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On the K-Property of Some Planar Hyperbolic Billiards

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Abstract. The K-property is demonstrated for a class of planar billiards satisfying Wojtkowski's principles. Their boundary may consist of convex-scattering, concave and linear pieces. Earlier Wojtkowski showed that these billiards had non-zero Lyapunov exponents.

1. Introduction

A highly intriguing and actual aim of the theory of Hamiltonian dynamical system is to understand the nature of coexistence of integrable and nonintegrable behaviors. There is a natural concensus that the simplest and most hopeful case to be studied is the two-dimensional one but there are different views as to whether which models are easier to attack. Two most concurrent families are billiards and standard maps. [In fact, an interesting example was constructed in Przytycki (1982).]

Billiards show a rich variety of phase portraits also encountered in general Hamiltonian dynamical systems. Nevertheless, their simpler geometrical properties might help to understand this variety more easily and serve as a starting point to learn more about general systems.

In 1979 Bunimovich proved the ergodicity of a billiard in a stadium. After that Wojtkowski (1986) constructed an extension of the class of billiards considered by Bunimovich. Later Markarian (1988) gave another extension of this class of billiards. Both Wojtkowski and Markarian proved that billiards in their classes had non-zero Lyapunov exponents. In the present note we demonstrate that billiards satisfying Wojtkowski's principles are, in fact, ergodic and even K-systems. A better understanding of the mechanism of ergodicity is hoped to provide tractable models where the nature of coexistence of integrable and non-integrable domains in the phase space can be revealed.

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2. Statement of Result

Our starting point is a simple observation according to which Bunimovich's or Wojtkowski's convex scattering billiards can be represented or behave as semi-dispersing billiards. We claim that

(i) by introducing a different, homothetic metric in the phase space depending on the curvature of the boundary;
(ii) and, by improving the technique of showing convergence and continuity of continued fractions;

the approach through the fundamental theorem for semi-dispersing billiards also works for the case when the convex components of the boundary satisfy the **convex-scattering property** of Wojtkowski [W (1986)]: $\frac{d^2r}{ds^2} \leqq 0$, *where* $r(s)$ *is the radius of curvature as the function of the arc length parameter on the curve.* As it was also mentioned in [W (1986)], examples of convex scattering curves include (epi) (hypo) cycloids, the cardioid, etc. For C^4 curves, an equivalent geometric formulation of the convex-scattering property is expressed by Eq. (5).

Let $\mathbf{Q}$ be a bounded, closed domain with a connected interior in the Euclidean plane $\mathbb{R}^2$ or on the 2-D torus $\mathbf{T}^2$ with the flat metric. We assume that the boundary $\partial\mathbf{Q}$ consists of a finite number of smooth non-selfintersecting curves $\partial\mathbf{Q}_i$, $i=1, \ldots, p$, the regular components of $\partial\mathbf{Q}$. The boundary is equipped with a field of unit outer normal vectors $n(q)$, $q \in \partial\mathbf{Q}$. We assume that the curvature of any regular component is either identically zero or does not vanish. Regular boundary components with positive, negative, and zero curvature will be called convex, concave, and neutral, respectively, and the union of all convex, concave, and neutral boundary components will be denoted by $\partial\mathbf{Q}^+$, $\partial\mathbf{Q}^-$, and $\partial\mathbf{Q}^0$.

The class of billiards considered in [W (1986)] is described by the following principles:

P_0: *The set of trajectories confined to neutral boundary components is of measure* 0 *(in particular,* $\mathbf{Q}$ *is not a polygon, i.e.* $\partial\mathbf{Q}^+ \cup \partial\mathbf{Q}^- \neq \emptyset$*).*
P_1: *All the pieces of* $\partial\mathbf{Q}^+$ *have to be convex-scattering.*
P_2: *Any piece of* $\partial\mathbf{Q}^+$ *has to be sufficiently far away from other non-neutral pieces of the boundary.*
P_3: *If two components of* $\partial\mathbf{Q}$ *meet at a vertex, then the internal angle at the vertex has to be bigger than* π *when both pieces are convex, not less than* π *when one piece is convex and the other concave, and bigger than* $\pi/2$ *when one piece is convex and the other is flat.*

The meaning of "sufficiently far away" in P_2 will be seen from the proof, in particular, from the remark after Lemma 4 and from the proofs of Lemmas 6 and 14. Here we just mention that, if applied to one convex piece of the boundary, then this condition is equivalent to requiring that it is, in fact, convex-scattering.

