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BRST Analysis of Physical States for 2D Gravity Coupled to $c \leqq 1$ Matter

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Abstract. We consider 2D gravity coupled to $c \leqq 1$ conformal matter in the conformal gauge. The Liouville system is represented by a free scalar field, ϕ^L, with background charge such that the BRST operator imposing reparametrization invariance is nilpotent. We compute the cohomology of this BRST charge on the product of the Fock space of ϕ^L with those of the ghosts and one other free scalar field, ϕ^M, representing the matter system. From this calculation the physical states of the full theory are determined. For the $c < 1$ case the further projection from the Fock space of ϕ^M to the irreducible representation, using Felder's resolution, reproduces the results of Lian and Zuckerman.

1. Introduction

Matrix model techniques appear to give a great deal of information on discretized gravity coupled to $c^M \leqq 1$ matter in two dimensions. To put these insights to use it is clearly necessary to understand the continuum theory, and thus the Liouville dynamics to which it reduces in the chiral gauge [1, 2], and conformal gauge [3, 4]. Upon gauge fixing, the continuum theory factorizes into matter, Liouville, and ghost sectors coupled by the BRST constraint imposing diffeomorphism invariance. Indeed in the same way the relevance of Liouville dynamics to string theory was already well understood [5], and several groups went on to study the Liouville theory in detail [6–9]. From the work in [8–10], and as is consistent with semiclassical calculations [11–14], free field techniques may be used to advantage in the Liouville theory. Further, the free field description of minimal matter theories is well developed [15–17]. Application of these techniques to the full theory requires at least the description of the physical spectrum, and construction of the correlators of physical operators (see e.g. [18–25] for some recent developments). In this paper we discuss the former.

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The physical states are identified with nontrivial cohomology classes of the BRST operator, d. We will explicitly compute this cohomology for a free boson of $c^M=1$, the "$D=1$ string," and for the minimal models of $c^M<1$. In the latter problem we employ the free field resolution [17], and thus both problems reduce to calculation of cohomology in a complex consisting of Fock spaces. The main technical result of this paper is the computation of this cohomology summarized in Theorem 3.3. It may be applied directly to obtain the physical states for the case $c^M=1$, and we find that they occur for at most three ghost numbers. For ghost number zero such states are known [24, 26], they occur at conformal weights corresponding to singular vectors in the matter Fock space. For $c^M<1$, upon imposing the projection implied by Felder's resolution [17], we reproduce precisely the results of [27]; i.e. there is an infinite set of physical states at different ghost numbers, appearing at the conformal weights where singular vectors arise in the Verma modules built on the matter primary states.

Our computation of the BRST cohomology on Fock spaces has its roots in the analogous problem for the critical string [28–35]. However, apart from similarities between the two computations which we indicate in the text, there is an important difference due to the presence of background charges. In particular, this novel feature is responsible for the absence of the so-called "vanishing theorem" [30–32], i.e. in Fock spaces with special discrete values of the momenta there are nontrivial cohomology states at different ghost numbers. We develop a rather simple and systematic method for computing this cohomology which exploits the presence of a grading of the Fock space in addition to that by the ghost number. One can view this either as a streamlined approach to the Kugo-Ojima quartets [36, 28], or a simple case of a spectral sequence analogous to the one discussed in more advanced analyses of the critical string theory [34, 35] or the BRST cohomology in general [30, 37]. For the sake of simplicity we decided not to introduce any machinery of homological algebra, and try to give elementary and explicit proofs.

The $c^M=1$ case has been recently discussed in [26]. They used methods developed in [33, 38], but found that the general case could not be analyzed this way. Thus our work for this case can be thought of as the "more complete classification of BRST cohomology" asked for in that paper.

It is interesting that the complications discussed above can be circumvented for $c^M<1$ – where the matter sector is taken to be the (p, p') minimal model [39] with the representation of the Virasoro algebra in the fundamental range $1 \leqq m \leqq p-1$, $1 \leqq m' \leqq p'-1$ [40] – provided one chooses a suitable free field resolution. We should note that this case has already been discussed as a problem of Fock space cohomology – albeit in chiral gauge – in [41] (see also [42, 43]). The results obtained there were not quite complete, as evidenced later by the work of [27]. As we will see this is precisely because the projection required by the Felder resolution was not enforced.

A complete analysis of the $c^M<1$ minimal model case has been given by Lian and Zuckerman [27], who first classified the BRST cohomology of the Verma modules (and their irreducible quotients), and then use these results to determine which Feigin-Fuchs modules of the Liouville sector afford the nontrivial cohomology. Our rederivation of their results is motivated by the desire to remain within the framework of Fock spaces, providing a discussion we believe to be more accessible to a physicist for this problem.

The paper is presented as follows. In Sect. 2 we gather the definitions of the basic objects we use, and recall the concept of relative cohomology of the BRST

operator, d. We go on to compute this on the product of Fock space modules in Sect. 3, and obtain from these results the total cohomology of d in Sect. 4. The previous two sections, together with Appendix A, constitute the technical part of the paper. In particular, Appendix A gives a straightforward and detailed explanation of how such calculations are done. The following two sections, 5 and 6, contain our applications; namely, the results indicated above for $c^M \leqq 1$ matter. In Appendix B we have collected a few facts about Schur polynomials which are useful for Sect. 5. We end with a couple of comments, and outlook for further work.

2. Notations and Conventions

For an arbitrary Virasoro module $\mathscr{V}$, the constraint $T(z) \sim 0$ can be implemented by the BRST operator

$$d = \oint \frac{dz}{2\pi i} : \left(T(z) + \frac{1}{2} T^G(z) \right) c(z) : , \tag{2.1}$$

acting on the tensor product module $\mathscr{V} \otimes \mathscr{F}^G$, where $\mathscr{F}^G$ is the Fock space of the spin $(2, -1)$ bc-ghosts, and $T^G(z)$ is the corresponding stress energy tensor. The BRST operator d is nilpotent provided the central charge of $\mathscr{V}$ is equal to 26 [28, 29, 44, 30], in which case we can study its cohomology. We will refer to the latter as the BRST cohomology of $\mathscr{V}$ [44, 30].

In this paper we will mainly be concerned with the case in which $\mathscr{V}$ is the product of two Fock spaces of free scalar fields with background charges, one corresponding to conformal matter and the other representing the Liouville field of 2D quantum gravity. For both the Liouville and the matter sector we will take the following convention for the stress energy tensor:

$$T(z) = -\tfrac{1}{2} : \partial\phi(z)\partial\phi(z) : + iQ\partial^2\phi(z), \tag{2.2}$$

where the scalar field has two-point function

$$\langle \phi(z)\phi(w) \rangle = -\ln(z-w). \tag{2.3}$$

The central charge is given by

$$c = 1 - 12Q^2. \tag{2.4}$$

We will denote the Fock space built on the vacuum state $|p\rangle$ with momentum p by $\mathscr{F}(p)$. The conformal dimension of the corresponding Virasoro representation is

$$\Delta(p) = \tfrac{1}{2} p(p-2Q). \tag{2.5}$$

In terms of modes $i\partial\phi(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1}$, $p = \alpha_0$,

$$L_n = \tfrac{1}{2} \sum_{m \in \mathbb{Z}} : \alpha_m \alpha_{n-m} : - (n+1) Q \alpha_n, \tag{2.6}$$

where

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0}. \tag{2.7}$$

In the remainder we will distinguish between the Liouville and matter fields by writing superscripts L and M respectively. Further, throughout the paper we are

using the normal ordering of operators with respect to the $SL(2,\mathbb{R})$ vacuum. The BRST operator d written in terms of modes is then

$$d=\sum_{n\in\mathbb{Z}} c_{-n}(L_n^M+L_n^L)-\tfrac{1}{2}\sum_{m,n\in\mathbb{Z}}(m-n):c_{-m}c_{-n}b_{m+n}:. \tag{2.8}$$

Requiring that the total central charge adds up to zero gives[1]

$$(Q^M)^2+(Q^L)^2=-2. \tag{2.9}$$

For convenience we will denote the (physical) vacuum of the Fock space $\mathscr{F}(p^M,p^L)\equiv\mathscr{F}^M(p^M)\otimes\mathscr{F}^L(p^L)\otimes\mathscr{F}^G$ by $|p^M,p^L\rangle$, i.e.

$$|p^M,p^L\rangle=|p^M\rangle_M\otimes|p^L\rangle_L\otimes c_1|0\rangle_G, \tag{2.10}$$

and normalize the ghost number (gh) such that $|p^M,p^L\rangle$ has ghost number zero and d has ghost number one.

