) = \rho_\sigma(b_0^N)$. Thus when acting on observables $a \in O_\infty(N)$, one has

$$\tilde{\rho}_\nu \circ \rho_\sigma(a) = \rho_\sigma(a), \quad (3.34)$$

while on observables in the vacuum representation,

$$\pi_0 \circ \rho_\sigma \circ \tilde{\rho}_\nu(a) = \pi_0 \circ \sigma_U \circ \rho_\sigma(a) \quad (3.35)$$

with the unitary

$$U = U^{-1} = \mathbf{1} - 2b_{-1/2}^N b_{1/2}^N. \quad (3.36)$$

4. Construction of $\tilde{\mathcal{L}}$

We would now like to identify a global Lie algebra $\tilde{\mathcal{L}}$ of observables such that each of the endomorphisms ρ_u for $u = v, s, c$ (N even), respectively $u = v, \sigma$ (N odd), of Maj induces a *-endomorphism of $\tilde{\mathcal{L}}$. Clearly, this endomorphism property is satisfied for any bilinear $b_p^i b_q^j$, and hence also for all finite sums thereof, i.e. in particular for all elements of the algebra $O_\infty(N)$. It remains to investigate the action of the endomorphisms on infinite sums of fermion bilinears. The infinite series encountered so far are the generators L_m of the Virasoro algebra and the generators J_m^a of the affine Kac-Moody algebra $\widehat{\mathfrak{so}}(N)$, which together with $O_\infty(N)$ span the subalgebra (2.21) of $\tilde{\mathcal{L}}$. As we will see, for the full algebra $\tilde{\mathcal{L}}$, we will essentially have to double the number of generators which are infinite sums.

Before going into details, let us make the following general remark. Let a Lie algebra g be realized in an associative algebra A in terms of commutators. Given an endomorphism ρ of A , we may then address the question of what is the Lie algebraic extension of g on which the corresponding endomorphism closes. It is clear that closure is obtained when adjoining to A appropriate successive commutators of the generators, but it is not at all obvious that a closed Lie algebra can be obtained after a finite number of steps (each step consisting either of application of the endomorphisms ρ or of taking commutators). In the case at hand, we will see that we do arrive at the final answer after a finite number of steps.

4.1. *Action of the Endomorphisms on $\text{Vir}_{N/2}$.* To start, consider now the action of the endomorphism ρ_ν on the Virasoro generators. It is given by

$$\rho_\nu(L_m) = L_m + \beta_m, \quad (4.1)$$

where

$$\beta_m = \frac{1}{2N} \sum_{i,j} [(m+1)b_{m+1/2}^i + (m-1)b_{m-1/2}^i](b_{1/2}^j + b_{-1/2}^j) + 4mb_m^i b_0^j. \tag{4.2}$$

Thus in order to close the algebra, we have to include the finite sums $\beta_m, m \in \mathbb{Z}$, of bilinears in the b_q^i into the global Lie algebra of observables. Moreover, it is straightforward to verify that

$$\rho_\nu(\beta_m) = -\beta_m, \tag{4.3}$$

for all $m \in \mathbb{Z}$, so that at this stage no further generators are needed.

Since we include all bilinears of the b_q^i into the global Lie algebra of observables anyway, from now on we will often suppress finite sums in these bilinears in order to make the formulæ more readable. We will write $a \cong b$ if a and b coincide up to such finite sums. Using this notation, the action of the spinor endomorphisms on the Virasoro algebra reads, for N even,

$$\rho_{s,c}(L_m) \cong L_m + F_m, \tag{4.4}$$

where

$$F_m := \frac{1}{2} \sum_i \sum'_{q>m/2} (-1)^{i+2q} b_{m-q}^i b_q^i, \tag{4.5}$$

and where for later convenience we defined

$$\sum'_{\substack{m < p < n \\ p \in \mathbb{Z}/2}} f(p) := \left(\sum_{\substack{m < p < n \\ p \in \mathbb{Z}/2}} f(p) \right) + \frac{1}{2} f(m) + \frac{1}{2} f(n), \tag{4.6}$$

and analogously for sums which are bounded only from one side or for which the summation is restricted to $p \in \mathbb{Z} + \varepsilon/2$ for $\varepsilon = 0, 1$ (in the latter case, the boundary terms are included only if $m, n \in \mathbb{Z} + \varepsilon/2$).

For N odd, the corresponding formulæ read

$$\rho_\sigma(L_m) \cong L_m + \tilde{F}_m + \hat{F}_m, \tag{4.7}$$

where \tilde{F}_m denotes the operator F_m at $\tilde{N} = N - 1 \in 2\mathbb{Z}$, and

$$\hat{F}_m := -\frac{1}{2} \sum'_{q>m/2} b_{m-q}^N b_q^N. \tag{4.8}$$

Also,

$$\rho_{s,c}(F_m) \cong -F_m \tag{4.9}$$

for N even, so that in particular $\rho_{s,c} \circ \rho_{s,c}(L_m) \cong L_m$. Analogously, $\rho_\sigma(\tilde{F}_m) \cong -\tilde{F}_m$ for N odd, whereas in this case

$$\rho_\sigma(\hat{F}_m) \cong \hat{F}_m. \tag{4.10}$$

In addition,

$$\begin{aligned} [F_m, L_n] &\cong mF_{m+n} \\ [F_m, F_n] &\cong 0, \end{aligned} \tag{4.11}$$

and analogously for \tilde{F}_m and \hat{F}_m . Thus we conclude that for even N we have to include $F_m, m \in \mathbb{Z}$, into the global observable algebra, while for odd N we need both \tilde{F}_m and \hat{F}_m .

4.2. *Action of the Endomorphisms on $\widehat{\mathfrak{so}}(N)_1$.* Next we come to the action of the endomorphism ρ_v on the Kac-Moody generators. It reads

$$\rho_v(J_m^a) = J_m^a + \beta_m^a \tag{4.12}$$

with

$$\beta_m^a = -\frac{1}{N} \sum_{i,j,k} (T^a)_{ij} (b_{m+1/2}^j + b_{m-1/2}^j)(b_{1/2}^k + b_{-1/2}^k) + 4b_m^j b_0^k. \tag{4.13}$$

Hence we must include the finite sums $\beta_m^a, m \in \mathbb{Z}$, of bilinears in order to close the algebra. Also,

$$\rho_v(\beta_m^a) = -\beta_m^a \tag{4.14}$$

for all $m \in \mathbb{Z}$, so that again no further generators are needed at this stage. Next we must consider the action of the spinor endomorphisms on the Kac-Moody algebra. To keep the formulæ short, we introduce some further notation. Denote

$$D_{(\varepsilon)}^{ij}(m) := \sum'_{\substack{q > m/2 \\ q \in \mathbb{Z} + \varepsilon/2}} b_{m-q}^i b_q^j \tag{4.15}$$

for $m \in \mathbb{Z}$ and $\varepsilon = 0, 1 \pmod 2$, and

$$D^{ij}(m) := D_{(0)}^{ij}(m) + D_{(1)}^{ij}(m), \tag{4.16}$$

$$E^{ij}(m) := D^{ij}(m) - D^{ji}(m). \tag{4.17}$$

In this notation, one has e.g.

$$F_m = \frac{1}{2} \sum_i (-1)^i (D_{(0)}^{ii}(m) - D_{(1)}^{ii}(m)), \tag{4.18}$$

and also

$$\begin{aligned} J_m^a &= \frac{1}{2} \sum_{i,j} (T^a)_{ij} \left(\sum_{q \geq 0} + \sum_{q > m} \right) b_{m-q}^i b_q^j \\ &\cong \frac{1}{2} \sum_{i,j} (T^a)_{ij} E^{ij}(m). \end{aligned} \tag{4.19}$$

In particular, the transformation properties of the Kac-Moody generators J_m^a follow immediately from those of the operators $E^{ij}(m)$, and hence from those of the $D^{ij}(m)$.

