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Finite Dimensional Representations of the Quantum Lorentz Group

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Abstract. All finite dimensional irreducible representations of the quantum Lorentz group $\mathrm{SL}_q(2,\mathbb{C})$ are described explicitly and it is proved all finite dimensional representations of $\mathrm{SL}_q(2,\mathbb{C})$ are completely reducible. The conjecture of Podleś and Woronowicz will be answered affirmatively.

0. Introduction

The quantum Lorentz group $\mathrm{SL}_q(2,\mathbb{C})$, where q is a real parameter $\neq 0, \pm 1$, was introduced by Podleś and Woronowicz [PW], and the Iwasawa decomposition and representation theory were studied. This quantum group is combined with the double group of $\mathrm{SU}_q(2)$, a q-analogue of the compact group SU(2) [W, MMNNU], through the Iwasawa decomposition. Let A_q (respectively B_q) be the *-Hopf algebra corresponding to the quantum group $\mathrm{SL}_q(2,\mathbb{C})$ (respectively $\mathrm{SU}_q(2)$). (A *-Hopf algebra means a Hopf algebra over $\mathbb{C}$ with a *-operation satisfying some properties. See Sect. 4.) The dual vector space $B'_q = \mathrm{Hom}_{\mathbb{C}}(B_q, \mathbb{C})$ has a topological Hopf algebra structure. By a *topological Hopf algebra*, we mean a topological analogue of the usual Hopf algebra, in which the underlying vector space is assumed to have a linear topology and the complete tensor product $\hat{\otimes}$ plays the role of the usual tensor product. (See Sect. 1.)

Podleś and Woronowicz have introduced some topological Hopf algebra structure as well as some *-operation on $B_q \hat{\otimes} B'_q$ and have proved there is an injective *-Hopf algebra map of A_q into $B_q \hat{\otimes} B'_q$. We call

$$\mathscr{E}_q = B_q \hat{\otimes} B'_q$$

the *quantum double* of B_q. This is the dual version of Drinfeld's quantum double [D], and corresponds to the double group of $\mathrm{SU}_q(2)$.

The topological Hopf algebra $\mathscr{E}_q$ has the largest (non-topological) Hopf subalgebra E_q, and what they have done is the construction of an injective *-Hopf algebra map of A_q into E_q. This is not surjective. There is a central group-like element $\tilde{\tau}$ of order 2 in E_q outside A_q. The conjecture 6.4 [PW] tells that we have

$$E_q = A_q \otimes A_q \tilde{\tau}.$$

The purpose of this paper is to prove this conjecture is true.

This conjecture has a close relation with representation theory. By a representation of the quantum Lorentz group $\mathrm{SL}_q(2, \mathbb{C})$ we mean a (right) comodule for $\mathscr{E}_q$. This concept involves a linear topology, since $\mathscr{E}_q$ is a topological Hopf algebra, but we consider only *discrete* comodules. Thus the structure map of a comodule V is a linear map $\rho: V \to V \hat{\otimes} \mathscr{E}_q$ with discrete topology on V. We see $\rho(V)$ is contained in $V \otimes E_q$ if and only if V is *locally finite*, i.e., it is the sum of finite dimensional subcomodules. In other words, locally finite representations of $\mathrm{SL}_q(2, \mathbb{C})$ are the same thing as comodules for E_q.

In the previous paper [T2], we showed how the finite dimensional representation theory of $U_q = \mathrm{U}_q(\mathfrak{sl}(2))$ leads to an explicit description of the dual Hopf algebra U_q° which is the largest (non-topological) Hopf subalgebra of the topological dual U_q'. One sees we are precisely in the same situation.

We will think A_q is a Hopf subalgebra of E_q. Comodules for A_q are called *smooth* representations of $\mathrm{SL}_q(2, \mathbb{C})$. The above conjecture literally tells that if V is a comodule for E_q, there are smooth representations V_1 and V_2 (uniquely determined up to isomorphisms) such that

$$V \cong V_1 \oplus (V_2 \otimes \mathbb{C}\tilde{\tau}).$$

One sees this is equivalent to the previous statement.

To prove the conjecture, first we observe that comodules for $\mathscr{E}_q$ are the same thing as *crossed bimodules* for B_q in the sense of Yetter [Y]. Next, we note that there is an isomorphism of *coalgebras* (but not Hopf algebras)

$$B_q \otimes B_{q^{-1}} \cong A_q.$$

This means all comodules for A_q, i.e., all smooth representations, are completely reducible, and if V_k (respectively V_k^*) denotes the $k+1$-dimensional simple comodule for B_q (respectively $B_{q^{-1}}$), then $V_k \otimes V_l^*$, $k, l \in \mathbb{N}$, give all mutually non-isomorphic simple comodules for A_q. (See [PW, Theorem 6.3].)

Therefore, the conjecture reduces to the following theorems.

Theorem A. *$V_k \otimes V_l^*$ and $V_k \otimes V_l^* \otimes \mathbb{C}\tilde{\tau}$, $k, l \in \mathbb{N}$, give a complete set of representatives for the isomorphism classes of all finite dimensional simple crossed bimodules for B_q.*

Theorem B. *All finite dimensional crossed bimodules for B_q are completely reducible.*

The paper is outlined as follows. In Sect. 1, we state basic facts on topological Hopf algebras. In Sect. 2, we review the construction of the quantum double. In Sect. 3, we establish the correspondence between comodules for the quantum double and crossed bimodules. In Sect. 4, we review the construction of the quantum Lorentz group $\mathrm{SL}_q(2, \mathbb{C})$ as well as the description of *-Hopf algebras A_q and B_q and the embedding of A_q into the quantum double of B_q. In Sect. 5, we express the crossed axiom for B_q in terms of (U_q, B_q)-bimodules. Using this expression, we prove Theorem A in Sect. 6 and Theorem B in Sect. 7. The final section, Sect. 8, which is an appendix, describes the braid structure on A_q arising from the braid structure of the category of crossed bimodules.

The first three sections deal with generalities, and we work over a general field k. In the rest of the sections we let $k=\mathbb{C}$ the complex numbers.

The symbol * is limited to mean the *-operation. Hence we denote by V' the dual vector space $\mathrm{Hom}_k(V,k)$ (not by V^*).

1. Topological Hopf Algebras

We state basic facts on topological Hopf algebras. Generalities on topological coalgebras are developed in [T1].

We work over a field k. A topology on a vector space V is *linear* if all translation $v+$ $(v\in V)$ is continuous and if there is a fundamental system of neighborhoods of 0 $\{V_\alpha\}$ consisting of vector subspaces. In this paper, we consider only *Hausdorff* topological vector spaces. This is equivalent to saying $\bigcap_\alpha V_\alpha=0$. The completion of V is defined to be

$$\hat{V}=\varprojlim_{\alpha} V/V_\alpha$$

which has the prodiscrete topology.

If V and W are topological vector spaces with fundamental systems of neighborhoods of 0 $\{V_\alpha\}$ and $\{W_\beta\}$ we give the linear topology on $V\otimes W$ such that $\{V\otimes W_\beta+V_\alpha\otimes W\}$ is a fundamental system of neighborhoods of 0. $V\otimes W$ is Hausdorff if V and W are Hausdorff. The completion of $V\otimes W$ will be denoted $V\hat{\otimes}W$, and this plays a basic role in the paper. Note that we have

$$V\hat{\otimes}W=\varprojlim_{\alpha,\beta} V/V_\alpha\otimes W/W_\beta .$$

We always give the discrete topology on k.

If M is a vector space, we give the linear topology on the dual space M' such that $L^\perp$ for all finite dimensional subspaces L of M form a fundamental system of neighborhoods of 0. More generally, if M and N are vector spaces, $\mathrm{Hom}_k(M,N)$ will be given the linear topology such that $\mathrm{Hom}_k(M/L,N)$ for all finite dimensional L form a fundamental system of neighborhoods of 0. There is a natural isomorphism of topological vector spaces [T1, p. 513],

$$N\hat{\otimes}M'\cong \mathrm{Hom}_k(M,N),$$

where N is given the discrete topology.

By a *topological coalgebra* we mean a topological vector space $\mathscr{C}$ with linear continuous maps

$$\Delta:\mathscr{C}\to\mathscr{C}\hat{\otimes}\mathscr{C},\quad \varepsilon:\mathscr{C}\to k$$

satisfying the coassociativity and the counit condition [T1, p. 510].

A *topological algebra*, its dual concept, consists of a topological vector space $\mathscr{A}$ and linear continuous maps

$$m:\mathscr{A}\hat{\otimes}\mathscr{A}\to\mathscr{A},\quad u:k\to\mathscr{A}.$$

If A is an algebra and C is a coalgebra, then the dual topological vector spaces A' and C' have natural structures of a topological coalgebra and a topological algebra respectively. More generally, the topological vector space $\mathscr{A} = \mathrm{Hom}_k(C, A)$ has the following topological algebra structure:

$$\tilde{m} = \mathrm{Hom}_k(\Delta, m): \mathscr{A} \hat{\otimes} \mathscr{A} \to \mathscr{A},$$

$$\tilde{u} = \mathrm{Hom}_k(\varepsilon, u): k \to \mathscr{A},$$

where we identify $\mathscr{A} \hat{\otimes} \mathscr{A} = \mathrm{Hom}_k(C \otimes C, A \otimes A)$ [T1, 1.14]. The multiplication $\tilde{m}$ is called the *convolution product* [S, pp. 69–72]. If we consider A is a discrete topological algebra, we have an isomorphism of topological algebras

$$\mathscr{A} \cong A \hat{\otimes} C'.$$

If H is a bialgebra, $\mathrm{End}_k(H)$ has two algebra structures. The convolution product will be written as $f \cdot g$, while the composition will be written as $f \circ g$ or fg. Note that we have an isomorphism of topological algebras (with convolution products)

$$\mathrm{End}_k(H \otimes H) \cong \mathrm{End}_k(H) \hat{\otimes} \mathrm{End}_k(H).$$

Let $\mathscr{C}$ be a topological coalgebra. Since we assume it is Hausdroff, $\mathscr{C} \otimes \mathscr{C}$ is a subspace of $\mathscr{C} \hat{\otimes} \mathscr{C}$. A subspace C of $\mathscr{C}$ is called a *subcoalgebra* if we have $\Delta(C) \subset C \otimes C$. (We consider no topology on C.) Obviously the sum of all subcoalgebras is the largest subcoalgebra of $\mathscr{C}$. For example, if A is an algebra, the largest subcoalgebra of the topological coalgebra A' is denoted by A° [S, Chap. VI].