The structure of the BRST operator (2.8) is similar to that in the usual bosonic string [28–30], in particular we can decompose it with respect to the ghost zero modes as follows:

$$d=c_0L_0-b_0M+\hat{d}, \tag{2.11}$$

where

$$\begin{aligned} L_0&=L_0^M+L_0^L+L_0^G, \qquad M=\sum_{n\neq 0} n:c_{-n}c_n:,\\ \hat{d}&=\sum_{n\neq 0} c_{-n}(L_n^M+L_n^L)-\frac{1}{2}\sum_{\substack{m,n\neq 0\\ m+n\neq 0}}(m-n):c_{-m}c_{-n}b_{m+n}:. \end{aligned} \tag{2.12}$$

The nilpotency of d is equivalent to the following set of identities:

$$\hat{d}^2=L_0M, \qquad [\hat{d},L_0]=[\hat{d},M]=[L_0,M]=0. \tag{2.13}$$

Since $L_0=\{b_0,d\}$ it is clear, by the same reasoning as for the bosonic string [30–32], that the cohomology of d must be contained in the zero eigenspace of L_0. In fact it is convenient to reduce this subspace further by restricting to the states which are annihilated by the antighost zero mode b_0. On this space, which we denote $\mathscr{F}_0(p^M,p^L)$, the restriction of d coincides with $\hat{d}$, and the cohomology states ψ satisfying

$$L_0\psi=0, \qquad b_0\psi=0, \tag{2.14}$$

correspond to the so-called *relative* cohomology of d [30], the computation of which will be the subject of the next section. We will return to the *absolute*, i.e. full, cohomology of d later.

3. The Relative Cohomology of d on $\mathscr{F}^M(p^M)\otimes\mathscr{F}^L(p^L)\otimes\mathscr{F}^G$

The basic tool in computing the relative cohomology of d is a suitable grading of the Fock space $\mathscr{F}(p^M,p^L)$, which allows to reduce the problem to that of studying the cohomology of a simpler operator. The basic results on cohomology of such complexes are briefly discussed in Appendix A.

[1] Note that in our conventions both p^L and Q^L are purely imaginary

It is convenient to first introduce "lightcone-like" linear combinations of the scalar fields [28, 41] with the modes

$$q^{\pm}=\sqrt{\tfrac{1}{2}}(q^M \pm iq^L), \qquad p^{\pm}=\sqrt{\tfrac{1}{2}}((p^M-Q^M)\pm i(p^L-Q^L)),$$
$$\alpha_n^{\pm}=\sqrt{\tfrac{1}{2}}(\alpha_n^M \pm i\alpha_n^L), \qquad n\neq 0. \tag{3.1}$$

The nonvanishing commutation relations are

$$[q^{\pm},p^{\mp}]=i, \qquad [\alpha_m^{\pm},\alpha_n^{\mp}]=m\delta_{m+n,0}. \tag{3.2}$$

Furthermore we define a set of generalized momentum variables

$$P^{\pm}(n)=\sqrt{\tfrac{1}{2}}((p^M-(n+1)Q^M)\pm i(p^L-(n+1)Q^L)). \tag{3.3}$$

In particular, $p^{\pm}=P^{\pm}(0)$. In terms of these operators we have

$$L_0=p^+p^-+\hat{L}_0=p^+p^-+\sum_{n\neq 0}(:\alpha_{-n}^+\alpha_n^-:+n:c_{-n}b_n:)+1, \tag{3.4}$$

and

$$\hat{d}=\sum_{n\neq 0} c_{-n}(\alpha_n^- P^+(n)+\alpha_n^+ P^-(n))$$
$$+\sum_{\substack{n,m\neq 0\\ m+n\neq 0}} :c_{-n}(\alpha_{-m}^+\alpha_{m+n}^- +\tfrac{1}{2}(m-n)c_{-m}b_{m+n}):. \tag{3.5}$$

We introduced here $\hat{L}_0$ to denote the level operator for the oscillators $\{\alpha_{-n}^-, \alpha_{-n}^+, b_{-n}, c_{-n}\}$, $n>0$, with respect to the physical vacuum.

From (3.4) we see that $\mathscr{F}_0(p^M,p^L)$ (recall that this is the subspace annihilated by L_0 and b_0) is nontrivial provided p^+p^- takes a nonpositive integer value, and is always finite-dimensional. For the special case $p^+p^-=0$, $\mathscr{F}_0(p^M,p^L)$ consists of a single state, the vacuum, which is the relative cohomology state.

In general to compute the cohomology of $\hat{d}$ on $\mathscr{F}_0(p^M,p^L)$ we must consider two cases:

I. Either $P^+(n)\neq 0$ or $P^-(n)\neq 0$ for all $n\in\mathbb{Z}$, $n\neq 0$;
II. There exist $r, s\in\mathbb{Z}$ such that $P^+(r)=0$ and $P^-(s)=0$.

Case I. We may suppose $P^+(n)\neq 0$ for all $n\neq 0$. The other case, $P^-(n)\neq 0$, can be analyzed similarly. The Fock space $\mathscr{F}(p^M,p^L)$ can be decomposed into a direct sum of subspaces of definite *degree*. We define the degree (deg) of the vacuum state $|p^M,p^L\rangle$ to be zero and assign the following degree to the oscillators;

$$\deg(\alpha_n^+)=\deg(c_n)=+1,$$
$$\deg(\alpha_n^-)=\deg(b_n)=-1. \tag{3.6}$$

The decomposition of $\hat{d}$ into components of definite degree is [compare (A.2) in Appendix A]

$$\hat{d}=\hat{d}_0+\hat{d}_1+\hat{d}_2, \tag{3.7}$$

where

$$\hat{d}_0=\sum_{n\neq 0} P^+(n)c_{-n}\alpha_n^-,$$
$$\hat{d}_1=\sum_{\substack{n,m\neq 0\\ m+n\neq 0}} :c_{-n}(\alpha_{-m}^+\alpha_{m+n}^- +\tfrac{1}{2}(m-n)c_{-m}b_{m+n}):, \tag{3.8}$$
$$\hat{d}_2=\sum_{n\neq 0} P^-(n)c_{-n}\alpha_n^+,$$

satisfy, when acting on $\mathscr{F}_0(p^M, p^L)$,

$$\hat{d}_0^2 = \hat{d}_2^2 = 0, \quad \{\hat{d}_0, \hat{d}_1\} = \{\hat{d}_1, \hat{d}_2\} = 0, \quad \hat{d}_1^2 + \{\hat{d}_0, \hat{d}_2\} = 0. \tag{3.9}$$

Our strategy now is to compute first the cohomology of $\hat{d}_0$ and then use the results of Appendix A to determine the relative cohomology of d. Note that $\hat{d}_0$ is nilpotent on the entire Fock space, and moreover $[\hat{d}_0, L_0] = \{\hat{d}_0, b_0\} = 0$. Thus one can as well first compute the cohomology of $\hat{d}_0$ on $\mathscr{F}(p^M, p^L)$ and afterwards restrict it to the subspace of the relative cohomology.