Finally, we have to introduce a mild, additional postulate that can be easily checked in the particular models. It excludes counterexamples of the type of [W (1986)] where several ergodic components appear for the Chernov-Sinai Ansatz is hurt. (In fact, the method of our proof also provides that in Wojtkowski's stadium with semi-ellipses the number of ergodic components is exactly 3 if the stadium is sufficiently long.)

P_4: *The set of trajectories where the dynamics is not defined forms a zero-measure subset of the set of the singular trajectories (with respect to the Riemannian measure).*

The theorem we claim is the following:

Theorem. *Any planar billiard satisfying the principles P_0–P_4 is a K-system.*

3. Proof

For economy of exposition, we rely upon [BUN (1990)] by accepting its notions and notations and by only indicating those points where new ideas are involved.

Denote by $(M, S^{\mathbf{R}}, \mu)$ the flow and by $(\partial M, T^{\mathbf{Z}}, \mu_1)$ its Poincaré section map. In fact, for simplifying the discussion, the Poincaré section is defined by the curved parts of the boundary ∂Q, only, and thus $\partial M := \partial M^+ \cup \partial M^-$, where $\partial M^\pm := \bigcup_{\partial Q_i \in \partial Q^\pm} \partial Q_i \times [-\pi/2, \pi/2]$. The choice of the angular interval indicates our preference to work with incoming velocities. Both the omission of neutral boundary components and the choice of precollision linear elements differs from the usage of [BUN (1990)], but the initial idea is intact: the flow does not possess any good invariance or dilation property for orthogonal curves, but – as a consequence of the principles – the Poincaré section map does!

It is easy to see that from among the conditions of the fundamental theorem of [BUN (1990)], notably conditions **G1–G6** and **A**, conditions **G1–G3** and **A** also hold for the billiards considered in [W (1986)]. First of all, **G4**, requiring that convex arcs be focusing in the sense of [DON (1991)], is a trivial consequence of the convex-scattering property [see (5) below]. Thus our task is to verify conditions **G5** on the local uniform dilation and **G6** on the convergence and continuity of the continued fractions determining the curvatures of the local invariant manifolds. As to the necessary results on absolute continuity, we refer to [K-S (1986)].

The necessary arguments, however, should be preceded by certain general comments on the idea of the proof of the fundamental theorem. It was first proved in the papers of Sinai [S (1970), S (1979)] for dispersing billiards, then by Sinai and Chernov [S-CH (1987)] for semi-dispersing billiards and later modified in [K-S-SZ (1990)]. We want to use the fundamental theorem in a more general form than formulated in [BUN (1990)]. The first remark is that the basic objects of the whole proof are suitably chosen local orthogonal manifolds whose tangents lie in the corresponding element of the invariant cone-field.

The first, and most important deviation from the exposition of [BUN (1990)] is that, on components of ∂M^+, we want to introduce a different metric. For that we have to make an important remark related to the proof of the fundamental theorem. The original proof [S-CH (1987), K-S-SZ (1990)] works with convex local orthogonal manifolds. In the version of [BUN (1990)] these are replaced by expanding curves, i.e. with (projections to the phase space of the Poincaré section map of) local orthogonal curves whose differentials belong to the introduced invariant cone field. Let Σ denote an infinitesimal local orthogonal curve of the flow through a point $x = (q, v^-) \in \partial M$ that is incoming, i.e. is just before collision. For simplicity, we will denote in the same way its natural projection to $\mathbf{Q}$ [the curve in the phase space can be reconstructed from its projection by attaching orthogonal velocities to its (configuration) points]. If $q\in\partial\mathbf{Q}^-$, then we only require that Σ be convex in the classical sense. (In this case, its projection to ∂M is an increasing curve.) If $q\in\partial\mathbf{Q}^+$, then beside convexity we also assume that its curvature B_Σ be bounded: $0\leqq B_\Sigma\leqq\frac{1}{h}$, where $h=k(q)^{-1}\cos\phi$ and $k(q)$ denotes the curvature of the boundary at q and $\cos\phi=(n(q),v^-)$. The projections of these curves to ∂M are called in [BUN (1990)] *expanding ones.* A further important remark is that by changing the metric angles will also change. Having got acquainted with the proof of the fundamental theorem, the reader can, however, convince himself easily that here, too, the continuity of the continued fractions implies the necessary bounds on the angles (these are local!; see Lemma 4.9 in [K-S-SZ (1990)]).