By a *comodule* for $\mathscr{C}$, we mean a discrete vector space V with a linear map

$$\rho: V \to V \hat{\otimes} \mathscr{C}$$

satisfying the usual axiom:

$$(I \hat{\otimes} \Delta) \circ \rho = (\rho \hat{\otimes} I) \circ \rho \quad \text{as maps} \quad V \to V \hat{\otimes} \mathscr{C} \hat{\otimes} \mathscr{C}$$
$$\text{and} \quad (I \hat{\otimes} \varepsilon) \circ \rho \quad \text{is the canonical map} \quad V \to V \hat{\otimes} k.$$

All commodules are *right* in this paper. If $\mathscr{C} = A'$ the dual topological coalgebra of an algebra A, then the comodules for $\mathscr{C}$ are naturally identified with the left A-modules [T1, Theorem 1.19].

1.1 Proposition. *Let $\mathscr{C}$ be a complete topological coalgebra with the largest subcoalgebra C. Let (V, ρ) be a comodule for $\mathscr{C}$. We have $\rho(V) \subset V \otimes C$ if and only if V is locally finite*, i.e., *it is the sum of finite dimensional subcomodules.*

Proof. The "only if" part follows from [S, Theorem 2.1.3]. To prove the "if" part, we may assume V is finite dimensional. Note that $V \otimes \mathscr{C} = V \hat{\otimes} \mathscr{C}$, since $\mathscr{C}$ is complete. Take a base $v_1, \dots, v_n$ for V and write

$$\rho(v_j) = \sum_i v_i \otimes x_{ij}.$$

It is easy to see

$$\Delta(x_{ij}) = \sum_s x_{is} \otimes x_{sj},$$

hence x_{ij} span a subcoalgebra and we have $\rho(V) \subset V \otimes C$. □

A *topological bialgebra* is the topological version of the concept of a bialgebra. The underlying vector space $\mathscr{H}$ is assumed to have a linear topology and the structure maps which are linear continuous

$$m:\mathscr{H}\hat{\otimes}\mathscr{H}\to\mathscr{H},\quad u:k\to\mathscr{H},$$
$$\Delta:\mathscr{H}\to\mathscr{H}\hat{\otimes}\mathscr{H},\quad \varepsilon:\mathscr{H}\to k$$

are assumed to satisfy the same axioms as a usual bialgebra. (Read $\hat{\otimes}$ for $\otimes$ in the diagrams of [S, p. 52].)

If H is a bialgebra, then the topological dual vector space H' has a natural structure of a topological bialgebra.

For a topological bialgebra $\mathscr{H}$, let $E=\mathrm{End}_{\mathrm{cont}}(\mathscr{H})$ be the vector space of all linear continuous endomorphisms. This space is closed relative to the convolution product

$$f\cdot g=m\circ(f\hat{\otimes}g)\circ\Delta.$$

E is an algebra with this product and unit $u\varepsilon$. If the identity I has an inverse S in the algebra E, we say $\mathscr{H}$ is a *topological Hopf algebra* and S the *antipode*. Just as the non-topological Hopf algebras [S, Proposition 4.0.1], one sees the antipode is an anti-endomorphism of the topological bialgebra.

1.2 Proposition. *The largest subcoalgebra H of a topological Hopf algebra $\mathscr{H}$ is a Hopf algebra.*

This follows easily, since k, HH, and $S(H)$ are subcoalgebras, hence they are contained in H.

2. Quantum Double

Podleś and Woronowicz [PW, Sect. 4] introduced the dual concept of Drinfeld's quantum double [D] under the name "double group." The construction is reviewed in the context of topological Hopf algebras. We include all proofs for self-containedness.

Let H be a Hopf algebra with structure maps $m, u, \Delta, \varepsilon$. Assume the antipode S of H is *bijective*. We will make the topological algebra $\mathscr{E}=\mathrm{End}_k(H)$ into a topological Hopf algebra. Let $\tilde{m}=\mathrm{Hom}_k(\Delta, m)$ and $\tilde{u}=\mathrm{Hom}_k(\varepsilon, u)$ be the structure maps of $\mathscr{E}$, and recall we have

$$\mathscr{E}\hat{\otimes}\mathscr{E}=\mathrm{End}_k(H\otimes H)$$

as topological algebras (Sect. 1).

The comultiplication $\tilde{\Delta}:\mathscr{E}\to\mathscr{E}\hat{\otimes}\mathscr{E}$ is defined to be the following continuous homomorphism of topological algebras:

$$\tilde{\Delta}:\mathrm{End}_k(H)\to\mathrm{End}_k(H\otimes H)$$
$$\tilde{\Delta}(f)=\sigma\cdot(\Delta\circ f\circ m^{op})\cdot\sigma^{-1},$$

where $\sigma=(u\otimes I)\circ(I\otimes\varepsilon)$, hence $\sigma^{-1}=(u\otimes I)\circ S\circ(I\otimes\varepsilon)$. We have

$$\tilde{\Delta}(f)(x\otimes y)=\sum(1\otimes x_{(1)})\Delta(f(yx_{(2)}))(1\otimes S(x_{(3)})),\quad x, y\in H,$$

where and in the following we use the sigma notation [S, p. 10].

2.1 Lemma. *$\tilde{\Delta}$ is coassociative.*

Proof. If $F \in \mathrm{End}_k(H \otimes H)$, then $x \otimes y \otimes z$ in $H \otimes H \otimes H$ is mapped by $(I \hat{\otimes} \tilde{\Delta})(F)$ to

$$\sum (1 \otimes 1 \otimes y_{(1)})(I \otimes \Delta)(F(x \otimes z y_{(2)}))(1 \otimes 1 \otimes S(y_{(3)})),$$

and by $(\tilde{\Delta} \hat{\otimes} I)(F)$ to

$$\sum (1 \otimes x_{(1)} \otimes 1)(\Delta \otimes I)(F(y x_{(2)} \otimes z))(1 \otimes S(x_{(3)}) \otimes 1).$$

If $F = \tilde{\Delta}(f)$, both are equal to

$$\sum (1 \otimes x_{(1)} \otimes y_{(1)} x_{(2)}) \Delta_2(f(z y_{(2)} x_{(3)}))(1 \otimes S(x_{(5)}) \otimes S(y_{(3)} x_{(4)})),$$

where $\Delta_2 : H \to H \otimes H \otimes H$ the iterated comultiplication. □

Define the counit $\tilde{\varepsilon} : \mathscr{E} = \mathrm{End}_k(H) \to k$,

$$\tilde{\varepsilon}(f) = \varepsilon(f(1)).$$

2.2 Theorem. *$\mathscr{E}$ is a topological Hopf algebra.*

Proof. The counit condition is easy to check. Define a continuous linear map $\tilde{S} : \mathscr{E} \to \mathscr{E}$,

$$\tilde{S}(f) = S \cdot (S \circ f \circ S^{-1}) \cdot I$$

which maps x in H to $\sum S(x_{(1)}) S f S^{-1}(x_{(2)}) x_{(3)}$. We show $\tilde{S}$ is an antipode of $\mathscr{E}$. If $F \in \mathrm{End}_k(H \otimes H)$, then $(I \hat{\otimes} \tilde{S})(F)$ maps $x \otimes y$ in $H \otimes H$ to

$$\sum (1 \otimes S(y_{(1)}))(I \otimes S) F(x \otimes S^{-1}(y_{(2)}))(1 \otimes y_{(3)}).$$

If $F = \tilde{\Delta}(f)$, this becomes

$$\sum (1 \otimes S(y_{(1)}) S^2(x_{(3)}))(I \otimes S) \Delta(f(S^{-1}(y_{(2)}) x_{(2)}))(1 \otimes S(x_{(1)}) y_{(3)}).$$

Hence $\tilde{m}(I \hat{\otimes} \tilde{S}) \tilde{\Delta}(f)$ maps x in H to

$$m\left(\sum (1 \otimes S(x_{(4)}) S^2(x_{(3)}))(I \otimes S) \Delta(f(S^{-1}(x_{(5)}) x_{(2)}))(1 \otimes S(x_{(1)}) x_{(6)})\right)$$

which reduces to $\varepsilon(f(1))\varepsilon(x)$. This means we have

$$\tilde{m}(I \hat{\otimes} \tilde{S}) \tilde{\Delta} = \tilde{u} \tilde{\varepsilon}.$$

Similarly, $(\tilde{S} \hat{\otimes} I) \tilde{\Delta}(f)$ maps $x \otimes y$ to

$$\sum (S(x_{(1)}) \otimes S^{-1}(x_{(4)}))(S \otimes I) \Delta(f(y S^{-1}(x_{(3)})))(x_{(5)} \otimes x_{(2)}).$$

Hence $\tilde{m}(\tilde{S} \hat{\otimes} I) \tilde{\Delta}(f)$ maps x to

$$\sum S(x_{(1)}) S(f(x_{(6)} S^{-1}(x_{(3)}))_{(1)}) x_{(5)} S^{-1}(x_{(4)}) f(x_{(6)} S^{-1}(x_{(3)}))_{(2)} x_{(2)}$$

which is equal to $\varepsilon(f(1))\varepsilon(x)$. This means we have

$$\tilde{m}(\tilde{S} \hat{\otimes} I) \tilde{\Delta} = \tilde{u} \tilde{\varepsilon}.$$

Hence $\tilde{S}$ is the antipode of $\mathscr{E}$. □

2.3 Definition. *The topological Hopf algebra $\mathscr{E}$ is called the quantum double of H.*

If we identify $\mathscr{E} = H \hat{\otimes} H'$ as a topological algebra, the structure maps $\tilde{\Delta}$, $\tilde{\varepsilon}$, and $\tilde{S}$ have the following expressions:

$$\begin{aligned}
&\tilde{\Delta}: H \hat{\otimes} H' \xrightarrow{\Delta \otimes \Delta^{\mathrm{op}}} (H \otimes H) \hat{\otimes} (H' \hat{\otimes} H') \xrightarrow{I \otimes \omega \otimes I} \\
&\quad (H \otimes H) \hat{\otimes} (H' \hat{\otimes} H') \xrightarrow{I \otimes tw \otimes I} (H \hat{\otimes} H') \hat{\otimes} (H \hat{\otimes} H'), \\
&\tilde{\varepsilon}: H \hat{\otimes} H' \xrightarrow{\varepsilon \otimes \varepsilon} k \hat{\otimes} k = k, \\
&\tilde{S}: H \hat{\otimes} H' \xrightarrow{\omega} H \hat{\otimes} H' \xrightarrow{S \hat{\otimes} S^{-1}} H \hat{\otimes} H',
\end{aligned} \tag{2.4}$$

where ω denotes the inner action $f \mapsto I \cdot f \cdot S$. (We think I, S are elements in $H \hat{\otimes} H'$.) One sees our construction of the quantum double coincides with the double group [PW, Sect. 4].

