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On Multivortices in the Electroweak Theory II: Existence of Bogomol'nyi Solutions in $\mathbb{R}^2$

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Abstract. In this paper we study the Bogomol'nyi equations of the electroweak theory in the full plane. We will show that, for any distribution of the vortices, there exists a two parameter family of gauge-distinct solutions. Moreover, we also establish some sharp decay rate estimates for these solutions.

1. Introduction

In Part I of this paper [7], we have proven the existence of Abrikosov like periodic vortices in the bosonic sector model proposed by Ambjorn and Olesen [3, 4] of the Glashow–Salam–Weinberg theory. These solutions were found from a Bogomol'nyi system of first order equations which take on a more complicated form than in the classical abelian case due to the anti-screening of the magnetic field. As a result, this system further reduces to a semilinear elliptic system of nonstandard type and we showed in Part I that the number of such vortices is bounded above in terms of the relevant physical parameters, although the locations may be prescribed arbitrarily.

The goal of the present paper is to study this Bogomol'nyi system for the self-dual electroweak interactions in the full space $\mathbb{R}^2$. These solutions are necessarily of infinite energy and thus the method of Part I cannot be directly applied. Our main strategy then, is to combine the method of weighted Sobolev spaces, used by McOwen [6] in his study of conformal deformation equations, with the crucial change of variables introduced in Part I to reduce our elliptic system to a lower diagonal form. As a result, we are able to show (Theorem 3.3) that for any distribution of vortex locations there is a two parameter family of

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gauge-distinct solutions. Furthermore, we are also able to obtain some results concerning the asymptotic behavior of these solutions which may provide information about the blow-up rate of the energy. One of the interesting things is that it can be shown that the Higgs field φ and the W field vanish at infinity with entirely different speeds: φ decays faster than any exponential function of the type $\exp(-\sigma r)$ $(\sigma > 0)$ while W obeys a power law of the form r^{-a} $(a > 0)$. These decay estimates are shown to be sharp.

In order to fix the ideas, we first illustrate this method applied to the simplified $SO(3)$ or $SU(2)$ theory of Ambjorn and Olesen [2] (see also Yang [9]) in which the W-bosons acquire mass through a Higgs mechanism but the Higgs field are neglected from the Lagrangian. Here the system of Bogomol'nyi equations can be reduced to a single semilinear elliptic equation very closely related to the equation of prescribed Gaussian curvature. Thus in Sect. 2, we apply McOwen's method to study the existence of these massive $SO(3)$ vortices. We then go on in Sect. 3 to study the full electroweak theory and prove our main existence theorem for multivortices. In Sect. 4 we present a variant of the existence theorem in a different parameter regime. The method used is similar to that adopted in Sect. 3 but the results is of independent interest. In Sect. 5 we obtain the detailed asymptotic behavior of our solutions.

2. The Massive $SO(3)$ Multivortices

According to the discussion of Ambjorn and Olesen [2], the reduced energy density for vortex-line solutions of the massive $SO(3)$ gauge field theory is given by

$$\mathscr{E} = \tfrac{1}{2}P_{12}^2 + |D_1 W + iD_2 W|^2 + 2m_W^2|W|^2 - 2eP_{12}|W|^2 + 2e^2|W|^4, \tag{2.1}$$

where W is a complex scalar field, $P_j (j = 1, 2)$ is a vector field, $P_{12} = \partial_1 P_2 - \partial_2 P_1$, and

$$D_j W = \partial_j W - ieP_j W.$$

The model (2.1) can also be viewed as describing the pure gauge photon and W-boson interactions of the full electroweak theory defined by the expression (2.5) in Part I in the limit $g \to e$, $\varphi \to \varphi_0$, $Z_j \to 0$. $m_W^2 > 0$ gives rise to massive W-particles.

By virtue of the relation $(D_j D_k - D_k D_j)W = -ieP_{jk}W$, the Euler–Lagrange equations associated with (2.1) may be written as

$$\begin{cases} D_j D_j W = 2m_W^2 W - 3eP_{12}W + 4e^2|W|^2 W, \\ \partial_j P_{jk} = ie(W^\dagger(D_k W) - W(D_k W)^\dagger) + 3e\varepsilon_{jk}(W^\dagger(D_j W) + W(D_j W)^\dagger). \end{cases} \tag{2.2}$$

The linearized version of (2.2) has been studied by Ambjorn, Nielsen, and Olesen [1] in view of stability.

By rewriting the energy density $\mathscr{E}$ as

$$\mathscr{E} = |D_1 W + iD_2 W|^2 + \frac{1}{2}\left(P_{12} - \left[\frac{m_W^2}{e} + 2e|W|^2\right]\right)^2 + \frac{m_W^2}{e}\left(P_{12} - \frac{m_W^2}{2e}\right), \tag{2.3}$$

it can be seen that the 't Hooft boundary condition [8] implies that the magnetic flux through a periodic cell domain Ω is quantized and the energy minima are saturated by the solutions of the following Bogomol'nyi equations:

$$\begin{cases} D_1 W + iD_2 W = 0, \\ P_{12} - \left(\dfrac{m_W^2}{e} + 2e|W|^2\right) = 0. \end{cases} \tag{2.4}$$

Of course (2.4) implies (2.2) in Ω. In [9], one of us (Y. Y.) has shown that (2.4) possesses a periodic N-vortex solution if

$$\frac{2\pi(N-2)}{|\Omega|} < m_W^2 < \frac{2\pi N}{|\Omega|}.$$

Here we are interested in solutions of (2.4) over the full $\mathbb{R}^2$. It is easily checked that solutions of (2.4) are also solutions (2.2) on $\mathbb{R}^2$. What is the energy of such solutions? Using (2.4) in (2.3) we have

$$\mathscr{E} = \frac{m_W^2}{e}\left(P_{12} - \frac{m_W^2}{2e}\right) = \frac{1}{2}\frac{m_W^4}{e^2} + 2m_W^2|W|^2 \geqq \frac{m_W^4}{2e^2}.$$

Therefore the total energy $\int_{\mathbb{R}^2} \mathscr{E}\, dx$ is necessarily infinite.

For convenience, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and use z to denote a point in $\mathbb{C}$. Let $z_0 \in \mathbb{C}$ be a zero of W. The first equation in (2.4) implies that in a neighborhood of $z = z_0$,

$$W(z) = (z - z_0)^{n_0} h_0(x_1, x_2),$$

where n_0 is an integer and h_0 is a smooth nonvanishing function. Thus the zero set $Z(W)$ of W is discrete. If $Z(W) = \{z_1, \ldots, z_m\}$ and the multiplicity of the zero $z = z_l$ of W is n_l, then the replacement $u = \ln|W|^2$ reduces (2.4) to

$$\Delta u = -2m_W^2 - 4e^2 \exp(u) + 4\pi \sum_{l=1}^{m} n_l \delta(z - z_l). \tag{2.5}$$

Define

$$u_0(z) = \sum_{l=1}^{m} \ln|z - z_l|^{2n_l} - \tfrac{1}{2}m_W^2|z|^2.$$

Then

$$\Delta u_0 = 4\pi \sum_{l=1}^{m} n_l \delta(z - z_l) - 2m_W^2$$

and

$$u_1 = u - u_0$$

satisfies

$$\Delta u_1 = -4e^2 U_0 \exp(u_1),$$

where

$$U_0 = \exp(u_0) = \prod_{l=1}^{m} |z - z_l|^{2n_l} \exp(-\tfrac{1}{2}m_W^2 r^2), \quad r = |z| = |x|. \tag{2.6}$$

We now introduce the function $u_2 \in C^\infty(\mathbb{R}^2)$ so that

$$u_2 = -\alpha \ln r, \quad r \geqq 1,$$

where $\alpha > 0$ is a constant. Let $\eta = u_1 - u_2$. Then (2.5) is reduced to

$$\Delta\eta + K\exp(\eta) = -\Delta u_2 \equiv f, \tag{2.7}$$

where

$$K = 4e^2 U_0 \exp(u_2).$$

Because of (2.6), the function K satisfies:

$$K \geqq 0, \quad K = O(\exp[-r]) \quad \text{for large} \quad r > 0. \tag{2.8}$$