Now the proof of the fundamental theorem uses dilation of expanding curves. To obtain this we are going to introduce a new measure of arc lengths of such infinitesimal curves. The relevant metric of the proofs is always the one defined on curves orthogonal to an incoming vector. Denote this arc length of orthogonal curves by dl_* that at the same time also defines a length for projections of these curves to ∂M (the ratio of the length of an infinitesimal orthogonal curve and of its projection to ∂M is, of course, $\cos\phi$ where ϕ is the incoming angle, cf. [S (1970)]). This arc length dl_* has been used in the earlier papers, e.g. in [S (1970), B-S (1973), S-CH (1987), K-S-SZ (1990), BUN (1990)]. Now the new arc length dl is defined as follows: on infinitesimal, orthogonal curves incoming to components of ∂M^- let $dl=dl_*$ while on those incoming to components of ∂M^+

$$dl=\frac{1}{h}dl_*,$$

where h is as above.

Next recall some necessary notations from [BUN (1990)]: for $x\in\partial M$,

$$\tau(x):=\sup\{t>0:\text{ for all } s,\ 0\leqq s\leqq t,\ S^t x\notin\partial M\},$$

i.e. $Tx=S^{\tau(x)}x$. Further, if Σ denotes a local orthogonal curve, and $x\in\Sigma$, then we denote by $\kappa_0(x)$ its curvature at the point x. If $\Sigma_t=S^t\Sigma$, then we denote by $\kappa_t(x)$ the curvature of Σ_t at the point $S^t x$. We recall the evolution equations for the curvatures of local orthogonal curves: if $S^{[0,t]}x\cap\partial M=\emptyset$, then

$$\kappa_t(x)=\frac{\kappa_0(x)}{1+t\kappa_0(x)},\tag{2-a}$$

while if $S^t x\in\partial M$, then

$$\kappa_{t+}(x)=\kappa_{t-}(x)-\frac{2k(\pi S^t x)}{\cos\phi(S^t x)},\tag{2-b}$$

where $\kappa_{t\pm}(x)$ denote the curvatures of the local orthogonal curves before and after the collision [cf. Sinai (1970)], and, as usual, $\pi: M\to\mathbf{Q}$ denotes the natural projection onto the configuration space.

For an $x\in\partial M$, let Σ denote an incoming expanding curve (just before collision), let Σ_+ denote its image just after reflection and let Σ_1 denote the image of Σ_+ just before it reaches again ∂M: of course, $\Sigma_1=S^{\tau(x)}\Sigma$. Denote finally the curvatures of Σ and Σ_+ at x by κ and κ_+ and the curvature of Σ_1 at Tx by κ_1. Then by (2),

$$\kappa_+=\kappa-\frac{2k(\pi x)}{\cos\phi(x)}$$

and

$$\kappa_1 = \cfrac{1}{\tau + \cfrac{1}{\kappa - \cfrac{2k}{\cos\phi}}}, \tag{3}$$

where we omitted to indicate the dependences on the point x.

Lemma 4. *T maps (pre-collision) expanding curves into (pre-collision) expanding ones.*

Proof was given in [W (1986)]. Since we are starting our proof *in medias res*, for warming up with the notations it is, nonetheless, worth briefly recalling. The statement is trivial if x, $Tx \in \partial M^-$, or if $x \in \partial M^+$ and $Tx \in \partial M^-$ and $\tau(x)$ is sufficiently large (see P_2). If x, $Tx \in \partial M_i^+$, then the condition that Σ is expanding says that $0 \leqq \kappa \leqq \frac{1}{h}$. Thus $-\frac{2}{h} \leqq \kappa_+ \leqq -\frac{1}{h}$, and finally by (3),

$$\frac{1}{\tau - h/2} \leqq \kappa_1 \leqq \frac{1}{\tau - h} \leqq \frac{1}{h_1},$$

where $h_1 = k(\pi Tx)^{-1} \cos\phi(Tx)$, and the inequality

$$\tau \geqq h + h_1 \tag{5}$$

is just another way of expressing the convex-scattering property (cf. [W *(1986)*]). The last inequality should hold again if $x \in \partial M_i^+$, $Tx \in \partial M_j^+$, $i \neq j$ that can be attained on behalf of P_2. The same property trivially settles the case $x \in \partial M^-$, $Tx \in \partial M^+$, too. □