Lemma 3.1. *Suppose $P^+(n) \neq 0$ for all $n \neq 0$, then*

$$H^{(n)}(\mathscr{F}(p^M, p^L), \hat{d}_0) = \begin{cases} \mathbb{C} & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \neq 0 \end{cases},$$

where the nontrivial cohomology is represented by $|p^M, p^L\rangle$. This state survives the projection onto the $L_0 = 0$ subspace iff $p^+ p^- = 0$.

Proof. Define an operator

$$K = \sum_{n \neq 0} \frac{1}{P^+(n)} \alpha^+_{-n} b_n. \tag{3.10}$$

Then one easily checks that K satisfies

$$\{\hat{d}_0, K\} = \hat{L}_0, \tag{3.11}$$

i.e. it is a *contracting homotopy operator* for $\hat{d}_0$. This shows in particular that any $\hat{d}_0$-closed state of nonzero level is $\hat{d}_0$-exact. □

Case II. Suppose there exist integers $r, s \neq 0$ such that $P^+(r) = 0$ and $P^-(s) = 0$. It follows that

$$\begin{aligned} p^M - Q^M &= \tfrac{1}{2}((r+s)Q^M + i(r-s)Q^L), \\ i(p^L - Q^L) &= \tfrac{1}{2}((r-s)Q^M + i(r+s)Q^L), \end{aligned} \tag{3.12}$$

hence

$$\begin{aligned} P^+(n) &= \sqrt{\tfrac{1}{2}}(Q^M + iQ^L)(r-n), \\ P^-(n) &= \sqrt{\tfrac{1}{2}}(Q^M - iQ^L)(s-n). \end{aligned} \tag{3.13}$$

In particular

$$p^+ p^- = P^+(0) P^-(0) = -rs, \tag{3.14}$$

from which we conclude that there exist states with $L_0 = 0$ only if $rs > 0$. From (2.9) we find $Q^M \neq \pm iQ^L$ which shows that $P^+(n) \neq 0$ for $n \neq r$, and $P^-(n) \neq 0$ for $n \neq s$.

Lemma 3.2. *Let $P^+(r) = P^-(s) = 0$ for some integers $r, s \neq 0$, $rs > 0$. The cohomology of $\hat{d}_0$ on $\mathscr{F}_0(p^M, p^L)$ is nontrivial for precisely two ghost numbers and, depending on the sign of r and s, is represented by the states:*
(i) *for $r, s > 0$,*

$$(\alpha^+_{-r})^s |p^M, p^L\rangle \quad \text{and} \quad c_{-r}(\alpha^+_{-r})^{s-1} |p^M, p^L\rangle; \tag{3.15}$$

(ii) *for* $r, s<0$,

$$(\alpha_r^-)^{-s}|p^M, p^L\rangle \quad \textit{and} \quad b_r(\alpha_r^-)^{-s-1}|p^M, p^L\rangle. \tag{3.16}$$

Proof. To prove (i) consider $K_r = \sum_{n\neq 0, r} \frac{1}{P^+(n)} \alpha_{-n}^+ b_n$. Then $\hat{L}_{0,r} \equiv \{\hat{d}_0, K_r\}$ is the level operator for all the oscillators except α_{-r}^+ and c_{-r} with which it commutes. It also commutes with $\hat{d}_0$. Thus the cohomology of $\hat{d}_0$ must be contained within the subspace spanned by the states in (3.15). One verifies by inspection that these states are indeed nontrivial cohomology states. The proof of (ii) is analogous.[2] □

Lemmas 3.1 and 3.2 completely classify the nontrivial cohomology of $\hat{d}_0$. We observe that this cohomology, for each ghost number in which it is nontrivial, occurs for precisely one degree. Thus we can use the general result of Theorem A.3 in Appendix A to conclude that there is a one to one correspondence between the cohomology states of $\hat{d}_0$ and the relative cohomology of d. We can summarize our computation of the relative cohomology as follows:

Theorem 3.3. *We distinguish three different cases in which the* $H_{\text{rel}}^{(*)}(\mathscr{F}(p^M, p^L), d)$ *is nontrivial*

i) *If* $P^+(r)\neq 0$ *or* $P^-(s)\neq 0$ *for all* $r, s\in\mathbb{Z}$, $r, s\neq 0$, *and* $p^+p^-=0$ *then*

$$H_{\text{rel}}^{(n)}(\mathscr{F}(p^M, p^L), d) = \begin{cases} \mathbb{C} & \textit{for} \quad n=0, \\ 0 & \textit{otherwise}. \end{cases}$$

ii) *If there exist* $r, s\in\mathbb{Z}_+$ *such that* $P^+(r)=0$ *and* $P^-(s)=0$ *then*

$$H_{\text{rel}}^{(n)}(\mathscr{F}(p^M, p^L), d) = \begin{cases} \mathbb{C} & \textit{for} \quad n=0, 1, \\ 0 & \textit{otherwise}. \end{cases}$$

iii) *If there exist* $r, s\in\mathbb{Z}_-$ *such that* $P^+(r)=0$ *and* $P^-(s)=0$ *then*

$$H_{\text{rel}}^{(n)}(\mathscr{F}(p^M, p^L), d) = \begin{cases} \mathbb{C} & \textit{for} \quad n=0, -1, \\ 0 & \textit{otherwise}. \end{cases}$$

In all other cases $H_{\text{rel}}^{(*)}(\mathscr{F}(p^M, p^L), d)=0$.

At this stage it is worth comparing to the analogous result for the critical bosonic string [28–35], for which the generic situation is case *I* and the relative cohomology states can be built using purely transverse oscillators. The only exception is when the momentum is zero, which is the precursor to the cases ii) and iii) in Theorem 3.3. However, since $L_0=0$ this is just a vacuum state, and can be treated easily as a special case.

The characterization of the relative cohomology in Theorem 3.3 is rather abstract and for specific applications it may be desirable to construct explicit representatives of these cohomology classes. In case i) the relative cohomology is clearly generated by the vacuum state $|p^M, p^L\rangle$. In the remaining two cases the proof of Theorem A.3 provides a general – although probably not the most

[2] It might appear that the above analysis also holds for $P^+(r)=0$ and $P^-(n)\neq 0$, $\forall n\in\mathbb{Z}$. However one easily checks that the equation $L_0=0$ cannot be satisfied for the nontrivial $\hat{d}_0$ states, so that there is no contradiction with the calculation in *I*

efficient – procedure for computing these representatives. We will now briefly summarize this method by specifying it to the present situation.

Let ψ_0 be the $\hat{d}_0$ cohomology state listed in Lemma 3.2 corresponding to particular r, s, and n from case ii) or iii). Then $\hat{d}_1\psi_0$ is $\hat{d}_0$-closed, i.e. we have $\hat{d}_1\psi_0 = -\hat{d}_0\psi_1$. To construct ψ_1 we first note that $K_r\hat{d}_1\psi_0$ is an eigenstate of $\hat{L}_{0,r}$ with a nonzero eigenvalue, and we take

$$\psi_1 = -\hat{L}_{0,r}^{-1}K_r\hat{d}_1\psi_0 . \tag{3.17}$$

We then proceed by induction. Having constructed $\psi_0, \psi_1, \ldots, \psi_k$ we define

$$\psi_{k+1} = -\hat{L}_{0,r}^{-1}K_r(\hat{d}_1\psi_k + \hat{d}_2\psi_{k-1}) . \tag{3.18}$$

A more detailed analysis shows that all the steps are well defined and that the procedure terminates after a finite number of steps. One then verifies that the state $\psi = \psi_0 + \psi_1 + \ldots$ is the desired representative of the relative cohomology class.