It is easily seen that f is of compact support. Also,

$$\begin{aligned}\int_{\mathbb{R}^2} f\,dx &= \int_{|x| \leqq 1} f\,dx = -\int_{|x| \leqq 1} \Delta u_2\,dx \\ &= -\int_{|x|=1} \frac{\partial u_2}{\partial r}\,ds = 2\pi\alpha.\end{aligned} \tag{2.9}$$

As in McOwen [6] we define the functionals

$$I(\eta) = \int_{\mathbb{R}^2} [\tfrac{1}{2}|\nabla\eta|^2 + f\eta]\,dx,$$

$$J(\eta) = \int_{\mathbb{R}^2} K\exp(\eta)\,dx.$$

In order that these functionals be defined properly, we need to consider a suitable weighted Sobolev space. Let $d\mu = h_0\,dx$, where h_0 is a positive C^∞ function with

$$h_0(x) = r^{-\kappa} \quad \text{for} \quad r = |x| \geqq 1.$$

Here and in the sequel, $\kappa > 4$.

Use the notation $L^p(d\mu) = L^p(\mathbb{R}^2, d\mu)$. Let $\mathscr{H}$ denote the Hilbert space of L^2_{loc} functions for which

$$\|\eta\|^2_{\mathscr{H}} = \|\nabla\eta\|^2_{L^2(dx)} + \|\eta\|^2_{L^2(d\mu)} < \infty.$$

Notice that $\mathscr{H}$ contains the constants and thus

$$\eta \mapsto \int_{\mathbb{R}^2} \eta\,d\mu$$

is a continuous linear functional on $\mathscr{H}$ so that

$$\tilde{\mathscr{H}} = \left\{\eta \in \mathscr{H} : \int_{\mathbb{R}^2} \eta\,d\mu = 0\right\}$$

is a closed subspace of $\mathscr{H}$. Therefore we have for each $\eta \in \mathscr{H}$ the decomposition:

$$\eta = \bar{\eta} + \eta', \quad \bar{\eta} = \text{constant}, \quad \eta' \in \tilde{\mathscr{H}}. \tag{2.10}$$

The following results may be found in McOwen [6]:

Lemma 2.1. *For any* $0 < \varepsilon < 4\pi$, *there is* $C(\varepsilon) > 0$ *so that*

$$\int_{\mathbb{R}^2} \exp(a|\eta|)\,d\mu \leqq C(\varepsilon)\exp\left[\frac{a^2}{4(4\pi - \varepsilon)}\|\nabla\eta\|^2_{L^2(dx)}\right], \quad \eta \in \tilde{\mathscr{H}}$$

for any $a \in \mathbb{R}$.

Lemma 2.2. *The Poincaré inequality holds on* $\tilde{\mathscr{H}}$: *there is a constant* $C > 0$ *so that*

$$\|\eta\|^2_{L^2(d\mu)} \leqq C\|\nabla\eta\|^2_{L^2(dx)}, \quad \eta\in\tilde{\mathscr{H}}.$$

Lemma 2.3. *The injection* $\tilde{\mathscr{H}} \to L^2(d\mu)$ *is a compact embedding.*

Thus we see that both $I(\eta)$ and $J(\eta)$ are well defined on $\mathscr{H}$. Consider now the optimization problem

$$\min\{I(\eta)|J(\eta) = 2\pi\alpha, \eta\in\mathscr{H}\}. \tag{2.11}$$

Lemma 2.4. *The problem (2.11) has a solution provided* $0 < \alpha < 4$.

Proof. For $\eta\in\mathscr{H}$, let us use the decomposition (2.10). If $J(\eta) = 2\pi\alpha$, then

$$\exp(\bar{\eta}) \int_{\mathbb{R}^2} K\exp(\eta')dx = 2\pi\alpha,$$

or

$$\bar{\eta} = \ln 2\pi\alpha - \ln\left[\int_{\mathbb{R}^2} K\exp(\eta')dx\right]. \tag{2.12}$$

As a consequence,

$$\begin{aligned} I(\eta) &= \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\eta'|^2 dx + \int_{\mathbb{R}^2} (f\bar{\eta} + f\eta')dx \\ &= \frac{1}{2}\|\nabla\eta'\|^2_{L^2(dx)} + \int_{\mathbb{R}^2} f\eta' dx + 2\pi\alpha\left[\ln 2\pi\alpha - \ln\left(\int_{\mathbb{R}^2} K\exp(\eta')dx\right)\right]. \end{aligned} \tag{2.13}$$

On the other hand, using Lemma 2.1, we find

$$\begin{aligned} \int_{\mathbb{R}^2} K\exp(\eta')dx &= \int_{\mathbb{R}^2} Kh_0^{-1}\exp(\eta')h_0 dx \leqq C_1 \int_{\mathbb{R}^2} \exp(\eta')d\mu \\ &\leqq C_1' \exp\left(\frac{1}{4(4\pi-\varepsilon)}\|\nabla\eta'\|^2_{L^2(dx)}\right), \end{aligned} \tag{2.14}$$

and

$$\left|\int_{\mathbb{R}^2} f\eta' dx\right| = \left|\int_{\mathbb{R}^2} fh_0^{-1/2}\eta' h_0^{1/2} dx\right| \leqq \varepsilon^{-1}C_2 + \varepsilon\|\eta'\|^2_{L^2(d\mu)}. \tag{2.15}$$

Substituting (2.14)–(2.15) into (2.13) and using Lemma 2.2 yield the lower bound

$$I(\eta) \geqq \frac{1}{2}\left(1 - \frac{\pi\alpha}{4\pi-\varepsilon} - \varepsilon C'\right)\|\nabla\eta'\|^2_{L^2(dx)} - C''(\varepsilon), \tag{2.16}$$

where C' is a constant independent of $\varepsilon, \alpha > 0$.

Since $0 < \alpha < 4$, we can fix $\varepsilon > 0$ sufficiently small to make

$$\sigma \equiv 1 - \frac{\pi\alpha}{4\pi - \varepsilon} - \varepsilon C' > 0.$$

Let $\{\eta_j\}$ be a minimizing sequence of (2.11). Then (2.16) says that

$$\|\nabla\eta'_j\|^2_{L^2(dx)} \leqq M, \quad j=1,2,\ldots,$$

where $M>0$ is a constant.

By virtue of (2.12) and (2.14), it is seen that $\{\bar\eta_j\}$ is bounded as well. So we may assume

$$\eta'_j \to \eta' \in \tilde{\mathscr{H}} \quad \text{weakly},$$
$$\bar\eta_j \to \bar\eta \in \mathbb{R}.$$

Hence from Lemma 2.3, we may assume that $\eta_j \to \eta = \bar\eta + \eta' \in \mathscr{H}$ strongly in $L^2(d\mu)$. Therefore,

$$\left|\int_{\mathbb{R}^2} f\eta_j dx - \int_{\mathbb{R}^2} f\eta dx\right| \leqq \int_{\mathbb{R}^2} |f| h_0^{-1/2} |\eta_j - \eta| h_0^{1/2} dx$$
$$\leqq C\|\eta_j - \eta\|_{L^2(d\mu)} \to 0 \quad \text{as} \quad j\to\infty,$$

and

$$\left|\int_{\mathbb{R}^2} K\exp(\eta_j)dx - \int_{\mathbb{R}^2} K\exp(\eta)dx\right|$$
$$\leqq \int_{\mathbb{R}^2} K\exp(|\eta_j| + |\eta|)|\eta_j - \eta| dx$$
$$\leqq C\int_{\mathbb{R}^2} K\exp(|\eta'_j| + |\eta'|)h_0^{-3/4} h_0^{1/4} |\eta_j - \eta| h_0^{1/2} dx$$
$$\leqq C\left(\int_{\mathbb{R}^2} K^4 \exp(4|\eta'|)h_0^{-3} dx\right)^{1/4} \left(\int_{\mathbb{R}^2} \exp(4|\eta'_j|)d\mu\right)^{1/4} \left(\int_{\mathbb{R}^2} |\eta_j - \eta|^2 d\mu\right)^{1/2}$$
$$\leqq C'\exp\left[\frac{1}{4\pi - 1}\|\nabla\eta_j\|^2_{L^2(dx)}\right]\|\eta_j - \eta\|_{L^2(d\mu)} \to 0 \quad \text{as} \quad j\to\infty.$$