It follows we have the following homomorphisms of topological Hopf algebras:

$$\begin{aligned}
\pi_1 &= I \hat{\otimes} \varepsilon: \mathscr{E} = H \hat{\otimes} H' \to H, \\
\pi_2 &= \varepsilon \hat{\otimes} I: \mathscr{E} = H \hat{\otimes} H' \to H'^{\mathrm{cop}},
\end{aligned} \tag{2.5}$$

where H is given the discrete topology and cop means the coalgebra opposite.

2.6 Proposition. *The composite*

$$\mathscr{E} \xrightarrow{\tilde{\Delta}} \mathscr{E} \hat{\otimes} \mathscr{E} \xrightarrow{\pi_1 \hat{\otimes} \pi_2} H \hat{\otimes} H'$$

is equal to the identity.

Proof. If we identify $\mathscr{E} \hat{\otimes} \mathscr{E} = \mathrm{End}_k(H \otimes H)$ and $H \hat{\otimes} H' = \mathrm{End}_k(H)$, then $\pi_1 \hat{\otimes} \pi_2$ maps F in $\mathrm{End}_k(H \otimes H)$ to the composite $(I \otimes \varepsilon) \circ F \circ (u \otimes I)$. If $F = \tilde{\Delta}(f)$, this composite reduces to f. □

3. Crossed Bimodules

This section gives a correspondence between comodules for the quantum double and crossed bimodules in the sense of Yetter [Y].

Let H be a Hopf algebra and let $\mathscr{E} = \mathrm{End}_k(H)$ its quantum double.

By a right *bimodule* for H, we mean a right H-module and a right H-comodule V. The coaction will be denoted [S, p. 32]

$$v \mapsto \sum v_{(0)} \otimes v_{(1)}, \quad v \in V.$$

3.1 Crossed Axiom [Y, Def. 3.6].

$$\sum v_{(0)} h_{(1)} \otimes v_{(1)} h_{(2)} = \sum (v h_{(2)})_{(0)} \otimes h_{(1)} (v h_{(2)})_{(1)}, \quad v \in V, \quad h \in H.$$

A *crossed bimodule* for H means a right bimodule for H satisfying the crossed axiom.

Let V be a comodule for the topological bialgebra $\mathscr{E}$ with structure

$$\rho: V \to V \hat{\otimes} \mathscr{E}.$$

By means of the topological bialgebra maps π_1, π_2 (2.5), we have the following comodule structures:

$$\rho_1: V \xrightarrow{\rho} V \hat{\otimes} \mathscr{E} \xrightarrow{I \hat{\otimes} \pi_1} V \otimes H,$$

$$\rho_2: V \xrightarrow{\rho} V \hat{\otimes} \mathscr{E} \xrightarrow{I \hat{\otimes} \pi_2} V \hat{\otimes} H'^{\mathrm{cop}}.$$

The second comodule structure ρ_2 can be thought of as a right H-module structure. It follows from 2.6 Proposition that we have

$$\rho: V \xrightarrow{\rho_2} V \hat{\otimes} H'^{\mathrm{cop}} \xrightarrow{\rho_1 \hat{\otimes} I} V \otimes H \hat{\otimes} H'^{\mathrm{cop}} = V \hat{\otimes} \mathscr{E}.$$

Conversely, when V is a right H-bimodule, define ρ as the above composite.

3.2 Proposition. *(V, ρ) is a right comodule for $\mathscr{E}$ if and only if the crossed axiom holds for the bimodule V.*

Proof. The counit condition for ρ is always true. Identify

$$V \hat{\otimes} \mathscr{E} = V \hat{\otimes} \mathrm{End}_k(H) = \mathrm{Hom}_k(H, V \otimes H)$$

as topological vector spaces. To say (V, ρ) is a comodule means that we have the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho} & \mathrm{Hom}_k(H, V \otimes H) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho \hat{\otimes} I} \\ \mathrm{Hom}_k(H, V \otimes H) & \xrightarrow{I \hat{\otimes} \tilde{\Delta}} & \mathrm{Hom}_k(H \otimes H, V \otimes H \otimes H) \end{array}$$

with comultiplication $\tilde{\Delta}$ of $\mathscr{E}$. Let $f = \rho(v)$ with $v \in V$. Then f maps $h \in H$ to $\sum (vh)_{(0)} \otimes (vh)_{(1)}$. One sees by definition of $\tilde{\Delta}$

$$(\rho \hat{\otimes} I)(f)(x \otimes y) = \textstyle\sum \rho_1((vy)_{(0)} x) \otimes (vy)_{(1)},$$
$$(I \hat{\otimes} \tilde{\Delta})(f)(x \otimes y) = \textstyle\sum (vyx_{(2)})_{(0)} \otimes (1 \otimes x_{(1)}) \Delta((vyx_{(2)})_{(1)})(1 \otimes S(x_{(3)}))$$

for $x, y \in H$. Note that both are functions of vy and x. So, we may assume $y = 1$, and the comodule condition tells that we have

$$\textstyle\sum (v_{(0)} x_{(1)})_{(0)} \otimes (v_{(0)} x_{(1)})_{(1)} \otimes v_{(1)} x_{(2)} = \sum (vx_{(2)})_{(0)} \otimes (vx_{(2)})_{(1)} \otimes x_{(1)} (vx_{(2)})_{(2)}$$

for all $v \in V$, $x \in H$. This follows from the crossed axiom by application of the map $\rho_1 \otimes I$, and conversely, the crossed axiom follows from this by application of the map $I \otimes \varepsilon \otimes I$. Hence the comodule condition is equivalent to the crossed axiom. □

It follows that the crossed bimodules for H are the same thing as the comodules for the quantum double $\mathscr{E}$. In particular, if E denotes the largest (non-topological) subcoalgebra (which is a subbialgebra) of $\mathscr{E}$, then the category of right E-comodules is completely identified with the category of locally finite crossed H-bimodules.

The following remarks will be useful in later discussions.

3.3 Remark. The crossed axiom is enough to check for generators of the algebra H.

In fact, the set of $h \in H$ satisfying the crossed axiom for all $v \in V$ forms a subalgebra.

Assume H is a Hopf subalgebra of U° for some Hopf algebra U. For $x \in U$ and $h \in H$, define the actions [S, pp. 46, 100]

$$x \rightharpoonup h = \sum h_{(1)} \langle x, h_{(2)} \rangle, \quad h \leftharpoonup x = \sum \langle x, h_{(1)} \rangle h_{(2)}.$$

Recall [S, Sect. 2.1] that every right H-comodule has a natural left U-module structure. Hence, if V is a right H-bimodule, we can think we have a bimodule ${}_U V_H$ for the algebras U, H.

3.4 Remark. With the above assumptions, the crossed axiom is expressed

$$\sum (x_{(1)} v)(x_{(2)} \rightharpoonup h) = \sum x_{(2)} (v(h \leftharpoonup x_{(1)}))$$

for $v \in V$, $x \in U$, $h \in H$.

3.5 Remark. With the above assumptions, if the algebra U is generated by a subcoalgebra C and the algebra H by a subset Λ, then the crossed axiom of 3.4 is enough to check for all $v \in V$, $x \in C$, and $h \in \Lambda$.

In fact, the set of $h \in H$ satisfying the crossed axiom in 3.4 for all $v \in V$ and $x \in C$ is a subalgebra, and the set of $x \in U$ satisfying it for all $v \in V$ and $h \in H$ is also a subalgebra.

4. Quantum Lorentz Group

We review the construction of the quantum Lorentz group $SL_q(2, \mathbb{C})$ introduced by Podleś and Woronowicz [PW, Sects. 1, 5].

Hereafter, we let $k = \mathbb{C}$ the complex numbers. A **-algebra* means an algebra over $\mathbb{C}$ with an involutive conjugate linear automorphism $*$ such that $(ab)^* = b^* a^*$, $a, b \in A$. The tensor product of two *-algebras has a natural *-structure. A **-Hopf algebra* means a Hopf algebra over $\mathbb{C}$ and a *-algebra H such that the comultiplication Δ and the counit ε are *-homomorphisms, i.e., $\Delta(h)^* = \Delta(h^*)$ and $\overline{\varepsilon(h)} = \varepsilon(h^*)$, $h \in H$. If this is the case, the algebra opposite H^{op} or the coalgebra opposite H^{cop} is a Hopf algebra, too. This implies that the antipode S of H is bijective and one has

$$* \circ S = S^{-1} \circ *, \quad \text{or} \quad (S \circ *)^2 = I.$$

A topological *-algebra and a topological *-Hopf algebra are defined similarly.