Remark. Since the fundamental theorem is only applicable in neighborhoods of sufficient points in the terminology of [S-CH (1987)] and of [K-S-SZ (1990)] (or in the language of cones one has eventually strict inclusion (cf. [W (1986)]), or else the cones are perfect, cf. [BUN (1990)]), it will be useful to survey which cords lead to immediate sufficiency. This is always the case if a) x, $Tx \in \partial M^-$ or b) x, $Tx \in \partial M_i^+$ and $\tau > h + h_1$. On the other hand, the property P_2 ensures sufficiency whenever c) $x \in \partial M^+$ and $Tx \in \partial M^-$ or d) $x \in \partial M_i^+$ and $Tx \in \partial M_j^+$ $(i \neq j)$ or e) $x \in \partial M^-$ and $Tx \in \partial M^+$.

Lemma 6. *The l-length of (pre-collision) expanding curves does not decrease under T, and it always increases if we have strict inequality in* (5).

Proof. Case a) is trivial. The most interesting case is b). Assume x, $Tx \in \partial M_i^+$. Then, by elementary geometry, we have for the infinitesimal arc-lengths

$$\frac{dl_*(\Sigma_1)}{dl_*(\Sigma)} = \frac{\tau - \frac{1}{|\kappa_+|}}{\frac{1}{|\kappa_+|}}$$

and, consequently, by taking the projections to ∂M and changing the distances from l_* to l, the necessary inequality will be

$$\frac{\dfrac{\tau-\dfrac{1}{|\kappa_+|}}{h_1}}{\dfrac{\dfrac{1}{|\kappa_+|}}{h}} \geqq 1 .$$

Note that, for the pre-collision expanding curves we are working with, the law (1) leads to multiplying their Euclidean arc-length by $\frac{1}{h}$! By simplifying the left-hand side (note that $|\kappa^+|h \geqq 1$!) we obtain

$$\frac{\tau-h}{h_1} \geqq 1 ,$$

but this is again the convex scattering property (5).

Next consider case c). Then

$$\frac{dl(\Sigma_1)}{dl(\Sigma)} = h\frac{dl_*(\Sigma_1)}{dl_*(\Sigma)} = h\tau|\kappa_+| - h \geqq \tau - h > 1 ,$$

where the last, strict inequality can be reached on behalf of P_2. Case d) is similar to b). Finally, in case e)

$$\frac{dl(\Sigma_1)}{dl(\Sigma)} = \frac{1}{h_1}\,\frac{dl_*(\Sigma_1)}{dl_*(\Sigma)} \geqq \frac{dl_*(\Sigma_1)}{dl_*(\Sigma)} > 1$$

whenever $h_1 \leqq r_{\max} \leqq 1$ where

$$r_{\max} := \max\{k(q)^{-1} \colon q \in \partial\mathbf{Q}^+\} .$$

The inequality $r_{\max} < 1$ can be postulated by choosing the length unit appropriately and only then we are allowed to choose $\tau > r_{\max} + 1$ by virtue of P_2 [as needed in case c)]. □

Important Remark. Since we have changed the metric, we should also check whether Lemma 1 of [BUN (1990)] keeps its validity if now the function $z_1 : \partial M \to \mathbb{R}_+$ measuring the distance from the singularities is defined in terms of our new metric. Before giving the precise definition of z_1 we should introduce some notations. Let $T^+ : M \to \partial M$ be given by $T^+(x) := S^{\tau(x)-0}x$ and $T^- : M \to \partial M$ be given by $T^-(x) := -S^{\tau(-x)+0}(-x)$, where $-x = -(q,v) = (q,-v)$. Let, moreover, $\mathscr{SR}^+ \subset \partial M$ be the set of singular reflections (see [K-S-SZ (1990), p. 540]). For $x = (q,v) \notin \partial M$ define the function $w(x)$ as follows. First we want to determine the sets L_r, where $r > 0$ is small ($r \leqq r_0$, say). Set $L_0 := \pi[T^-x, T^+x] \subset \mathbf{Q}$ and assume that, for some small and fixed $\varepsilon > 0$ and some $k \in \mathbf{Z}_+$, $L_{k\varepsilon} \in \mathbf{Q}$ is defined ($k\varepsilon < r_0$). Then, if moreover $(k+1)\varepsilon < r_0$, set $L_{(k+1)\varepsilon} := \bigcup \pi[T^-y, T^+y]$, where the union is taken for every y such that $p(y) = p(x) = v$ and $\pi(y) \in L_{k\varepsilon}^{[\varepsilon]}$ (here A^ε denotes the ε-neighborhood of the set A in $\mathbf{Q}$ measured in Euclidean metric). Since ε is arbitrarily small, this recursion determines $L_r : 0 < r \leqq r_0$ uniquely. Now let