Thus $I(\eta) \leqq \liminf_{j\to\infty} I(\eta_j)$ and $J(\eta) = \lim_{j\to\infty} J(\eta_j) = 2\pi\alpha$. In other words, η solves (2.11). □

Lemma 2.5. *The minimizer η of (2.11) obtained in Lemma 2.4 is a solution to (2.7)*

Proof. By the Lagrange multiplier rule, $\exists\,\lambda\in\mathbb{R}$ so that

$$\int_{\mathbb{R}^2} (\nabla\eta\cdot\nabla\chi + f\chi)dx = \lambda \int_{\mathbb{R}^2} K\exp(\eta)\chi dx, \quad \forall\,\chi\in\mathscr{H}. \tag{2.17}$$

Taking the test function $\chi \equiv 1$ in (2.17), we get

$$2\pi\alpha = \lambda J(\eta) = 2\pi\alpha\lambda.$$

Hence $\lambda = 1$, and η is a weak solution of (2.7). The elliptic regularity theory then implies that η is a C^∞ solution of (2.7). □

Under the notation of this section,

$$u = u_0 + u_1 = u_0 + u_2 + \eta$$

is a solution of (2.5). From the function u we can construct as in Jaffe and Taubes [5] a solution pair (W, P_j) of (2.4) so that $|W|^2 = \exp(u)$. Of course, different values of α in (2.11) correspond to different solutions of (2.5). Do these solutions give rise to different (gauge-distinct) solutions of the Bogomol'nyi system (2.4)? To answer this, we recall that

$$K = 4e^2 U_0 \exp(u_2) = 4e^2 \exp(u_0 + u_2).$$

Hence

$$\begin{aligned} 2\pi\alpha &= \int_{\mathbb{R}^2} K \exp(\eta) dx = \int_{\mathbb{R}^2} 4e^2 \exp(u_0 + u_2 + \eta) dx \\ &= 4e^2 \int_{\mathbb{R}^2} \exp(u) dx = 4e^2 \int_{\mathbb{R}^2} |W|^2 dx. \end{aligned} \tag{2.18}$$

But (2.18) is invariant under the (residual) gauge symmetry

$$W \mapsto \exp(i\chi) W, \quad P_j \mapsto P_j + \frac{1}{e} \partial_j \chi.$$

Therefore different α's give rise to gauge-distinct solutions of the Bogomol'nyi system (2.4). We have thus shown

Theorem 2.6. *Let $z_1, \ldots, z_m \in \mathbb{C} = \mathbb{R}^2$ and $n_1, \ldots, n_m \in \mathbb{Z}_+$. Then, for any $0 < \alpha < 4$, the Bogomol'nyi equations (2.4) have a solution $(W^{(\alpha)}, P_j^{(\alpha)})$ satisfying*

$$\int_{\mathbb{R}^2} |W^{(\alpha)}| dx = \frac{\pi\alpha}{2e^2},$$

$Z(W^{(\alpha)}) = \{z_1, \ldots, z_m\}$, and the multiplicity of the zero $z = z_l$ of $W^{(\alpha)}$ is n_l $(l = 1, \ldots, m)$. In other words, for any distribution of zero locations $z_1, \ldots, z_m \in \mathbb{R}^2$, (2.4) have a continuous family of gauge-distinct solutions, labelled by the parameter $0 < \alpha < 4$, which realize these zeros.

We now turn to the full electroweak theory.

3. Multivortices in the Full Electroweak Theory

In the unitary gauge and under the vortex ansatz of Ambjorn and Olesen [4], the energy density of the electroweak theory is [7]:

$$\begin{aligned} \mathscr{E} &= |\mathscr{D}_1 W + i\mathscr{D}_2 W|^2 + \tfrac{1}{2} P_{12}^2 + \tfrac{1}{2} Z_{12}^2 - 2g(P_{12} \sin\theta + Z_{12} \cos\theta)|W|^2 + 2g^2 |W|^4 \\ &\quad + (\partial_j \varphi)^2 + \frac{1}{4\cos^2\theta} g^2 \varphi^2 Z_j^2 + g^2 \varphi^2 |W|^2 + \lambda(\varphi_0^2 - \varphi^2)^2, \end{aligned} \tag{3.1}$$

where W is a complex scalar field, φ is a real scalar field, P_j, Z_j are real-valued vector fields,

$$\mathscr{D}_j W = \partial_j W - ig(P_j \sin\theta + Z_j \cos\theta) W,$$

and $P_{jk} = \partial_j P_k - \partial_k P_j$, $Z_{jk} = \partial_j Z_k - \partial_k Z_j$, $j, k = 1, 2$. The Euler–Lagrange equations of (3.1) take the form

$$\begin{cases} \mathscr{D}_j\mathscr{D}_j W = g^2\varphi^2 W - 3g(P_{12}\sin\theta + Z_{12}\cos\theta)W + 4g^2|W|^2W, \\ \Delta\varphi = \left(g^2|W|^2 + \dfrac{g^2}{4\cos^2\theta}Z_j^2\right)\varphi + 2\lambda(\varphi^2-\varphi_0^2)\varphi, \\ \partial_j P_{jk} = ig\sin\theta(W^\dagger[\mathscr{D}_k W] - W[\mathscr{D}_k W]^\dagger) \\ \qquad + 3g\sin\theta\varepsilon_{jk}(W^\dagger[\mathscr{D}_j W] + W[\mathscr{D}_j W]^\dagger), \\ \partial_j Z_{jk} = \dfrac{g^2}{2\cos^2\theta}\varphi^2 Z_j + ig\cos\theta(W^\dagger[\mathscr{D}_k W] - W[\mathscr{D}_k W]^\dagger) \\ \qquad + 3g\cos\theta\varepsilon_{jk}(W^\dagger[\mathscr{D}_j W] + W[\mathscr{D}_j W]^\dagger). \end{cases} \tag{3.2}$$

In the critical coupling where

$$\lambda = \frac{g^2}{8\cos^2\theta},$$

we have seen in Sect. 3 of Part I that, when a 't Hooft type periodic boundary condition is imposed, the energy minima are attained by the solutions of the Bogomol'nyi equations

$$\begin{cases} \mathscr{D}_1 W + i\mathscr{D}_2 W = 0, \\ P_{12} = \dfrac{g}{2\sin\theta}\varphi_0^2 + 2g\sin\theta|W|^2, \\ Z_{12} = \dfrac{g}{2\cos\theta}(\varphi^2 - \varphi_0^2) + 2g\cos\theta|W|^2, \\ Z_j = -\dfrac{2\cos\theta}{g}\varepsilon_{jk}\partial_k \ln\varphi. \end{cases} \tag{3.3}$$