If H is a *-Hopf algebra, we give the following *-structure to the dual topological Hopf algebra H':

$$x^*(a) \underset{\text{def}}{=} \overline{x(S(a)^*)}, \quad x \in H', \quad a \in H.$$

For a *-Hopf algebra H, let $\mathscr{E} = \mathrm{End}_{\mathbb{C}}(H)$ be the quantum double. As a topological algebra, this has a natural *-structure isomorphic to $H \hat{\otimes} H'$. If $f \in \mathscr{E}$, we have $f^* = * \circ f \circ * \circ S$, or

$$f^*(a) = f(S(a)^*)^*, \quad a \in H.$$

4.1 Proposition. *$\mathcal{E}$ is a topological *-Hopf algebra.*

Proof. We show the comultiplication $\tilde{\Delta}$ commutes with $*$. If $f \in \mathcal{E}$, $\tilde{\Delta}(f)^*$ maps $x \otimes y$ in $H \otimes H$ to

$$[\tilde{\Delta}(f)(S(x)^* \otimes S(y)^*)]^*$$

which is

$$[\sum(1 \otimes S(x_{(3)})^*)\Delta(f(S(y)^* S(x_{(2)})^*))(1 \otimes x_{(1)}{}^*)]^*$$
$$= \sum(1 \otimes x_{(1)})[\Delta(f(S(yx_{(2)})^*))]^*(1 \otimes S(x_{(3)})).$$

Since Δ and $*$ commute, it follows that $\tilde{\Delta}(f)^* = \tilde{\Delta}(f^*)$. □

Note that the identity I is *unitary* in the *-algebra $\mathcal{E}$ in the sense that $I^* = I^{-1}(=S)$.

Fix a real parameter $q \neq 0, \pm 1$ in the rest of the paper. The following notations will be used throughout.

4.2 Definition. *Let B_q be the $\mathbb{C}$-algebra defined by generators a, b, c, d and the following relations:*

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd,$$
$$cb = bc, \quad ad - q^{-1}bc = 1 = da - qbc.$$

These relations tell that $\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$ is the inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The algebra B_q has the following *-Hopf algebra structure:

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes 1 & b\otimes 1 \\ c\otimes 1 & d\otimes 1 \end{pmatrix}\begin{pmatrix} 1\otimes a & 1\otimes b \\ 1\otimes c & 1\otimes d \end{pmatrix},$$

$$\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

The *-Hopf algebra B_q corresponds to the compact quantum group $\mathrm{SU}_q(2)$ [W] [MMNNU].

Let $\mathcal{E}_q = \mathrm{End}_{\mathbb{C}}(B_q)$ the quantum double. The topological *-Hopf algebra $\mathcal{E}_q$ represents the *double group* of $\mathrm{SU}_q(2)$ [PW, Sect. 4]. The quantum Lorentz group $\mathrm{SL}_q(2,\mathbb{C})$ was introduced by Podleś and Woronowicz as a *-Hopf subalgebra of $\mathcal{E}_q$. We define a *-Hopf subalgebra C_q of the topological dual B'_q, and then a *-subalgebra A_q of $B_q \otimes C_q$. We show A_q is a Hopf subalgebra of $\mathcal{E}_q = B_q \hat{\otimes} B'_q$. One sees later the *-Hopf algebra A_q corresponds to the quantum Lorentz group $\mathrm{SL}_q(2,\mathbb{C})$.

Begin with two algebra maps $B_q \to \mathbb{C}$,

$$p: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix},$$

$$\tau: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We think p and τ are group-like elements on B'_q. The actions $\tau\rightharpoonup$, $\leftharpoonup\tau$ on B_q (see above 3.4) are the algebra automorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix},$$

hence the group-like τ is central in B'_q. We have

$$\tau^* = \tau, \quad p^* = \begin{cases} p & \text{if} \quad q>0, \\ \tau p & \text{if} \quad q<0 \end{cases} \tag{4.3}$$

in the *-algebra B'_q. Here, and in the following, we note that the involution of B_q, $x \mapsto S(x)^*$, preserves a and d, and exchanges b and c.

For any $e \in \mathbb{C}$, there is an *opposite* algebra map $\pi_e : B_q \to M_2(\mathbb{C})$

$$a \mapsto \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$c \mapsto \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}, \quad d \mapsto \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}.$$

Let us express

$$\pi_e(x) = \begin{pmatrix} p(x) & n_e(x) \\ 0 & p^{-1}(x) \end{pmatrix}, x \in B_q$$

with $n_e \in B'_q$. The fact that π_e is an opposite representation means we have in the topological Hopf algebra B'_q,

$$\Delta(n_e) = p^{-1} \otimes n_e + n_e \otimes p, \quad \varepsilon(n_e) = 0.$$

In other words, n_e is a *(p^{-1}, p)-primitive.*

4.4 Lemma. *In the topological *-algebra B'_q, we have*

(1) $p^*p = pp^*$,

(2) $n_e p = qpn_e$,

(3) $n_e p^* = qp^* n_e$,

(4) $n_e^* n_e - n_e n_e^* = \dfrac{e\bar{e}}{|q|(1-q^{-2})}(pp^* - p^{-1}(p^{-1})^*)$.

Proof. (1) follows from (4.3), and (3) from (2) plus (4.3). To prove (2) and (4), note that $pn_e p^{-1}$ is a (p^{-1}, p)-primitive and $[n_e^*, n_e]$ is a $(p^{-1}(p^{-1})^*, pp^*)$-primitive. (The latter follows from the fact that $p^{-1} \otimes n_e$ and $n_e^* \otimes p^*$ commute. This fact is a consequence of (3).) In general, if x, y are two (g, γ)-primitives in B'_q with group-likes g, γ, then a linear relation $y = \lambda x$ ($\lambda \in \mathbb{C}$) will hold if it does on the generators a, b, c, d. The linear relation (2) becomes at the generating matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} = q \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix},$$

which is true. Similarly, one can check (4). □

Let $n = n_e$ for $e = q^{1/2}(1 - q^{-2})$ so that we have

$$[n^*, n] = (1 - q^{-2})(pp^* - p^{-1}(p^{-1})^*).$$

4.5 Definition. Let C_q be the *-subalgebra of B'_q generated by p, p^{-1}, and n.

It is a *-Hopf subalgebra. It will be natural to consider C_q as a *-Hopf subalgebra of B'^{cop}_q (but not of B'_q), since we started with anti-representation π_e.

We have the following inclusions of *-algebras:

$$B_q \otimes C_q \subset B_q \otimes B'_q \subset B_q \hat{\otimes} B'_q = \mathscr{E}_q.$$

4.6 Definition. *Let A_q be the *-subalgebra of $B_q \otimes C_q$ generated by the elements $\alpha, \beta, \gamma, \delta$ defined as follows:*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & n \\ 0 & p^{-1} \end{pmatrix}.$$

Note that we have

$$\begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \begin{pmatrix} p^* & 0 \\ n^* & (p^{-1})^* \end{pmatrix} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

4.7 Proposition. *The following* 17 *relations hold among $\alpha, \beta, \gamma, \delta$:*

(1) $\beta\alpha = q\alpha\beta$, (2) $\gamma\alpha = q\alpha\gamma$, (3) $\delta\beta = q\beta\delta$, (4) $\delta\gamma = q\gamma\delta$,
(5) $\gamma\beta = \beta\gamma$, (6) $\delta\alpha - q\beta\gamma = 1$, (7) $\alpha\delta - q^{-1}\beta\gamma = 1$,
(8) $\alpha^*\delta = \delta\alpha^*$, (9) $\beta^*\gamma = \gamma\beta^*$, (10) $\gamma^*\gamma = \gamma\gamma^*$,
(11) $\alpha^*\gamma = q\gamma\alpha^*$, (12) $\gamma^*\delta = q^{-1}\delta\gamma^*$,
(13) $\alpha^*\alpha = \alpha\alpha^* + (q^{-2} - 1)\gamma\gamma^*$, (14) $\delta^*\delta = \delta\delta^* + (1 - q^{-2})\gamma\gamma^*$;
(15) $\alpha^*\beta = q^{-1}\beta\alpha^* + q^{-1}(q^{-2} - 1)\delta\gamma^*$,
(16) $\beta^*\delta = q\delta\beta^* + q(1 - q^{-2})\gamma\alpha^*$,
(17) $\beta^*\beta = \beta\beta^* + (1 - q^{-2})(\alpha\alpha^* - \delta\delta^*) - (1 - q^{-2})^2\gamma\gamma^*$.

Proof. By using the previous Lemma 4.4, one checks that a, b, c, d and $p, n, 0, p^{-1}$ satisfy the 17 relations. Since they commute "doubly," the claim will follow from [PW, Proposition 1.1, p. 389]. □

4.8 Proposition. *We have*

$$\tilde{\Delta}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha \otimes 1 & \beta \otimes 1 \\ \gamma \otimes 1 & \delta \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes \alpha & 1 \otimes \beta \\ 1 \otimes \gamma & 1 \otimes \delta \end{pmatrix}, \quad \tilde{\varepsilon}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\tilde{\Delta}, \tilde{\varepsilon}$ the coalgebra structures of $\mathscr{E}_q$.

Proof. Define the following coaction on a 2-dimensional vector space $\mathbb{C}u \oplus \mathbb{C}v$

$$u \mapsto u \otimes \alpha + v \otimes \gamma, \quad v \mapsto u \otimes \beta + v \otimes \delta.$$

The statement means that this satisfies the comodule condition for $\mathscr{E}_q$. In fact, this is precisely the $\mathscr{E}_q$-comodule corresponding to the 2-dimensional crossed B_q-bimodule V_1 (5.2). □

If $\tilde{S}$ denotes the antipode of $\mathscr{E}_q$, we have

$$\tilde{S}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}^{-1}=\begin{pmatrix}\delta & -q\beta\\ -q^{-1}\gamma & \alpha\end{pmatrix}.$$

It follows that A_q is a *-Hopf subalgebra of $\mathscr{E}_q$.