In case ii) this construction simplifies dramatically because $\hat{d}_2$ annihilates ψ_0 and, subsequently, also ψ_1, ψ_2, etc. Introduce an operator

$$T_r = \hat{L}_{0,r}^{-1}K_r\hat{d}_1 . \tag{3.19}$$

Then we verify that $(T_r)^n\psi_0$ is well defined for $n>0$ and vanishes for $n>rs$. The result of the calculation above can be summarized by

Lemma 3.4. *The relative cohomology state in case* ii) *of Theorem 3.3 is given by*

$$\psi = \sum_{n=0}^{\infty} (-1)^n (T_r)^n \psi_0 = \frac{1}{1+T_r}\psi_0 , \tag{3.20}$$

where, depending on the ghost number, ψ_0 is one of the states in (3.15). *For ghost number zero we can take*

$$T_r = \hat{L}_{0,r}^{-1} \sum_{\substack{i,j>0 \\ 1<i+j\leq r}} \frac{1}{P^+(i)} \alpha_{-i}^+ \alpha_{-j}^+ \alpha_{i+j}^- . \tag{3.21}$$

4. The Absolute Cohomology of d on $\mathscr{F}^M(p^M)\otimes\mathscr{F}^L(p^L)\otimes\mathscr{F}^G$

In general the relative and absolute cohomologies are related via a long exact cohomology sequence [30]. In the present case, given the relative cohomology obtained in the previous section, the computation of the full cohomology of d on $\mathscr{F}^M(p^M)\otimes\mathscr{F}^L(p^L)\otimes\mathscr{F}^G$ is straightforward and we will keep our discussion elementary. First we prove a technical lemma

Lemma 4.1. *If $L_0\psi = b_0\psi = 0$ then $\hat{d}\psi = 0$ implies that $M\psi = \hat{d}\chi$, for some χ in $\mathscr{F}_0(p^M, p^L)$.*

Proof. Using (2.12) and (2.13) we deduce that $\hat{d}M\psi = 0$ and $\mathrm{gh}(\hat{d}M\psi) = \mathrm{gh}(\psi) + 3$ while by Theorem 3.3 a nontrivial relative cohomology can occur in at most two subsequent ghost numbers. □

Theorem 4.2.

$$H_{\mathrm{abs}}^{(*)}(\mathscr{F}(p^M, p^L), d) \cong H_{\mathrm{rel}}^{(*)}(\mathscr{F}(p^M, p^L), d) \oplus c_0 H_{\mathrm{rel}}^{(*-1)}(\mathscr{F}(p^M, p^L), d) . \tag{4.1}$$

Proof. Each ψ in the relative cohomology gives rise to two d-closed elements, the first, ψ_I, being ψ itself and the second $\psi_{II}=c_0\psi-\chi$, where $\chi\in\mathscr{F}_0(p^M,p^L)$ satisfies $M\psi=\hat{d}\chi$. By Lemma 4.1 such a χ always exists. Next, we will show that ψ_I and ψ_{II} are *not* d-exact when ψ is not $\hat{d}$-exact. Indeed, suppose first that $\psi=d\phi$, where $\phi=\phi_0+c_0\phi_1$ with $L_0\phi_0=L_0\phi_1=0$ and $b_0\phi_0=b_0\phi_1=0$. Then $\hat{d}\phi_0-M\phi_1=\psi$ and $\hat{d}\phi_1=0$. From the latter equation and the difference of two between the ghost numbers of ψ and ϕ_1, we find $\phi_1=\hat{d}\chi_1$, which substituted into the first equations yields a contradiction. In the case of the second state we would find similarly the contradiction $\psi=-\hat{d}\phi_1$.

We must still show that the resulting d-cohomology does not depend on the choice of ψ and χ. For ψ_I this is clear. If $\chi\to\chi+\hat{d}\phi$ then $\psi_{II}\to\psi_{II}+d\phi$, since $b_0\phi=L_0\phi=0$. For $\psi=\hat{d}\phi$ we simply have $\psi_{II}=d(-c_0\phi)$. This proves that $H^{(*)}_{\text{abs}}(\mathscr{F}(p^M,p^L),d)$ contains the right-hand side of (4.1).

Now let $\psi=\psi_0+c_0\psi_1$, $\psi_i\in\mathscr{F}_0(p^M,p^L)$, represent a nontrivial cohomology class of d. Using Lemma 4.1 we deduce from $d\psi=0$ that

$$\psi=(\psi_0-\chi_1)+(\chi_1+c_0\psi_1)\,,\qquad \hat{d}\chi_1=M\psi_1\,, \tag{4.2}$$

where $\hat{d}(\psi_0-\chi_1)=0$. This corresponds to the decomposition of ψ according to the right-hand side of (4.1), with $\psi_0-\chi_1$ and ψ_1 representing relative cohomology classes which depend only on the cohomology class of ψ. Indeed, if $\psi\to\psi+d\phi$, $\phi=\phi_0+c_0\phi_1$ then $\psi_0-\chi_1\to\psi_0-\chi_1+\hat{d}\phi_0$ and $\psi_1\to\psi_1-\hat{d}\phi_1$. Finally, by the first part of the proof they cannot be simultaneously trivial. □

From the discussion below Theorem 3.3, it is clear that for the critical bosonic string the relative cohomology states are annihilated by M, and the result is as above with $\chi=0$ [30–32].

5. 2D-Gravity Coupled to $c=1$ Matter

In this section we will specialize the result of the previous sections to the case $c^M=1$, i.e. the one-dimensional noncritical string. Note that in general all cases in Theorem 3.3 can arise, and thus physical states will appear in three different ghost numbers, -1, 0, and 1. We will derive explicit expressions for a certain set of physical states and compare the results to those recently obtained in [26] (see also [45]). Some useful background material on Schur polynomials $S_k(x)$ is collected in Appendix B.

In the following we will take $Q^M=0$ and $Q^L=i\sqrt{2}$.[3] The equations $P^+(r)=0=P^-(s)$ for $r,s\in\mathbb{Z}$ in this case read [see (3.12)]

$$p^M=-\sqrt{\tfrac{1}{2}}(r-s)\,,\qquad i(p^L-Q^L)=-\sqrt{\tfrac{1}{2}}(r+s)\,. \tag{5.1}$$

Define

$$\eta(p^L)=\operatorname{sign} i(p^L-Q^L)\,, \tag{5.2}$$

[3] The choice of square root for Q^L is not essential here

such that cases (i) and (ii) in Lemma 3.2 correspond to $\eta(p^L)<0$ and $\eta(p^L)>0$, respectively. We note that this distinction also shows up in the structure of the Liouville-Fock space $\mathscr{F}^L(p^L)$ at $c^L=25$. Specifically, it turns out that $\mathscr{F}^L(p^L)$ is of type $III^0_+(-)$ for $\eta(p^L)<0$ and of type $III^0_+(+)$ for $\eta(p^L)>0$, in the notation of Feigin and Fuchs [15]. In the case $III^0_+(-)$ the null vectors in the Verma module $M(\Delta(p^L))$ of highest weight $\Delta(p^L)$ never vanish identically when expressed in terms of oscillators, i.e. there exists an isomorphism $M(\Delta(p^L))\cong\mathscr{F}^L(p^L)$, while in the case $III^0_+(+)$ all null vectors in $M(\Delta(p^L))$ vanish identically when expressed in oscillators [15] (see also [26]).

We will now show that in the case $\eta(p^L)<0$ it is easy to give explicit expressions for the ghost number zero nontrivial cohomology states.

Theorem 5.1. *Suppose $P^+(r)=0=P^-(s)$ for some $r,s\in\mathbb{Z}_+$, i.e. $p^M=-\sqrt{\tfrac{1}{2}}(r-s)$ and $i(p^L-Q^L)=-\sqrt{\tfrac{1}{2}}(r+s)$, then the state $(S_r(\alpha^+_{-j}/j))^s\,|p^M,p^L\rangle$ spans $H^{(0)}_{\mathrm{rel}}(\mathscr{F}(p^M,p^L),d)$.*

Proof. The theorem follows as a direct application of the results in Lemma 3.4. We will, however, present a more elementary proof below.