It turns out that all the operators $D_{(0)}^{ij}(m)$ and $D_{(1)}^{ij}(m)$ (and not just their particular antisymmetric combinations $E^{ij}(m)$) must be included into the global observable algebra. (In particular, owing to (4.18), this includes the generators F_m .) The corresponding formulæ are obtained as follows. First we observe that

$$[D_{(\varepsilon)}^{ij}(m), D_{(\varepsilon')}^{kl}(n)] \cong \delta_{\varepsilon,\varepsilon'} [\delta^{jk} D_{(\varepsilon)}^{il}(m+n) - \delta^{il} D_{(\varepsilon)}^{kj}(m+n)]. \tag{4.20}$$

This implies

$$[D^{ij}(m), D^{kl}(n)] \cong \delta^{jk} D^{il}(m+n) - \delta^{il} D^{kj}(m+n). \tag{4.21}$$

Note that, neglecting finite sums in the bilinears $b_p^i b_q^j$, the latter is the loop algebra associated to $\mathfrak{sl}(N)$. Including also the generators F_m , one obtains a structure similar to the loop algebra of $\mathfrak{gl}(N) \cong \mathfrak{sl}(N) \oplus \mathfrak{u}(1)$. As a consequence, we will from now on denote the vector space spanned by the generators $D_{(\varepsilon)}^{ij}(m)$ by $\widetilde{\mathfrak{gl}}(N)$.

Consider now first the case $N \in 2\mathbb{Z}$. Then

$$\rho_{s,c}(D_{(\varepsilon)}^{ij}(m)) \cong D_{(\varepsilon+1)}^{ij}(m - \frac{1}{2}(-1)^\varepsilon[(-1)^i - (-1)^j]), \tag{4.22}$$

and hence

$$\rho_{s,c}(D^{ij}(m)) \cong D_{(0)}^{ij}(m + \frac{1}{2}[(-1)^i - (-1)^j]) + D_{(1)}^{ij}(m - \frac{1}{2}[(-1)^i - (-1)^j]), \tag{4.23}$$

so that in particular

$$\pi_0 \circ \rho_{s,c}(D^{ij}(m)) \cong \pi_0(D^{ij}(m - \frac{1}{2}[(-1)^i - (-1)^j])). \tag{4.24}$$

Note that the shift in the mode number of D^{ij} that is induced by $\rho_{s,c}$ is not symmetric in i and j ; as a consequence, $\rho_{s,c}$ does not close on the antisymmetric combinations $E^{ij}(m)$. From (4.24) it also follows that (in the vacuum representation) taking further commutators does not introduce any new generators. (More explicitly, one has

$$[D_{(\varepsilon)}^{ij}(m), \rho_{s,c}(D_{(\varepsilon')}^{jk}(n))] \cong \delta_{\varepsilon,\varepsilon'+1} D_{(\varepsilon)}^{ik}(m+n + \frac{1}{2}(-1)^\varepsilon[(-1)^j - (-1)^k]) \tag{4.25}$$

and an analogous formula for $[D^{ij}(m), \rho_{s,c}(D^{kl}(n))]$. Finally,

$$[L_m, D_{(\varepsilon)}^{ij}(n)] \cong -nD_{(\varepsilon)}^{ij}(m+n), \tag{4.26}$$

$$[F_m, D_{(\varepsilon)}^{ij}(n)] \cong \frac{1}{2}(-1)^\varepsilon((-1)^i + (-1)^j)D_{(\varepsilon)}^{ij}(m+n), \tag{4.27}$$

i.e.

$$[L_m, D^{ij}(n)] \cong -nD^{ij}(m+n)$$

$$\pi_0([F_m, D^{ij}(n)]) \cong \frac{1}{2}((-1)^i + (-1)^j)\pi_0(D^{ij}(m+n)). \tag{4.28}$$

For N odd, the calculations are identical with the ones above as long as $i, j \neq N$. For i or j equal to N , one has

$$\begin{aligned} \rho_\sigma(D_{(\varepsilon)}^{iN}(m)) &\cong \eta^{-1} D_{(\varepsilon+1)}^{iN}(m + \frac{1}{2}[1 - (-1)^{i+\varepsilon}]), \\ \rho_\sigma(D_{(\varepsilon)}^{Ni}(m)) &\cong \eta D_{(\varepsilon+1)}^{Ni}(m - \frac{1}{2}[1 - (-1)^{i+\varepsilon}]), \end{aligned} \tag{4.29}$$

where $\eta = 1$ for $N \in 4\mathbb{Z} + 1$, $\eta = i$ for $N \in 4\mathbb{Z} + 3$. From this one can infer the transformation behavior and commutation relations of $D^{iN}(m)$ and $D^{Ni}(m)$. These are very similar to the previous results and we omit the details.

4.3. The Lie Algebra $\widetilde{\mathcal{L}}$ of Global Observables. We summarize the results above in the

Theorem. *The global observable algebra $\widetilde{\mathcal{L}}$ is the Lie algebra*

$$\widetilde{\mathcal{L}} = O_\infty(N) \oplus \text{Vir}_{N/2} \oplus \widetilde{\mathfrak{gl}}(N), \tag{4.30}$$

where $O_\infty(N)$ is generated by the identity $\mathbf{1}$ and the bilinears $b_p^i b_q^j$ ($i, j \in \{1, 2, \dots, N\}$, $p, q \in \frac{1}{2}\mathbb{Z}$, $p - q \in \mathbb{Z}$), $\text{Vir}_{N/2}$ is generated by the infinite sums L_m ($m \in \mathbb{Z}$), and (for N even, and similarly for N odd) $\widetilde{\mathfrak{gl}}(N)$ is generated by the infinite sums $D_{(\varepsilon)}^{ij}(m)$ ($i, j \in \{1, 2, \dots, N\}$, $\varepsilon \in \{0, 1\}$, $m \in \mathbb{Z}$).

Up to finite sums, the commutation relations among the infinite sums of bilinears are given by the formulæ (2.19), (4.20) and (4.26).

4.4. *Some More Details.* In the equations above, we have not written out the terms involving finite sums of the bilinears $b_p^i b_q^j$. The explicit form of these terms is not very illuminating; for completeness let us however present the full result, including finite sums of fermion bilinears, for the following objects. First, we give the complete version of (4.21) for the case $j = k$ with i, j, l all different; it reads

$$[D^{ij}(m), D^{il}(n)] = D^{il}(m+n) + A^{il}(m, n) \tag{4.31}$$

with

$$A^{ij}(m, n) := \begin{cases} - \sum'_{0 < p < n/2} b_{(m+n)/2-p}^i b_{(m+n)/2+p}^j & \text{for } m+n > 0, n > 0, \\ \sum'_{n/2 < p < 0} b_{(m+n)/2-p}^i b_{(m+n)/2+p}^j & \text{for } m+n > 0, n < 0, \\ \sum'_{-m/2 < p < 0} b_{(m+n)/2-p}^i b_{(m+n)/2+p}^j & \text{for } m+n < 0, m > 0, \\ - \sum'_{0 < p < -m/2} b_{(m+n)/2-p}^i b_{(m+n)/2+p}^j & \text{for } m+n < 0, m < 0. \end{cases} \tag{4.32}$$

