Commun. Math. Phys. 142, 67–98 (1991)

Communications in
Mathematical
Physics

Propagation Estimates for N-body Schroedinger Operators

Erik Skibsted

Matematisk Institut, Aarhus Universitet, DK-8000 Aarhus C, Denmark

Received October 1, 1990; in revised form February 22, 1991

Abstract. We prove propagation estimates (of strong type) for long-range N-body Hamiltonians. Emphasis is on phase–space analysis in the free channel region.

1. Introduction

In this paper we prove various propagation estimates for a fairly large class of long-range N-body Schroedinger operators (denoted by H). The form of these estimates (in the configuration space representation) is

$$B(t)e^{-itH}f(H)\langle x\rangle^{-s'} = O(t^{-s}) \tag{1.1}$$

for $t \to +\infty$, and with $\langle x\rangle$ given by multiplication by $(1+|x|^2)^{1/2}$, $f \in C_0^\infty(\mathbf{R})$, $0 \leqq s < s'$, and finally with $\{B(t)\}_{t>0}$ a family of pseudodifferential operators (typically of non-negative order).

Among our estimates are the minimal and large velocity estimates of Sigal and Soffer [S–S] (obtained by putting $B(t) = \chi\left(\frac{x^2}{4t^2} - E < -\varepsilon\right)$ or $B(t) = \langle x\rangle^{s'-s}\chi\left(E' - \frac{x^2}{4t^2} < -\varepsilon\right)$, with $0 < E \ll E'$ and assuming f to be supported in a small neighbourhood of E, respectively), however obtained for arbitrary s and s' as above and for a larger class of potentials.

We shall also prove maximal velocity estimates for the free channel, which are obtained by localizing further in the configuration space namely to regions where the potential "goes to zero." Finally if χ_{fr} denotes such localization operator, we shall prove the estimate with $B(t) = P_-(X, D)\chi_{fr}$, where the symbol $p_-(x, \xi)$ vanishes in a certain conical neighbourhood of the forward direction: $x = c\xi, c > 0$. The latter result was established for $N = 2$ by Isozaki [I] and independently by Jensen [J] (in both cases under more restrictive conditions on the potential). For $N > 2$ the most resembling results in the literature seem to be due to Mourre

[M1], however these are given under very restrictive conditions (which cover the pure Coulomb case though).

The estimates indicated above are stated as Theorems 3.3, 3.4, 4.4 and 4.5, respectively. All hold under the conditions on the potential as given in Assumption 3.1.

In Sect. 2 we extend and refine the abstract theory of Sigal and Soffer [S–S], which shows how to turn positivity of certain Heisenberg derivatives into estimates of a type similar to (1.1). Main new things (crucial for applications) are the removal of a certain commutativity assumption of [S–S] and that approximate positivity is sufficient. We remark that the abstract theory is stated only for time-independent Hamiltonians although one can extend it to time-dependent ones (cf. [S–S]).

In Sect. 3 we give examples. We are dependent on those of [S–S] and in particular on a recent beautiful paper by Graf [G]. The results needed from [G] are stated in Lemma 3.2.

Consequences of the examples are the free channel propagation estimates indicated above. This is shown in Sect. 4. The main ingredient of the proofs is the "global" estimate appearing as Example 4 in Sect. 3.

In Appendix A we prove that the Graf vector field is nicely behaved in the forward direction in a certain sense, and in Appendix B we establish certain Mourre-type estimates (depending on this vector field).

Under the same conditions on the potential one can use our estimates together with the method in [H–S1] to obtain the precise asymptotic behaviour in the free channel region (as $|x| \to \infty$) of the boundary values (from above) at positive energies of the resolvent. This will be done in [H–S2], where also a relationship to scattering theory will be shown.

2. Abstract Theory

Notations and Assumptions. By $\mathbf{R}$, $\mathbf{R}^+$ and $\mathbf{N}$ we mean the real numbers, the positive real numbers and the positive integers, respectively.

Given a Hilbert space $\mathscr{H}$, $\mathscr{B}(\mathscr{H})$ denotes the set of bounded linear operators on $\mathscr{H}$. Given a linear operator H on $\mathscr{H}$, $\mathscr{D}(H)$ denotes its domain. For a selfadjoint operator A on $\mathscr{H}\langle A\rangle := (I + A^2)^{1/2}$.

For $x \in \mathbf{R}$ we put $(x)_- = -\min\{0, x\}$ and $(x)_+ = \max\{0, x\}$.

By $C^\infty(\mathbf{R}^n)$ and $C_0^\infty(\mathbf{R}^n)$ we mean the smooth, respectively the smooth compactly supported, functions on $\mathbf{R}^n$; the support of a function f is denoted $\operatorname{supp} f$. For $f \in C_0^\infty(\mathbf{R}^n)$ $\hat{f}$ denotes the Fourier transform of $f: \hat{f}(k) = (2\pi)^{-n} \int \exp(-ikx) f(x)\, dx$.

The standard L^p-spaces on $\mathbf{R}^n$ are denoted by $L^p(\mathbf{R}^n)$.

Definition 2.1. *Given $\beta, \alpha \geqq 0$ and $\varepsilon > 0$ let $\mathscr{F}_{\beta,\alpha,\varepsilon}$ denote the set of functions $g, g(x,\tau) = g_{\beta,\alpha,\varepsilon}(x,\tau) = -\tau^{-\beta}(-x)^\alpha \chi\left(\frac{x}{\tau}\right)$, defined for $(x,\tau) \in \mathbf{R} \times \mathbf{R}^+$ and for $\chi \in C^\infty(\mathbf{R})$ with the following properties*:

$$\chi(x) = 1 \quad \text{for} \quad x < -2\varepsilon, \chi(x) = 0 \quad \text{for} \quad x > -\varepsilon,$$

$$\frac{d}{dx}\chi(x) \leqq 0 \quad \text{and} \quad \alpha\chi(x) + x\frac{d}{dx}\chi(x) = \tilde{\chi}^2(x),$$

where $\tilde{\chi}(x) \geqq 0$ and $\tilde{\chi} \in C^\infty(\mathbf{R})$.

We shall sometimes (in Sects. 3 and 4) use the notation $\chi(A < -\varepsilon) = \chi(A)$ for a selfadjoint operator A and with χ smooth and satisfying the first three properties enlisted above.

For any inputs $\beta, \alpha, \varepsilon$ as above $\mathscr{F}_{\beta,\alpha,\varepsilon} \neq \emptyset$ and with $g^{(n)}(x,\tau) = \left(\frac{\partial}{\partial x}\right)^n g(x,\tau)$, $n \in \mathbf{N} \cup \{0\}$, $(g^{(1)}(x,\tau))^{1/2} = (\tau^{-\beta}(-x)^{\alpha-1})^{1/2}\tilde{\chi}\left(\frac{x}{\tau}\right)$ is smooth.

Assumption 2.2. *Let $n_0 \in \mathbf{N}$ with $n_0 \geqq 2$, $t_0 > 0$, $\kappa_0 \geqq 0$, $\beta_0 > 0$, $n_0 - \frac{1}{2} > \alpha_0 > 0$, $f, f_2 \in C_0^\infty(\mathbf{R})$, f_2 realvalued with $f_2 f = f$, and $H, A(\tau), B$ be selfadjoint operators on a Hilbert space $\mathscr{H}$. Assume with $\tau = t + t_0$, $t \geqq 0$, that the operators $A(\tau)$ have a common domain $\mathscr{D}$, $\mathscr{D}(H) \cap \mathscr{D}$ is dense in $\mathscr{D}(H)$, $B \geqq I$, H is bounded from below, and with $A = A(t_0)$ that $\langle A \rangle^{n_0/2} B^{-n_0/2} \in \mathscr{B}(\mathscr{H})$.*

Assume moreover

(1) *With $\mathrm{ad}^0_{A(\tau)}(H) = H$ and $1 \leqq n \leqq n_0$ the (commutator-)form (defined iteratively) $i^n \mathrm{ad}^n_{A(\tau)}(H) = i[i^{n-1}\mathrm{ad}^{n-1}_{A(\tau)}(H), A(\tau)]$ on $\mathscr{D}(H) \cap \mathscr{D}$ extends to a symmetric operator with domain $\mathscr{D}(H)$.*

(2) *If A is unbounded $\sup_{|s|<1} \| H e^{iA(\tau)s}\psi \| < \infty$ for any $\psi \in \mathscr{D}(H)$ and $\tau \geqq t_0$.*

(3) *For any $\tau_1, \tau_2 \geqq t_0$, $A(\tau_1) - A(\tau_2)$ is bounded, and the derivative $d_\tau A(\tau) = \frac{d}{dt}A(\tau)$ exists in $\mathscr{B}(\mathscr{H})$.*

For $n \leqq n_0 - 1$ and $\tau \geqq t_0$ the form (defined iteratively)

$$i^n \mathrm{ad}^n_{A(\tau)}(d_\tau A(\tau)) = i[i^{n-1}\mathrm{ad}^{n-1}_{A(\tau)}(d_\tau A(\tau)), A(\tau)]$$

on $\mathscr{D}$ extends to a bounded selfadjoint operator on $\mathscr{H}$.