Let us write $x_j=\alpha^+_{-j}/j$ for $j\geqq 1$, and thus identify α^-_j with $\dfrac{\partial}{\partial x_j}$ for $j\geqq 1$. From Lemma B.1(i) it follows that, up to terms vanishing on $|p^M,p^L\rangle$ and commuting with c_{-n} and α^+_{-n},

$$\begin{aligned}
[\hat{d}_0,S_r(x)]&=\left[\sum_{n\geqq 1}c_{-n}\frac{\partial}{\partial x_n}P^+(n),S_r(x)\right]\\
&=\sum_{n=1}^{r-1}c_{-n}P^+(n)S_{r-n}(x),\\
[\hat{d}_1,S_r(x)]&=\left[\sum_{j=1}^{r}\sum_{n=1}^{j-1}c_{-n}(j-n)x_{j-n}\frac{\partial}{\partial x_j},S_r(x)\right]\\
&=\sum_{n=1}^{r-1}c_{-n}\left(\sum_{j=n+1}^{r}(j-n)x_{j-n}S_{r-j}(x)\right),
\end{aligned}\tag{5.3}$$

while

$$[\hat{d}_2,S_r(x)]=0\,.\tag{5.4}$$

It now follows from $P^+(n)=-(r-n)$ [see (3.13)], and Lemma B.1(ii) of Appendix B, that the state $(S_r(\alpha^+_{-j}/j))^s|p^M,p^L\rangle$ is indeed in $\operatorname{Ker} d$. The term of lowest degree is proportional to $(\alpha^+_{-r})^s|p^M,p^L\rangle$ which, in view of Lemma 3.2, proves its nontriviality. □

The proof above also shows that for $P^+(r)=0=P^-(s)$, r, $s\in\mathbb{Z}_+$, explicit expressions for representatives of $H^{(1)}_{\mathrm{rel}}(\mathscr{F}(p^M,p^L),d)$ can be found in the form

$$(S_r(\alpha^+_{-j}/j))^{s-1}(c_{-r}+\ldots)\,|p^M,p^L\rangle\,,\tag{5.5}$$

where the … stand for terms of $(\deg)\geqq 2$, *independent* of s.

As a corollary we will rederive the result of [26], proved by different means (using results of [38, 33]), that for $\eta(p^L)<0$ the space $H^{(0)}_{\mathrm{rel}}(\mathscr{F}(p^M,p^L),d)$ is spanned by states of the form $|\psi\rangle_M\otimes|p^L\rangle_L\otimes c_1|0\rangle_G$, where $|\psi\rangle_M$ is a singular vector in the matter Fock space module $\mathscr{F}^M(p^M)$. This type of physical states were first

discovered in [24] (in the matrix model in [46]) and are sometimes referred to as "Polyakov states." To be able to formulate the result we recall

Theorem 5.2 [47, 48]. *Consider the $c=1$ Fock space $\mathscr{F}(p)$. We have*

i) *$\mathscr{F}(p)$ is irreducible iff $\frac{1}{2}p^2 \neq \frac{m^2}{4}$ for all $m \in \mathbb{Z}$.*

ii) *If $\frac{1}{2}p^2 = \frac{m^2}{4}$ for some $m \in \mathbb{Z}$ then for all $k \in \mathbb{Z}_+$ such that $m+k \geqq 0$ there are singular vectors*

$$|m,k\rangle = \underbrace{S_{k+m,k+m,\ldots,k+m}}_{k}(\sqrt{2}\alpha_{-j}/j)\,|p=-m/\sqrt{2}\rangle$$

(i.e. rectangular Young tableaux) of weight $\frac{1}{4}(m+2k)^2$.

We now have

Corollary 5.3. *Suppose that p^M and p^L are as in Theorem 5.1, then the following state spans $H^{(0)}_{\mathrm{rel}}(\mathscr{F}(p^M,p^L),d)$*

$$\underbrace{S_{r,r,\ldots,r}}_{s}(\sqrt{2}\alpha^M_{-j}/j)\,|p^M,p^L\rangle\,. \tag{5.6}$$

Proof. First observe that, as a consequence of Theorem 5.2, the state (5.6) is BRST closed. For $p^M = -\sqrt{\frac{1}{2}}(r-s)$, r, $s \in \mathbb{Z}$, the Fock space $\mathscr{F}(p^M)$ decomposes into a direct sum of irreducible Virasoro modules [15],

$$\mathscr{F}(p^M = -\sqrt{\tfrac{1}{2}}(r-s)) = \bigoplus_{k=\max(0,r-s)}^{\infty} L(\tfrac{1}{4}(s-r+2k)^2,1)\,, \tag{5.7}$$

corresponding to the singular vectors in Theorem 5.2. To verify that the state (5.6) is not BRST exact we may restrict to any submodule of $\mathscr{F}(p^M)$ which contains this state. In particular we may take

$$\bigoplus_{k=r}^{\infty} L(\tfrac{1}{4}(s-r+2k)^2,1) \cong \mathscr{F}(p^{M'} = \sqrt{\tfrac{1}{2}}(r+s))\,,$$

in which the state (5.6) is the vacuum state! A simple computation using Eq. (3.3) shows that $\mathscr{F}(p^{M'},p^L)$ falls into category i) of Theorem (3.3), since $P^+(n)=n$ and $P^-(n)=(r+s)-n$. Thus the nontrivial cohomology in this case is precisely the vacuum state, which by construction is exactly the state (5.6), and the corollary has been proven. □

For the opposite case, r, $s<0$, although the general procedure to construct the cohomology states as outlined in Sect. 3 may be applied, the result does not seem to have as succinct a presentation.

6. 2D-Gravity Coupled to $c<1$ Minimal Models

As the next application of the results in Sect. 3 and 4, we will rederive the classification of the space of physical states, as recently presented by Lian and Zuckerman [27], for 2D gravity coupled to a $c^M<1$ minimal model. In this case

the matter system has central charge $c^M = c(p,p') = 1 - 6\frac{(p-p')^2}{pp'}$, and is represented by a set of irreducible highest weight modules $L(\Delta(m,m'))$ [40] with conformal dimensions given by

$$\Delta_M = \Delta(m,m') = \frac{(mp'-m'p)^2-(p-p')^2}{4pp'}. \tag{6.1}$$

Here p and p' are relatively prime positive integers, $p'>p$, whilst m and m' are positive integers satisfying $1 \leqq m \leqq p-1$, $1 \leqq m' \leqq p'-1$, $mp' \geqq m'p$. To ensure the vanishing of the total central charge we take[4]

$$iQ^L = \frac{(p+p')}{\sqrt{2pp'}}, \qquad \Delta_L = \Delta(p^L) = \frac{1}{2}p^L(p^L - 2Q^L), \tag{6.2}$$

where the momentum p^L is pure imaginary and otherwise arbitrary.

The computation of the BRST cohomology on $L(\Delta(m,m')) \otimes \mathscr{F}^L(p^L)$ can be reduced to that of a product of Fock spaces using the free field resolution of $L(\Delta(m,m'))$ [15, 17] which we will now briefly discuss.