Second, let us present the action of the vector automorphism on the generators $D_{(e)}^{ij}(m)$:

$$\rho_v(D_{(e)}^{ij}(m)) = D_{(e)}^{ij}(m) + g_{(e)}^{ij}(m), \quad \rho_v(g_{(e)}^{ij}(m)) = -g_{(e)}^{ij}(m), \tag{4.33}$$

where $g_{(e)}^{ij}(m)$ denotes the finite sums

$$\begin{aligned} g_{(0)}^{ij}(m) &= \frac{2}{N} [\theta(m)b_m^j - \theta(-m)b_m^i] \sum_l b_l^i, \\ g_{(1)}^{ij}(m) &= \frac{1}{N} [\theta(m-1)b_{m-1/2}^j + \theta(m+1)b_{m+1/2}^j \\ &\quad - \theta(-m+1)b_{m-1/2}^i - \theta(-m-1)b_{m+1/2}^i] \sum_l (b_{1/2}^l + b_{-1/2}^l), \end{aligned} \tag{4.34}$$

with

$$\theta(m) = \begin{cases} 0 & \text{for } m < 0, \\ \frac{1}{2} & \text{for } m = 0, \\ 1 & \text{for } m > 0. \end{cases} \tag{4.35}$$

Let us also discuss the extended algebra $\bar{\mathcal{L}}$ in a different guise. First note that according to (4.19), the Kac-Moody generators J_m^a are essentially the antisymmetric combinations $E^{ij}(m)$ of the operators $D^{ij}(m)$; they can also be considered as appropriately normal ordered versions of the $D^{ij}(m)$. It is therefore of interest to re-express the relations of $\bar{\mathcal{L}}$ in terms of the $E^{ij}(m)$. The results are most easily described in terms of the operators

$$E_{\pm}^{ij}(m, n) := D_{(0)}^{ij}(m+n) - D_{(1)}^{ji}(m+n) \pm [D_{(1)}^{ij}(m-n) - D_{(0)}^{ji}(m-n)] \tag{4.36}$$

which carry an additional mode number n . As a consequence of (4.22), one must

include into $\bar{\mathcal{L}}$ the operators $E_{\pm}^{ij}(m, n)$ for $m \in \mathbb{Z}$ and $n = \pm 1$. More precisely, by application of $\rho_{s,c}$ one obtains these operators with $i \in 2\mathbb{Z}$, $j \in 2\mathbb{Z} + 1$, or vice versa. However, owing to (4.25), one has

$$[E^{ij}(m), \rho_{s,c}(E^{jk}(n))] \cong E_{\pm}^{ik}(m + n, \frac{1}{2}[(-1)^j - (-1)^k]), \tag{4.37}$$

implying that these generators also appear with both i and j even or both odd. Finally, taking further commutators, one sees that also the parameter n introduced in (4.36) can take arbitrary integer values. Thus we have to include the $E_{\pm}^{ij}(m, n)$ with arbitrary i, j and $m, n \in \mathbb{Z}$ into the global observable algebra. Under $\rho_{s,c}$, these generators transform as

$$\rho_{s,c}(E_{\pm}^{ij}(m, n)) \cong E_{\pm}^{ij}(m, -n + \frac{1}{2}[(-1)^i - (-1)^j]) \tag{4.38}$$

(so that in particular $\rho_{s,c} \circ \rho_{s,c}(E_{\pm}^{ij}(m, n)) \cong E_{\pm}^{ij}(m, n)$). Finally one computes the commutation relations

$$[E_{\pm}^{ij}(m, n), E_{\pm}^{kl}(m', n')] \cong \delta^{jk} E_{\pm}^{il}(m + m', n + n') - \delta^{il} E_{\pm}^{kj}(m + m', n + n') + \delta^{ik} E_{\pm}^{lj}(m + m', n - n') - \delta^{jl} E_{\pm}^{ik}(m + m', n - n'), \tag{4.39}$$

$$[E_{\pm}^{ij}(m, n), E_{\mp}^{kl}(m', n')] \cong \delta^{jk} E_{\mp}^{il}(m + m', n + n') - \delta^{il} E_{\mp}^{kj}(m + m', n + n') - \delta^{ik} E_{\mp}^{lj}(m + m', n - n') + \delta^{jl} E_{\mp}^{ik}(m + m', n - n'), \tag{4.40}$$

$$[E_{\pm}^{ij}(m, n), L_l] \cong m E_{\pm}^{ij}(m + l, n) + n E_{\mp}^{ij}(m + l, n), \tag{4.41}$$

$$\pi_0([E_{\pm}^{ij}(m, n), F_l]) \cong \frac{1}{2}((-1)^i + (-1)^j)\pi_0(E_{\pm}^{ij}(m + l, n)). \tag{4.42}$$

From (4.41), one deduces that one indeed has to include not only $E_{\pm}^{ij}(m, n)$, but also $E_{\mp}^{ij}(m, n)$. Also note that according to (4.41) for $n \neq 0$ the operators $E_{\pm}^{ij}(m, n)$ cannot be interpreted as the Laurent modes of conformal fields.

For N odd, one has in addition similar relations for the corresponding quantities $E^{iN}(m, n)$ and $E^{Ni}(m, n)$ (as well as $E^{NN}(m, n)$ for $n \neq 0$; note that $E_{\pm}^{ij}(m, n)$ is not anti-symmetric in i and j , but rather $E_{\pm}^{ij}(m, n) = -E_{\pm}^{ji}(m, -n)$).

4.5. *The Maximal Abelian Subalgebra $\bar{\mathcal{L}}_0$.* Let us also consider the “zero modes” L_0 and $D_{(\varepsilon)}^{ij}(0)$. Together with the finite bilinears

$$b_0^{ij} := b_0^i b_0^j \tag{4.43}$$

they close under commutation, and in particular all these generators commute with L_0 :

$$[L_0, D_{(\varepsilon)}^{ij}(0)] = [L_0, b_0^{ij}] = 0, \tag{4.44}$$

$$[b_0^{ij}, b_0^{kl}] = \delta^{jk} b_0^{il} - \delta^{il} b_0^{kj} + \delta^{ik} b_0^{lj} - \delta^{jl} b_0^{ik}, \tag{4.45}$$

$$[D_{(\varepsilon)}^{ij}(0), D_{(\varepsilon)}^{kl}(0)] = \delta_{\varepsilon, \varepsilon'} \{ \delta^{jk} D_{(\varepsilon)}^{il}(0) - \delta^{il} D_{(\varepsilon)}^{kj}(0) + \frac{1}{4} \delta_{\varepsilon, 0} [b_0^{ij}, b_0^{kl}] \}. \tag{4.46}$$

In particular, for $\varepsilon = 1$ the $D_{(\varepsilon)}^{ij}(0)$ generate the simple Lie algebra $\mathfrak{sl}(N)$, and for $\varepsilon = 0$ they do so up to the bilinears b_0^{ij} . Including also the action of the automorphism ρ_v , one still gets a finite number of modes which close upon commutation. First, according to (4.1), (4.3), (4.12), (4.14) and (4.33) one has

$$\begin{aligned} \rho_v(L_0) &= L_0 + \beta_0, & \rho_v(\beta_0) &= -\beta_0, \\ \rho_v(D_{(0)}^{ij}(0)) &= D_{(0)}^{ij}(0) + \alpha_0^{ij}, & \rho_v(\alpha_0^{ij}) &= -\alpha_0^{ij}, \end{aligned} \tag{4.47}$$

$$\begin{aligned} \rho_\nu(D_{(1)}^{ij}(0)) &= D_{(1)}^{ij}(0) + \beta_0^{ij}, \quad \rho_\nu(\beta_0^{ij}) = -\beta_0^{ij}, \\ \rho_\nu(b_0^{ij}) &= b_0^{ij} + 2\alpha_0^{ij}, \end{aligned}$$

with

$$\begin{aligned} \alpha_0^{ij} &:= \frac{1}{N} \sum_l (b_0^{jl} - b_0^{il}), \\ \beta_0^{ij} &:= \frac{1}{N} (b_{1/2}^j - b_{-1/2}^j) \sum_l (b_{1/2}^l + b_{-1/2}^l), \\ \beta_0 &= \frac{1}{2} \sum_j \beta_0^{jj} = \frac{1}{2N} \sum_{i,j} [b_{1/2}^i, b_{-1/2}^j]. \end{aligned} \tag{4.48}$$

The generators β_0^{ij} do not yet close, but if one counts the contributions

$$\beta_{\pm 1/2}^i := b_{\pm}^i \sum_l b_{\pm 1/2}^l \tag{4.49}$$

($i = 1, \dots, N$) and $b_{1/2}^i b_{-1/2}^j$ ($i, j = 1, \dots, N$) to β_0^{ij} independently, then a closed algebra is obtained. In contrast, after acting on the operators $D_{(e)}^{ij}(0)$ also with a spinor automorphism, closure can only be obtained with an infinite number of operators.