(4) *For $n \leqq n_0$ $\mathrm{ad}^n_{A(\tau)}(H)(H-i)^{-1}$ and $\mathrm{ad}^{n-1}_{A(\tau)}(d_\tau A(\tau))$ are continuous $\mathscr{B}(\mathscr{H})$-valued functions of $\tau \geqq t_0$.*

(5)

(a) $\mathrm{ad}^{n_0-1}_{A(\tau)}(d_\tau A(\tau)) = O(\tau^{\kappa_0})$ *in $\mathscr{B}(\mathscr{H})$ as $\tau \to \infty$.*
(b) *For $n \leqq n_0$,*
$\mathrm{ad}^{n-1}_{A(\tau)}(d_\tau A(\tau))(H-i)^{-1} = O(1)$ *in $\mathscr{B}(\mathscr{H})$ as $\tau \to \infty$.*
(c) *For $n \leqq n_0$,*
$\mathrm{ad}^n_{A(\tau)}(H)(H-i)^{-1} = O(1)$ *in $\mathscr{B}(\mathscr{H})$ as $\tau \to \infty$.*

(6) $q(\beta_0, \alpha_0)$:

With $\alpha_0' = \max\{m \in \mathbf{N} \mid m < \alpha_0\}$ (for $1 < \alpha_0$) the following (positivity) estimate holds for $(\beta, \alpha) = (0,1), \ldots, (0, \alpha_0'), (\beta_0, \alpha_0) (= (\beta_0, \alpha_0)$ if $0 < \alpha_0 \leqq 1)$:

Let $DA(\tau)$ denote the symmetric operator $i[H, A(\tau)] + d_\tau A(\tau)$. Given $\varepsilon > 0$ and $g(x,\tau) \in \mathscr{F}_{\beta,\alpha,\varepsilon}$ there exists $C > 0$ such that with $\zeta(t) = (g^{(1)}(A(\tau),\tau))^{1/2} e^{-itH} f(H) B^{-\alpha/2}\phi$,

$$\int_0^\infty dt (\langle \zeta(t), f_2(H) DA(\tau) f_2(H)\zeta(t)\rangle)_- \leqq C\|\phi\|^2, \quad \forall \phi \in \mathscr{H}.$$

Remark 2.3. (1) As for (6) $e^{-itH}f(H)B^{-\alpha/2}\phi \in \mathscr{D}((g^{(1)}(A(\tau),\tau))^{1/2})$ for any $\phi \in \mathscr{H}$ and $\tau \geqq t_0$. A sufficient condition is obviously $f_2(H)DA(\tau)f_2(H) \geqq 0$. (See Corollary 2.6 for a more refined one.)

(2) If $i^n \mathrm{ad}^n_{A(\tau)}(H)$ extends to a bounded selfadjoint operator, continuous in τ and $O(1)$ for $\tau \to \infty$, we don't need H to be bounded from below. (In particular one can get results for Stark Hamiltonians.)

(3) As proved by Mourre [M2, Proposition II.1] if A is unbounded and Assumption 2.2(2) holds one can verify Assumption 2.2(1) by showing the existence of some subspace $\mathscr{S} \subseteq \mathscr{D}(H) \cap \mathscr{D}$ such that $e^{iA(\tau)s}\mathscr{S} \subseteq \mathscr{S}(|s|<1)$, $\mathscr{S}$ is dense in $\mathscr{D}(H)$ and the assumption holds with $\mathscr{D}(H) \cap \mathscr{D}$ replaced by $\mathscr{S}$. This remark is useful for Exs. 1 and 3 in Sect. 3.

Main Results

Theorem 2.4. *Suppose Assumption 2.2 and in addition*

$$\kappa_0 + \alpha_0 + \tfrac{3}{2} < \beta_0 + n_0, \tag{2.1}$$

$$\kappa_0 + \alpha_0' + \tfrac{3}{2} < n_0 \quad (\textit{if } 1 < \alpha_0), \tag{2.2}$$

$$\frac{\alpha_0}{2} + 2 < n_0 + \beta_0, \tag{2.3}$$

$$\frac{\alpha_0'}{2} + 2 \leqq n_0 \ (\textit{implied by } (2.2)). \tag{2.4}$$

Then for $(\beta, \alpha) = (0, 1), \ldots, (0, \alpha_0'), (\beta_0, \alpha_0)$, *any* $\varepsilon > 0$ *and* $g(x, \tau) \in \mathscr{F}_{\beta,\alpha,\varepsilon}$,

$$(-g_{\beta,\alpha,\varepsilon}(A(\tau), \tau))^{1/2} e^{-itH} f(H) B^{-\alpha/2} = O(1) \quad \textit{in} \quad \mathscr{B}(\mathscr{H}) \textit{ for } \tau \to \infty.$$

The proof, given at the very end of this section, requires a series of preliminaries.

Corollary 2.5. *With the situation as in Theorem 2.4 for* $(\beta, \alpha) = (0, 1), \ldots, (0, \alpha_0')$, (β_0, α_0), *any* $\varepsilon > 0$, $g(x, \tau) \in \mathscr{F}_{\beta,\alpha,\varepsilon}$ *and* $1 \geqq \theta \geqq 0$,

$$(-g_{0,\alpha(1-\theta),\varepsilon}(A(\tau), \tau))^{1/2} e^{-itH} f(H) B^{-\alpha/2} = O(\tau^{(\beta - \alpha\theta)/2}) \quad \textit{in} \quad \mathscr{B}(\mathscr{H}) \textit{ for } \tau \to \infty.$$

Proof. We use that for $0 \leqq \theta \leqq 1$, $-g_{\beta,\alpha,\varepsilon}(x, \tau) \geqq -\tau^{-\beta}(\varepsilon\tau)^{\alpha\theta} g_{0,\alpha(1-\theta),2\varepsilon}(x, \tau)$ and the spectral theorem. $\square$

In two applications (Exs. 2 and 4 in Sect. 3) the following result will be very useful in verifying Assumption 2.2(6):

Corollary 2.6. *Suppose Assumption 2.2(1)–(5) and that* $q(\beta_0, \alpha_0)$ *is replaced for some* $\delta > 0$ *by* $q(\beta_0, \alpha_0, \delta)$:

There exist bounded operators $B_1(\tau)$ *and* $B_2(\tau)$ *on* $\mathscr{H}$ *such that*

$$\begin{aligned} &f_2(H) D A(\tau) f_2(H) \geqq B_1(\tau) + B_2(\tau), \\ &B_1(\tau) = O(\tau^{-\delta}) \quad \textit{for} \quad \tau \to \infty, \end{aligned} \tag{2.5}$$

and for $(\beta, \alpha) = (0, 1), \ldots, (0, \alpha_0'), (\beta_0, \alpha_0)$ *the following estimate holds:*

Given $\varepsilon > 0$ *and* $g(x, \tau) \in \mathscr{F}_{\beta,\alpha,\varepsilon}$ *there exists* $C > 0$ *such that with* $\zeta(t) = (g^{(1)}(A(\tau), \tau))^{1/2} e^{-itH} f(H) B^{-\alpha/2} \phi$,

$$\int_0^\infty dt |\langle \zeta(t), B_2(\tau) \zeta(t) \rangle| \leqq C \|\phi\|^2, \quad \forall \phi \in \mathscr{H}. \tag{2.6}$$

Suppose in addition (2.1–4).

Then $q(\beta_0, \alpha_0)$, *and hence in particular the conclusion of Theorem 2.4, hold.*

Idea of proof. Suppose Assumption 2.2(1)–(5), $q(\beta_0, \alpha_0, \delta)$, (2.1–4) and $\alpha_0 \leqq 1$. Then for any $\delta' < \delta$ and with $\beta_0' = \max\{\beta_0, 1-\delta'\}$, $q(\beta_0', \alpha_0)$ holds. Hence the conclusions of Theorem 2.4 and Corollary 2.5 hold with the input $(\beta, \alpha) = (\beta_0', \alpha_0)$. By iterating we obtain $q(\beta_0', \alpha_0)$ with $\beta_0' = \max\{\beta_0, 1 - n\delta'\}$ for any $n \in \mathbf{N}$. So $q(\beta_0, \alpha_0)$ holds with the constraint $m - 1 < \alpha_0 \leqq m$, $m = 1$, imposed.

The rest of the proof goes by induction in m (cf. the proof of Theorem 2.4), and involves a similar iteration argument. □

Preliminary Results

Lemma 2.7. *Suppose A and P are linear operators on a Hilbert space, A selfadjoint and P bounded, and $n \in \mathbf{N}$. Suppose that for $1 \leqq m \leqq n$ the form (defined iteratively) $\mathrm{ad}_A^m(P) = [\mathrm{ad}_A^{m-1}(P), A]$ on $\mathscr{D}(A)$ (by $\langle \phi, \mathrm{ad}_A^m(P)\psi \rangle = \langle \phi, \mathrm{ad}_A^{m-1}(P)A\psi \rangle - \langle A\phi, \mathrm{ad}_A^{m-1}(P)\psi \rangle$) extends to a bounded operator.*

Then for any $g \in C_0^\infty(\mathbf{R})$,

(1)

$$Pg(A) = \sum_{m=0}^{n-1} \frac{g^{(m)}(A)}{m!} \mathrm{ad}_A^m(P) + \int_{-\infty}^{\infty} dk \widehat{g^{(n)}}(k) e^{ikA} R_{n,A,P}^r(k),$$

$$R_{n,A,P}^r(k) = \int_0^k dl \frac{(k-l)^{n-1}}{(n-1)!k^n} e^{-ilA} \mathrm{ad}_A^n(P) e^{ilA},$$

(2)

$$g(A)P = \sum_{m=0}^{n-1} \mathrm{ad}_A^m(P) \frac{(-1)^m}{m!} g^{(m)}(A) + \int_{-\infty}^{\infty} dk \widehat{g^{(n)}}(k) R_{n,A,P}^l(k) e^{ikA},$$

$$R_{n,A,P}^l(k) = -\int_0^k dl \frac{(l-k)^{n-1}}{(n-1)!k^n} e^{ilA} \mathrm{ad}_A^n(P) e^{-ilA},$$

(3) *The norm of the integral is (in both cases) bounded by*

$$(n!)^{-1} \| \widehat{g^{(n)}} \|_{L^1(\mathbf{R})} \| \mathrm{ad}_A^n(P) \|.$$

Proof. As a form on $\mathscr{D}(A)$

$$e^{-ikA} P e^{ikA} - P = ik\, \mathrm{ad}_A(P) + i \int_0^k dl \{ e^{-ilA} \mathrm{ad}_A(P) e^{ilA} - \mathrm{ad}_A(P) \}. \tag{2.7}$$

This identity extends to an identity between bounded operators.