Introduce a scalar field with a background charge

$$Q^M = \frac{(p'-p)}{\sqrt{2pp'}} = \sqrt{\frac{1}{2}}(\alpha_+ + \alpha_-), \tag{6.3}$$

where, as usual, $\alpha_+ = \sqrt{p'/p}$ and $\alpha_- = -\sqrt{p/p'}$. Just as there is an isomorphism between the highest weight representations $L(\Delta(m,m'))$ and $L(\Delta(p-m,p'-m'))$, there are also two complexes $(\mathscr{F}_+^{(n)}(m,m'), d')$ and $(\mathscr{F}_-^{(n)}(m,m'), d')$ of Fock spaces of this scalar field which provide a resolution of $L(\Delta(m,m'))$. To describe these we introduce for arbitrary $l, l' \in \mathbb{Z}$ the two sets of momenta $\{p_+^{(n)}(l,l'),\ n \in \mathbb{Z}\}$ and $\{p_-^{(n)}(l,l'),\ n \in \mathbb{Z}\}$, where

$$p_\pm^{(n)}(l,l') - Q^M = \begin{cases} \pm\sqrt{\frac{1}{2}}((l \pm np)\alpha_+ + l'\alpha_-) & \text{if } n \text{ is even}, \\ \pm\sqrt{\frac{1}{2}}((-l \pm (n\pm 1)p)\alpha_+ + l'\alpha_-) & \text{if } n \text{ is odd}, \end{cases} \tag{6.4}$$

and the corresponding sets of conformal dimensions

$$\Delta_\pm^{(n)}(l,l') = \tfrac{1}{2}p_\pm^{(n)}(l,l')(p_\pm^{(n)}(l,l') - 2Q^M). \tag{6.5}$$

Note that for $n=0$, $p_+^{(0)}(m,m')$ and $p_-^{(0)}(m,m') = p_+^{(0)}(p-m, p'-m')$ are exactly those two momenta corresponding to the same conformal dimension $\Delta(m,m')$. Then the following result is due to Felder [17].

Theorem 6.1. *Let $m, m' \in \mathbb{Z}$ such that $1 \leqq m \leqq p-1$, $1 \leqq m' \leqq p'-1$, $mp' \geqq m'p$. There exist (two-sided) resolutions $(\mathscr{F}_+^{(n)}(m,m'), d')$ and $(\mathscr{F}_-^{(n)}(m,m'), d')$ of the irreducible module $L(\Delta(m,m'))$, i.e.*

$$H^{(n)}(\mathscr{F}_\pm^{(n)}(m,m'), d') \cong \delta_{n,0} L(\Delta(m,m')), \tag{6.6}$$

where

$$\mathscr{F}_\pm^{(n)}(m,m') = \mathscr{F}(p_\pm^{(n)}(m,m')), \qquad n \in \mathbb{Z}, \tag{6.7}$$

[4] Again, the sign in Q^L, and in Q^M later on, is irrelevant

and the differential $d' : \mathscr{F}_{\pm}^{(n)}(m,m') \to \mathscr{F}_{\pm}^{(n+1)}(m,m')$ *is given in terms of appropriately integrated products of the screening operator* $s^+(z) = \exp(i\sqrt{2}\alpha_+\phi)(z)$.

For convenience we collect in the following lemma several elementary facts which follow directly from the definitions above.

Lemma 6.2.
(i) *For* $1 \leqq m \leqq p-1$ *and* $1 \leqq m' \leqq p'-1$, *the sets of momenta* $\{p_+^{(n)}(m,m'), n \in \mathbb{Z}\}$ *and* $\{p_-^{(n)}(m,m'), n \in \mathbb{Z}\}$ *are disjoint.*
(ii) *For* $n, l, l' \in \mathbb{Z}$,

$$p_+^{(n)}(l,l') = p_-^{(n+1)}(l,-l'), \tag{6.8}$$

$$\Delta_+^{(n)}(l,l') = \Delta_-^{(-n)}(l,l'). \tag{6.9}$$

Following [27] we set $\tilde{E}_{m,m'}(p,p') = \{1 - \Delta_+^{(n)}(m,m'),\ n \in \mathbb{Z}\}$ and for $\Delta = 1 - \Delta_+^{(n)}(m,m')$ define $d(\Delta) = |n|$. Recall (5.2), $\eta(p^L) = \operatorname{sign}(i(p^L - Q^L))$. The main result of Lian and Zuckerman on the space of physical states of for 2D-gravity coupled to a $c<1$ minimal model is [27, Theorem 3].

Theorem 6.3. *Let* $m, m' \in \mathbb{Z}$ *such that* $1 \leqq m \leqq p-1$, $1 \leqq m' \leqq p'-1$, $mp' \geqq m'p$, *then*
a) $H_{\mathrm{rel}}^{(*)}(L(\Delta(m,m')) \otimes \mathscr{F}^L(p^L) \otimes \mathscr{F}^G, d) \neq 0$ *iff* $\Delta(p^L) \in \tilde{E}_{m,m'}(p,p')$.
b) *For* $\Delta(p^L) \in \tilde{E}_{m,m'}(p,p')$,

$$\dim H_{\mathrm{rel}}^{(n)}(L(\Delta(m,m')) \otimes \mathscr{F}^L(p^L) \otimes \mathscr{F}^G, d) = \delta_{n,\eta(p^L)d(\Delta(p^L))}.$$

c)

$$H_{\mathrm{abs}}^{(*)}(\ldots) = H_{\mathrm{rel}}^{(*)}(\ldots) \oplus c_0 H_{\mathrm{rel}}^{(*-1)}(\ldots).$$

Proof. Note that for given p^L we can always choose a resolution (corresponding to the + or − sign in Theorem 6.1) of $L(\Delta(m,m'))$ such that for all the Fock spaces $\mathscr{F}^{(n)}(m,m') \otimes \mathscr{F}^L(p^L) \otimes \mathscr{F}^G$, $n \in \mathbb{Z}$, the equations $P^+(r) = 0 = P^-(s)$ [see (3.12)] cannot be simultaneously satisfied for $r, s \in \mathbb{Z}$, $rs > 0$, i.e. we are in Case I of Sect. 3. Indeed, for the choice of background charges (6.2) and (6.3), these equations read

$$i(p^L - Q^L) = \sqrt{\tfrac{1}{2}}(r\alpha_+ - s\alpha_-), \tag{6.10}$$

$$p_{\pm}^{(n)}(m,m') - Q^M = \sqrt{\tfrac{1}{2}}(r\alpha_+ + s\alpha_-). \tag{6.11}$$

Now (6.10) determines r and s up to $r \to r + tp$ and $s \to s - tp'$, where t is an arbitrary integer. In (6.11) this corresponds to $p_{\pm}^{(n)}(m,m') \to p_{\pm}^{(n \pm 2t)}(m,m')$. Therefore using Lemma 6.2(i) we can always choose the resolution which does not contain these particular matter momenta. By Theorem 3.3 the relative BRST cohomology of $\mathscr{F}^{(n)}(m,m') \otimes \mathscr{F}^L(p^L)$ will be either trivial or one-dimensional.

Consider the tensor product of Fock spaces $\mathscr{F}(m,m') \otimes \mathscr{F}^L(p^L) \otimes \mathscr{F}^G$ in which d' acts on the first factor while the BRST operator d is defined as in (2.8). It is evident that the differential d' commutes with d because, by construction, d' commutes with the generators of the matter Virasoro algebra and does not contain Liouville or ghost oscillators. Thus we may now form a double complex

$$(\mathscr{F}(m,m') \otimes \mathscr{F}^L(p^L) \otimes \mathscr{F}^G, d, d')$$

graded by the ghost number and the order in Felder's complex. Moreover, if we restrict to the subcomplex of the relative cohomology then both gradings are bounded from above and below. Since both the cohomology of d and of d' on the

relative complex are nontrivial in at most one dimension, we can use a standard result on the cohomology of a double complex[5] to conclude that

$$\begin{aligned} &H^{(n)}_{\mathrm{rel}}(H^{(0)}(\mathscr{F}(m,m')\otimes\mathscr{F}^L(p^L)\otimes\mathscr{F}^G,d'),d) \\ &\quad\cong H^{(n)}(H^{(0)}_{\mathrm{rel}}(\mathscr{F}(m,m')\otimes\mathscr{F}^L(p^L)\otimes\mathscr{F}^G,d),d'). \end{aligned} \tag{6.12}$$

Since by Theorem 6.1

$$\begin{aligned} &H^{(*)}_{\mathrm{rel}}(L(\Delta(m,m'))\otimes\mathscr{F}^L(p^L)\otimes\mathscr{F}^G,d) \\ &\quad\cong H^{(*)}_{\mathrm{rel}}(H^{(0)}(\mathscr{F}(m,m')\otimes\mathscr{F}^L(p^L)\otimes\mathscr{F}^G,d'),d), \end{aligned} \tag{6.13}$$

the left-hand side of (6.12) is precisely the quantity we wish to compute. Further, the right-hand side of (6.12) is exactly the problem we studied in Sect. 3, and the results there suffice to complete proof, as we now see.