It is also natural to look for maximal abelian subalgebras of $\bar{\mathcal{L}}$. Among these, the interesting ones are those which contain the Virasoro zero mode L_0 . We will present one of these algebras, which we will denote by $\bar{\mathcal{L}}_0$ (other maximal abelian subalgebras can then be obtained by acting on $\bar{\mathcal{L}}_0$ with any automorphism). Let us start by extending the definition (4.43) to arbitrary moding:

$$b_q^{ij} := b_{-q}^i b_q^j \quad \text{for } q \geq 0. \tag{4.50}$$

These bilinears obey the commutation relations (4.45) and

$$[b_p^{ij}, b_q^{kl}] = \delta_{pq} (\delta^{jk} b_p^{il} - \delta^{il} b_p^{kj}) \quad \text{for } q > 0. \tag{4.51}$$

It follows that a maximal commuting subset of the linear span of these operators is spanned by the central generators $\mathbf{1}$ and Y and the combinations

$$\begin{aligned} &b_0^{2j-1, 2j}, \\ &b_q^{2j-1, 2j-1} + b_q^{2j, 2j}, \quad q > 0, \\ &b_q^{2j-1, 2j} - b_q^{2j, 2j-1}, \quad q > 0, \end{aligned} \tag{4.52}$$

with $j \in \{1, \dots, N\}$ for any N , and in addition by

$$b_q^{NN}, \quad q \geq 0 \tag{4.53}$$

for N odd. It can be checked that there are no other elements of $O_\infty(N)$ which commute with these combinations, i.e. they span indeed a maximal abelian subalgebra of $O_\infty(N)$. Next we look for those independent infinite sums of the generators $b_p^i b_q^j$ of $O_\infty(N)$ which are contained in $\bar{\mathcal{L}}$ and which commute with the operators (4.52), (4.53). They are

$$L_0 = \sum_j \left(\frac{1}{8} b_0^{jj} + \sum_{q>0} q b_q^{jj} \right),$$

$$\begin{aligned}
 H_0^j &= -i \left(b_0^{2j-1,2j} + \sum_{q>0} (b_q^{2j-1,2j} - b_q^{2j,2j-1}) \right), \quad j = 1, \dots, [N/2], \quad (4.54) \\
 G_0^j &= \sum_{q>0} (b_q^{2j-1,2j-1} + b_q^{2j,2j}), \quad j = 1, \dots, [N/2],
 \end{aligned}$$

for any N , and in addition

$$G_0^{(N+1)/2} = \sum_{q>0} b_q^{NN} \quad (4.55)$$

for N odd. Together with the finite linear combinations given previously, these operators span a maximal abelian subalgebra \mathcal{L}_0 of \mathcal{L} . Note that (in the notation of (4.5), (4.8)), \mathcal{L}_0 does not contain the zero mode F_0 , while for odd N it does contain $\widehat{F}_0 = -\frac{1}{4}b_0^{NN} + \frac{1}{2}G_0^{(N-1)/2}$.

One might expect \mathcal{L}_0 to be closed under the action of the endomorphisms ρ_u which represent the superselection sectors, but inspection shows that (except for the special case $N = 1$ where e.g. the list (4.54), (4.55) reduces to L_0 and $G_0 \propto F_0 \equiv \widehat{F}_0$) this is not the case, and it also does not hold for any other maximal abelian subalgebra of \mathcal{L} . While this is a bit surprising, it does not violate any of the principles of algebraic quantum field theory.

5. Fusion Rules

If ρ_1 and ρ_2 are any endomorphisms of Maj which both project onto inner automorphisms of Maj_{NS} and Maj_{R} , i.e. $\rho_k(b_q^i) = U_k \cdot b_q^i \cdot U_k^*$ with $U_k, k = 1, 2$, unitary, and if either both of them are made out of odd or both out of even polynomials in the b_q^i , then ρ_1 and ρ_2 induce equivalent *-automorphisms of the global observable algebra, namely via

$$\rho_2 = \rho_1 \circ \sigma_{U_{12}}, \quad (5.1)$$

where $\sigma_{U_{12}}$ is defined according to (1.3), with

$$U_{12} = iU_1^*U_2 \in \mathcal{A}. \quad (5.2)$$

In particular, if the unitary U is even in the b_q^i , then $\sigma_U \in [\text{id}]$, and if it is odd in the b_q^i , then $\sigma_U \in [\rho_v]$.

Combining this observation with the results of the previous sections, one concludes that $\rho_s \circ \rho_c \cong \rho_v \cong \rho_c \circ \rho_s$ for $N \in 4\mathbb{Z}$ etc. Thus we arrive at the fusion rules

$$\begin{aligned}
 &[\rho_v \circ \rho_v] = [\text{id}], \\
 &\left. \begin{aligned}
 &[\rho_s \circ \rho_s] = [\rho_c \circ \rho_c] = [\text{id}] \\
 &[\rho_s \circ \rho_c] = [\rho_c \circ \rho_s] = [\rho_v]
 \end{aligned} \right\} \text{ for } N \in 4\mathbb{Z}, \\
 &\left. \begin{aligned}
 &[\rho_s \circ \rho_c] = [\rho_c \circ \rho_s] = [\text{id}] \\
 &[\rho_s \circ \rho_s] = [\rho_c \circ \rho_c] = [\rho_v]
 \end{aligned} \right\} \text{ for } N \in 4\mathbb{Z} + 2. \quad (5.3)
 \end{aligned}$$

Using these results (and in addition, for $N \in 4\mathbb{Z} + 2$, the fact that $\sigma_1 \cdot S = R^{-1}$ and $\sigma_2 \cdot R = -S^{-1}$), one also deduces immediately the remaining non-trivial fusion rules

for $N \in 2\mathbb{Z}$:

$$\begin{aligned} [\rho_s \circ \rho_v] &= [\rho_v \circ \rho_s] = [\rho_c], \\ [\rho_c \circ \rho_v] &= [\rho_v \circ \rho_c] = [\rho_s]. \end{aligned} \quad (5.4)$$

Thus the composition of equivalence classes of endomorphisms reproduces the fusion rules (1.6) and (1.7), as it should be.