By iterating (2.7)

$$e^{-ikA} P e^{ikA} - P = \sum_{m=1}^{n-1} (ik)^m (m!)^{-1} \mathrm{ad}_A^m(P) + i^n \int_0^k dl_1 \dots \int_0^{l_{n-1}} dl_n e^{-il_nA} \mathrm{ad}_A^n(P) e^{il_nA}.$$

By the latter identity

$$\begin{aligned}[P, g(A)] &= \int dk \hat{g}(k) e^{ikA} \{ e^{-ikA} P e^{ikA} - P \} \\ &= \sum_{m=1}^{n-1} (m!)^{-1} g^{(m)}(A) \mathrm{ad}_A^m(P) + \int dk \widehat{g^{(n)}}(k) e^{ikA} R_{n,A,P}^r(k).\end{aligned}$$

We have proved (1). Statement (2) follows similarly. Statement (3) is obvious. □

Remark 2.8. (1) Under suitable assumptions and modifications the expansions hold for P unbounded. This will not be needed.

(2) The expansions can also be generalized to larger classes of functions than $C_0^\infty(\mathbf{R})$. This will be done/used (cf. the proof of Lemma 2.16).

Lemma 2.9. *With the situation given by Assumption 2.2(1) and (2) $(A(\tau)\pm i\lambda)^{-1}$ leave $\mathscr{D}(H)$ invariant for sufficiently large real λ, and $(H+i)i\lambda(A(\tau)+i\lambda)^{-1}(H+i)^{-1}$ converges strongly to I as $|\lambda|\to\infty$.*

Proof. The proof of [M2, Proposition II.3] goes through. (When $A(\tau)$ is bounded we do not need the estimate of Assumption 2.2(2).) □

Lemma 2.10. *Suppose Assumption 2.2(1)–(5). For large enough C, for $1\leqq n\leqq n_0$ and with $\tilde{H}=(H+C)^{-1}$ the form (defined iteratively) $i^n\operatorname{ad}^n_{A(\tau)}(\tilde{H})=i[i^{n-1}\operatorname{ad}^{n-1}_{A(\tau)}(\tilde{H}), A(\tau)]$ on $\mathscr{D}$ extends to a bounded selfadjoint operator. Moreover $(H+i)i^n\operatorname{ad}^n_{A(\tau)}(\tilde{H})$ is a continuous $\mathscr{B}(\mathscr{H})$-valued function, and $O(1)$ for $\tau\to\infty$.*

Proof. As a form on $\mathscr{D}$ and by a repeated application of Lemma 2.9,

$$\begin{aligned}[\tilde{H},A(\tau)] &= \lim_{\lambda\to\infty}\left[\tilde{H},\frac{i\lambda A(\tau)}{A(\tau)+i\lambda}\right]=\lim_{\lambda\to\infty}\left[\tilde{H},\frac{\lambda^2}{A(\tau)+i\lambda}\right]\\ &= \lim_{\lambda\to\infty}-\tilde{H}\left[H,\frac{\lambda^2}{A(\tau)+i\lambda}\right]\tilde{H}=\lim_{\lambda\to\infty}\tilde{H}\frac{\lambda}{A(\tau)+i\lambda}\operatorname{ad}_{A(\tau)}(H)\frac{\lambda}{A(\tau)+i\lambda}\tilde{H}\\ &= -\tilde{H}\operatorname{ad}_{A(\tau)}(H)\tilde{H}.\end{aligned}$$

Hence $i\operatorname{ad}_{A(\tau)}(\tilde{H})$ extends to the bounded selfadjoint operator $-\tilde{H}i\operatorname{ad}_{A(\tau)}(H)\tilde{H}$, which is continuous in τ and $O(1)$ for $\tau\to\infty$. The same holds for $Hi\operatorname{ad}_{A(\tau)}(\tilde{H})$.

We shall show by induction in n' that the statement of Lemma 2.10 holds for $1\leqq n\leqq n'\leqq n_0$ and also (for the same values of n and as bounded operators) that

$$\operatorname{ad}^n_{A(\tau)}(\tilde{H})=\sum_{\substack{n_1+n_2+n_3=n-1\\ n_i\geqq 0}}-\frac{(n-1)!}{n_1!n_2!n_3!}\operatorname{ad}^{n_1}_{A(\tau)}(\tilde{H})\operatorname{ad}^{n_2+1}_{A(\tau)}(H)\operatorname{ad}^{n_3}_{A(\tau)}(\tilde{H}). \tag{2.8}$$

We have proved the above statements for $n'=1$. Suppose they hold for n'. We shall prove them for $n'+1$ (provided $n'+1\leqq n_0$).

Using (2.8) for $n=n'$ we compute as a form on $\mathscr{D}$,

$$\begin{aligned}[i^{n'}\operatorname{ad}^{n'}_{A(\tau)}(\tilde{H}),A(\tau)] &= \lim_{\lambda\to\infty}\left[i^{n'}\operatorname{ad}^{n'}_{A(\tau)}(\tilde{H}),\frac{i\lambda A(\tau)}{A(\tau)+i\lambda}\right]\\ &= \lim_{\lambda\to\infty}\left[i^{n'}\operatorname{ad}^{n'}_{A(\tau)}(\tilde{H}),\frac{\lambda^2}{A(\tau)+i\lambda}\right]\\ &= -i^{n'}\lim_{\lambda\to\infty}\sum\frac{(n'-1)!}{n_1!n_2!n_3!}\left\{\left[\operatorname{ad}^{n_1}_{A(\tau)}(\tilde{H}),\frac{\lambda^2}{A(\tau)+i\lambda}\right]\right.\\ &\quad\cdot\operatorname{ad}^{n_2+1}_{A(\tau)}(H)\operatorname{ad}^{n_3}_{A(\tau)}(\tilde{H})+\operatorname{ad}^{n_1}_{A(\tau)}(\tilde{H})\left[\operatorname{ad}^{n_2+1}_{A(\tau)}(H),\frac{\lambda^2}{A(\tau)+i\lambda}\right]\\ &\quad\left.\cdot\operatorname{ad}^{n_3}_{A(\tau)}(\tilde{H})+\operatorname{ad}^{n_1}_{A(\tau)}(\tilde{H})\operatorname{ad}^{n_2+1}_{A(\tau)}(H)\left[\operatorname{ad}^{n_3}_{A(\tau)}(\tilde{H}),\frac{\lambda^2}{A(\tau)+i\lambda}\right]\right\}\\ &= -i^{n'}\sum_{\substack{n_1+n_2+n_3=n'\\ n_i\geqq 0}}\frac{n'!}{n_1!n_2!n_3!}\operatorname{ad}^{n_1}_{A(\tau)}(\tilde{H})\operatorname{ad}^{n_2+1}_{A(\tau)}(H)\operatorname{ad}^{n_3}_{A(\tau)}(\tilde{H}).\end{aligned}$$

Hence (2.8) and as an easy consequence (together with the induction hypothesis) the statement of Lemma 2.10 hold for $n = n' + 1$. The induction is complete. $\square$

Lemma 2.11. *Suppose Assumption 2.2(1)–(5).*

(1) *For any $g \in C_0^\infty(\mathbf{R})$ and $1 \leqq n \leqq n_0$ the form (defined iteratively)*

$$\mathrm{ad}^n_{A(\tau)}(g(H)) = [\mathrm{ad}^{n-1}_{A(\tau)}(g(H)), A(\tau)] \text{ on } \mathscr{D}$$

extends to a bounded operator on $\mathscr{H}$. Moreover $(H+i)\,\mathrm{ad}^n_{A(\tau)}(g(H))$ and $\mathrm{ad}^n_{A(\tau)}(g(H))(H+i)$ are continuous $\mathscr{B}(\mathscr{H})$-valued functions of τ, and $O(1)$ for $\tau \to \infty$.

(2) *For $1 \leqq n \leqq n_0$ $\mathrm{ad}^n_A(e^{-itH}f(H))$ is a continuous $\mathscr{B}(\mathscr{H})$-valued function of $t \in \mathbf{R}$.*

(3) *For any real s with $0 \leqq s \leqq n_0$ $\langle A\rangle^s e^{-itH} f(H) \langle A\rangle^{-s}$ is a continuous $\mathscr{B}(\mathscr{H})$-valued function of $t \in \mathbf{R}$.*

Proof. As a form on $\mathscr{D}$,

$$\begin{aligned}[e^{ik\tilde{H}}, A(\tau)] &= \lim_{\lambda \to \infty} e^{ik\tilde{H}} i \int_0^k dl\, e^{-il\tilde{H}} \left[\tilde{H}, \frac{i\lambda A(\lambda)}{A(\lambda)+i\lambda}\right] e^{il\tilde{H}} \\ &= i e^{ik\tilde{H}} \int_0^k e^{-il\tilde{H}}\, \mathrm{ad}_{A(\tau)}(\tilde{H}) e^{il\tilde{H}}\, dl,\end{aligned}$$

which by Lemma 2.10 is bounded. The same holds when multiplied by $(H+i)$. Moreover $(H+i)\,\mathrm{ad}_{A(\tau)}(e^{ik\tilde{H}})$ and $\mathrm{ad}_{A(\tau)}(e^{ik\tilde{H}})(H+i)$ are jointly norm continuous in k and τ with normbounds of the form $C\langle k\rangle$ ($\langle k\rangle = (1+k^2)^{1/2}$).