Recall the result of Theorem 3.3 that

$$H^{(0)}_{\mathrm{rel}}(\mathscr{F}^{(n)}_{\pm}(m,m')\otimes\mathscr{F}^L(p^L)\otimes\mathscr{F}^G,d)$$

is nonvanishing iff $p^{(n)+}p^{(n)-}=0$. In fact, using (2.9), this condition is equivalent to $\Delta(p^L)\in\tilde{E}_{m,m'}(p,p')$. For $\Delta(p^L)=1-\Delta^{(n)}(m,m')$ we must have [see, (6.5) and (6.9)]

$$i(p^L-Q^L)+Q^M=\begin{cases} p^{(n)}_{+}(m,m') & \text{if} \quad \eta(p^L)>0, n\geqq 0 \quad \text{or} \quad \eta(p^L)<0, n<0, \\ p^{(-n)}_{-}(m,m') & \text{if} \quad \eta(p^L)>0, n<0 \quad \text{or} \quad \eta(p^L)<0, n\geqq 0. \end{cases} \tag{6.14}$$

Then (6.14) and Lemma 6.2(i) imply that, depending on the case, p^L and $p^{(n+1)}_{-}(m,m')$ or $p^{(n-1)}_{+}(m,m')$, respectively, generate solutions to (6.10) and (6.11). Thus (6.12) holds for precisely one resolution. The nontrivial cohomology occurs when the matter Fock space has the momentum given in (6.14) and is generated by the vacuum state which clearly survives upon taking the d'-cohomology. The order of this state in the Felder's complex is $\eta(p^L)d(\Delta(p^L))$, which by (6.12) translates into the same ghost number in the BRST cohomology of $L(\Delta(m,m'))\otimes\mathscr{F}^L(p^L)$.

The proof of the part c) of the theorem is exactly the same as in Sect. 4. □

7. Conclusions

In this paper we have presented, with various degrees of explicitness, the physical states of 2D-gravity coupled to $c^M\leqq 1$ conformal matter. The technical result required was the computation of the BRST cohomology on products of Fock spaces, and we gave a complete analysis for the cases which arose.

It should be noted that, although we have used the conformal gauge description, the results can be derived equally well in the chiral gauge. There the Liouville theory is effectively replaced by an $sl(2)$ current algebra [1], which again has a free field description in terms of a $\beta\gamma$ system and a scalar field with background charge [52–54]. The BRST operator in this case may be written [42, 43] $d=d_T+d_J$, where $d_T^2=d_J^2=\{d_T,d_J\}=0$. Here d_J is of the form "$\hat{c}\beta$," where $\hat{c},\hat{b}$ is an additional ghost system, and d_T is just as in (2.1), where T includes all of

[5] See [49] for the mathematical background and [50] or [51] for an elementary review

the $\beta, \gamma, \phi, \hat{c}, \hat{b}$, and matter contributions. Thus the calculation is again reduced to a Fock space problem, exactly as studied in [41]. The same arguments as in our paper will give the full result. In fact it may be observed that d_J is effecting a "Hamiltonian reduction" from the current algebra to the Liouville system [50, 55, 56].

The major assumption in applying this work is really that the Liouville system may be represented by a free field with background charge. Although this treatment is supported by the evidence we have cited, it is still at the level of an ansatz.

It is very important to study further the consequences of this assumption in light of our results. As stated in the introduction, this leads in particular to the construction of correlators of physical operators. Indeed, there is a growing literature on the computation of correlators for these models, and comparison to the results of matrix model calculations [18–25]. We expect that the careful application of free field techniques will be a useful tool in clarifying these problems.

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Appendix A. Cohomology of a Filtered Complex

In this Appendix we derive some basic results on the cohomology of filtered complexes which have been used in Sect. 3. They can be obtained by a standard computation using the general formalism of spectral sequences associated with a finite filtration (see e.g. [49]). However, we found it rather useful, and perhaps more accessible, to follow the elementary approach presented here.

Consider a complex $(\mathscr{C}, d)$, where $\mathscr{C} = \bigoplus_n \mathscr{C}^{(n)}$ and the differential $d: \mathscr{C}^{(n)} \to \mathscr{C}^{(n+1)}$. We assume that there is an additional gradation, such that for each *order* n,

$$\mathscr{C}^{(n)} = \bigoplus_{k \in \mathbb{Z}} \mathscr{C}_k^{(n)}. \tag{A.1}$$

We will refer to the integer k as the *degree*, and denote the projection onto the subspace of degree k by $(\cdot)_k$. Complexes discussed in the main part of the paper have the property that both the set of orders and the set of degrees for which $\mathscr{C}_k^{(n)}$ is nontrivial are finite. However, in the discussion below it is sufficient to require a weaker property; namely, that for each order the range of degrees for which $\mathscr{C}_k^{(n)}$ are nontrivial is bounded. Furthermore, we assume that the decomposition of d with respect to the gradation by the degree is of the form

$$d = d_0 + d_1 + \ldots d_N = d_0 + d_>, \tag{A.2}$$

where

$$d_i: \mathscr{C}_k^{(n)} \to \mathscr{C}_{k+i}^{(n+1)}, \tag{A.3}$$

and N is some fixed nonnegative integer. Clearly, $d^2 = 0$ implies

$$\sum_{\substack{i,j \\ i+j=k}} d_i d_j = 0, \qquad k = 0, \ldots, 2N, \tag{A.4}$$

and in particular

$$d_0^2 = 0. \tag{A.5}$$

Thus we can consider another complex $(\mathscr{C}, d_0)$, with the same underlying space $\mathscr{C}$ and d_0 as the differential. Note that $(\mathscr{C}, d_0)$ is in fact a direct sum of complexes labelled by the degree, and therefore its cohomology is much easier to investigate – indeed for the examples discussed in this paper it can be computed precisely. This observation may be put to practical use once we know sufficient relations between the cohomologies of both complexes. In the following we construct such relations.

Lemma A.1. *If $H^{(n)}(\mathscr{C}, d_0) = 0$, then $H^{(n)}(\mathscr{C}, d) = 0$.*

Proof. Let $\psi = \psi_k + \psi_{k+1} + \ldots + \psi_p$, $\psi_i \in \mathscr{C}_i^{(n)}$, represent a nontrivial cohomology class of d, where p is the maximal degree at order n. Then $d\psi = 0$ implies $d_0 \psi_k = 0$ and thus $\psi_k = d_0 \chi_k$, where $\chi_k \in \mathscr{C}_k^{(n)}$. Consider $\psi' = \psi - d\chi_k$ which is cohomologous to ψ. Clearly the first term in the decomposition of ψ' has degree at least $k+1$. Proceeding by induction we construct elements $\chi_k, \ldots, \chi_p$ such that $\psi = d(\chi_k + \ldots + \chi_p)$, which proves the Lemma. □

In fact we have proven a stronger result

Lemma A.2. *We can always choose representative $\psi = \psi_k + \ldots + \psi_p$ of a nontrivial cohomology class in $H^{(n)}(\mathscr{C}, d)$ such that the lowest degree term ψ_k in ψ represents a nontrivial cohomology class in $H^{(n)}(\mathscr{C}, d_0)$.*

In the cases we are interested in this paper there is the further simplification that for each n the cohomology $H^{(n)}(\mathscr{C}_k, d_0)$ is nontrivial for at most one $k = k(n)$. This allows us to characterize completely the cohomology of d in terms of the cohomology of d_0.