For $N \in 2\mathbb{Z} + 1$, the situation is more complicated. First from (3.34), (3.35) and the fact that $\tilde{\rho}_v \in [\rho_v]$ we see that

$$[\rho_\sigma \circ \rho_v] = [\rho_v \circ \rho_\sigma] = [\rho_\sigma]. \quad (5.5)$$

To determine also $[\rho_\sigma \circ \rho_\sigma]$, let us to define the operator

$$\Pi_0 := b_{1/2}^N b_{-1/2}^N. \quad (5.6)$$

This is a projector, $\Pi_0 \Pi_0 = \Pi_0$, on \mathcal{H}_0 . Moreover, Π_0 (and hence also the orthogonal projector $\mathbf{1} - \Pi_0$) lie in the commutant of $\rho_\sigma \circ \rho_\sigma(\mathcal{L})$ since $b_{1/2}^N$ and $b_{-1/2}^N$ are not contained in the image by $\rho_\sigma \circ \rho_\sigma$ of Maj. In the following we show that the invariant subspaces $\Pi_0 \mathcal{H}_0$ and $(\mathbf{1} - \Pi_0) \mathcal{H}_0$ with respect to $\rho_\sigma \circ \rho_\sigma$ carry representations in the equivalence class of π_0 and $\pi_0 \circ \tilde{\rho}_v$ for $N \in 4\mathbb{Z} + 1$, and of $\pi_0 \circ \tilde{\rho}_v$ and π_0 for $N \in 4\mathbb{Z} + 3$, respectively. Let us define the observables

$$\begin{aligned} \mathcal{I} &= \left(\prod_{n=1}^{\infty} a_n \right) \Pi_0, \\ \tilde{\mathcal{I}} &= \tilde{\rho}_v(\mathcal{I}) \end{aligned} \quad (5.7)$$

with

$$a_n := b_{n+1/2}^N b_{-n-1/2}^N + b_{-n+1/2}^N b_{n+1/2}^N \quad \text{for } n \in \mathbb{Z} \quad (5.8)$$

(and with the product defined iteratively as $\prod_{n=1}^m a_n = a_m \prod_{n=1}^{m-1} a_n$). It is not difficult to verify (compare the similar calculation in [4] for $N = 1$) that they obey

$$\begin{aligned} \mathcal{I} \mathcal{I}^* &= \tilde{\mathcal{I}} \tilde{\mathcal{I}}^* = \mathbf{1}, \\ \mathcal{I}^* \mathcal{I} &= \Pi_0, \\ \tilde{\mathcal{I}}^* \tilde{\mathcal{I}} &= \mathbf{1} - \Pi_0, \end{aligned} \quad (5.9)$$

so that the maps

$$\begin{aligned} \mathcal{I} &: \Pi_0 \mathcal{H}_0 \rightarrow \mathcal{H}_0, \\ \tilde{\mathcal{I}} &: (\mathbf{1} - \Pi_0) \mathcal{H}_0 \rightarrow \mathcal{H}_0 \end{aligned} \quad (5.10)$$

are bijective. Moreover, the operator \mathcal{I} possesses the following intertwining properties for $N \in 4\mathbb{Z} + 1$: $b_q^j \mathcal{I} = \mathcal{I} b_q^j$ for $j < N$, and $b_q^N \mathcal{I} = \mathcal{I} b_{q \pm 1}^N$ for $q \in \mathbb{Z} \pm \frac{1}{2}$; for $N \in 4\mathbb{Z} + 3$ similar relations are valid. As a consequence, by comparison with the properties (3.28) to (3.31), and defining σ_U by

$$U = b_{1/2}^{N-1} + i b_{-1/2}^{N-1}, \quad (5.11)$$

one has

$$\begin{aligned} a \mathcal{I} &= \mathcal{I} \rho_\sigma \circ \rho_\sigma(a) \quad \text{for } N \in 4\mathbb{Z} + 1, \\ \sigma_U(a) \mathcal{I} &= \mathcal{I} \rho_\sigma \circ \rho_\sigma(a) \quad \text{for } N \in 4\mathbb{Z} + 3 \end{aligned} \quad (5.12)$$

for all bilinears $a = b_p^i b_q^j$ in the vacuum representation (here we write a for $\pi_0(a) \cdots$) and more generally for all generators of $\pi_0(\tilde{\mathcal{L}})$. Similarly, using $\tilde{\rho}_v \circ \rho_\sigma(a) = \rho_\sigma(a)$ for \tilde{U} as given in (3.32), it follows that

$$\begin{aligned} \tilde{\rho}_v(a)\tilde{\mathcal{F}} &= \tilde{\mathcal{F}}\rho_\sigma \circ \rho_\sigma(a) \quad \text{for } N \in 4\mathbb{Z} + 1, \\ \tilde{\rho}_v \circ \sigma_U(a)\tilde{\mathcal{F}} &= \tilde{\mathcal{F}}\rho_\sigma \circ \rho_\sigma(a) \quad \text{for } N \in 4\mathbb{Z} + 3. \end{aligned} \tag{5.13}$$

Next we notice that U is linear in the fermion modes so that σ_U is a representative of the $[\rho_v]$ equivalence class. Analogously, the fact that $\tilde{U} \cdot U$ is quadratic in the fermion modes implies $[\tilde{\rho}_v \circ \sigma_U] = [\sigma_{\tilde{U}U}] = [\text{id}]$. Putting all these results together, we can conclude that

$$[\rho_\sigma \circ \rho_\sigma] = [\text{id}] + [\rho_v]. \tag{5.14}$$

Thus we see that also for odd N the composition of equivalence classes of endomorphisms reproduces the relevant fusion rules, namely (1.8).

6. Discussion

In this paper we have analysed the level one $so(N)$ Wess–Zumino–Witten conformal field theories from the point of view of algebraic quantum field theory. We have constructed the endomorphisms which represent the superselection sectors of the theory on the global observable algebra (Sect. 3, (3.1), (3.11)–(3.18)), and checked that they reproduce the fusion rules of the WZW theory (Sect. 5). A large part of our paper is of a technical nature, expressing the fact that most of our results are obtainable through calculations which are tedious but straightforward. What is less straightforward is the basic task of constructing the endomorphisms. This is the main point where the information coming from conformal field theory, namely on the chiral symmetry algebra, the spectrum of primary fields and on their fusion rules, becomes essential. Without these guidelines from conformal field theory the search for the endomorphisms would be a rather hopeless task.

The endomorphisms do not close on the algebra \mathcal{L} (2.21) which is the natural first guess for the Lie algebra of global observables as it corresponds to the chiral symmetry algebra of the WZW theory, supplemented by the fermion bilinears (including the two central elements $\mathbf{1}$ and Y). The complete Lie algebra $\tilde{\mathcal{L}}$ of global observables on which the endomorphisms close is given by (4.30); it contains as a subspace the vector space $\tilde{\mathfrak{gl}}(N)$ whose generators have commutation relations which up to finite sums of fermion bilinears coincide with those of the $\widehat{\mathfrak{sl}}(N) \oplus \hat{\mathfrak{u}}(1)$ Kac–Moody algebra; it contains the subalgebra of functions on the circle which is similar to $\hat{\mathfrak{u}}(1)$. A deeper understanding of the $\tilde{\mathfrak{gl}}(N)$ part of the global observable algebra is still lacking. Certainly it should be one of the main issues of research in this area to learn more about the appropriate way of extending the naive global observable algebra, and in particular about the relation with Haag duality. It is also tempting to try to identify the Lie group which has $\tilde{\mathcal{L}}$ as its Lie algebra.

The generators $D_{(e)}^{ij}(m)$ of $\tilde{\mathfrak{gl}}(N)$ close under commutation only up to finite sums of fermion bilinears, i.e. form a closed Lie algebra only when combined with the infinite orthogonal Lie algebra $O_\infty(N)$ (2.14). (Nevertheless $\tilde{\mathfrak{gl}}(N)$ contains the $\widehat{\mathfrak{so}}(N)_1$ Kac–Moody algebra. This is so because, after taking antisymmetric combinations of the $\tilde{\mathfrak{gl}}(N)$ -generators, one is left with a closed algebra generated only by infinite

sums, provided that one properly adjusts a finite number of terms in the sums. The operators obtained this way are precisely the generators J_m^a of the affine algebra $\widehat{\mathfrak{so}}(N)_1$. Note that there is no way to obtain a closed algebra by adjusting the finite contributions to the non-antisymmetric generators $D_{(e)}^{ij}(m)$. It would in fact be rather disturbing if this were possible, because then the operators $D_{(e)}^{ij}(m)$ could be interpreted as the modes of a primary conformal field of conformal dimension 1. Such a field would necessarily be a Kac-Moody current, and as a consequence the chiral symmetry algebra of the theory would have to be larger than the semidirect sum of $\widehat{\mathfrak{so}}(N)_1$ and $\text{Vir}_{N/2}$, which is, however, just the correct maximal chiral algebra of the WZW theory.