By induction (cf. the proof of Lemma 2.10) for any $1 \leqq n \leqq n_0$ the form (defined iteratively)

$$\mathrm{ad}^n_{A(\tau)}(e^{ik\tilde{H}}) \text{ extends to the operator}$$

$$i \int_0^k dl \sum_{\substack{n_1+n_2+n_3=n-1 \\ n_i \geqq 0}} \frac{(n-1)!}{n_1!n_2!n_3!} \mathrm{ad}^{n_1}_{A(\tau)}(e^{i(k-l)\tilde{H}})\, \mathrm{ad}^{n_2+1}_{A(\tau)}(\tilde{H})\, \mathrm{ad}^{n_3}_{A(\tau)}(e^{il\tilde{H}}), \tag{2.9}$$

as an operator is bounded, jointly continuous and satisfies

$$\|\mathrm{ad}^n_{A(\tau)}(e^{ik\tilde{H}})\| \leqq C\langle k\rangle^n, \quad \forall \tau, k;$$

and similarly when multiplied by $(H+i)$.

Since $g(H) = \int dk\, \hat{\tilde{g}}(k) e^{ik\tilde{H}}$ with $\tilde{g}(x) = g\left(\frac{1}{x} - C\right)$ ($\in C_0^\infty(\mathbf{R})$ for C large enough) inductively

$$i^n\, \mathrm{ad}^n_{A(\tau)}(g(H)) = i^n \int dk\, \hat{\tilde{g}}(k)\, \mathrm{ad}^n_{A(\tau)}(e^{ik\tilde{H}}), \tag{2.10}$$

and thus bounded as are $(H+i)\,\mathrm{ad}^n_{A(\tau)}(g(H))$ and $\mathrm{ad}^n_{A(\tau)}(g(H))(H+i)$, and all are continuous in τ and $O(1)$ for $\tau \to \infty$. This proves (1).

As for (2) we use (2.9), (2.10) and the fact that $e^{-itH}f(H) = \tilde{g}_t(\tilde{H})$, $\tilde{g}_t(x) = e^{-it(1/x - C)} f\left(\frac{1}{x} - C\right)$: For $t, t_0 \in \mathbf{R}$,

$$\begin{aligned}&\|\mathrm{ad}^n_A(e^{-itH}f(H)) - \mathrm{ad}^n_A(e^{-it_0H}f(H))\| \\ &\leqq C \int dk \langle k\rangle^n |\hat{\tilde{g}}_t(k) - \hat{\tilde{g}}_{t_0}(k)| \\ &\leqq C_1(\|\tilde{g}_t - \tilde{g}_{t_0}\|_{L^2(\mathbf{R})} + \|\tilde{g}^{(n+1)}_t - \tilde{g}^{(n+1)}_{t_0}\|_{L^2(\mathbf{R})}).\end{aligned}$$

The right-hand side $\to 0$ for $t \to t_0$.

As for (3) it suffices (by an interpolation argument) to show that $A^{n_0}e^{-itH}f(H)\langle A\rangle^{-n_0}$ is a continuous $\mathscr{B}(\mathscr{H})$-valued function of $t\in\mathbf{R}$. This is done by showing inductively for $1\leqq n\leqq n_0$, that $e^{-itH}f(H)\langle A\rangle^{-n_0}\phi\in\mathscr{D}(A^n)$ for any $\phi\in\mathscr{H}$, and that (cf. Lemma 2.7(2))

$$A^n e^{-itH}f(H)\langle A\rangle^{-n_0}=\sum_{m=0}^{n} c_m \operatorname{ad}_A^m(e^{-itH}f(H))A^{n-m}\langle A\rangle^{-n_0},$$

the right-hand side being bounded and continuous by (2). □

Lemma 2.12. *Suppose Assumption 2.2(1)–(4).*

(1) *Then for $1\leqq n\leqq n_0$, $A(\tau)^n\langle A\rangle^{-n}$ is a continuous $\mathscr{B}(\mathscr{H})$-valued function.*

(2) *For any $1\leqq\alpha\leqq n_0, 0<\varepsilon$ and $g_{0,n_0,\varepsilon}(x,\tau)=-(-x)^{n_0}\chi\left(\frac{x}{\tau}\right)\in\mathscr{F}_{0,n_0,\varepsilon}$, $(-A(\tau))^\alpha\chi\left(\frac{A(\tau)}{\tau}\right)\langle A\rangle^{-\alpha}$ is bounded, locally uniformly in $\tau\geqq t_0$.*

In particular the same statement holds for $(-g_{\beta,\alpha,\varepsilon}(A(\tau),\tau))^{1/2}\langle A\rangle^{-\alpha/2}$ with $0\leqq\beta, 0<\varepsilon, 0<\alpha<n_0-\frac{1}{2}$ and $g_{\beta,\alpha,\varepsilon}\in\mathscr{F}_{\beta,\alpha,\varepsilon}$.

Proof. We prove by induction in $n', 1\leqq n'\leqq n_0$, that for $1\leqq n\leqq n'$, $A(\tau)^n\langle A\rangle^{-n}$ is a continuously differentiable $\mathscr{B}(\mathscr{H})$-valued function with

$$\frac{d}{d\tau}\{A(\tau)^n\langle A\rangle^{-n}\}=\sum_{m=0}^{n-1} c_m \operatorname{ad}_{A(\tau)}^m(d_\tau A(\tau))A(\tau)^{n-1-m}\langle A\rangle^{-n}.$$

Clearly this statement $p(n')$ holds for $n'=1$.

Suppose $p(n')$ for $n'\leqq n_0-1$. Then

$$\langle A\rangle^{-1}A(\tau)^{n'+1}\langle A\rangle^{-n'-1}=(A(\tau)\langle A\rangle^{-1})^*A(\tau)^{n'}\langle A\rangle^{-n'-1}$$

is continuously differentiable and (by a commutation)

$$\begin{aligned}&\frac{d}{d\tau}\langle A\rangle^{-1}A(\tau)^{n'+1}\langle A\rangle^{-n'-1}\\&\quad=\langle A\rangle^{-1}d_\tau A(\tau)A(\tau)^{n'}\langle A\rangle^{-n'-1}\\&\qquad+\langle A\rangle^{-1}A(\tau)\sum_{m=0}^{n'-1} c_m \operatorname{ad}_{A(\tau)}^m(d_\tau A(\tau))A(\tau)^{n'-1-m}\langle A\rangle^{-n'-1}\\&\quad=\langle A\rangle^{-1}\sum_{m=0}^{n'} c'_m \operatorname{ad}_{A(\tau)}^m(d_\tau A(\tau))A(\tau)^{n'-m}\langle A\rangle^{-n'-1}.\end{aligned}$$

It follows that

$$\begin{aligned}&\langle A\rangle^{-1}A(\tau)^{n'+1}\langle A\rangle^{-n'-1}\\&\quad=A^{n'+1}\langle A\rangle^{-n'-2}+\langle A\rangle^{-1}\int_{t_0}^{\tau}\sum_{m=0}^{n'} c'_m \operatorname{ad}_{A(\tau')}^m(d_\tau A(\tau'))A(\tau')^{n'-m}\langle A\rangle^{-n'-1}d\tau'.\end{aligned}$$

By multiplying by $\langle A\rangle$ on both sides we obtain that $A(\tau)^{n'+1}\langle A\rangle^{-n'-1}$ is bounded and that $p(n'+1)$ holds. This proves (1).

As for (2), with $\alpha=n_0$,

$$\left\|(-A(\tau))^\alpha\chi\left(\frac{A(\tau)}{\tau}\right)\langle A\rangle^{-\alpha}\right\|\leqq\|A(\tau)^{n_0}\langle A\rangle^{-n_0}\|\leqq C$$

locally uniformly in $\tau \geqq t_0$ (by (1)). A similar statement holds with $\alpha = 0$. We obtain (2) by interpolating these estimates. $\square$

Definition 2.13. *Suppose the situation as in Lemma 2.12.*

Let $f_1 \in C_0^\infty(\mathbf{R})$ be real-valued and satisfy $f_1 f_2 = f_2$. Put $D_1 A(\tau) = i\,\mathrm{ad}_{A(\tau)}(f_1(H)H) + d_\tau A(\tau)$ (bounded by Lemma 2.11(1)).

Lemma 2.14. *With the situation as above*

$$f_2(H) D_1 A(\tau) f_2(H) = f_2(H) D A(\tau) f_2(H).$$

Proof. By Lemmas 2.11(1) and 2.9

$$\begin{aligned} &f_2(H)[f_1(H)H, A(\tau)] f_2(H) \\ &= s - \lim_{\lambda\to\infty} f_2(H)\left[f_1(H)H, \frac{i\lambda A(\tau)}{A(\tau)+i\lambda}\right] f_2(H) \quad \text{(strong convergence)} \\ &= s - \lim_{\lambda\to\infty} f_2(H)\left[H, \frac{i\lambda A(\tau)}{A(\tau)+i\lambda}\right] f_2(H) \\ &= f_2(H)[H, A] f_2(H). \quad \square \end{aligned}$$