Theorem A.3. *If for each n, $H^{(n)}(\mathscr{C}_k, d_0) \neq 0$ for at most one degree k, then $H^{(*)}(\mathscr{C}, d_0)$ and $H^{(*)}(\mathscr{C}, d)$ are isomorphic.*

Proof. First we prove that each element in $H^{(n)}(\mathscr{C}, d_0)$ gives rise to an element in $H^{(n)}(\mathscr{C}, d)$. Take ψ_k representing a nontrivial element in $H^{(n)}(\mathscr{C}, d_0)$. Then $d\psi_k = d_> \psi_k$ has the lowest degree at least $k+1$. Using (A.4) we verify that $d_0(d_> \psi_k)_{k+1} = 0$, thus $(d_> \psi_k)_{k+1} = d_0 \chi_{k+1}$. Then $d(\psi_k - \chi_{k+1})$ has terms of degree at least $k+2$, and, using once more (A.4), we find $d_0(d(\psi_k - \chi_{k+1}))_{k+2} = 0$. In this manner we construct in finite number of steps a set of elements $\chi_{k+1}, \ldots, \chi_p$, of degree $k+1, \ldots, p$, respectively, such that $\psi = \psi_k - \chi_{k+1} - \ldots - \chi_p$ is closed under d. Denote the correction term $\chi_{k+1} + \ldots + \chi_p$ in ψ by $\chi_>$, i.e. $\psi = \psi_k - \chi_>$. Clearly $\chi_>$ is not uniquely specified by this construction, since at each step there is an ambiguity of adding terms that are d_0 exact. Let $\psi' = \psi_k - \chi'_>$ be another extension of ψ_k to a d-closed element, where the lowest degree in $\chi'_>$ is greater than k. Since $d(\psi - \psi') = d(\chi'_> - \chi_>) = 0$, and there is no d_0 cohomology in degrees greater than k, a calculation similar to the one in the proof of Lemma A.1 shows that $\chi'_> - \chi_> = d\phi$ which in turn implies that ψ and ψ' correspond to the same cohomology class of d. Moreover, if $\psi_k = d_0 \phi_k$ is d_0 exact then we can take $\psi = d\phi_k$ as the extension. To summarize we have shown that each element in $H^{(n)}(\mathscr{C}, d_0)$ extends to a unique element in $H^{(n)}(\mathscr{C}, d)$.

Next we must show the opposite; namely, that each element in $H^{(n)}(\mathscr{C}, d)$ projects onto a unique element in $H^{(n)}(\mathscr{C}, d_0)$. Using Lemma A.2 we see that each representative $\psi = \psi_k + \chi_>$, $\deg(\chi_>) > k$, of a nontrivial cohomology class of d gives rise to a cohomology class of d_0 represented by ψ_k. The latter should not depend on the choice of ψ, or, equivalently, if $\psi = d\phi$ then ψ_k must be d_0 exact. Indeed, let $\psi = d\phi$, where $\phi = \phi_l + \ldots \phi_k + \phi_>$, $l \leqq k$, and $\phi_>$ denotes the sum of components with degree greater than k. By inspecting the degrees present in ψ we obtain

$$\sum_{\substack{i,j \\ i+j<k}} d_i \phi_j = 0, \tag{A.6}$$

$$\sum_{\substack{i,j \\ i+j=k}} d_i \phi_j = \psi_k. \tag{A.7}$$

The first equation in (A.6) is $d_0 \phi_l = 0$ so that $\phi_l = d_0 \chi_l$. Substituting this into the next equations and using (A.4) we find $\phi_{l+1} = d_0 \chi_{l+1} + d_1 \chi_l$. By induction we prove that there exist $\chi_l, \ldots, \chi_{k-1}$ such that

$$\phi_m = \sum_{i+j=m} d_i \chi_j, \qquad m = l, \ldots, k-1. \tag{A.8}$$

Using (A.8) and (A.4) we can rewrite (A.7) as follows

$$\begin{aligned} \psi_k &= \sum_{\substack{i,j \\ i+j=k}} d_i \phi_j \\ &= d_0 \phi_k + \sum_{\substack{i>0,j \\ i+j=k}} \sum_{\substack{m,n \\ m+n=j}} d_i d_m \chi_n \\ &= d_0 \phi_k + \sum_{\substack{i>0,m,n \\ i+m+n=k}} d_i d_m \chi_n \\ &= d_0 \left(\phi_k - \sum_{l \leqq n < k} d_{k-n} \chi_n \right). \end{aligned} \tag{A.9}$$

Since the two maps constructed above are clearly inverse of each other, the theorem has been proved. □

One should note that the above proof provides, at least in principle, an explicit construction of representatives of the cohomology of d starting with the cohomology of d_0. This construction is completely straightforward, except for the computation of the coboundaries of d_0 when we determine the corrections χ. In the cases of interest the latter can usually be achieved using a suitable contracting homotopy operator [49].

For completeness, we may rephrase the result of Theorem A.3 in the language of a spectral sequence as follows. Introduce a filtration $\mathscr{K}_p = \bigoplus_{k \geqq p} \mathscr{C}_k$ of the complex $(\mathscr{C}, d)$ by the degree. The first term of the spectral sequence [49] associated with this filtration is

$$E_1 \cong H^{(*)}(\mathscr{C}, d_0). \tag{A.10}$$

Under the assumptions of Theorem A.3 we verify that this sequence collapses after the first term which yields

$$H^{(*)}(\mathscr{C}, d) \cong E_\infty \cong E_1 \cong H^{(*)}(\mathscr{C}, d_0). \tag{A.11}$$

Appendix B. Schur Polynomials

The elementary Schur polynomials $S_k(x)$ are defined through their generating function

$$\sum_{k\geqq 0} S_k(x)z^k = \exp\left(\sum_{k\geqq 1} x_k z^k\right). \tag{B.1}$$

For convenience we put $S_k(x)=0$ for $k<0$. More explicitly we have

$$S_k(x) = \sum_{k_1+2k_2+\ldots=k} \frac{x_1^{k_1} x_2^{k_2}}{k_1!\,k_2!}\cdots \tag{B.2}$$

To any partition (Young tableaux) $\lambda=\{\lambda_1\geqq\lambda_2\geqq\ldots\}$ is associated a Schur polynomial

$$S_{\lambda_1,\lambda_2\ldots}(x) = \det(S_{\lambda_i+j-i}(x))_{i,j}. \tag{B.3}$$

For later use we list some properties of Schur polynomials

Lemma B.1.

i)

$$\frac{\partial}{\partial x_j} S_k(x) = S_{k-j}(x), \tag{B.4}$$

ii)

$$\sum_{m=j+1}^{k} (m-j)x_{m-j}S_{k-m}(x) - (k-j)S_{k-j}(x) = 0, \tag{B.5}$$

iii)

$$S_k(x+y) = \sum_{j=0}^{k} S_j(x)S_{k-j}(y). \tag{B.6}$$

Proof. All the statements are proved through the generating function technique. As an illustration we will give the proof of ii). We have

$$\begin{aligned}
&\sum_{k\geqq j} (k-j)S_{k-j}(x)z^{k-j} \\
&= z\frac{d}{dz}\left(\sum_{k\geqq j} S_{k-j}(x)z^{k-j}\right) \\
&= z\frac{d}{dz}(\exp(\textstyle\sum x_k z^k)) = \left(\sum_{k\geqq 1} kx_k z^k\right)\exp(\textstyle\sum x_k z^k) \\
&= \sum_{k\geqq 1}\sum_{l\geqq 0} kx_k S_l(x)z^{k+l} = \sum_{k\geqq j}\left(\sum_{m=j+1}^{k}(m-j)x_{m-j}S_{k-m}(x)\right)z^{k-j}. \quad \square
\end{aligned} \tag{B.7}$$

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