Hence (3.3) implies (3.2) on a periodic cell. In fact, it is straightforward to verify that such an implication does not depend on the domain of the equations and, in particular, solutions of (3.3) over $\mathbb{R}^2$ are also solutions of (3.2). The purpose of this section is to obtain multivortex solutions of (3.3) in the full plane. As in the case of the massive $SO(3)$ vortices discussed in Sect. 2, the solutions of (3.3) are also of infinite energy because there holds the following energy lower bound estimate in view of (3.3):

$$\begin{aligned} \mathscr{E} &\geqq \tfrac{1}{2}P_{12}^2 + \tfrac{1}{2}Z_{12}^2 - 2g(P_{12}\sin\theta + Z_{12}\cos\theta)|W|^2 \\ &\quad + 2g^2|W|^4 + g^2\varphi^2|W|^2 + \lambda(\varphi_0^2 - \varphi^2)^2 \\ &= \frac{1}{2}\left(\frac{g^2}{4\sin^2\theta}\varphi_0^4 + 2g^2\varphi_0^2|W|^2 + 4g^2\sin^2\theta|W|^4\right) \\ &\quad + \frac{1}{2}\left(\frac{g^2}{4\cos^2\theta}(\varphi^2-\varphi_0^2)^2 + 4g^2\cos^2\theta|W|^4 + 2g^2(\varphi^2-\varphi_0^2)|W|^2\right) \end{aligned}$$

$$
\begin{aligned}
&-2g\left(\frac{g}{2}(\varphi^2-\varphi_0^2)+2g\cos^2\theta|W|^2+\frac{g}{2}\varphi_0^2+2g\sin^2\theta|W|^2\right)|W|^2 \\
&+2g^2|W|^4+g^2\varphi^2|W|^2+\lambda(\varphi_0^2-\varphi^2)^2 \\
\geqq{}& \frac{g^2}{8\sin^2\theta}\varphi_0^2.
\end{aligned}
$$

The first equation in (3.3) implies that the zero set $Z(W)$ of W is discrete and each zero has an integral multiplicity. Let $Z(W)=\{z_1,\dots,z_m\}$ so that the multiplicity of the zero $z=z_l$ of W is $n_l>0$, $l=1,\dots,m$. Define as before the new variables

$$u=\ln|W|^2,\quad w=\ln\varphi^2.$$

Then Eqs. (3.3) are transformed as in Part I into the system:

$$
\begin{cases}
\Delta u=-g^2\exp(w)-4g^2\exp(u)+4\pi\sum\limits_{l=1}^{m}n_l\delta(z-z_l),\\[2ex]
\Delta w=\dfrac{g^2}{2\cos^2\theta}(\exp(w)-\varphi_0^2)+2g^2\exp(u).
\end{cases}
\tag{3.4}
$$

Let

$$
\begin{cases}
u_0=\sum\limits_{l=1}^{m}\ln|z-z_l|^{2n_l},\\[2ex]
w_0=-\dfrac{g^2}{8\cos^2\theta}\varphi_0^2|z|^2;
\end{cases}
$$

$$
\begin{cases}
u_1=u-u_0,\\
w_1=w-w_0.
\end{cases}
$$

Then u_1,w_1 satisfy

$$
\begin{cases}
\Delta u_1=-g^2\exp(w_0+w_1)-4g^2\exp(u_0+u_1),\\[1ex]
\Delta w_1=\dfrac{g^2}{2\cos^2\theta}\exp(w_0+w_1)+2g^2\exp(u_0+u_1).
\end{cases}
\tag{3.5}
$$

The term $\exp(u_0+u_1)$ is a bad term while $\exp(w_0+w_1)$ is a good term, because $\exp(w_0)$ decays exponentially fast.

As in the periodic case, we introduce the change of dependent variables as follows:

$$
\begin{cases}
u_2=u_1+2w_1,\\
w_2=u_1.
\end{cases}
$$

Then Eqs. (3.5) become

$$
\begin{cases}
\Delta u_2=g^2\tan^2\theta\exp(w_0)\exp(\tfrac{1}{2}[u_2-w_2]),\\
\Delta w_2=-g^2\exp(w_0)\exp(\tfrac{1}{2}[u_2-w_2])-4g^2\exp(u_0+w_2).
\end{cases}
\tag{3.6}
$$

As in Sect. 2, we make the translations

$$\begin{cases} u_2 = u_3 + \xi, \\ w_2 = w_3 + \zeta, \end{cases}$$

where u_3, w_3 are smooth functions so that

$$\begin{cases} u_3 = \alpha \ln r, \\ w_3 = -\beta \ln r, \end{cases} \quad r \geqq 1$$

with $\alpha, \beta > 0$. Hence $\Delta u_3, \Delta w_3$ have compact supports and Eqs. (3.6) become

$$\begin{cases} \Delta\xi = g^2 \tan^2 \theta U \exp(\tfrac{1}{2}[\xi - \zeta]) + f, \\ \Delta\zeta = -g^2 U \exp(\tfrac{1}{2}[\xi - \zeta]) - 4g^2 V \exp(\zeta) + h, \end{cases} \tag{3.7}$$

where

$$U \equiv \exp(w_0 + \tfrac{1}{2}[u_3 - w_3]), \quad V = \exp(u_0 + w_3),$$

$$f \equiv -\Delta u_3 \quad h \equiv -\Delta w_3.$$

As before (see (2.9)), we have

$$\int_{\mathbb{R}^2} f dx = -2\pi\alpha, \quad \int_{\mathbb{R}^2} h dx = 2\pi\beta.$$

In view of the above expressions, let us now impose for a solution pair of (3.7) the constraints

$$g^2 \tan^2 \theta \int_{\mathbb{R}^2} U \exp(\tfrac{1}{2}[\xi - \zeta]) dx = 2\pi\alpha, \tag{3.8}$$

and

$$g^2 \int_{\mathbb{R}^2} U \exp(\tfrac{1}{2}[\xi - \zeta]) dx + 4g^2 \int_{\mathbb{R}^2} V \exp(\zeta) dx = 2\pi\beta,$$

or

$$4g^2 \int_{\mathbb{R}^2} V \exp(\zeta) dx = 2\pi\left(\beta - \frac{\alpha}{\tan^2 \theta}\right). \tag{3.9}$$

In order to make sense out of (3.9), we require:

$$\beta > \frac{\alpha}{\tan^2 \theta}. \tag{3.10}$$

There holds

$$V = \exp(u_0 + w_3) = O(r^{2N-\beta}) \quad \text{for large } r > 0,$$

where $N = n_1 + \cdots + n_m$. Hence, if

$$\beta > 2N + 4, \tag{3.11}$$

then we can choose a suitable $\kappa > 4$ so that

$$V = O(r^{-\kappa}) \quad \text{for large } r > 0. \tag{3.12}$$

This property is important in our discussions to follow.

On the other hand, since

$$\exp(w_0) = O\left(\exp\left[-\frac{g^2\varphi_0^2}{8\cos^2\theta}r^2\right]\right),$$

we have

$$U = \exp(w_0 + \tfrac{1}{2}[u_3 - w_3]) = O(\exp[-r]) \quad \text{for large } r > 0. \tag{3.13}$$

We now consider the following optimization problem as in the periodic case:

$$\min\{I(\xi,\zeta) \mid \xi, \zeta \in \mathscr{H}, (\xi,\zeta) \text{ satisfies the constraints (3.8)–(3.9)}\}, \tag{3.14}$$

where

$$I(\xi,\zeta) = \int_{\mathbb{R}^2} dx \left\{\frac{1}{2}|\nabla\xi|^2 + \frac{1}{2}\sigma|\nabla\zeta|^2 + f\xi + \sigma h\zeta\right\}.$$

From (3.12)–(3.13) and Lemma 2.1, it is easily seen that (3.8), (3.9) are well-defined over $\mathscr{H}$.