Another line of research is to extend the results to more complicated theories. The WZW theories considered here provide an infinite number of conformal field theories, but all of them are rather simple theories, manifested by the fact that the level of the relevant affine algebra ($\widehat{\mathfrak{so}}(N)$) is equal to one. As WZW theories are believed to be the building blocks of all rational conformal field theories, one should next try to extend the results to higher level WZW theories. Since the progress made in the present paper relies on the realization of the affine algebra in terms of free fermions, one may first look for higher level theories which share this property. Such theories exist [19]; in fact, they are in one to one correspondence with the conformal embeddings [17] in the classical affine algebras $\widehat{\mathfrak{so}}(N)_1$ and $\widehat{\mathfrak{sl}}(N)_1$ [19, 20], so that one could apply the results of the present paper rather directly to these more complicated models. Note, however, that in the case of conformal embeddings the maximal chiral symmetry algebra (which is the algebra one is usually interested in in conformal field theory) contains the large level one classical affine algebra rather than only its higher level subalgebra. Nevertheless it seems to be possible [18] to obtain non-trivial results also for the superselection structure of the higher level WZW theory. Another possibility to go to higher level is to use constrained fermions or free fields coupled to a background charge. It is, however, not at all obvious how to describe such systems in terms of algebraic quantum field theory.

Let us also mention the following problem which will arise when more complicated theories are considered. For the theories considered in this paper, one can express the quantum dimensions \mathcal{D}_u of the primary fields (i.e. [6] the statistical dimensions of the corresponding superselection sectors) as $\mathcal{D}_u = 2^{\text{ind}([\rho_u])/4}$, where $\text{ind}([\rho_u])$ is the index of the representative endomorphisms $\rho_u \in [\rho_u]$ of the Majorana algebra (this follows in the same way as for the special case $N = 1$, i.e. the conformal Ising model [4]). For theories with more complicated fusion rules this is certainly no longer true because the quantum dimensions are generically not just fractional powers of 2.

Finally let us recall that one can construct the field algebra which describes the superselection structure in terms of the observable algebra and of additional operators which intertwine between the sectors. One then obtains a natural product in the field algebra, which should correspond to the operator product expansion of the conformal field theory description. Thus it may be possible to compute the operator product coefficients of the conformal field theory as the structure constants of the field algebra, which in principle can be determined from the explicit form of the intertwiners.

Acknowledgements. It is a pleasure to thank L. Alvarez-Gaumé, G. Mack and V. Schomerus for interesting discussions, and B. Schroer for providing a copy of ref. [18].

A. Appendix: The Zero Mode Currents

A.1. Cartan–Weyl Basis. The zero mode generators J_0^a of the current algebra $\widehat{\mathfrak{so}}(N)_1$ span the horizontal subalgebra $\mathfrak{so}(N) \subset \widehat{\mathfrak{so}}(N)_1$. Throughout this appendix, we will neglect the contributions with mode numbers different from 0 or $\pm \frac{1}{2}$ to these zero mode generators, because these do not contribute when acting on the states of zero grade in any of the affine highest weight modules; the conclusions concerning the action on higher grade states are also not altered. The Lie algebra $\mathfrak{so}(N)$ is then spanned, in the Ramond sector, by the operators $b_0^i b_0^j$, and in the Neveu–Schwarz sector by $[b_{1/2}^i, b_{-1/2}^j]$, with $i, j \in \{1, \dots, N\}$. Explicit expressions in terms of Majorana bilinears for an orthogonal basis of the Cartan subalgebra and for the elements corresponding to the simple roots of a Cartan–Weyl basis of $\mathfrak{so}(N)$ are given as follows.

We write $N = 2r$ and $N = 2r + 1$ for N even and odd, respectively. Then in the Ramond sector, one has

$$\begin{aligned} H_0^k &= -i b_0^{2k-1} b_0^{2k}, & k &= 1, \dots, r, \\ E_0^{\alpha_k} &= \frac{1}{2} (b_0^{2k-1} + i b_0^{2k}) (b_0^{2k+1} - i b_0^{2k+2}), & k &= 1, \dots, r-1, \\ E_0^{\alpha_r} &= \begin{cases} \frac{1}{2} (b_0^{2r-3} + i b_0^{2r-2}) (b_0^{2r-1} + i b_0^{2r}), & N = 2r, \\ (b_0^{2r-1} + i b_0^{2r}) b_0^{2r+1}, & N = 2r + 1, \end{cases} \end{aligned} \quad (\text{A.1})$$

and in the Neveu–Schwarz sector

$$\begin{aligned} H_0^k &= -i (b_{-1/2}^{2k-1} b_{1/2}^{2k} - b_{-1/2}^{2k} b_{1/2}^{2k-1}), & k &= 1, \dots, r, \\ E_0^{\alpha_k} &= \frac{1}{2} [(b_{-1/2}^{2k-1} + i b_{-1/2}^{2k}) (b_{1/2}^{2k+1} - i b_{1/2}^{2k+2}) \\ &\quad - (b_{-1/2}^{2k+1} + i b_{-1/2}^{2k+2}) (b_{1/2}^{2k-1} - i b_{1/2}^{2k+2})], & k &= 1, \dots, r-1, \\ E_0^{\alpha_r} &= \begin{cases} \frac{1}{2} [(b_{-1/2}^{2r-3} + i b_{-1/2}^{2r-2}) (b_{1/2}^{2r-1} + i b_{1/2}^{2k+2}) \\ \quad - (b_{-1/2}^{2r-1} + i b_{-1/2}^{2r}) (b_{1/2}^{2r-3} + i b_{1/2}^{2r-2})], & N = 2r, \\ (b_{-1/2}^{2r-1} + i b_{-1/2}^{2r}) b_{1/2}^{2r+1} - b_{-1/2}^{2r+1} (b_{1/2}^{2r-1} + i b_{1/2}^{2r}), & N = 2r + 1, \end{cases} \end{aligned} \quad (\text{A.2})$$

These operators satisfy

$$(H_0^k)^* = H_0^k, \quad (E_0^{\alpha_k})^* = E_0^{-\alpha_k}, \quad (\text{A.3})$$

and obey the commutation relations

$$\begin{aligned} [H_0^k, H_0^l] &= 0, \\ [H_0^k, E_0^{\alpha_l}] &= \alpha_l^k \cdot E_0^{\alpha_l}, \\ [E_0^{\alpha_k}, E_0^{-\alpha_k}] &= \frac{2(\alpha_k, H)}{(\alpha_k, \alpha_k)}. \end{aligned} \quad (\text{A.4})$$

Here H denotes the vector with components H_0^k . Also, the components of the simple roots $\alpha_1, \dots, \alpha_r$ have been chosen as

$$\begin{aligned} \alpha_k^l &= \delta_{k,l} - \delta_{k+1,l}, & k &= 1, \dots, r-1, \\ \alpha_r^l &= \begin{cases} \delta_{r-1,l} + \delta_{r,l}, & N = 2r, \\ \delta_{r,l}, & N = 2r + 1 \end{cases} \end{aligned} \quad (\text{A.5})$$

in an orthogonal basis on the weight lattice.