Lemma 2.15. *Suppose Assumption 2.2(1)–(5). Let $\beta \geqq 0$, $n_0 - \frac{1}{2} > \alpha > 0$, $\varepsilon > 0$, $\delta > 0$, $\phi \in \mathscr{H}$ and $g_{\beta,\alpha,\varepsilon} \in \mathscr{F}_{\beta,\alpha,\varepsilon}$. Put $\psi(t) = e^{-itH} f(H) B^{-\alpha/2}\phi$ and $g_\delta(x,\tau) = g_{\beta,\alpha,\varepsilon}(x,\tau) F^2(\delta x)$, where F is real, $F \in C_0^\infty(\mathbf{R})$ and $F(x) = 1$ for $|x| < 1$. Then, with the convention $\langle P \rangle_t = \langle \psi(t), P\psi(t)\rangle$ for an operator (or form) P, $\langle g_\delta(A(\tau),\tau)\rangle_t$ is continuously differentiable with $\frac{d}{dt}\langle g_\delta(A(\tau),\tau)\rangle_t = \langle D g_\delta(A(\tau),\tau)\rangle_t$, where*

$$\begin{aligned} D g_\delta(A(\tau),\tau) &= \left(\frac{\partial}{\partial\tau} g_\delta\right)(A(\tau),\tau) + \sum_{m=1}^{n_0-1} (m!)^{-1} g_\delta^{(m)}(A(\tau),\tau)\,\mathrm{ad}_{A(\tau)}^{m-1}(D_1 A(\tau)) \\ &\quad + \int dk\, \widehat{g_\delta^{(n_0)}}(k,\tau) e^{ikA(\tau)} R^r_{n_0,\tau}(k), \\ R^r_{n_0,\tau}(k) &= \int_0^k dl \frac{(k-l)^{n_0-1}}{(n_0-1)!\,k^{n_0}} e^{-ilA(\tau)}\,\mathrm{ad}_{A(\tau)}^{n_0-1}(D_1 A(\tau)) e^{ilA(\tau)}. \end{aligned}$$

Proof.

$$\begin{aligned} &\frac{d}{dt}\langle \psi(t), g_\delta(A(\tau),\tau)\psi(t)\rangle \\ &= \langle \psi(t), i[f_1(H)H, g_\delta(A(\tau),\tau)]\psi(t)\rangle + \left\langle \psi(t), \left(\frac{d}{dt} g_\delta(A(\tau),\tau)\right)\psi(t)\right\rangle, \end{aligned}$$

where (cf. the proof of Lemma 2.7)

$$\begin{aligned} \frac{d}{dt} g_\delta(A(\tau),\tau) &= \left(\frac{\partial}{\partial\tau} g_\delta\right)(A(\tau),\tau) + \int dk\, \widehat{g_\delta^{(1)}}(k,\tau) e^{ikA(\tau)} k^{-1} \int_0^k dl\, e^{-ilA(\tau)} d_\tau A(\tau) e^{ilA(\tau)} \\ &= \left(\frac{\partial}{\partial\tau} g_\delta\right)(A(\tau),\tau) + \sum_{m=1}^{n_0-1} (m!)^{-1} g_\delta^{(m)}(A(\tau),\tau)\,\mathrm{ad}_{A(\tau)}^{m-1}(d_\tau A(\tau)) \\ &\quad + \int dk\, \widehat{g_\delta^{(n_0)}}(k,\tau) e^{ikA(\tau)} \int_0^k \frac{(k-l)^{n_0-1}}{(n_0-1)!\,k^{n_0}} e^{-ilA(\tau)}\,\mathrm{ad}_{A(\tau)}^{n_0-1}(d_\tau A(\tau)) e^{ilA(\tau)} dl. \end{aligned}$$

We use Lemmas 2.7(1) and 2.11(1) to expand $i[f_1(H)H, g_\delta(A(\tau),\tau)]$. □

Lemma 2.16. *Suppose Assumption 2.2(1)–(5). With $\beta,\alpha,\varepsilon, g=g_{\beta,\alpha,\varepsilon}$ and $\psi(t)$ as in Lemma 2.15 $\langle g(A(\tau),\tau)\rangle_t$ is absolutely continuous with $\frac{d}{dt}\langle g(A(\tau),\tau)\rangle_t=\langle Dg(A(\tau),\tau)\rangle_t$, where $Dg(A(\tau),\tau)=E_1+\cdots+E_9$, the forms being locally integrable on $\psi(t)$ and given as follows (we use the notation of Lemmas 2.7 and 2.15):*

$$E_1=\left(\frac{\partial}{\partial\tau}g\right)(A(\tau),\tau),$$

$$E_2=\tilde g(A(\tau),\tau)f_2(H)DA(\tau)f_2(H)\tilde g(A(\tau),\tau),\quad \tilde g=(g^{(1)})^{1/2},$$

$$E_3=\sum_{m=1}^{n_0-1}(m!)^{-1}\tilde g^{(m)}(A(\tau),\tau)\,\mathrm{ad}^m_{A(\tau)}(f_2(H))D_1A(\tau)f_2(H)\tilde g(A(\tau),\tau),$$

$$E_4=R_0(\tau)\tilde g(A(\tau),\tau)$$

with

$$R_0(\tau)=\int dk\widehat{\tilde g^{(n_0)}}(k,\tau)e^{ikA(\tau)}R^r_{n_0,A(\tau),f_2(H)}(k)D_1A(\tau)f_2(H),$$

$$E_5=\tilde g(A(\tau),\tau)\sum_{m=1}^{n_0-2}\mathrm{ad}^m_{A(\tau)}(D_1A(\tau)f_2(H))\frac{(-1)^m}{m!}\tilde g^{(m)}(A(\tau),\tau),$$

$$E_6=\tilde g(A(\tau),\tau)R_1(\tau)$$

with

$$R_1(\tau)=-\int dk\widehat{\tilde g^{(n_0-1)}}(k,\tau)R^l_{n_0-1,A(\tau),D_1A(\tau)f_2(H)}(k)e^{ikA(\tau)},$$

$$E_7=\sum_{m=2}^{n_0-1}g_m(A(\tau),\tau)h_m(A(\tau),\tau)\sum_{m_1=0}^{n_0-m-1}\mathrm{ad}^{m_1}_{A(\tau)}(H_m)\frac{(-1)^{m_1}}{m_1!}g_m^{(m_1)}(A(\tau),\tau)$$

with

$$g_m=g_{\beta/2,((\alpha-m)/2)_+,\varepsilon/2},\quad h_m(x,\tau)=\frac{\tau^\beta}{m!}(-x)^{-(\alpha-m)_+}g^{(m)}(x,\tau)$$

and

$$H_m=\mathrm{ad}^{m-1}_{A(\tau)}(D_1A(\tau))f_2(H),$$

$$E_8=\sum_{m=2}^{n_0-1}g_m(A(\tau),\tau)h_m(A(\tau),\tau)R_m(\tau)$$

with

$$R_m(\tau)=-\int dk\widehat{g_m^{(n_0-m)}}(k,\tau)R^l_{n_0-m,A(\tau),H_m}(k)e^{ikA(\tau)},$$

$$E_9=R_{n_0}(\tau)=\int dk\widehat{g^{(n_0)}}(k,\tau)e^{ikA(\tau)}R^r_{n_0,\tau}(k).$$

Proof. By Lemma 2.15 for any $\delta, t'>0$,

$$\langle g_\delta(A(\tau'),\tau')\rangle_{t'}-\langle g_\delta(A(t_0),t_0)\rangle_0=\int_0^{t'}dt\langle Dg_\delta(A(\tau),\tau)\rangle_t, \tag{2.11}$$

where $\tau' = t' + t_0$ and

$$Dg_\delta(A(\tau), \tau) = \left(\frac{\partial}{\partial\tau} g_\delta\right)(A(\tau), \tau) + \sum_{m=1}^{n_0 - 1} (m!)^{-1} g_\delta^{(m)}(A(\tau), \tau)\, \mathrm{ad}_{A(\tau)}^{m-1}(D_1 A(\tau))$$
$$+ \int dk \widehat{g_\delta^{(n_0)}}(k, \tau) e^{ikA(\tau)} R^r_{n_0,\tau}(k). \tag{2.12}$$

We shall rewrite the right-hand side of (2.12) as a sum of forms $\sum_{l=1}^{9} E_{l,\delta}(\tau)$, each term being integrable on $\psi(t)$ and obeying, with $E_l(\tau) = E_l$ as introduced above,

$$\langle E_{l,\delta}(\tau)\rangle_t \to \langle E_l(\tau)\rangle_t \quad \text{for} \quad \delta \to 0 \tag{2.13}$$

and for any $0 < T$

$$|\langle E_{l,\delta}(\tau)\rangle_t| \leqq C_T \quad \forall 0 < t < T, \quad 0 < \delta < 1.$$

Since by the spectral theorem and Lemmas 2.11(3) and 2.12(2) the left-hand side of (2.11) goes to $\langle g(A(\tau'), \tau')\rangle_{t'} - \langle g(A(t_0), t_0)\rangle_0$ for $\delta \to 0$ (an argument to be used repeatedly in verifying (2.13)), the lemma follows from (2.13) and the Lebesgue theorem of dominated convergence.

We proceed to the proof of (2.13):

Let $E_{1,\delta}(\tau) = \left(\frac{\partial}{\partial\tau} g_\delta\right)(A(\tau), \tau)$. Then we can write (for any $g_{0,\alpha/2,\varepsilon/2} \in \mathscr{F}_{0,\alpha/2,\varepsilon/2}$)

$$\left(\frac{\partial}{\partial\tau} g_\delta\right)(A(\tau), \tau) = g_{0,\alpha/2,\varepsilon/2}(A(\tau), \tau) F^2(\delta A(\tau))$$
$$\cdot (-A(\tau))^{-\alpha} \left(\frac{\partial}{\partial\tau} g\right)(A(\tau), \tau) g_{0,\alpha/2,\varepsilon/2}(A(\tau), \tau).$$

By Lemmas 2.11(3) and 2.12(2) for any $0 < T$,

$$\| g_{0,\alpha/2,\varepsilon/2}(A(\tau), \tau)\psi(t) \| < C \quad \forall 0 < t < T.$$

Clearly by the above facts together with the spectral theorem (2.13) follows for $E_{1,\delta}(\tau)$.