Lemma 3.1. *If $\sigma = \tan^2\theta$, then a solution (ξ,ζ) of (3.14) is a solution of (3.7).*

Proof. For $\sigma > 0$, let (ξ,ζ) be a solution of (3.14). Since the Fréchet derivatives of the constraint functionals are linearly independent, the Lagrange multiplier rule implies there are constants $\lambda_\sigma, \mu_\sigma \in \mathbb{R}$ so that

$$\int_{\mathbb{R}^2} (\nabla\xi\cdot\nabla\chi_1 + f\chi_1)dx = \frac{1}{2}\lambda_\sigma g^2\tan^2\theta \int_{\mathbb{R}^2} U\exp\left(\frac{1}{2}[\xi-\zeta]\right)\chi_1 dx, \quad \chi_1 \in \mathscr{H}, \tag{3.15}$$

$$\begin{aligned}\int_{\mathbb{R}^2} (\sigma\nabla\zeta\cdot\nabla\chi_2 + \sigma h\chi_2)dx = &-\frac{1}{2}\lambda_\sigma g^2\tan^2\theta \int_{\mathbb{R}^2} U\exp\left(\frac{1}{2}[\xi-\zeta]\right)\chi_2 dx \\ &+ 4\mu_\sigma g^2 \int_{\mathbb{R}^2} V\exp(\zeta)\chi_2 dx, \quad \chi_2 \in \mathscr{H}.\end{aligned} \tag{3.16}$$

In (3.15), put $\chi_1 \equiv 1$. We obtain $-2\pi\alpha = \frac{1}{2}\lambda_\sigma 2\pi\alpha$. Hence $\lambda_\sigma = -2$ and the first equation in (3.7) is recoverded. Let $\chi_2 \equiv 1$ in (3.16). We get

$$2\pi\beta\sigma = 2\pi\alpha + \mu_\sigma\cdot 2\pi\left(\beta - \frac{\alpha}{\tan^2\theta}\right).$$

To obtain the second equation in (3.7), we choose $\sigma = \tan^2\theta$. Hence

$$\mu_\sigma = \frac{\beta\tan^2\theta - \alpha}{\beta - \alpha/\tan^2\theta} = \tan^2\theta.$$

Therefore the second equation in (3.7) is recovered as well. $\square$

From now on we fix $\sigma = \tan^2\theta$. Thus we see that it is sufficient to solve the constrained optimization problem (3.14). As in Sect. 2, we make the decomposition

$$\xi = \bar{\xi} + \xi', \quad \zeta = \bar{\zeta} + \zeta',$$

where $\bar{\xi},\bar{\zeta}\in\mathbb{R}$, $\xi',\zeta'\in\tilde{\mathscr{H}}$. Equation (3.9) says that

$$\exp(\bar{\zeta})\int_{\mathbb{R}^2} V\exp(\zeta')dx=\frac{\pi}{2g^2}\left(\beta-\frac{\alpha}{\tan^2\theta}\right),$$

or

$$\bar{\zeta}=\ln\left[\frac{\pi}{2g^2}\left(\beta-\frac{\alpha}{\tan^2\theta}\right)\right]-\ln\left[\int_{\mathbb{R}^2} V\exp(\zeta')dx\right]. \tag{3.17}$$

From (3.8), we get

$$\exp\left(\frac{1}{2}[\bar{\xi}-\bar{\zeta}]\right)\int_{\mathbb{R}^2} U\exp\left(\frac{1}{2}[\xi'-\zeta']\right)dx=\frac{2\pi\alpha}{g^2\tan^2\theta},$$

or

$$\bar{\xi}=\bar{\zeta}+2\ln\left(\frac{2\pi\alpha}{g^2\tan^2\theta}\right)-2\ln\left[\int_{\mathbb{R}^2} U\exp\left(\frac{1}{2}[\xi'-\zeta']\right)dx\right]. \tag{3.18}$$

As a consequence, the objective functional $I(\xi,\zeta)$ takes the form

$$\begin{aligned}I(\xi,\zeta)=&\int_{\mathbb{R}^2}\left\{\frac{1}{2}|\nabla\xi'|^2+\frac{1}{2}\tan^2\theta|\nabla\zeta'|^2\right\}dx\\&+\int_{\mathbb{R}^2}(f\xi'+\tan^2\theta h\zeta')dx-2\pi\alpha\bar{\xi}+2\pi\beta\tan^2\theta\bar{\zeta}.\end{aligned} \tag{3.19}$$

We first estimate in (3.19) the term

$$\Lambda=-2\pi\alpha\bar{\xi}+2\pi\beta\tan^2\theta\bar{\zeta}.$$

We have from (3.17)–(3.18), that

$$\begin{aligned}\Lambda&=-2\pi\alpha\left[\bar{\zeta}-2\ln\left(\int_{\mathbb{R}^2} U\exp\left(\frac{1}{2}[\xi'-\zeta']\right)dx\right)\right]+2\pi\beta\tan^2\theta\bar{\zeta}+C_1\\&=2\pi\tan^2\theta\left(\beta-\frac{\alpha}{\tan^2\theta}\right)\bar{\zeta}+4\pi\alpha\ln\left[\int_{\mathbb{R}^2} U\exp\left(\frac{1}{2}[\xi'-\zeta']\right)dx\right]+C_1.\end{aligned} \tag{3.20}$$

Let us find a lower bound for $\int_{\mathbb{R}^2} U\exp\left(\frac{1}{2}[\xi'-\zeta']\right)dx$. We have

$$\begin{aligned}\int_{\mathbb{R}^2} U\exp\left(\frac{1}{2}[\xi'-\zeta']\right)dx&=\int_{\mathbb{R}^2} h_0^{-1}U\exp\left(\frac{1}{2}[\xi'-\zeta']\right)d\mu\\&\geqq\varepsilon_0\int_{\mathbb{R}^2} U\exp\left(\frac{1}{2}[\xi'-\zeta']\right)d\mu\\&\geqq\varepsilon_0 C_2\exp\left(\int_{\mathbb{R}^2}\left[w_0+\frac{1}{2}(u_3-w_3)\right]d\mu\Big/\int_{\mathbb{R}^2}d\mu\right).\end{aligned}$$

Here we have used $h_0^{-1}\geqq\varepsilon_0$, $h_0=O(r^{-\kappa})$ $(\kappa>4)$ for large $r>0$, $w_0=O(r^2)$, u_3 and $w_3=O(\ln r)$, so that $w_0+\frac{1}{2}(u_3-w_3)\in L(d\mu)$; then the final inequality above follows from Jensen's inequality.

Thus (3.19) implies:

$$\Lambda \geqq 2\pi \tan^2\theta\left(\beta - \frac{\alpha}{\tan^2\theta}\right)\bar{\zeta} - C_3. \tag{3.21}$$

We next analyze (3.17). From (3.12) we see that $Vh_0^{-1} = O(1)$. Hence,

$$\begin{aligned}\int_{\mathbb{R}^2} V\exp(\zeta')dx &= \int_{\mathbb{R}^2} Vh_0^{-1}\exp(\zeta')d\mu \leqq C_4 \int_{\mathbb{R}^2} \exp(\zeta')d\mu \\ &\leqq C_5(\varepsilon)\exp\left[\frac{1}{4(4\pi-\varepsilon)}\|\nabla\zeta'\|^2_{L^2(dx)}\right] \quad \text{(using Lemma 2.1)}.\end{aligned} \tag{3.22}$$

Therefore (3.17), (3.22) yield the lower bound

$$\bar{\zeta} \geqq C_6 - \frac{1}{4(4\pi-\varepsilon)}\|\nabla\zeta'\|^2_{L^2(dx)}.$$

Thus, from (3.21), there holds

$$\Lambda \geqq -\pi\tan^2\theta\left(\beta - \frac{\alpha}{\tan^2\theta}\right)\cdot\frac{1}{2(4\pi-\varepsilon)}\|\nabla\zeta'\|^2_{L^2(dx)} - C_7. \tag{3.23}$$

Also, since f, h have compact supports, we easily obtain using Lemma 2.2, the inequalities:

$$\begin{cases} \int_{\mathbb{R}^2} |f\xi'|dx \leqq \varepsilon^{-1}C_8 + \varepsilon\int_{\mathbb{R}^2} |\xi'|^2 d\mu \leqq \varepsilon^{-1}C_8 + \varepsilon C\|\nabla\xi'\|^2_{L^2(dx)}, \\ \int_{\mathbb{R}^2} |h\zeta'|dx \leqq \varepsilon^{-1}C_9 + \varepsilon C\|\nabla\zeta'\|^2_{L^2(dx)}. \end{cases} \tag{3.24}$$