Let $\tilde{\tau}=1\otimes\tau$ which is a central group-like element of order 2 in $B_q\hat{\otimes}B'_q=\mathscr{E}_q$. The following theorem is the main result of the paper.

4.9 Theorem. (a) *The *-algebra A_q is defined by generators $\alpha,\beta,\gamma,\delta$ and the* 17 *relations of* 4.7.

(b) *We have $A_q+A_q\tilde{\tau}=A_q\oplus A_q\tilde{\tau}$.*

(c) *$A_q\oplus A_q\tilde{\tau}$ coincides with the largest subcoalgebra of the quantum double $\mathscr{E}_q=\mathrm{End}_{\mathbb{C}}(B_q)$.*

In the next section, the statement will be reduced to some representation-theoretic facts.

(a) means the *-Hopf algebra A_q corresponds to the quantum Lorentz group $\mathrm{SL}_q(2,\mathbb{C})$. One will see (c) answers the conjecture of Podleś and Woronowicz [PW, 6.4].

5. Crossed B_q-Bimodules

We recall the embedding of B_q into a Hopf subalgebra of U_q°, where $U_q=\mathrm{U}_q(\mathfrak{sl}(2))$ [T2], and express the crossed axiom in the form of 3.4 explicitly.

Let U_q be the $\mathbb{C}$-algebra defined by generators K,K^{-1},E,F and the relations

$$KK^{-1}=1=K^{-1}K,$$
$$KEK^{-1}=q^2E,\quad KFK^{-1}=q^{-2}F,$$
$$[E,F]=\frac{K-K^{-1}}{q-q^{-1}}.$$

We give it the following Hopf algebra structure:

$$\Delta(K)=K\otimes K,\quad \varepsilon(K)=1,\quad S(K)=K^{-1},$$
$$\Delta(E)=1\otimes E+E\otimes K,\quad \varepsilon(E)=0,\quad S(E)=-EK^{-1},$$
$$\Delta(F)=K^{-1}\otimes F+F\otimes 1,\quad \varepsilon(F)=0,\quad S(F)=-KF.$$

We can embed the Hopf algebra B_q (4.2) into U_q° as follows. Let $\lambda\colon U_q\to \mathrm{M}_2(\mathbb{C})$ be the basic representation

$$K\mapsto\begin{pmatrix}q & 0\\ 0 & q^{-1}\end{pmatrix},\quad E\mapsto\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad F\mapsto\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

and write

$$\lambda(x)=\begin{pmatrix}\tilde{a}(x) & \tilde{b}(x)\\ \tilde{c}(x) & \tilde{d}(x)\end{pmatrix},\quad x\in U_q$$

with $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in U_q^\circ$. Then, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ induces an injective Hopf algebra map $B_q \to U_q^\circ$, and more precisely, we have $U_q^\circ = B_q \oplus B_q \gamma$, where $\gamma: B_q \mapsto \mathbb{C}$ the algebra map $K \mapsto -1$, $E, F \mapsto 0$ (see [T2, 3.11]).

The actions $\rightharpoonup, \leftharpoonup$ (above 3.4) have the following descriptions:

$$K \rightharpoonup \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} qa & q^{-1}b \\ qc & q^{-1}d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftharpoonup K = \begin{pmatrix} qa & qb \\ q^{-1}c & q^{-1}d \end{pmatrix},$$

$$E \rightharpoonup \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftharpoonup E = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix},$$

$$F \rightharpoonup \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftharpoonup F = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}.$$

It follows from 3.5 Remark that the crossed B_q-bimodules admit the following description.

5.1 Proposition. *Let V be a right B_q-bimodule. If we consider V as a left U_q- and a right B_q-module, then the crossed axiom is equivalent to the following 12 commutation relations.*

(1) $K(va) = (Kv)a$, (2) $K(vb) = q^{-2}(Kv)b$,

(3) $K(vc) = q^2(Kv)c$, (4) $K(vd) = (Kv)d$,

(5) $E(va) = q(Ev)a - q^2(Kv)c$, (6) $E(vb) = q^{-1}(Ev)b + va - (Kv)d$,

(7) $E(vc) = q(Ev)c$, (8) $E(vd) = q^{-1}(Ev)d + vc$,

(9) $F(va) = q(Fv)a + q(K^{-1}v)b$, (10) $F(vb) = q(Fv)b$,

(11) $F(vc) = q^{-1}(Fv)c + q^{-1}(K^{-1}v)d - q^{-1}va$, (12) $F(vd) = q^{-1}(Fv)d - q^{-1}vb$,

where $v \in V$.

5.2 Example. Let $V_1 = \mathbb{C}u + \mathbb{C}v$ a 2-dimensional vector space. Give the following B_q-bimodule structure on V_1.

$$\text{comodule structure} \begin{cases} u \mapsto u \otimes a + v \otimes c, \\ v \mapsto u \otimes b + v \otimes d, \end{cases}$$

$$\text{module structure} \begin{cases} ux = p(x)u, \\ vx = n(x)u + p^{-1}(x)v, \end{cases} \quad x \in B_q.$$

The corresponding actions of K, E, F and a, b, c, d are as follows:

$$Ku = qu, \quad Kv = q^{-1}v, \quad Eu = 0, \quad Ev = u, \quad Fu = v, \quad Fv = 0,$$

$$ua = q^{1/2}u, \quad va = q^{-1/2}v, \quad ub = 0, \quad vb = 0,$$

$$uc = 0, \quad vc = q^{1/2}(1 - q^{-2})u, \quad ud = q^{-1/2}u, \quad vd = q^{1/2}v.$$

One verifies the 12 commutation relations hold. Hence V_1 is a crossed B_q-bimodule. (Note that the choice $e = q^{1/2}(1 - q^{-2})$ is compulsory.)

All finite dimensional U_q-modules are completely reducible, and for each integer $k \geqq 0$, there are two simple U_q-modules of dimension $k+1$. One is a comodule for B_q and the other for $B_q\gamma$. Let V_k be the simple B_q-comodule of dimension $k+1$. It has the highest weight q^k and the lowest weight q^{-k}. Every B_q-comodule is the direct sum of a set of V_k, $k \in \mathbb{N}$ [T2, Sects. 2, 3].

Let B_q^{op} the algebra opposite of B_q with the same coalgebra structure. It is identified with $B_{q^{-1}}$. The maps $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}$ induce Hopf algebra maps

$$B_q \to A_q \quad \text{and} \quad B_q^{\mathrm{op}} \to A_q .$$

Let Φ be the composite

$$B_q \otimes B_q^{\mathrm{op}} \to A_q \otimes A_q \xrightarrow{\text{mult}} A_q ,$$

which is a coalgebra map.

Let V_k^* be the simple B_q^{op}-comodule of dimension $k+1$. Then $V_k \otimes V_l^*$, $(k,l) \in \mathbb{N}^2$ give a complete set of representatives for the isomorphism classes of all simple comodules for $B_q \otimes B_q^{\mathrm{op}}$.

Every comodule for $B_q \otimes B_q^{\mathrm{op}}$ has an A_q-comodule structure through Φ, hence the structure of a crossed B_q-bimodule.

5.3 Definition. For integers $k, l \geqq 0$, let $V_{k,l}$ denote the A_q-comodule $V_k \otimes V_l^*$. It is also considered as a crossed B_q-bimodule.

An explicit description of the actions of K, E, F and a, b, c, d on $V_{k,l}$ will be given in the next section (6.4). One sees 5.2 Example gives a description of $V_{1,0}$.

When V is a crossed B_q-bimodule, let $V^{(-)}$ denote the B_q-comodule V on which a, b, c, d act by the operations of $-a$, $-b$, $-c$, $-d$. One verifies the 12 commutation relations for $V^{(-)}$. The corresponding comodule structure for the quantum double $\mathscr{E}_q = \mathrm{End}_{\mathbb{C}}(B_q)$ is obtained through the multiplication of $\tilde{\tau}$.

The following two theorems will be proved in the next two sections.

5.4 Theorem. *$V_{k,l}$ and $V_{k,l}^{(-)}$ give a complete set of representatives for the isomorphism classes of all simple finite dimensional crossed B_q-bimodules.*

5.5 Theorem. *All finite dimensional crossed B_q-bimodules are completely reducible.*

We deduce 4.9 Theorem from these results. Let $\tilde{A}_q$ be the *-algebra defined by generators $\alpha, \beta, \gamma, \delta$ and the 17 relations of 4.7. It has a *-Hopf algebra structure given by 4.8, and it corresponds to the quantum Lorentz group $\mathrm{SL}_q(2, \mathbb{C})$. There is a canonical surjective *-Hopf algebra map $\tilde{A}_q \to A_q$, and the coalgebra map Φ factors as

$$\Phi : B_q \otimes B_q^{\mathrm{op}} \to \tilde{A}_q \to A_q .$$

Let E_q be the largest subcoalgebra of $\mathscr{E}_q = \mathrm{End}_{\mathbb{C}}(B_q)$. 5.5 Theorem means E_q is cosemisimple, and 5.4 Theorem means it is the direct sum of the coefficient spaces of simple comodules $V_{k,l}$ and $V_{k,l}^{(-)}$, $k, l \in \mathbb{N}$. In other words, the map

$$\Phi + \Phi\tilde{\tau} : (B_q \otimes B_q^{\mathrm{op}}) \oplus (B_q \otimes B_q^{\mathrm{op}}) \to A_q + A_q\tilde{\tau} \subset \mathscr{E}_q$$

is injective and has image E_q. Since $A_q + A_q\tilde{\tau}$ is contained in E_q, it follows that

$\underset{\sim}{E}_q = A_q + A_q\tilde{\tau} = A_q \oplus A_q\tilde{\tau}$ and that Φ is an isomorphism. This implies the projection $\tilde{A}_q \to A_q$ is an isomorphism, and the statements (a), (b), (c) of 4.9 follow.