We look at the contributions from the second term $g_\delta^{(1)}(A(\tau), \tau) D_1 A(\tau)$ on the right-hand side of (2.12). It shall be proved that $g_\delta^{(1)}(A(\tau), \tau) D_1 A(\tau) = E_{2,\delta}(\tau) + \cdots + E_{6,\delta}(\tau)$ for some $E_{l,\delta}(\tau), 2 \leqq l \leqq 6$, satisfying (2.13): Using the abbreviations $g = g(A(\tau), \tau)$, $\tilde{g} = \tilde{g}(A(\tau), \tau)$, $F = F(\delta A(\tau))$ and $F' = \frac{\partial}{\partial x} F(\delta x)_{|x = A(\tau)}$ we expand

$$\begin{aligned}
& g_\delta^{(1)}(A(\tau), \tau) D_1 A(\tau) = D_1 + \cdots + D_5, \\
& D_1 = 2FF' g D_1(A(\tau)) \\
& D_2 = F\tilde{g} f_2(H) D_1 A(\tau) f_2(H) \tilde{g} F \\
& D_3 = F\tilde{g}(I - f_2(H)) D_1 A(\tau) f_2(H) \tilde{g} F \\
& D_4 = -F\tilde{g}[D_1 A(\tau) f_2(H), \tilde{g} F] \\
& D_5 = F^2 \tilde{g}^2 D_1 A(\tau)(I - f_2(H)).
\end{aligned}$$

As for D_1 we write $\left(\text{with an abuse of notation if } \frac{\alpha}{2} - 1 < 0\right)$

$$D_1 = 2F'(-A(\tau))^{1-\alpha/2} g\{-g_{0,\alpha/2-1,\varepsilon/2}(A(\tau),\tau)F\} D_1 A(\tau)$$

and use Lemma 2.7(2) to commute $\{\cdots\}$ with the last factor $D_1 A(\tau)$. The obtained expansion together with arguments used above give that D_1 does not contribute in the sense of (2.13), i.e. $\langle D_1 \rangle_t \to 0$ for $\delta \to 0$ and for any $0 < T$: $|\langle D_1 \rangle_t| \leqq C \ \forall t < T$, $0 < \delta < 1$.

As for D_2, by Lemma 2.14 (and Lemmas 2.11(3) and 2.12(2) + the spectral theorem) $\langle D_2 \rangle_t \to \langle E_2 \rangle_t$ for $\delta \to 0$ and the uniform bound of (2.13) holds.

In the following the estimate

$$\|\hat{h}\|_{L^1(\mathbf{R})} \leqq C(\|h\|_{L^2(\mathbf{R})} + \|h^{(1)}\|_{L^2(\mathbf{R})}) \tag{2.14}$$

(valid whenever the right-hand side is finite) is useful.

As for D_3 we notice that

$$\langle D_3 \rangle_t = -\langle [F\tilde{g}, f_2(H)] D_1 A(\tau) f_2(H) \tilde{g} F \rangle_t.$$

By Lemma 2.7(1)

$$-[F\tilde{g}, f_2(H)] = \sum_{m=1}^{n_0-1} (m!)^{-1} \left(\frac{\partial}{\partial x}\right)^m \{F(\delta x)\tilde{g}(x,\tau)\}_{|x=A(\tau)} \operatorname{ad}^m_{A(\tau)}(f_2(H)) + \text{remainder}.$$

This fact together with the bound (valid for any $m \in \mathbf{N}$)

$$\left|\left(\frac{\partial}{\partial x}\right)^m \{F(\delta x)\tilde{g}(x,\tau)\}\right| \leqq C(-x)^{\alpha/2-m-1/2}, \quad \forall 0 < \delta < 1, \quad t_0 < \tau, \quad x < 0, \tag{2.15}$$

Lemma 2.7(3) and (2.14), imply that $\langle D_3 \rangle_t \to \langle E_3 \rangle_t + \langle E_4 \rangle_t$ for $\delta \to 0$. Moreover the uniform estimate of (2.13) is satisfied for the relevant terms.

As for D_4 we apply Lemma 2.7, (2.14) and (2.15) similarly to obtain that $\langle D_4 \rangle_t \to \langle E_5 \rangle_t + \langle E_6 \rangle_t$ for $\delta \to 0$, still with the uniform estimate of (2.13) satisfied for the terms involved.

Clearly $\langle D_5 \rangle_t = 0$.

It remains to look at the "$m > 1$" terms and the last term on the right-hand side of (2.12). The latter contributes with E_9 in the limit $\delta \to 0$ (by an analogue of (2.15)). The others with E_7 and E_8 (by similar arguments as used above). □

Proof of Theorem 2.4. With the situation of Lemma 2.16 we have (by the conclusion) for any $t' > 0$ (with $\tau' = t' + t_0$)

$$\langle -g_{\beta,\alpha,\varepsilon}(A(\tau'),\tau') \rangle_{t'} = \langle -g_{\beta,\alpha,\varepsilon}(A(t_0),t_0) \rangle_0 - \int_0^{t'} dt \langle E_1 + \cdots + E_9 \rangle_t. \tag{2.16}$$

In various cases (to be specified below) we shall estimate the right-hand side of (2.16) from above. For that we notice that due to Assumption 2.2(5), Lemma 2.11(1) and (2.14) the following estimates hold for $\tau \to \infty$:

$$E_3 = \sum_{m=1}^{n_0-1} \tilde{g}^{(m)}(A(\tau),\tau) O(1) \tilde{g}(A(\tau),\tau),$$

$$E_4 = O(\tau^{-\beta/2+\alpha/2-n_0}) \tilde{g}(A(\tau),\tau),$$

$$E_5 = \tilde{g}(A(\tau), \tau) \sum_{m=1}^{n_0 - 2} O(1) \tilde{g}^{(m)}(A(\tau), \tau),$$

$$E_6 = \tilde{g}(A(\tau), \tau) O(\tau^{-\beta/2 + \alpha/2 - n_0 + 1}),$$

$$E_7 = \sum_{m=2}^{n_0 - 1} g_m(A(\tau), \tau) \sum_{m_1 = 0}^{n_0 - m - 1} O(\tau^{-(m-\alpha)_+}) g_m^{(m_1)}(A(\tau), \tau),$$

$$E_8 = \sum_{m=2}^{n_0 - 1} g_m(A(\tau), \tau) O(\tau^{-\beta/2 + ((\alpha - m)/2)_+ - n_0 + m + 1/2 - (m - \alpha)_+}),$$

$$E_9 = O(\tau^{-\beta + \alpha - n_0 + 1/2 + \kappa_0}). \tag{2.17}$$

We shall prove Theorem 2.4 by showing by induction in $n \in N$ the statement $p(n)$ that the theorem holds under the further restriction $n - 1 < \alpha_0 \leqq n$.

We start by proving $p(1)$: So suppose the conditions of Theorem 2.4 and in addition that $0 < \alpha_0 \leqq 1$. For $\phi \in \mathscr{H}$ let $\psi(t) = e^{-itH} f(H) B^{-\alpha_0/2} \phi$ (as in Lemmas 2.15 and 2.16). It suffices to verify the estimate $\langle -g_{\beta_0, \alpha_0, \varepsilon}(A(\tau), \tau) \rangle_t \leqq C \| \phi \|^2$ for any $\varepsilon > 0$. For that we use (2.16) and (2.17): Clearly (by Lemma 2.11(3)) the first term on the right-hand side of (2.16) satisfies such estimate. The contribution from E_1 trivially (since it is non-positive), the one from E_2 by Assumption 2.2(6), and the ones from $E_3, \ldots, E_9$, because by (2.17),

$$\begin{aligned}
&|\langle E_3 \rangle_t|, |\langle E_5 \rangle_t|, |\langle E_7 \rangle_t| \leqq C \tau^{-\beta_0 - 1} \| \phi \|^2, \\
&|\langle E_4 \rangle_t| \leqq C \tau^{-\beta_0 + \alpha_0/2 - n_0} \| \phi \|^2, \\
&|\langle E_6 \rangle_t| \leqq C \tau^{-\beta_0 + \alpha_0/2 - n_0 + 1} \| \phi \|^2, \\
&|\langle E_8 \rangle_t| \leqq C \tau^{-\beta_0 + \alpha_0 - n_0 + 1/2} \| \phi \|^2, \\
&|\langle E_9 \rangle_t| \leqq C \tau^{-\beta_0 + \alpha_0 - n_0 + 1/2 + \kappa_0} \| \phi \|^2.
\end{aligned} \tag{2.18}$$

By (2.1) and (2.3) the terms on the right-hand sides of the inequalities of (2.18) are integrable (to infinity). This proves $p(1)$.

Suppose now that $p(n)$ (with $1 \leqq n \leqq n_0 - 1$) is true. We shall verify the statement for $n + 1$. So suppose the conditions of Theorem 2.4 and that $n < \alpha_0 \leqq n + 1$. Then by $p(n)$, (2.2) and (2.4) (all assumed to hold)

$$(-g_{4/5, n, \varepsilon}(A(\tau), \tau))^{1/2} e^{-itH} f(H) B^{-n/2} = O(1) \quad \text{for} \quad \tau \to \infty$$

for any $\varepsilon > 0$ and $g_{4/5, n, \varepsilon} \in \mathscr{F}_{4/5, n, \varepsilon}$.