A.2. Highest Weight Modules. As a consequence of the choice (A.5), the highest weights of the trivial, the vector and the spinor representations of $\mathfrak{so}(N)$ have the following components in the Dynkin basis (second column) and in the orthogonal basis (third column), respectively:

$$\begin{aligned} (0): & (0, 0, \dots, 0) & (0, 0, \dots, 0), \\ (v): & (1, 0, \dots, 0) & (1, 0, \dots, 0), \\ (s): & (0, \dots, 0, 0, 1) & (\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}), \\ (c): & (0, \dots, 0, 1, 0) & (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2}), \\ (\sigma): & (0, \dots, 0, 0, 1) & (\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}), \end{aligned} \quad \left. \vphantom{\begin{aligned} (0): \\ (v): \\ (s): \\ (c): \\ (\sigma): \end{aligned}} \right\} \begin{array}{l} N = 2r, \\ N = 2r + 1. \end{array} \quad (\text{A.6})$$

The corresponding vectors in the NS sector that have these eigenvalues with respect to the chosen Cartan subalgebra are

$$\begin{aligned} (0): & |\text{NS}\rangle, \\ (v): & (b_{-1/2}^1 + ib_{-1/2}^2)|\text{NS}\rangle \end{aligned} \quad (\text{A.7})$$

for the trivial and the vector representation, respectively. Acting with all bilinears $b_{-1/2}^i b_{1/2}^j$ on $(b_{-1/2}^1 + ib_{-1/2}^2)|\text{NS}\rangle$, we generate the N -dimensional space spanned by the vectors (2.23), i.e. the vector module, while acting on $|\text{NS}\rangle$ we always get zero, i.e. $|\text{NS}\rangle$ spans the trivial one-dimensional module. Thus the $\mathfrak{so}(N)$ singlet and vector modules appear precisely once in the zero grade subspace of the Neveu–Schwarz sector. These multiplicities are also the multiplicities of the basic and vector representations of $\widehat{\mathfrak{so}}(N)$ in π_{NS} because bilinears do not make transitions between $\mathcal{H}_{\text{NS}}^{\text{even}}$ and $\mathcal{H}_{\text{NS}}^{\text{odd}}$.

The analysis of the R sector is more involved. We begin by introducing some notation. Define $B_\eta^{(\varepsilon)k}$, with $\varepsilon, \eta = 0, 1 \pmod{2}$, $k = 1, \dots, r$, as follows (abusing notation for η , we will use both the multiplicative notation \pm and the additive notation $0, 1 \pmod{2}$):

$$B_\pm^{0(k)} = \frac{1}{2} \mp ib_0^{2k-1} b_0^{2k}, \quad B_\pm^{1(k)} = \frac{1}{\sqrt{2}} (b_0^{2k-1} \pm ib_0^{2k}). \quad (\text{A.8})$$

Immediately we obtain

$$(B_\pm^{(\varepsilon)k})^2 = 0 = B_\pm^{(1)k} B_\pm^{(0)k}, \quad B_\pm^{(1)k} B_\mp^{(\varepsilon)k} = B_\pm^{(\varepsilon+1)k} \quad (\text{A.9})$$

and

$$\begin{aligned} [b_0^j, B_\pm^{(\varepsilon)k}] &= 0 & \text{for } j \neq 2k-1, 2k, \\ b_0^j B_\pm^{(\varepsilon)k} &= \xi B_\mp^{(\varepsilon+1)k} & \text{for } j = 2k-1, 2k \end{aligned} \quad (\text{A.10})$$

with $\sqrt{2}\xi \in \{1, i, -i\}$. The step operators from (A.1) are in this notation $E^{\alpha_k} = B_+^{(1)k} B_+^{(1)k+1}$, for $k = 1, \dots, r-1$, and $E^{\alpha_r} = B_+^{(1)r-1} B_+^{(1)r}$ for $N = 2r$, respectively $E^{\alpha_r} = \sqrt{2} B_+^{(1)r} b_0^{2r+1}$ for $N = 2r + 1$.

Now consider first the case $N = 2r$. Obviously the 2^N -dimensional space spanned by the basis (2.28) can equivalently be described by the basis consisting of the vectors

$$v(\vec{\alpha}, \vec{\beta}) := B_{\beta_1}^{(\alpha_1)1} \dots B_{\beta_r}^{(\alpha_r)r} |\mathbf{R}\rangle, \quad (\text{A.11})$$

with $\vec{\alpha}, \vec{\beta} \in (\mathbb{Z}_2)^r$. This big space breaks up into irreducible subspaces under the action of $\mathfrak{so}(N)$ or equivalently under the action of bilinears $b_0^i b_0^j$ with $i, j = 1, \dots, N$.

Using (A.10) the following is obvious. If both i and j are in $\{2k - 1, 2k\}$ for some k , then $b_0^i b_0^j v(\vec{\alpha}, \vec{\beta})$ is proportional to $v(\vec{\alpha}, \vec{\beta})$. If $i \in \{2k - 1, 2k\}$ while $j \in \{2k' - 1, 2k'\}$ with $k \neq k'$, then the elements of the two pairs (α_k, β_k) and $(\alpha_{k'}, \beta_{k'})$ are “flipped,” and because we are acting with bilinears we always flip two pairs. As a consequence the irreducible subspaces can be described as

$$V_N(\vec{\varepsilon}; \gamma) = \{v(\vec{\alpha}, \vec{\beta}) | \vec{\alpha} + \vec{\beta} = \vec{\varepsilon} \bmod(\mathbb{Z}_2)^r, |\beta| = \gamma \bmod 2\}, \tag{A.12}$$

where

$$\begin{aligned} \gamma &= 0 \bmod 2 && \text{for spinors,} \\ \gamma &= 1 \bmod 2 && \text{for conjugate spinors,} \end{aligned} \tag{A.13}$$

and $|\beta| = \beta_1 + \dots + \beta_r \bmod 2$. Each of these spaces $V_N(\vec{\varepsilon}; \gamma)$ has dimension 2^{r-1} . From (A.9) it is clear that the highest weight vector of the spinor module $V_N(\vec{\varepsilon}; 0)$ is $v(\vec{\varepsilon}; (+ \dots +))$ while the highest weight vector of the conjugate spinor module $V_N(\vec{\varepsilon}; 1)$ is $v(\vec{\varepsilon}; (+ \dots -))$.

Consider now the case of odd $N = 2r + 1$. The 2^N -dimensional space is spanned by

$$v(\vec{\alpha}; \vec{\beta}; \delta) := (b_0^{2r+1})^\delta v(\vec{\alpha}, \vec{\beta}) \tag{A.14}$$

with $\vec{\alpha}, \vec{\beta} \in (\mathbb{Z}_2)^r$ and $\delta \in \mathbb{Z}_2$. Acting with $\mathfrak{so}(N)$ on a vector like this it is clear that both $\vec{\alpha} + \vec{\beta}$ and $|\beta| + \delta$ remain unchanged, i.e., we have irreducible spaces

$$V_N(\vec{\varepsilon}; \gamma) = \{v(\vec{\alpha}; \vec{\beta}; \delta) | \vec{\alpha} + \vec{\beta} = \vec{\varepsilon} \bmod(\mathbb{Z}_2)^r, |\beta| + \delta = \gamma \bmod 2\}. \tag{A.15}$$

Each of these spaces has dimension 2^r . The highest weight vector in such a space is $v(\vec{\varepsilon}; (+ \dots +); \gamma)$.