Substituting (3.22)–(3.24) into (3.19) we get

$$\begin{aligned} I(\xi,\zeta) &\geqq \frac{1}{2}(1-\varepsilon C')\|\nabla\xi'\|^2_{L^2(dx)} \\ &\quad + \frac{1}{2}\tan^2\theta\left(1 - \frac{\pi}{4\pi-\varepsilon}\left[\beta - \frac{\alpha}{\tan^2\theta}\right] - \varepsilon C''\right)\|\nabla\zeta'\|^2_{L^2(dx)} - C_{10} \\ &\equiv \delta_1\|\nabla\xi'\|^2_{L^2(dx)} + \delta_2\|\nabla\zeta'\|^2_{L^2(dx)} - C_{10}. \end{aligned} \tag{3.25}$$

Impose now the condition

$$\beta - \frac{\alpha}{\tan^2\theta} < 4. \tag{3.26}$$

Then, if $\varepsilon > 0$ is sufficiently small, we get $\delta_1, \delta_2 > 0$. In particular, I is bounded from below on the admissible set

$$\mathscr{S} = \{\xi, \zeta \in \mathscr{H} \mid \xi, \zeta \text{ satisfy (3.8)–(3.9)}\}.$$

Let $\{(\xi_j, \zeta_j)\}$ be a minimizing sequence of (3.14). Using (3.25) we see that $\{(\xi_j', \zeta_j')\}$ is bounded in $\tilde{\mathscr{H}}$ (see also Lemma 2.2). From (3.17), (3.22), we see that $\{\bar{\zeta}_j\}$ is a bounded sequence in $\mathbb{R}$ as well. Using (3.18), we can show that $\{\bar{\xi}_j\}$ is also a bounded sequence in $\mathbb{R}$. For simplicity, we assume there are $\xi, \zeta \in \mathscr{H}$ so that

$$\xi_j' \xrightarrow{w} \xi', \quad \zeta_j' \xrightarrow{w} \zeta', \quad \bar{\xi}_j \to \bar{\xi}, \quad \bar{\zeta}_j \to \bar{\zeta}.$$

In other words, $\xi_j \to \xi, \zeta_j \to \zeta$ weakly in $\mathscr{H}$.

An obvious extension of Lemma 2.3 is:

Lemma 3.2. *The injection $\mathscr{H} \to L^2(d\mu)$ is a compact embedding.*

Hence (3.22) says that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} V \exp(\zeta_j) dx - \int_{\mathbb{R}^2} V \exp(\zeta) dx \right| \\
&\leqq C \int_{\mathbb{R}^2} \exp(|\zeta_j| + |\zeta|)|\zeta_j - \zeta| d\mu \\
&\leqq C' \left(\int_{\mathbb{R}^2} \exp(2[|\zeta_j'| + |\zeta'|]) d\mu \right)^{1/2} \|\zeta_j - \zeta\|_{L^2(d\mu)} \\
&\leqq C'' \exp\left[\frac{1}{4\pi - \varepsilon} (\|\nabla \zeta_j'\|^2_{L^2(dx)} + \|\nabla \zeta'\|^2_{L^2(dx)}) \right] \|\zeta_j - \zeta\|_{L^2(d\mu)} \\
&\to 0 \quad \text{as} \quad j \to \infty.
\end{aligned}$$

Similarly, we can show that

$$\int_{\mathbb{R}^2} U \exp\left(\frac{1}{2}[\xi_j - \zeta_j]\right) dx - \int_{\mathbb{R}^2} U \exp\left(\frac{1}{2}[\xi - \zeta]\right) dx \to 0 \quad \text{as} \quad j \to \infty.$$

Therefore (ξ, ζ) satisfies the constraints (3.8)–(3.9). Finally the comparison $I(\xi, \zeta) \leqq \liminf I(\xi_j, \zeta_j)$ is easily examined. Hence (ξ, ζ) solves (3.14).

For convenience, let us summarize the conditions imposed on $\alpha, \beta > 0$ as follows:

$$\begin{cases} \dfrac{\alpha}{\tan^2\theta} < \beta < \dfrac{\alpha}{\tan^2\theta} + 4 \quad \text{(see (3.10) and (3.26))}, \\ \beta > 2N + 4 \quad \text{(see (3.11))}. \end{cases} \tag{3.27}$$

So we have obtained a two parameter family of solutions to Eqs. (3.4). We can observe that these solutions give rise to gauge-distinct solutions of the Bogomol'nyi system (3.3).

In fact, in the notation of this section, we have

$$\begin{cases} u = u_0 + u_1 = u_0 + w_2 = u_0 + w_3 + \zeta, \\ w = w_0 + w_1 = w_0 + \frac{1}{2}(u_2 - w_2) = w_0 + \frac{1}{2}(u_3 - w_3) + \frac{1}{2}(\xi - \zeta). \end{cases} \tag{3.28}$$

As in Sect. 4 of Part I, a solution quartet (φ, W, P_j, Z_j) of (3.3) can be constructed in such a way that $\varphi^2 = \exp(w)$ and $|W|^2 = \exp(u)$. Hence (3.28), (3.8)–(3.9) imply the relations

$$\int_{\mathbb{R}^2} \varphi^2 dx = \frac{2\pi\alpha}{g^2 \tan^2\theta}, \tag{3.29}$$

$$\int_{\mathbb{R}^2} |W|^2 dx = \frac{\pi}{2g^2}\left(\beta - \frac{\alpha}{\tan^2\theta}\right). \tag{3.30}$$

Since the left-hand-sides of (3.29)–(3.30) are gauge-invariant under the residual gauge symmetry

$$W \mapsto \exp(i\chi)W, \quad P_j \mapsto P_j + \frac{1}{e}\partial_j \chi, \quad Z_j \mapsto Z_j, \quad \varphi \mapsto \varphi,$$

different values of α, β give rise to gauge-distinct solutions of (3.3)! We can thus summarize our results as follows.

Theorem 3.3. *Let $\{z_1, \ldots, z_m\} \subset \mathbb{R}^2 = \mathbb{C}$, $n_1, \ldots, n_m \in \mathbb{Z}_+$, $N = n_1 + \cdots + n_m$. For any $\alpha, \beta > 0$ satisfying (3.27), the Bogomol'nyi system (3.3) arising from the classical electroweak theory has a solution $(\varphi^{(\alpha,\beta)}, W^{(\alpha,\beta)}, P_j^{(\alpha,\beta)}, Z_j^{(\alpha,\beta)})$ so that $Z(W^{(\alpha,\beta)}) = \{z_1, \ldots, z_m\}$, the multiplicity of the zero $z = z_l$ of $W^{(\alpha,\beta)}$ is n_l, the integral averages of the squares of $\varphi^{(\alpha,\beta)}$ and $|W^{(\alpha,\beta)}|$ satisfy (3.29)–(3.30). These solutions are a two parameter family of gauge-distinct solutions of infinite energy.*

In particular, we have nonuniqueness of solutions for each distribution of vortex locations. There is again no restriction to the number of vortices in $\mathbb{R}^2$.

These infinite energy vortex solutions are "natural" in the sense that (2.4) or (3.3) does not allow any finite energy solutions.