As a corollary of 5.4 and 5.5, it follows that every locally finite crossed B_q-bimodule (or a locally finite representation of $\mathrm{SL}_q(2, \mathbb{C})$ in the terminology of [PW]) V is isomorphic to the direct sum $V_1 \oplus V_2^{(-)}$ for some uniquely determined (up to isomorphisms) A_q-comodules V_1, V_2. A_q-comodules are *smooth* representations of $\mathrm{SL}_q(2, \mathbb{C})$ in the terminology of [PW]. Thus our results give an affirmative answer to the Conjecture 6.4 of [PW].

6. Finite Dimensional Simple Crossed B_q-Bimodules

We prove 5.4 Theorem. We begin with describing the bimodule structure of $V_{k,l}$.

The bimodule structure of $V_1 = \mathbb{C}u \oplus \mathbb{C}v$ is described in 5.2. We can identify V_k as the subspace of $V_1 \otimes \cdots \otimes V_1$ (k copies spanned by $F^i v_k$. $0 \leqq i \leqq k$, with $v_k = u \otimes \cdots \otimes u$. It is a sub-bimodule, and the following description follows easily by induction using the 12 commutation relations of 5.1,

$$(F^i v_k)a = q^{(k/2)-i} F^i v_k, \quad (F^i v_k)b = 0,$$
$$(F^i v_k)c = q^{-(k/2)+i}(1-q^{-2})[i][k+1-i]F^{i-1}v_k, \quad (F^i v_k)d = q^{-(k/2)+i}F^i v_k, \tag{6.1}$$

for $0 \leqq i \leqq k$. Here and in the following, we use the notation

$$[i] = \frac{q^i - q^{-i}}{q - q^{-1}}.$$

The expression $(F^i v_k)c$ should be interpreted to mean 0 when $i = 0$, since $[0] = 0$ (though $F^{-1}v_k$ has no meaning). Such a convention will be used in the following.

The description of $V_1^*(= V_{0,1})$ involves the signature of q. Let η be the signature of q in the rest of the paper. By definition, the A_q-comodule V_1^* has a base u^*, v^* such that the coaction is given by

$$v^* \mapsto v^* \otimes \alpha^* + u^* \otimes \gamma^*, \quad u^* \mapsto v^* \otimes \beta^* + u^* \otimes \delta^*.$$

By using the expression below 4.6, one sees the actions of K, E, F and a, b, c, d on V_1^* are described as follows.

$$Ku^* = qu^*, \quad Kv^* = q^{-1}v^*, \quad Eu^* = 0, \quad Ev^* = -qu^*,$$
$$Fu^* = -q^{-1}v^*, \quad Fv^* = 0,$$
$$u^*a = \eta q^{-1/2}u^*, \quad v^*a = \eta q^{1/2}v^*, \quad u^*b = \eta q^{1/2}(1-q^{-2})v^*, \quad v^*b = 0,$$
$$u^*c = 0, \quad v^*c = 0, \quad u^*d = \eta q^{1/2}u^*, \quad v^*d = \eta q^{-1/2}v^*. \tag{6.2}$$

We can identify V_l^* as the span of $E^i v_{-l}$, $0 \leqq i \leqq l$ in $V_1^* \otimes \cdots \otimes V_1^*$ (l copies), where $v_{-l} = v^* \otimes \cdots \otimes v^*$. Similarly as before, one verifies the following expressions:

$$(E^i v_{-l})a = \eta^l q^{(l/2)-i} E^i v_{-l},$$
$$(E^i v_{-l})b = -\eta^l q^{1-(l/2)+i}(1-q^{-2})[i][l+1-i]E^{i-1}v,$$
$$(E^i v_{-l})c = 0, \quad (E^i v_{-l})d = \eta^l q^{-(l/2)+i} E^i v_{-l}. \tag{6.3}$$

It follows from (6.1) and (6.3) that the crossed bimodule $V_{k,l}$ has a base $v_{i,j}=F^i v_k \otimes E^j v_{-l}$, $0 \leqq i \leqq k$, $0 \leqq j \leqq l$, and admits the following description of the actions:

$$\begin{aligned}
v_{i,j}a &= \eta^l q^{((k+l)/2)-i-j} v_{i,j},\\
v_{i,j}b &= -\eta^l q^{1+((k-l)/2)+j-1}(1-q^{-2})[j][l+1-j]v_{i,j-1},\\
v_{i,j}c &= \eta^l q^{((l-k)/2)+i-j}(1-q^{-2})[i][k+1-i]v_{i-1,j},\\
v_{i,j}d &= \eta^l q^{-((k+l)/2)+i+j}(v_{i,j}-q(1-q^{-2})^2[i][j][k+1-i][l+1-j]v_{i-1,j-1}),\\
Kv_{i,j} &= q^{k-l+2(j-i)}v_{i,j},\\
Ev_{i,j} &= v_{i,j+1}+q^{2j-l}[i][k-i+1]v_{i-1,j},\\
Fv_{i,j} &= v_{i+1,j}+q^{2i-k}[j][l-j+1]v_{i,j-1}.
\end{aligned} \tag{6.4}$$

Here, we use the convention that $v_{i,j}$ means 0 unless $0 \leqq i \leqq k$ or $0 \leqq j \leqq l$.

The description (6.4) tells that the following modified crossed bimodules are more natural objects to study.

6.5 Definition. *For integers $k, l \geqq 0$, let $\tilde{V}_{k,l}$ mean $V_{k,l}$ if $q>0$ or l even, and $V^{(-)}_{k,l}$ otherwise.*

The crossed bimodule $\tilde{V}_{k,l}$ has an expression which is obtained from (6.4) by omitting the factor η^l.

6.6 Corollary. *If we put $\tilde{v}=v_{k,l}$ in $\tilde{V}_{k,l}$, then we have*

$$\tilde{v}a = q^{-(k+l)/2}\tilde{v}, \quad \tilde{v}b^{l+1}=0=\tilde{v}c^{k+1},$$

and $\tilde{v}b^ic^j$, $0 \leqq i \leqq l$, $0 \leqq j \leqq k$, form a base for $\tilde{V}_{k,l}$. We have

$$\begin{aligned}
K(\tilde{v}b^ic^j) &= q^{l-k+2(j-i)}\tilde{v}b^ic^j,\\
E(\tilde{v}b^ic^j) &= -q^{2+((l-k)/2)+j-i}(1-q^{-2})[i][l+1-i]\tilde{v}b^{i-1}c^j\\
&\quad + \frac{q^{-(k/2)+(3/2)l-3i+j}}{1-q^{-2}}\tilde{v}b^ic^{j+1},\\
E(\tilde{v}b^ic^j) &= q^{1+((k-l)/2)+i-j}(1-q^{-2})[j][k+1-j]\tilde{v}b^ic^{j-1}\\
&\quad - \frac{q^{-1-(l/2)+(3/2)k-3j+i}}{1-q^{-2}}\tilde{v}b^{i+1}c^j.
\end{aligned}$$

6.7 Proposition. *The crossed B_q-bimodules $V_{k,l}$ and $V^{(-)}_{k,l}$ are simple and pairwise non-isomorphic.*

Proof. We show $\tilde{V}_{k,l}$ is simple. The base element $v_{i,j}$ has weight $k-l+2(j-i)$ relative to K and weight $\dfrac{k+l}{2}-i-j$ relative to a. The pairs of weights are distinct for distinct (i,j). Hence a subbimodule of $\tilde{V}_{k,l}$ is spanned by a set of $v_{i,j}$. In particular, it contains some $v_{i,j}$ if it is non-zero. Applying b and c, one sees it contains $v_{0,0}$. It is easy to see $\tilde{V}_{k,l}$ is generated by $v_{0,0}$ over U_q. This implies $\tilde{V}_{k,l}$ (and $\tilde{V}^{(-)}_{k,l}$ also) is simple. To see they are mutually non-isomorphic, note that $\tilde{V}_{k,l}$ has highest weight $k+l$ relative to K, and the corresponding weight vector has weight $\frac{k-l}{2}$ relative to a. Since the pair $\left(k+l, \frac{k-l}{2}\right)$ determines (k,l), it follows that $V_{k,l}$ and $V_{k,l}^{(-)}$ for all k,l are mutually non-isomorphic. □

For a crossed B_q-bimodule V and $\lambda, \mu \in \mathbb{C}$, let $V_{\lambda,\mu}$ be the subspace of $v \in V$ such that $Kv = \lambda v$ and $va = \mu v$. It follows from the commutation relations of b, c with K, a (4.2 and 5.1), that we have

$$V_{\lambda,\mu} b \subset V_{q^{-2}\lambda, q\mu}, \quad V_{\lambda,\mu} c \subset V_{q^2\lambda, q\mu}.$$

6.8 Lemma. *If V is finite dimensional, $V_{\lambda,0} = 0$.*

Proof. We may assume $\lambda \neq 0$. Since $V_{\lambda,0} b \subset V_{q^{-2}\lambda,0}$, b is nilpotent on $V_{\lambda,0}$. Similarly, c is nilpotent, too. Since $1 = ad - q^{-1}bc$, we have $v = -q^{-1}vbc$ for $v \in V_{\lambda,0}$. This implies $v = 0$, □

6.9 Lemma. *Let $v \in V_{\lambda,\mu}$. We have*

(1) $vc - (1 - q^{-2})\lambda^{-1}\mu Ev \in V_{q^2\lambda, q^{-1}\mu}$,

(2) $vb + q(1 - q^{-2})\lambda\mu Fv \in V_{q^{-2}\lambda, q^{-1}\mu}$.

Proof. (1) Let w be the element of the left-hand side. Since

$$(Ev)a = q^{-1}E(va) + q^{-1}K(vc) = q^{-1}\mu Ev + q\lambda vc,$$

we have

$$\begin{aligned} wa &= q\mu vc - (1 - q^{-2})\lambda^{-1}\mu(q^{-1}\mu Ev + q\lambda vc) \\ &= q^{-1}\mu vc - q^{-1}(1 - q^{-2})\lambda^{-1}\mu^2 Ev = q^{-1}\mu w. \end{aligned}$$

Since vc and Ev have weight $q^2\lambda$ relative to K, so is w. (2) is similar. □

Let V be a finite dimensional simple crossed B_q-bimodule. We show V is isomorphic to $V_{k,l}$ or $V_{k,l}^{(-)}$ for some natural numbers k, l.