Finally let us consider the action of arbitrary fermion bilinears $b_p^i b_q^j$, $i, j = 1, \dots, N$, $p, q \in \mathbb{Z}$. Obviously for $N = 2r$ the fermionic Fock space $\text{Maj}|\mathbf{R}\rangle$ is spanned by $(b_<)^k v(\vec{\alpha}, \vec{\beta})$ with $\vec{\alpha}, \vec{\beta} \in (\mathbb{Z}_2)^r$, $k = 0, 1, 2, \dots$, and $(b_<)^k$ being any product $b_{p_1}^{i_1} b_{p_2}^{i_2} \dots b_{p_k}^{i_k}$ with all $p_i < 0$, while for $N = 2r + 1$ the Fock space is spanned by $(b_<)^k v(\vec{\alpha}; \vec{\beta}; \delta)$. In a way analogous to the finite-dimensional case we may argue that

$$\tilde{V}_{2r}(\vec{\varepsilon}; \gamma) = \{(b_<)^k v(\vec{\alpha}, \vec{\beta}) | \vec{\alpha} + \vec{\beta} = \vec{\varepsilon} \bmod(\mathbb{Z}_2)^r, |\beta| + k = \gamma \bmod 2\} \tag{A.16}$$

provide the irreducible spaces of the spinor ($\gamma = 0$) and the conjugate spinor ($\gamma = 1$) representations for even N , and

$$\tilde{V}_{2r+1}(\vec{\varepsilon}; \gamma) = \{(b_<)^k v(\vec{\alpha}; \vec{\beta}; \delta) | \vec{\alpha} + \vec{\beta} = \vec{\varepsilon} \bmod(\mathbb{Z}_2)^r, |\beta| + \delta + k = \gamma \bmod 2\} \tag{A.17}$$

give the irreducible spaces of the spinor representations for odd N . In fact it is readily checked that these subspaces are irreducible under the action of any (finite or infinite) linear combination of fermion bilinears. As a consequence, they provide irreducible subspaces of the whole Lie algebra $\tilde{\mathcal{L}}$ of global observables.

As an illustration, let us describe the $\mathfrak{so}(N)$ spinor modules explicitly for the zero grade subspace in the case $N = 3$. Passing to the notation

$$|(-1)^\eta\rangle_{ab} := v(a + \eta; \eta; b + \eta) \tag{A.18}$$

for $a, b, \eta \in \mathbb{Z}_2$, the highest weight vectors are given by

$$|+\rangle_{00} := B_+^{(0)1} |\mathbf{R}\rangle, \quad |+\rangle_{10} := B_+^{(1)1} |\mathbf{R}\rangle, \quad |+\rangle_{01} := b_0^3 |+\rangle_{00}, \quad |+\rangle_{11} := b_0^3 |+\rangle_{10}. \tag{A.19}$$

They are annihilated by the single raising operator $E = (b_0^1 + ib_0^2)b_0^3$, while the lowering operator $F = (-b_0^1 + ib_0^2)b_0^3$ acts as

$$F|+\rangle_{ab} \propto |-\rangle_{ab} \tag{A.20}$$

for $a, b = 0, 1$, where

$$|-\rangle_{01} := B_-^{(1)1}|\mathbf{R}\rangle, \quad |-\rangle_{11} := B_-^{(0)1}|\mathbf{R}\rangle, \quad |-\rangle_{00} := b_0^3|-\rangle_{01}, \quad |-\rangle_{10} := b_0^3|-\rangle_{11}. \tag{A.21}$$

Finally, the states $|-\rangle_{ab}$ are annihilated by F , and the Cartan subalgebra generator $H = -ib_0^1b_0^2$ acts as

$$H|\pm\rangle_{ab} = \pm\frac{1}{2}|\pm\rangle_{ab}. \tag{A.22}$$

Thus there are four irreducible two-dimensional modules $V_3(a; b)$ each of which is spanned by $|+\rangle_{ab}$ and $|-\rangle_{ab}$ for fixed $a, b \in \{0, 1\}$.

A.3. Action of the Endomorphisms. Let us also check that $\rho_v(J_0^a)$ and $\rho_{s,c}(J_0^a)$ respectively $\rho_\sigma(J_0^a)$ indeed provide the vector and spinor representations of $\mathfrak{so}(N)$ on the space \mathcal{H}_0 . We start with the endomorphism ρ_v . To save space we consider first the antisymmetric basis

$$J_0^{ij} = b_{-1/2}^i b_{1/2}^j - b_{-1/2}^j b_{1/2}^i \tag{A.23}$$

of $\mathfrak{so}(N)$. One finds

$$\rho_v(J_0^{ij})|0\rangle = |i\rangle - |j\rangle, \tag{A.24}$$

where

$$|i\rangle := \frac{1}{N} \sum_i b_{-1/2}^i b_{1/2}^i |0\rangle, \tag{A.25}$$

and

$$\rho_v(J_0^{ij})|l\rangle = \left(\frac{1}{N} - \delta^{jl}\right)|i\rangle - \left(\frac{1}{N} - \delta^{il}\right)|j\rangle + \frac{1}{N}(\delta^{il} - \delta^{jl})|0\rangle. \tag{A.26}$$

Thus $\rho_v(J_0^{ij})$ closes on the vacuum and the states (A.25). Also, since

$$\sum_i |i\rangle = \frac{1}{2N} \sum_{i,j} \{b_{-1/2}^i, b_{-1/2}^j\} |0\rangle = 0, \tag{A.27}$$

only $N - 1$ of the states $|i\rangle$ are independent, and hence together with the vacuum $|0\rangle$ there are N independent states. Thus these states span the vector module of $\mathfrak{so}(N)$.

Moreover there is a representative of the vector endomorphisms, namely $\tilde{\rho}_v := \sigma_{\tilde{v}}$ with

$$\tilde{U} := \frac{1}{\sqrt{2}}(b_{1/2}^1 - ib_{1/2}^2 + b_{-1/2}^1 + ib_{-1/2}^2), \tag{A.28}$$

such that the map

$$\tilde{U}: \mathcal{H}_v \rightarrow \mathcal{H}_0 \tag{A.29}$$

maps the highest weight vector of the vector module to the highest weight vector

of the singlet,

$$\tilde{U}|\Lambda\rangle_v \equiv \frac{1}{\sqrt{2}}\tilde{U}(b^1_{-1/2} + ib^2_{-1/2})|\text{NS}\rangle = |\text{NS}\rangle, \tag{A.30}$$

and obeys the intertwining property

$$\pi_0 \circ \tilde{\rho}_v(a)\tilde{U} = \tilde{U}\pi_v(a). \tag{A.31}$$

Similarly one verifies that

$$\begin{aligned} \rho_s(H^i_0)|0\rangle &= \frac{1}{2}|0\rangle && \text{for } i = 1, \dots, r, \\ \rho_c(H^i_0)|0\rangle &= \begin{cases} \frac{1}{2}|0\rangle & \text{for } i = 1, \dots, r-1, \\ -\frac{1}{2}|0\rangle & \text{for } i = r \end{cases} \end{aligned} \tag{A.32}$$

for $N = 2r$, and

$$\rho_a(H^i_0)|0\rangle = \frac{1}{2}|0\rangle \quad \text{for } i = 1, \dots, r \tag{A.33}$$

for $N = 2r + 1$, as well as

$$\rho_s(E^{a_i})|0\rangle = \rho_c(E^{a_i})|0\rangle = \rho_a(E^{a_i})|0\rangle = 0 \quad \text{for } i = 1, \dots, r. \tag{A.34}$$

Thus the vacuum state in the NS sector provides us with the highest weight states of the various spinor representations through the action of the corresponding spinor endomorphisms. Therefore the identification maps

$$i_u: \mathcal{H}_u \rightarrow \mathcal{H}_0 \tag{A.35}$$

$$a|\Lambda_u\rangle \mapsto \rho_u(a)|0\rangle$$

for $u = s, c, \sigma$ map the equivalent highest weight modules into each other and obey the properties that the image of the highest weight states are again highest weight states.

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Communicated by J. Fröhlich