$$z_*(x) := \min\{r_0, \sup\{r \colon r \leqq r_0,\ L_r \cap \mathscr{SR}^+ = \emptyset\}\}$$

and for $x \in \partial M$ extend this equation by assuming that $z_*(S^t x)$ is left-continuous in t. In plain words, $z_*(x)$ is the width of the maximal tubular configurational neighborhood of a trajectory segment $\pi[T^- x, T^+ x]$ disjoint of any singularities if lifted to the phase space with velocities parallel to $p(x)$. Finally, set $z(x) := z_*(x)$ if $x \in \partial M^-$ and $z(x)$ = width of the same maximal tubular neighborhood as before but measured in the metric dl rather than in dl_* if $x \in \partial M^+$. It is now easy to see that

$$\mu_1\{z(x) < \varepsilon\} \leqq \mu_1\{z_*(x) < \max\{1, r_{\max}\} \cdot \varepsilon\},$$

implying that the aforementioned Lemma 1 still holds. We suggest that the reader convince himself that this inequality is, indeed, the one that we need in the proof of the Tail Bound (cf. [S-CH (1987, K-S-SZ (1990)]). □

Assume now for a moment that $x \in \partial M$ is fixed and denote by Σ and Σ' any pair of (pre-collision) expanding local orthogonal curves through x and by Σ_n and Σ'_n their (pre-collision) images under T^n. Introduce some shorthand notations:

$$\begin{aligned} &\tau_n := \tau(T^n x), \qquad \cos\phi_n := (n(\pi T^n x), v^-(T^n x)), \\ &k_n := k(\pi T^n x), \qquad \chi_n := \frac{k_n}{\cos\phi_n}, \\ &\kappa_n^{(\prime)} := \text{curvature of } \Sigma_n^{(\prime)} \quad (n \in \mathbf{Z}). \end{aligned} \tag{7}$$

If $n=0$, then the index 0 will be omitted, e.g. for $x \in \partial M^+$ we write $\chi = -\frac{1}{h}$.

Lemma 8. *For any* $x \in \partial M$

$$\kappa_1 - \kappa'_1 = \frac{1}{[1+\tau(2\chi+\kappa)][1+\tau(2\chi+\kappa')]}(\kappa - \kappa'). \tag{9}$$

Proof follows directly from (3). □

Corollary 10. *If* $x \in \partial M^-$, *then*

$$|\kappa_1 - \kappa'_1| \leqq \frac{1}{(1+2\tau\chi_{\min})^2}|\kappa - \kappa'|,$$

where $\chi_{\min} := \min\{k(\pi x) \colon x \in \partial M^-\}$.

Corollary 11. *If* $x \in \partial M^+$, *then*

$$|\kappa_1 - \kappa'_1| \leqq \frac{h^2}{(\tau - h)^2}|\kappa - \kappa'|.$$

Proof. For the denominator of the fraction on the right-hand side of (9) we have

$$\left(\tau\frac{2}{h} - \tau\kappa - 1\right)\left(\tau\frac{2}{h} - \tau\kappa' - 1\right) \geqq \left(\frac{\tau}{h} - 1\right)^2, \tag{12}$$

since $0 \leqq \kappa, \kappa' \leqq \frac{1}{h}$. Hence the statement. □

Since Corollary 11, unlike Corollary 10, does not ensure the contraction we would like to use, it is again useful to change the units on ∂M^+. For a (pre-collision) expanding local orthogonal curve through x let

$$\zeta(x) := \begin{cases} \kappa(x) & \text{if} \quad x \in \partial M^-, \\ h(x)^2 \cdot \kappa(x) & \text{if} \quad x \in \partial M^+. \end{cases} \tag{13}$$

All other forthcoming notations will be used with natural modifications of the previous ones.