4. A Variant of the Existence Theorem

In this section we shall modify the method used in the last section to establish another existence result for the electroweak multivortices in a different parameter region. Since the main ingredients of our approach have been illustrated above, here we will be brief. To proceed, we consider the governing equations (3.5). Introduce a change of variables

$$\begin{cases} u_2 = \dfrac{u_1}{2\cos^2\theta} + w_1, \\ w_2 = w_1. \end{cases}$$

Then we have

$$\begin{cases} \Delta u_2 = -2g^2 \tan^2\theta \exp(u_0) \exp(2\cos^2\theta[u_2 - w_2]), \\ \Delta w_2 = \dfrac{g^2}{2\cos^2\theta} \exp(w_0)\exp(w_2) + 2g^2 \exp(u_0) \exp(2\cos^2\theta[u_2 - w_2]). \end{cases} \tag{4.1}$$

Choose $u_3, w_3 \in C^\infty(\mathbb{R}^2)$ so that $u_3 = -\alpha \ln r$, $w_3 = \beta \ln r$, $r = |x| \geqq 1$. Set $f = -\Delta u_3$, $h = \Delta w_3$. Then we have as before the results

$$\int_{\mathbb{R}^2} f\,dx = 2\pi\alpha, \quad \int_{\mathbb{R}^2} h\,dx = 2\pi\beta. \tag{4.2}$$

With the translations $u_2 = u_3 + \xi$, $w_2 = w_3 + \zeta$, we obtain from (4.1) the modified equations

$$\begin{cases} \Delta\xi = -2g^2\tan^2\theta U \exp(2\cos^2\theta[\xi - \zeta]) + f, \\ \Delta\zeta = \dfrac{g^2}{2\cos^2\theta} V \exp(\zeta) + 2g^2 U \exp(2\cos^2\theta[\xi - \zeta]) - h, \end{cases} \tag{4.3}$$

where

$$U=\exp(u_0+2\cos^2\theta[u_3-w_3]),\quad V=\exp(w_0+w_3).$$

From the definition of u_0, u_3, w_0, w_3 it is seen that

$$U=O(r^{2N-2\cos^2\theta[\alpha+\beta]}),$$
$$V=O\left(r^\beta\exp\left[-\frac{g^2\varphi_0^2}{8\cos^2\theta}r^2\right]\right),\qquad r=|x|. \tag{4.4}$$

In this section, we assume the condition

$$\alpha+\beta>\frac{N+2}{\cos^2\theta}. \tag{4.5}$$

Thus there is a $\kappa{:}4<\kappa\leqq 2\cos^2\theta[\alpha+\beta]-2N$ so that $U=O(r^{-\kappa})$. Note that (4.5) is a new condition for the parameters.

In view of (4.2) we formally put the following constraints for the solution pair of (4.3):

$$2g^2\tan^2\theta\int_{\mathbb{R}^2}U\exp(2\cos^2\theta[\xi-\zeta])dx=2\pi\alpha, \tag{4.6}$$

and

$$\frac{g^2}{2\cos^2\theta}\int_{\mathbb{R}^2}V\exp(\zeta)dx+2g^2\int_{\mathbb{R}^2}U\exp(2\cos^2\theta[\xi-\zeta])dx=2\pi\beta,$$

namely (as a consequence of (4.6)),

$$\frac{g^2}{2\cos^2\theta}\int_{\mathbb{R}^2}V\exp(\zeta)dx=2\pi\left(\beta-\frac{\alpha}{\tan^2\theta}\right). \tag{4.7}$$

From (4.6)–(4.7), it is seen that we need to assume the additional condition

$$\alpha>0,\quad \beta>\frac{\alpha}{\tan^2\theta}, \tag{4.8}$$

which looks the same as that in Sect. 3 (see (3.10)).

Define

$$I(\xi,\zeta)=\int_{\mathbb{R}^2}dx\{\tfrac{1}{2}|\nabla\xi|^2+\tfrac{1}{2}\sigma|\nabla\zeta|^2+f\xi-\sigma h\zeta\}.$$

Consider the minimization problem

$$\min\{I(\xi,\zeta)\,|\,\xi,\zeta\in\mathscr{H},\ (\xi,\zeta)\text{ satisfies the constraints (4.6)–(4.7)}\}. \tag{4.9}$$

Lemma 4.1. *With the choice* $\sigma=\tan^2\theta$, *a minimizer of (4.9) is a solution of (4.3).*

The proof of this simple result is similar to that for Lemma 3.1 and, hence, omitted. In the rest of this section, we always assume $\sigma=\tan^2\theta$.

Next, we shall find conditions under which (4.9) has a solution. We make the decomposition for $\xi,\zeta\in\mathscr{H}$ as before: $\xi=\bar{\xi}+\xi', \zeta=\bar{\zeta}+\zeta'$ with $\bar{\xi},\bar{\zeta}\in\mathbb{R}, \xi',\zeta'\in\tilde{\mathscr{H}}$.

Then, by virtue of (4.6)–(4.7), we have

$$\bar{\xi}=\bar{\zeta}-\frac{1}{2\cos^2\theta}\ln\left[\int_{\mathbb{R}^2} U\exp(2\cos^2\theta[\xi'-\zeta'])dx\right]+C_1, \tag{4.10}$$

$$\bar{\zeta}=-\ln\left[\int_{\mathbb{R}^2} V\exp(\zeta')dx\right]+C_2. \tag{4.11}$$

By (4.2) and (4.10)–(4.11), the functional I takes the convenient form

$$\begin{aligned}I(\xi,\zeta)=&\int_{\mathbb{R}^2} dx\left\{\frac{1}{2}|\nabla\xi'|^2+\frac{\sigma}{2}|\nabla\zeta'|^2\right\}+\int_{\mathbb{R}^2} dx\{f\xi'-\sigma h\zeta'\}\\&+2\pi\sigma\left(\beta-\frac{\alpha}{\sigma}\right)\ln\left[\int_{\mathbb{R}^2} V\exp(\zeta')dx\right]\\&-\frac{\pi\alpha}{\cos^2\theta}\ln\left[\int_{\mathbb{R}^2} U\exp(2\cos^2\theta[\xi'-\zeta'])dx\right]+C_3.\end{aligned} \tag{4.12}$$

The fact that $V=\exp(w_0+w_3)$, $w_0=O(r^2), w_3=O(\ln r)$, $h_0=O(r^{-\kappa})$ $(\kappa>4)$ and Jensen's inequality again imply the lower bound

$$\int_{\mathbb{R}^2} V\exp(\zeta')dx\geqq C_4, \tag{4.13}$$

where $C_4>0$ is a constant. Besides, using (4.5), $U=O(r^{2N-2\cos^2\theta[\alpha+\beta]})=O(r^{-\kappa})$ (see (4.4)), and Lemma 2.1, we have

$$\begin{aligned}&\int_{\mathbb{R}^2} U\exp(2\cos^2\theta[\xi'-\zeta'])dx\\&\leqq C_5\int_{\mathbb{R}^2}\exp(2\cos^2\theta[\xi'-\zeta'])d\mu\\&\leqq C_5\left(\int_{\mathbb{R}^2}\exp(2p\cos^2\theta|\xi'|)d\mu\right)^{1/p}\left(\int_{\mathbb{R}^2}\exp(2q\cos^2\theta|\zeta'|)d\mu\right)^{1/q}\\&\leqq C(\varepsilon)\exp\left(\frac{\cos^4\theta}{4\pi-\varepsilon}[p\|\nabla\xi'\|^2_{L^2(dx)}+q\|\nabla\zeta'\|^2_{L^2(dx)}]\right),\end{aligned} \tag{4.14}$$

where $p,q>1, 1/p+1/q=1$. Therefore, using (4.8), (4.13)–(4.14), and Lemma 2.2 in (4.12), we get

$$I(\xi,\zeta)\geqq\delta_1\|\nabla\xi'\|^2_{L^2(dx)}+\delta_2\|\nabla\zeta'\|^2_{L^2(dx)}-C(\varepsilon,p,q), \tag{4.15}$$

where

$$\delta_1=\frac{1}{2}-\frac{\pi\alpha\cos^2\theta}{4\pi-\varepsilon}p-\varepsilon,\quad \delta_2=\frac{\tan^2\theta}{2}-\frac{\pi\alpha\cos^2\theta}{4\pi-\varepsilon}q-\varepsilon.$$

We now require the condition

$$\begin{aligned}1&>\frac{\alpha}{2}p\cos^2\theta,\\ \tan^2\theta&>\frac{\alpha}{2}q\cos^2\theta.\end{aligned} \tag{4.16}$$

Lemma 4.2. *There are $p, q > 1, 1/p + 1/q = 1$ so that (4.16) holds if and only if*

$$\alpha < 2\tan^2\theta. \tag{4.17}$$

Proof. Suppose (4.16) is verified. Adding the two inequalities in (4.16) gives (4.17). On the other hand, assume (4.17) is true. Set $p = 1/\sin^2\theta$, $q = 1/\cos^2\theta$. Then it is seen that (4.16) holds. $\square$

Let (4.17) hold. Then it follows from (4.16) that an $\varepsilon > 0$ can be chosen suitably to make $\delta_1, \delta_2 > 0$ in (4.15). As in Sect. 3, then we may prove that the optimization problem (4.9) has a solution.