In particular (cf. Corollary 2.5)

$$(-g_{0, n-1, \varepsilon}(A(\tau), \tau))^{1/2} e^{-itH} f(H) B^{-n/2} = O(\tau^{-1/10}) \quad \text{for} \quad \tau \to \infty. \tag{2.19}$$

We shall prove that

$$(-g_{0, n, \varepsilon}(A(\tau), \tau))^{1/2} e^{-itH} f(H) B^{-n/2} = O(1) \quad \text{for} \quad \tau \to \infty. \tag{2.20}$$

For that we use (2.16) and (2.17) (as above), now in conjunction with (2.19). We obtain the following analogue to (2.18):

$$\begin{aligned}
&|\langle E_3 \rangle_t|, |\langle E_5 \rangle_t|, |\langle E_7 \rangle_t| \leqq C \tau^{-6/5} \| \phi \|^2, \\
&|\langle E_4 \rangle_t| \leqq C \tau^{n/2 - n_0 - 1/10} \| \phi \|^2, \\
&|\langle E_6 \rangle_t| \leqq C \tau^{9/10 + n/2 - n_0} \| \phi \|^2,
\end{aligned}$$

$$|\langle E_8\rangle_t| \leqq C\tau^{-11/10}\|\phi\|^2,$$
$$|\langle E_9\rangle_t| \leqq C\tau^{n-n_0+1/2+\kappa_0}\|\phi\|^2.$$

By (2.2) and (2.4) the right-hand sides of these inequalities are integrable. Thus (2.20) holds.

It remains to prove that

$$(-g_{\beta_0,\alpha_0,\varepsilon}(A(\tau),\tau))^{1/2}e^{-itH}f(H)B^{-\alpha_0/2}=O(1) \quad \text{for} \quad \tau\to\infty. \tag{2.21}$$

Again we use (2.16) and (2.17), now in conjunction with (2.20). We obtain the following estimates:

$$|\langle E_3\rangle_t|, |\langle E_5\rangle_t|, |\langle E_7\rangle_t| \leqq C\tau^{-\beta_0-1}\|\phi\|^2,$$
$$|\langle E_4\rangle_t| \leqq C\tau^{-\beta_0+\alpha_0/2-n_0}\|\phi\|^2,$$
$$|\langle E_6\rangle_t|, |\langle E_8\rangle_t| \leqq C\tau^{-\beta_0+\alpha_0/2-n_0+1}\|\phi\|^2,$$
$$|\langle E_9\rangle_t| \leqq C\tau^{-\beta_0+\alpha_0-n_0+1/2+\kappa_0}\|\phi\|^2.$$

The integrability follows from (2.1) and (2.3). This proves (2.21) and hence $p(n+1)$. □

3. Applications to N-Body Schroedinger Operators

In this section we shall give four examples. In all cases $H=-\Delta+V$ on $\mathscr{H}=L^2(X)$, where X is the $C.$ M-configuration space $\left\{x=(x^1,\dots,x^N)\,|\,x^i\in\mathbf{R}^\nu,\ \sum_{i=1}^N m_ix^i=0\right\}$ of N ν-dimensional particles with masses m_i. The inner product in X is given by $x\cdot y=\sum_{i=1}^N 2m_ix^i\cdot y^i$. The operator $-\Delta$ denotes the Laplacian.

Put for any cluster decomposition a

$$X_a=\{x\in X\,|\,x^i=x^j \text{ if } i,j\in C \text{ for some } C\in a\}$$

and $X^a=$ the orthogonal complement in X.

The corresponding orthogonal projections are denoted Π_a and Π^a, respectively. The cluster decomposition $(1)\cdots(\hat{\imath})\cdots(\hat{\jmath})\cdots(N)(ij)$, where ^ indicates omission is denoted by (ij).

The momentum operator $-i\nabla$ is denoted by p. We put $p_a=\Pi_ap$ and $p^a=\Pi^ap$, and similarly for any $x\in X$ we define $x_a=\Pi_ax$ and $x^a=\Pi^ax$. For further N-body notation we refer to [G].

We assume throughout this section and Sect. 4 that the potential $V(x)=\sum_{(ij)}V_{ij}(x^{(ij)})$, where $V_{ij}(y)$ are real-valued and as operators on $L^2(X^{(ij)})$ respectively as functions on $X^{(ij)}$ satisfy

Assumption 3.1. (1) $V_{ij}(-\Delta+1)^{-1}$ *are compact.*

(2) $\exists R_0>0\,\exists\, 1>\varepsilon_0>0$:

$$V_{ij}(y) \textit{ are smooth in the regions } |y|>R_0$$

and

$$\partial_y^\alpha V_{ij}(y)=O(|y|^{-|\alpha|-\varepsilon_0}) \quad \textit{for} \quad |y|\to\infty, \quad \forall \textit{ multiindices } \alpha.$$

(3) *H as well as all sub Hamiltonians H^a(defined by $-\Delta + \sum_{(ij)\subset a} V_{ij}(\Pi^{(ij)}\cdot)$ on $L^2(X^a)$) have no positive eigenvalues.*

Remark. (1) Due to [F–H2] Assumption 3.1(3) is a rather weak additional assumption. For instance it is superfluous for $v=3$.

(2) As for Assumption 3.1(1) infinitesimal smallness with respect to the Laplacians (see [R–S, p. 162] for the definition) would suffice as for the theorems of Sects. 3 and 4.

We shall use various properties of a vector field and a partition of unity constructed recently by Graf [G]. These are enlisted in the following

Lemma 3.2. *Given $k>1$. Then $\exists r_1, r_2>0 \exists \omega\in C^\infty(X,X)$ (a smooth vector field) with the derivative ω_* symmetric, $\exists C^\infty(X)$-partition of unity $\{\tilde{j}_a\}$, indexed by the cluster decompositions a, such that*

(1) $\omega_*(x) \geqq \sum_a \tilde{j}_a(x)\Pi_a$,
(2) $\omega^{(ij)}(x)=0$ *if* $|x^{(ij)}|<r_1$,
(3) $\tilde{j}_a(kx)\omega^a(x)=0$,
(4) $|x^b|>kr_1$ *on* $\operatorname{supp}\tilde{j}_a$ *if* $b\not\subset a$,
(5) $\omega(x)=x$ *if* $|x^a|>r_2$ *for all* $a\neq(1)\cdots(N)$,
(6) $\tilde{j}_a(kx)\tilde{j}_b(x)=0$ *if* $a\not\subset b$,
(7) $|x^a|<r_2$ *on* $\operatorname{supp}\tilde{j}_a$,
(8) *For any* $\alpha \exists C>0: |\partial_x^\alpha \tilde{j}_a(x)|\leqq C$,
(9) *For any* α *and* $n\in N\cup\{0\}\exists C>0: |(x\cdot\nabla)^n\partial_x^\alpha(\omega(x)-x)|\leqq C$.

The property (2) follows from (3) and (4) (but is also contained explicitly in [G, Lemma 3.7]). Property (5) will play an important role in Sect. 4, however it is not used in the discussion below. It follows readily from the definition of ω in [G]. A similar remark is due for (7). The property (8) is contained in [G, Lemma 3.1]. As for (3) the statement follows from an application of [G, Lemma 3.2] (not to be discussed). Similarly (4) follows easily from [G, Lemma 2.1]. (It is a generalization of [G, Lemma 2.3]). The statements (1) and (6) are contained in [G, Lemmas 3.7 and 3.4 respectively]. As for the remaining property (9), it will be proved in Appendix A.

We shall only apply Lemma 3.2 with the input $k=2$. Moreover in all examples Assumption 2.2 can be verified for n_0 arbitrary. It is in the following tacitly assumed that n_0 is chosen large.

Example 1. Fix $0<E'<E$. Choose f and f_2 as in Assumption 2.2 and supported in a small neighbourhood of E. Put $t_0=1$, $\kappa_0=0$ and let $\beta_0, \alpha_0>0$ arbitrarily. Let for any $R>0$ $A(\tau)=\dfrac{R\omega(x/R)p+p\omega(x/R)R}{2}-2E'\tau(\tau=t+1)$ and $B=\langle A\rangle$ (By an application of Lemma 3.2(9) and [R–S, Theorem X.37] $A(\tau)$ is essentially selfadjoint on $C_0^\infty(X)$). The action of the group $e^{iA(\tau)s}$ can be expressed explicitly in terms of the flow associated with the vector field $R\omega\left(\dfrac{x}{R}\right)$ in X. Using this expression one verifies readily Assumption 2.2(2). As for Assumption 2.2(1) and (3)–(5) we need to have R large. Then the statements follow by using Remark

2.3(3), Lemma 3.2(2) and (9). (In computing the commutators in Assumption 2.2(1) we replace repeatedly $R\omega\left(\frac{x}{R}\right)$ by $R\left(\omega\left(\frac{x}{R}\right)-\frac{x}{R}\right)+x$ and treat the contributions from the two terms separately.)

As for Assumption 2.2(6) we shall prove in Appendix B that for given large R and for any f_2 supported in a small neighbourhood of $E, f_2(H)DA(\tau)f_2(H)\geqq 0$.

So the conclusions of Theorem 2.4 and Corollary 2.5 hold.

In particular for any $s\geqq\frac{1}{2}$ and $\varepsilon>0$,

$$\left(\frac{-A(\tau)}{\tau}\right)^{1/2}\chi\left(\frac{A(\tau)}{\tau}<-\varepsilon\right)e^{-itH}f(H)\langle A\rangle^{-s}=O(\tau^{-s})\quad\text{for}\quad\tau\to\infty. \tag{3.1}$$

We remark that all the above statements hold upon replacing $R\omega\left(\frac{x}{R}\right)$ by x, however with further smoothness assumptions on the potential (by the usual Mourre estimate cf. [F–H1]).

Example 2. Fix $0<E''<E'<E$. Let $f, f_2, t_0, \kappa_0, \beta_0$ and α_0 be as in Example 1. For $\varepsilon''>0$ let $g\in\mathscr{F}_{0,1,\varepsilon''}$ and $A(\tau)=$ multiplication by $g(-\tau M,\tau)$, $M=M(x,\tau)=\left(E''-\frac{x^2}{4\tau^2}\right)^{1/2}$. Let $A(\tau)'=$ the $A(\tau)$ considered in Example 1 (in terms of R large and the given E') and $B=\langle A(t_0)'\rangle^{1+\kappa}$ for some $\kappa>0$.