Lemma 14. *Under the constraints used in the proof of Lemma 6, for any pair of (pre-collision) expanding local orthogonal curves we have*

$$|\zeta_1-\zeta_1'|<|\zeta-\zeta'|.$$

Proof. Consider again the cases of the remark after Lemma 4 separately. Case a) is trivial by Corollary 10 and (13). In cases b) and d) a simple calculation yields that

$$|\zeta_1-\zeta_1'|<\frac{h_1^2}{(\tau-h)^2}|\zeta-\zeta'|.$$

This inequality implies for case c) the relation

$$|\zeta_1-\zeta_1'|<\frac{1}{(\tau-h)^2}|\zeta-\zeta'|,$$

while for case e) we deduce from Corollary 10 that

$$|\zeta_1-\zeta_1'|<h_1^2|\zeta-\zeta'|.$$

Now exactly then, when the expansion bounds provide the statement of Lemma 6, one also obtains the contraction claimed in our lemma. □

Proposition 15. *Except for a finite number of phase points $x\in\partial M$ the following is true: For every $\varepsilon>0$ there exists a positive integer n_ε and a suitably small neighborhood $U_\varepsilon(x)$ such that for every $n>n_\varepsilon$ and almost every $y\in U_\varepsilon(x)$ and every (pre-collision) expanding local orthogonal curve Σ $(T^{-n}y\in\Sigma)$,*

$$|\kappa^{(u)}(x)-\kappa_{T^n\Sigma}(y)|<\varepsilon.$$

If $T^{\mathbf{Z}^-}x$ intersects a singularity (singularities), then the statement is true on each component of the neighborhood $U_\varepsilon(x)$ where all non-positive powers of T are smooth.

Proof. It is sufficient to prove the statement for phase points x whose negative semitrajectories do not intersect singularities. By (3) we have

$$\kappa_{T^n\Sigma}(y)=\cfrac{1}{\tau_{-1}(y)+\cfrac{1}{\cfrac{2k_{-1}(y)}{\cos\phi_{-1}(y)}+\cfrac{1}{\tau_{-2}(y)+\cfrac{1}{\cfrac{2k_{-2}(y)}{\cos\phi_{-2}(y)}+\cfrac{1}{\tau_{-n}(y)+\cfrac{1}{\kappa_\Sigma(T^{-n}y)}}}}}}. \tag{18}$$

First we claim that there exists an n_ε such that for any (pre-collision) expanding local orthogonal curve Σ $(T_nx\in\Sigma)$

$$|\kappa^{(u)}(x)-\kappa_{T^n\Sigma}(x)|<\varepsilon$$

holds. A glance at Corollary 10 and Lemma 14 should convince the reader about the truth of the claim whenever $T^{\mathbf{Z}^-}x$ possesses the following property: *there exist no index i and no time moment $\tilde{n}$ such that for any $n \geqq \tilde{n}$, $T^{-n}x \in \partial M_i^+$*. Indeed, since $\chi_{\min} > 0$, the contraction claimed in Corollary 10 and Lemma 14 is also uniform except for the case when τ becomes arbitrarily small. This is only possible close to the intersections of components of $\partial \mathbf{Q}^-$, and even then for a bounded number of consecutive collisions after which τ should become larger than a suitable small threshold (cf. [B-S (1973)]). Hence the claim.

The claim, however, implies the statement of the lemma, by the continuity of the finite continued fractions (18). □

Since Conditions **G5** and **G6** of [BUN (1990)] are ensured by our Lemmas 6 and 15, the theorem now follows from the appropriately modified version of the fundamental theorem of the aforementioned work. Indeed, if once we have local ergodicity apart from a finite number of points, then global ergodicity – also implying the K-property – trivially follows.

4. Remarks

1. A further interesting question is whether and when generalizations [by Markarian (1988 and 1991), Donnay (1991), and Bunimovich (1991)] of Bunimovich's and Wojtkowski's billiards known to have non-zero Lyapunov-exponents are, moreover, ergodic. There are two recent announcements by Bunimovich (1991) and Markarian (1991) promising progress in this direction. The first work also announces an interesting converse statement.

It is worth stressing that the billiards considered here are not everywhere smooth. In fact, examples of 2-dimensional billiards in smooth convex domains had been known to be integrable well before Lazutkin (1973) [and later Douady (1982)] established that they, in general, possess caustics, topological objections for their being ergodic. [Here the role of dimension is quite important. According to a recent result of Berger (1990), if a local surface in the three-dimensional Euclidean space admits a caustic, then it is necessarily part of an ellipsoid.]

2. A further phenomenon to be understood is the role of zero curvatures in convex-scattering components of the boundary. If this happens, then our method breaks down, though as we know from Mather (1982), zero curvature excludes the possibility of having invariant tori close to the boundary, an effect that should, in fact, promote chaotic behavior.

3. We expect that, by using the methods of [CH (1991)] and our Proposition 17, Sinai's entropy formula can also be established for the billiards investigated here.

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