We summarize the restrictions (4.5), (4.8), and (4.17) on the parameters as follows:

$$\begin{cases} 2\tan^2\theta > \alpha > 0, \\ \beta > \dfrac{\alpha}{\tan^2\theta}, \\ \alpha + \beta > \dfrac{N+2}{\cos^2\theta}. \end{cases} \tag{4.18}$$

Let (φ, W, P_j, Z_j) be the solution of (3.3) constructed from the solution pair (ξ, ζ) of the system (4.3). We easily see that

$$\int_{\mathbb{R}^2} |W|^2 dx = \int_{\mathbb{R}^2} U\exp(2\cos^2\theta[\xi - \zeta])dx = \frac{\pi\alpha}{g^2\tan^2\theta}, \tag{4.19}$$

$$\int_{\mathbb{R}^2} \varphi^2 dx = \int_{\mathbb{R}^2} V\exp(\zeta)dx = \frac{4\pi\cos^2\theta}{g^2}\left(\beta - \frac{\alpha}{\tan^2\theta}\right). \tag{4.20}$$

So again different values of α, β give rise to gauge-distinct multivortex solutions of (3.3). Hence we can state

Theorem 4.3. *Under the condition (4.18), the Bogomol'nyi system (3.3) has a smooth solution quartet which verifies all the properties stated in Theorem 3.3 except that (3.29)–(3.30) are now replaced by (4.19)–(4.20).*

Finally we turn to an investigation of the asymptotic behavior of the solutions.

5. Asymptotic Decay Estimates

Using some suitable weighted Sobolev spaces, McOwen [6] has studied the decay rate of the solutions of the conformal deformation equations in $\mathbb{R}^2$. Our approach here to (3.7) and (4.3) follows the main line in his work.

For $\delta \in \mathbb{R}$ and $s \in \mathbb{N}$ (the set of nonnegative integers), define $W^2_{s,\delta}$ to be the closure of the set of C^∞ functions over $\mathbb{R}^2$ with compact supports in the norm

$$\|\eta\|^2_{W^2_{s,\delta}} = \sum_{|\gamma| \leqq s} \|(1+|x|)^{\delta+|\gamma|} D^\gamma \eta\|^2_{L^2(dx)}.$$

Let $C_0(\mathbb{R}^2)$ be the set of continuous functions on $\mathbb{R}^2$ vanishing at infinity. The following lemmas are cited from [6].

Lemma 5.1. *If* $s > 1$ *and* $\delta > -1$, *then* $W^2_{s,\delta} \subset C_0(\mathbb{R}^2)$.

Lemma 5.2. *For* $-1 < \delta < 0$, *the Laplace operator* $\Delta: W^2_{2,\delta} \to W^2_{0,\delta+2}$ *is 1–1 and the range of* Δ *has the following characterization*:

$$\Delta(W^2_{2,\delta}) = \left\{ f \in W^2_{0,\delta+2} \,\middle|\, \int_{\mathbb{R}^2} f\,dx = 0 \right\}.$$

Lemma 5.3. *If* $\xi \in \mathscr{H}$ *and* $\Delta\xi = 0$, *then* $\xi = \text{const}$.

Lemma 5.4. *Let* (ξ, ζ) *be a solution pair of* (3.7) *which is obtained in Sect. 3 as a minimizer of the problem* (3.14). *Then* ξ, ζ *approach some constants at infinity.*

Proof. Let the right-hand-sides of the two equations in (3.7) be denoted by f_1 and h_1 respectively. Then $f_1, h_1 \in L(dx)$ and

$$\int_{\mathbb{R}^2} f_1\,dx = \int_{\mathbb{R}^2} h_1\,dx = 0$$

in view of (3.8)–(3.9). Besides, using Lemma 2.1 and (3.11), it is straightforward to examine that $f_1, h_1 \in W^2_{0,\delta+2}$ for $-1 < \delta < 0$. Hence, by Lemma 5.2, there are unique $\xi_1, \zeta_1 \in W^2_{2,\delta}$ so that $\Delta\xi_1 = f_1, \Delta\zeta_1 = h_1$. From Lemma 5.1 we see that both ξ_1 and ζ_1 vanish at infinity. In particular, $\xi_1, \zeta_1 \in L^2(d\mu)$. Furthermore, since $\nabla\xi_1, \nabla\zeta_1 \in W^2_{0,\delta+1}$ and $\delta > -1$, so $\nabla\xi_1, \nabla\zeta_1 \in L^2(dx)$. As a consequence, we have obtained that $\xi_1, \zeta_1 \in \mathscr{H}$. Finally, by Lemma 5.3 and $\Delta(\xi - \xi_1) = \Delta(\zeta - \zeta_1) = 0$, we see that $\xi - \xi_1$ and $\zeta - \zeta_1$ are constants. □

Thus the discussion of Sect. 3 and Lemma 5.4 lead to

Theorem 5.5. *The solution* $(\varphi^{(\alpha,\beta)}, W^{(\alpha,\beta)}, P_j^{(\alpha,\beta)}, Z_j^{(\alpha,\beta)})$ *obtained in Theorem 3.3 enjoys the following sharp decay estimates*:

$$\begin{aligned} &(\varphi^{(\alpha,\beta)})^2 = O\left(r^{1/2[\alpha+\beta]}\exp\left[-\frac{g^2\varphi_0^2}{8\cos^2\theta}r^2\right]\right), \\ &|W^{(\alpha,\beta)}|^2 = O(r^{-(\beta-2N)}), \end{aligned} \qquad r = |x|. \tag{5.1}$$

A similar investigation on the system (4.3) can be carried out which enables us to conclude with

Theorem 5.6. *The solution* $(\varphi^{(\alpha,\beta)}, W^{(\alpha,\beta)}, P_j^{(\alpha,\beta)}, Z_j^{(\alpha,\beta)})$ *obtained in Theorem 4.3 vanishes at infinity according the rate*

$$\begin{aligned} &(\varphi^{(\alpha,\beta)})^2 = O\left(r^{\beta}\exp\left[-\frac{g^2\varphi_0^2}{8\cos^2\theta}r^2\right]\right), \\ &|W^{(\alpha,\beta)}|^2 = O(r^{-2([\alpha+\beta]\cos^2\theta - N)}), \end{aligned} \qquad r = |x|. \tag{5.2}$$

Remark 5.1. Using (5.1)–(5.2) and the Bogomol'nyi equations (3.3), the asymptotic behavior of the magnetic and weak field strengths, P_{12} and Z_{12}, can easily be described.

Remark 5.2. Let $(W^{(\alpha)}, P^{(\alpha)})$ be the solution pair of the $SO(3)$ Bogomol'nyi equations (2.4) obtained in Sect. 2. It can be proved that there holds the decay estimate

$$|W^{(\alpha)}|^2 = O(r^{2N-\alpha}\exp[-\tfrac{1}{2}m_W^2 r^2]), \quad r = |x|.$$

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