There are $\lambda, \mu \in \mathbb{C} - \{0\}$ such that $V_{\lambda,\mu} \neq 0$, since the actions of K and a commute with each other. We can choose λ, μ so that $V_{\nu, q^{-s}\mu} = 0$ for all $\nu \in \mathbb{C}$ and integers $s > 0$. Take a non-zero element v in $V_{\lambda,\mu}$. Since $V_{q^2\lambda, q^{-1}\mu} = 0 = V_{q^{-2}\lambda, q^{-1}\mu}$, it follows from 6.9 that

$$vb = -q(1 - q^{-2})\lambda\mu Fv \quad \text{and} \quad vc = (1 - q^{-2})\lambda^{-1}\mu Ev. \tag{6.10}$$

Note that vb^ic^j is in $V_{q^{2(j-i)}\lambda, q^{i+j}\mu}$. Hence they are linearly independent if non-zero.

6.11 Lemma. $vb^ic^jd = \mu^{-1}q^{-i-j}(vb^ic^j + q^{-1}vb^{i+1}c^{j+1})$.

Proof. Apply $1 = ad - q^{-1}bc$ to vb^ic^j. □

6.12 Lemma. *We have*

(1) $E(vb^i) = (\mu - \mu^{-1}\lambda q^{2(1-i)})[i]vb^{i-1} + \lambda\mu^{-1}q^{-3i}(1 - q^{-2})^{-1}vb^ic$,

(2) $F(vc^j) = -q^{-1}(\mu - \lambda^{-1}\mu^{-1}q^{2(1-j)})[j]vc^{j-1} - \lambda^{-1}\mu^{-1}q^{-1-3j}(1 - q^{-2})^{-1}vbc^j$.

Proof. (1) If $i = 0$, this is (6.10). (We use the convention that $[0]vb^{-1}$ means 0).

We have by (6) of 5.1,

$$E(vb^i) = q^{-1}(E(vb^{i-1}))b + vb^{i-1}a - K(vb^{i-1}d),$$

where $vb^{i-1}a = q^{i-1}\mu vb^{i-1}$ and $vb^{i-1}d = \mu^{-1}q^{1-i}(vb^{i-1} + q^{-1}vb^ic)$ by 6.11. Hence $K(vb^{i-1}d) = \mu^{-1}q^{3(1-i)}\lambda(vb^{i-1} + q^{-1}vb^ic)$. Using this, the claim follows by induction. (2) is similar. □

There are only finitely many pairs of integers (i,j) such that $vb^ic^j \neq 0$, since they are linearly independent. Take the largest integers $k, l \geqq 0$ such that $vc^k \neq 0$ and $vb^l \neq 0$. It follows from (6.10) and (7) (respectively (10)) of 5.1 that the vector E^jv (respectively F^iv) is proportional to vc^j (respectively vb^i). Hence,

$$E^kv \neq 0, \quad E^{k+1}v = 0; \quad F^lv \neq 0, \quad F^{l+1}v = 0.$$

It follows from [APW, 1.11 Lemma] that

$$\lambda = q^{l-k}.$$

Letting $i = l+1$ in (1) of 6.12, we conclude that

$$\mu^2 = \lambda q^{-2l} = q^{-k-l}, \quad \text{so} \quad \mu = \pm q^{-(k+l)/2}.$$

Assume we have $\mu = q^{-(k+l)/2}$.

6.13 Proposition. *The correspondence $\tilde{v}b^ic^j \mapsto vb^ic^j$ gives an isomorphism of B_q-bimodules $\tilde{V}_{k,l} \cong V$.*

Proof. Call the above correspondence $\phi: \tilde{V}_{k,l} \to V$. This map commutes with the actions of K and a, since $\tilde{v}$ and v have the same weights relative to K and a. Obviously, it commutes with the actions of b, c. By using 6.12 Lemma and the fact that

$$E(vb^ic^j) = q^j(E(vb^i))c^j, \quad F(vb^ic^j) = q^i(F(vc^j))b^i,$$

one sees easily that the last two identities of 6.6 Corollary hold with $\tilde{v}$ replaced by v. Thus, ϕ commutes with a, b, c and K, E, F. It commutes with d, too, since 6.11 Lemma (with $\mu = q^{-(k+l)/2}$) holds for $\tilde{v}$. It follows that ϕ is a homomorphism of B_q-bimodules. Since both $\tilde{V}_{k,l}$ and V are simple crossed bimodules, one concludes that ϕ is an isomorphism. □

In case $\mu = -q^{-(k+l)/2}$, one gets an isomorphism $\tilde{V}^{(-)}_{k,l} \cong V$. This concludes the proof of 5.4 Theorem.

7. Complete Reducibility

We prove 5.5 Theorem.

A Hopf algebra H over a general field k is called *co-semisimple* if all (right or left) H-comodules are completely reducible [S, XIV]. The following criterion seems fairly well-known among specialists, but it is difficult to find an explicit literature.

7.1 Proposition. *The following are equivalent.*

(1) *The Hopf algebra H is cosemisimple.*

(2) *All exact sequences of H-comodules of the form*

$$0 \to W \to V \to k \to 0$$

split, where k denotes the trivial 1-dimensional comodule.

(3) (2) *is true for all simple comodules W.*

Proof. (3) ⇒ (2) is an easy exercise. To prove (2) ⇒ (1), let V be a comodule for H and W a subcomodule of V. We show W is a direct summand of V. We may assume V is finite dimensional. Then $\mathrm{Hom}_k(V, W)$ has a comodule structure isomorphic to $W \otimes V'$. The restriction induces a surjective comodule map

$$\mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(W, W) \to 0.$$

Let $\tilde{V}$ be the inverse image of kI which is a subcomodule of $\mathrm{Hom}_k(W, W)$. We have a surjective comodule map $\tilde{V} \to k \to 0$ which splits by (2). If $1 \mapsto \psi$, $k \to \tilde{V}$ is a section, ψ is a comodule map $V \to W$ such that $\psi | W = I$. □

We show every exact sequence of crossed B_q-bimodules

$$0 \to \tilde{V}_{k,l} \to V \to \mathbb{C} \to 0$$

splits. The same method applies to $\tilde{V}_{k,l}^{(-)}$, too. It will follow from 7.1 that the largest subcoalgebra E_q of the quantum double $\mathscr{E}_q = \mathrm{End}_{\mathbb{C}}(B_q)$ is cosemisimple, yielding 5.5 Theorem.

The above exact sequence splits as B_q-comodules, since B_q is cosemisimple. Let $V = \tilde{V}_{k,l} \oplus \mathbb{C}\zeta$ be the decomposition as B_q-comodules. We have $K\zeta = \zeta$, $E\zeta = 0 = F\zeta$ and there are elements w_a, w_b, w_c, w_d in $\tilde{V}_{k,l}$ such that

$$\zeta a = \zeta + w_a, \quad \zeta b = w_b, \quad \zeta c = w_c, \quad \zeta d = \zeta + w_d.$$

7.2 Lemma. *We have*

(1) $Kw_a = w_a$, (2) $Kw_b = q^{-2} w_b$, (3) $Kw_c = q^2 w_c$, (4) $Kw_d = w_d$,

(5) $Ew_a = -q^2 w_c$, (6) $Ew_b = w_a - w_d$, (7) $Ew_c = 0$, (8) $Ew_d = w_c$,

(9) $Fw_a = qw_b$, (10) $Fw_b = 0$, (11) $Fw_c = q^{-1}(w_d - w_a)$, (12) $Fw_d = -q^{-1} w_b$,

(13) $w_b a = q(w_b + w_a b)$, (14) $w_c a = q(w_c + w_a c)$,

(15) $w_b d = q^{-1}(w_b + w_d b)$, (16) $w_c d = q^{-1}(w_c + w_d c)$,

(17) $w_b c = w_c b$,

(18) $w_a + w_d a = q w_b c$, (19) $w_d + w_a d = q^{-1} w_b c$.

Proof. The first 12 relations are consequences of 5.1 applied to $v = \zeta$. The remaining 7 relations follow from the commutation relations of 4.2. □

7.3 Corollary. $w_b = 0$ *if and only if* $w_c = 0$.

Proof. If $w_b = 0$, $w_a = w_d$ by (6), and $w_c = -q^2 w_c$ by (5), (8). Hence $w_c = 0$, since q is real. Similarly, $w_b = 0$ if $w_c = 0$. □

We will find all non-trivial pairs of elements w_b, w_c in $\tilde{V}_{k,l}$ satisfying conditions (2), (3), (7), (10), and (17).

The Clebsch–Gordan rule [T2, Proposition 2.4] tells that

$$\tilde{V}_{k,l} \cong V_{k+l} \oplus V_{k+l-2} \oplus \cdots \oplus V_{|k-l|}$$

as B_q-comodules or U_q-modules. This means the U_q-module $\tilde{V}_{k,l}$ has a highest weight vector of weight 2 (or a lowest weight vector of weight -2) if and only if

$$k \equiv l \bmod 2 \quad \text{and} \quad |k-l| \leqq 2 \leqq k+l.$$

This condition is satisfied if and only if (k, l) is of the form $(r+1, r+1)$, $(r, r+2)$, or $(r+2, r)$ for some $r \geqq 0$. In any case, we have

$$r = \frac{k+l}{2} - 1.$$

We have a highest weight vector $\sum_{i=0}^{r} x_i v_{r-i,l-i}$ of weight 2 (respectively a lowest weight vector $\sum_{i=0}^{r} y_i v_{k-i,r-i}$ of weight -2), where

$$\begin{aligned} x_i &= (-q^l)(-q^{l-2})\cdots(-q^{l-2(i-1)}) \\ &\quad \cdot [r][r-1]\cdots[r-(i-1)][k-r+1][k-r+2]\cdots[k-r+i], \\ y_i &= (-q^k)(-q^{k-2})\cdots(-q^{k-2(i-1)}) \\ &\quad \cdot [r][r-1]\cdots[r-(i-1)][l-r+1][l-r+2]\cdots[l-r+i]. \end{aligned}$$

Let

$$w_c = \lambda \sum_{i=0}^{r} x_i v_{r-i,l-i}, \quad w_b = \mu \sum_{i=0}^{r} y_i v_{k-i,r-i}.$$

We have by (6.4),

$$\begin{aligned} w_c b &= -q\lambda(q-q^{-1})\sum x_i[i+1][l-i]v_{r-i,l-i-1}, \\ w_b c &= \mu(q-q^{-1})\sum y_i[i+1][k-i]v_{k-i-1,r-i}. \end{aligned}$$

Case $(k, l) = (r+1, r+1)$. Condition $w_b c = w_c b$ implies $-q\lambda x_i = \mu y_i$, $0 \leqq i \leqq r$. Since $x_i = y_i$ in this case, this is equivalent to $\mu = -q\lambda$.

Case $(k, l) = (r, r+2)$ or $(r+2, r)$. In this case, one of $w_b c$ and $w_c b$ has length r, and the other $r+1$. Hence the condition $w_b c = w_c b$ will imply both are zero. If $r > 0$, it follows that $w_b = w_c = 0$. If $r = 0$, we have

$$\begin{aligned} \lambda &= 0 \quad \text{if} \quad k < l, \\ \mu &= 0 \quad \text{if} \quad l < k. \end{aligned}$$

Summarizing the above, we get the following.

7.4 Proposition. *The following list gives all non-trivial pairs of elements w_b, w_c in $\tilde{V}_{k,l}$ satisfying conditions (2), (3), (7), (10), and (17) of 7.2.*

(a) $k = l = r+1$ *with* $r \geqq 0$ *and*

$$w_b = -q\lambda \sum_{i=0}^{r} z_i v_{r+1-i,r-i}, \quad w_c = \lambda \sum_{i=0}^{r} z_i v_{r-i,r+1-i},$$

where

$$z_i = (-q^{r+1})(-q^{r-1})\cdots(-q^{r+1-2(i-1)})[r][r-1]\cdots[r-(i-1)][1][2]\cdots[i+1],$$

(b) $k=0$, $l=2$, $w_b=\lambda v_{0,0}$, $w_c=0$,
(c) $k=2$, $l=0$, $w_b=0$, $w_c=\lambda v_{0,0}$,

where $\lambda \in \mathbb{C}$.

To show that the extension $V=\tilde{V}_{k,l}\oplus\mathbb{C}\zeta$ is trivial, first we reduce to the case $w_b=w_c=0$. By 7.3 and 7.4, it is enough to consider the case (a) of 7.4. In this case, let

$$\omega=\sum_{i=0}^{r+1} s_i v_{r+1-i,r+1-i},$$

where

$$s_i=(-q^{r+1})(-q^{r-1})\cdots(-q^{r+1-2(i-1)})[r+1][r]\cdots[r+1-(i-1)][1][2]\cdots[i].$$

Then $K\omega=\omega$, $E\omega=0=F\omega$. Since we have

$$z_i=[r+1]^{-1}s_i[r+1-i][i+1],$$

it follows from (6.4) that

$$\omega b=-q(1-q^{-2})[r+1]\sum_{i=0}^{r} z_i v_{r+1-i,r-i},$$

$$\omega c=(1-q^{-2})[r+1]\sum_{i=0}^{r} z_i v_{r-i,r+1-i}.$$

Hence, if we use

$$\zeta'=\zeta-\lambda(1-q^{-2})^{-1}[r+1]^{-1}\omega$$

instead of ζ, then we have $\zeta' b=0=\zeta' c$.

We may assume $w_b=w_c=0$. We have $w_a=w_d$ by (6) or (11) of 7.2. Call this w. Then $Kw=w$ by (1), $Ew=Fw=0$ by (5), (9), $wb=wc=0$ by (13), (14), and $wa=wd=-w$ by (18), (19). (We refer to conditions of 7.2). If we put

$$\tilde{\zeta}=\zeta+\tfrac{1}{2}w,$$

it follows that we have $V=\tilde{V}_{k,l}\oplus\mathbb{C}\tilde{\zeta}$, where $\mathbb{C}\tilde{\zeta}$ is a subbimodule isomorphic to the trivial bimodule $\mathbb{C}$. This shows that every extension of $\mathbb{C}$ by $\tilde{V}_{k,l}$ is trivial. The same argument applies to $\tilde{V}_{k,l}^{(-)}$, too. This will finish the proof of 5.5 Theorem.

8. Appendix. The Braiding Structure

In this appendix, we discuss the braiding structure on A_q. Mostly, we work over a general field k.

The category of (right) crossed bimodules for a bialgebra H is *pre-braided* [Y, Theorem 5.2]. This means if X, Y are crossed bimodules, then the map

$$\mathrm{s}_{X,Y}: X\otimes Y\to Y\otimes X,\ x\otimes y\mapsto \sum y_{(0)}\otimes xy_{(1)} \tag{8.1}$$

is a bimodule map satisfying the coherence condition

$$\mathrm{s}_{X\otimes Y,Z}=(\mathrm{s}_{X,Y}\otimes I_Y)(I_X\otimes \mathrm{s}_{Y,Z}),$$

$$\mathrm{s}_{X,Y\otimes Z}=(I_Y\otimes \mathrm{s}_{X,Z})(\mathrm{s}_{X,Y}\otimes I_Z).$$

The map $s_{X,Y}$ is an isomorphism for all crossed bimodules X, Y if H has a *twisted antipode*, i.e., if H^{op} or H^{cop} is a Hopf algebra [Y, Theorem 7.2]. Then the category of crossed bimodules is a *braided* category. This is the case if H has a bijective antipode.

On the other hand, the concept of a *braided bialgebra* was introduced by Larson–Towber [LT] and independently by Hayashi [H]. Let A be a bialgebra. By a *braiding* on A, we mean a unit in the algebra $(A \otimes A)'$ which we identify with a bilinear map

$$\langle\, ,\, \rangle : A \otimes A \to k$$

satisfying the following conditions:

$$\begin{aligned} \sum \langle x_{(1)}, y_{(1)} \rangle x_{(2)} y_{(2)} &= \sum \langle x_{(2)}, y_{(2)} \rangle y_{(1)} x_{(1)}, \\ \langle xy, z \rangle &= \sum \langle x, z_{(1)} \rangle \langle y, z_{(2)} \rangle, \\ \langle x, yz \rangle &= \sum \langle x_{(2)}, y \rangle \langle x_{(1)}, z \rangle, \\ \langle x, 1 \rangle &= \langle 1, x \rangle = \varepsilon(x) \end{aligned} \tag{8.2}$$

for all x, y, z in A.

If A is a braided bialgebra, then Com^{-A}, the category of (right) A-comodules, is braided. If V, W are A-comodules, then the braiding is given by

$$s_{V,W} : V \otimes W \to W \otimes V, v \otimes w \mapsto \sum \langle v_{(1)}, w_{(1)} \rangle w_{(0)} \otimes v_{(0)}. \tag{8.3}$$

Conversely, if the category Com^{-A} is braided, there is a unique braiding on A such that the braiding of Com^{-A} is given by (8.3) [LT, Proposition 2.13]. In other words, there is a 1–1 correspondence between braidings on A and braidings on Com^{-A}.

Let H be a Hopf algebra with bijective antipode, and let A be a subbialgebra of $\mathscr{E} = \mathrm{End}_k(H)$, the quantum double. The bialgebra A has the following braiding

$$\langle x, y \rangle = \langle \pi_2(x), \pi_1(y) \rangle, \quad x, y \in A, \tag{8.4}$$

where we use the topological Hopf algebra maps of (2.5) and the canonical pairing between H' and H. Note that Com^{-A} is identified with a sub-monodial category of the category of crossed H-bimodules.

8.5 Proposition. *The braiding on* Com^{-A} *given by (8.4) and (8.3) coincides with the one induced from the braiding structure of the category of crossed H-bimodules.*

Proof. Let V, W be A-comodules. Note that

$$v \mapsto \sum v_{(0)} \otimes \pi_2(v_{(1)}), \quad v \in V,$$

(respectively

$$w \mapsto \sum w_{(0)} \otimes \pi_1(w_{(1)}), \; w \in W)$$

gives the H-module structure on V (respectively the H-comodule structure on W). Hence we have

$$\sum \langle \pi_2(v_{(1)}), \pi_1(w_{(1)}) \rangle w_{(0)} \otimes v_{(0)} = \sum w_{(0)} \otimes v \pi_1(w_{(1)}),$$

$v \in V$, $w \in W$. This means the braiding $s_{V,W}$ (8.3) coincides with the braiding $s_{X,Y}$ (8.1) if X, Y denote the crossed bimodules identified with V, W. □

We are in this situation if we take $H = B_q$ and $A = A_q$. In this case, the braiding (8.4) is defined by using the Hopf algebra maps

$$\pi_1 : A_q \to B_q, \quad \pi_2 : A_q \to C_q$$

and the canonical pairing between C_q and B_q. Since π_1 and π_2 are *-algebra maps, too, it follows from the definition of $*$ on B'_q (above 4.1) that we have

$$\langle x^*, y \rangle = \overline{\langle x, S(y)^* \rangle}, \quad x, y \in A_q.$$

Since $\langle x, y \rangle = \langle S(x), S(y) \rangle$ [LT, Proposition 2.9] and $S(y)^* = S^{-1}(y^*)$, it follows that

$$\langle x^*, y^* \rangle = \overline{\langle S(x), y \rangle} = \overline{\langle x, S^{-1}(y) \rangle}, \quad x, y \in A_q.$$

In a word, A_q is a *braided *-Hopf algebra.*

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