Then (1)–(5) of Assumption 2.2 hold. As for (6) we verify the condition $q(\beta_0,\alpha_0,\delta)$ of Corollary 2.6 with $\delta=1$: By using Lemma 3.2(9) we compute (cf. Lemma 2.15)

$$DA(\tau)=\left(\frac{\partial}{\partial\tau}g\right)(-\tau M,\tau)+\frac{1}{2}(g^{(1)}(-\tau M,\tau))^{1/2}M^{-1/2}$$
$$\cdot\left\{\frac{A(\tau)'}{\tau}+2(E'-E'')+O(\tau^{-1})\right\}M^{-1/2}(g^{(1)}(-\tau M,\tau))^{1/2}.$$

The first term is non-negative. As for the second we observe that

$$\frac{A(\tau)'}{\tau}+2(E'-E'')\geqq\frac{A(\tau)'}{\tau}\chi^2\left(\frac{A(\tau)'}{\tau}<-\varepsilon'\right)\quad\text{with}\quad\varepsilon'=E'-E''.$$

Hence it suffices to show that

$$\left(\frac{-A(\tau)'}{\tau}\right)^{1/2}\chi\left(\frac{A(\tau)'}{\tau}<-\varepsilon'\right)M^{-1/2}(g^{(1)}(-\tau M,\tau))^{1/2}$$
$$f_2(H)(g^{(1)}_{\beta,\alpha,\varepsilon}(A(\tau),\tau))^{1/2}e^{-itH}f(H)B^{-\alpha/2}=O(t^{-(1/2)-\alpha\kappa/2}).$$

If we commute the first factor $\left(\frac{-A(\tau)'}{\tau}\right)^{1/2}\chi\left(\frac{A(\tau)'}{\tau}<-\varepsilon'\right)$ in front of the last factors $e^{-itH}f(H)B^{-\alpha/2}$ it follows from (3.1) and the fact that $\|A(\tau)\|=O(\tau)$ for $\tau\to\infty$, that the indicated estimate holds. It remains to control the commutator. This can be done by using Lemmas 2.7(2) (or rather an extension cf. Remark 2.8(2)), 2.11(1) and 3.2(9) together with similar arguments.

We conclude that Assumption 2.2(6) holds.

Since $g_{\beta,\alpha,\varepsilon}(A(\tau),\tau)=g_{\beta,\alpha,\varepsilon}(-\tau M,\tau)$ for any $\varepsilon>2\varepsilon''$ (in this case the two operators are given by multiplication by the same function) and $\varepsilon''>0$ is arbitrary, we obtain for any $s\geqq 0$ and $\varepsilon>0$ the estimate

$$\chi\left(\frac{x^2}{4t^2}-E''<-\varepsilon\right)e^{-itH}f(H)B^{-s}=O(t^{-s}). \tag{3.2}$$

But since κ, E'' and E' are arbitrary (up to some relations) and since $(A(t_0)')^n(H-i)^{-n}\langle x\rangle^{-n}$ ($\langle x\rangle=(1+x^2)^{1/2}$) is bounded for any n (cf. [J–M–P]), a consequence of (3.2) is the following.

Theorem 3.3 (*minimal velocity estimates*). *Let $E,\varepsilon>0$ be given. Then for any $f\in C_0^\infty(\mathbf{R})$ supported in a small neighbourhood of E and any $s'>s>0$,*

$$\chi\left(\frac{x^2}{4t^2}-E<-\varepsilon\right)e^{-itH}f(H)\langle x\rangle^{-s'}=O(t^{-s}) \quad \textit{for} \quad t\to+\infty.$$

Remark. By the same method one can obtain similar results for negative non-threshold energies not eigenvalues (cf. Appendix B).

Example 3. Fix $0<E$. Let $f,f_2,t_0,\kappa_0,\beta_0$ and α_0 be as in Example 1. For $v>0$ let $A(\tau)=v\tau-\langle x\rangle$ and $B=\langle A\rangle$. For v large enough

$$f_2(H)DA(\tau)f_2(H)=f_2(H)\left\{v-\frac{x}{\langle x\rangle}p-p\frac{x}{\langle x\rangle}\right\}f_2(H)\geqq 0. \tag{3.3}$$

For such v Assumption 2.2 holds, and consequently (by Corollary 2.5)

Theorem 3.4 (*large velocity estimate*). *Let $E,\varepsilon>0$ be given. Then $\exists E'\geqq E$: For any $f\in C_0^\infty(\mathbf{R})$ supported in a small neighbourhood of E and any $s\geqq l\geqq 0$,*

$$\langle x\rangle^l\chi\left(E'-\frac{x^2}{4t^2}<-\varepsilon\right)e^{-itH}f(H)\langle x\rangle^{-s}=O(t^{-s+l}) \quad \textit{for} \quad t\to+\infty.$$

Remark. As noted by Sigal and Soffer [S–S] one can refine (3.3) as to obtain Theorem 3.4 with the explicit value $E'=E-\inf\sigma_c(H)$ ($\sigma_c(H)=$ the continuous spectrum of H). There exists a different proof along the line of the proof of Theorem 4.5. As before there are similar statements below zero.

Example 4. We shall apply Lemma 3.2 (again with $k=2$). Let

$$\bar{j}_a(x)=\tilde{j}_a(x)\left(\sum_b \tilde{j}_b(x)^4\right)^{-1/4}.$$

Then $\sum_a \bar{j}_a(x)^4=1$.

Let $v,t_0>0$ be given such that (with R_0 given in Assumption 3.1(2) and r_1 in accordance with Lemma 3.2)

$$vt_0r_1>R_0. \tag{3.4}$$

Let $\kappa_0>0$ be given. We shall verify the condition of Corollary 2.6 with $E>0$, f,f_2,β_0 and α_0 given as in Example 1, t_0 and κ_0 above, $\delta=\min\{\varepsilon_0,2\kappa_0\}$, $B=\langle x\rangle^{1+\kappa_0}$, and $A(\tau)$ given as follows:

With $\tau = t + t_0$ and $E'' = E' + 4, E' \geqq E$ large enough (cf. (3.28)), (and with some abuse of notation) let $\bar{j}_a(\tau)$ and $\chi(\tau)$ be the operators given by multiplication by $\bar{j}_a\left(\frac{2x}{v\tau}\right)$ and $\chi\left(\frac{x^2}{4\tau^2} - E'' < -1\right)$, respectively.

Put

$$\tilde{A}(\tau) = \frac{\tau}{2}\left\{\omega\left(\frac{x}{v\tau}\right)\left(p - \frac{x}{2\tau}\right) + \left(p - \frac{x}{2\tau}\right)\omega\left(\frac{x}{v\tau}\right)\right\},$$

$$I_a(\tau) = \tau^{2\kappa_0}(\tau^{2\kappa_0} + (p_a)^2)^{-1}, \quad B_a(\tau) = \bar{j}_a(\tau) I_a(\tau) \bar{j}_a(\tau),$$

$$A(\tau) = \sum_a \chi(\tau) B_a(\tau) \chi(\tau) \tilde{A}(\tau) \chi(\tau) B_a(\tau) \chi(\tau). \tag{3.5}$$

The operator $A(\tau)$ should be thought of as a regularisation of $\tilde{A}(\tau)$. We claim that (when properly interpreted) $A(\tau)$ is bounded and that the conditions of Corollary 2.6 are fulfilled.

The boundedness holds since by Lemma 3.2(3),

$$p\omega\left(\frac{x}{v\tau}\right)\bar{j}_a(\tau) = p_a\omega\left(\frac{x}{v\tau}\right)\bar{j}_a(\tau), \tag{3.6}$$

and similarly for the adjoint expression.

Since $p_a I_a(\tau) = O(\tau^{\kappa_0})$ for $\tau \to \infty$ we also have that

$$A(\tau) = O(\tau^{1+\kappa_0}) \quad \text{for} \quad \tau \to \infty. \tag{3.7}$$

We compute (using (3.6) again)

$$i[H, A(\tau)] = \sum_a [(p_a)^2 + (p^a)^2 + Q_a, \chi(\tau) B_a(\tau) \chi(\tau) \tilde{A}(\tau) \chi(\tau) B_a(\tau) \chi(\tau)], \tag{3.8}$$

where $Q_a = \sum_{(ij) \not\subset a} V_{ij}(x^{(ij)})$. Notice that if $(ij) \subset a$ then p_a and $V_{ij}(x^{(ij)})$ commutes.

On the other hand if $(ij) \not\subset a$ we have by Lemma 3.2(4) that $|x^{(ij)}| > v\tau r_1$ on the support of $V_{ij}(x^{(ij)})\bar{j}_a\left(\frac{2x}{v\tau}\right)$. So for any $\varepsilon > 0$ we can write

$$V_{ij}(x^{(ij)})\bar{j}_a\left(\frac{2x}{v\tau}\right)\left(1 - \chi\left((vr_1)^2 - \varepsilon - \frac{(x^{(ij)})^2}{\tau^2} < -\frac{\varepsilon}{2}\right)\right) = 0. \tag{3.9}$$

By choosing $\varepsilon > 0$ small enough we have (remember (3.4)) by Assumption 3.1(2) and with $\chi(\cdot)$ as on the left-hand side of (3.9), that $\chi(\cdot)V_{ij}(x^{(ij)})$ is smooth and satisfies the uniform estimates

$$|\partial^\alpha_{(\tau,x)}\{\chi(\cdot)V_{ij}(x^{(ij)})\}| \leqq C_\alpha \tau^{-|\alpha|-\varepsilon_0}. \tag{3.10}$$

The similar bounds, obtained by replacing ε_0 on the right-hand side by 0 (cf. (3.12)) and $\chi(\cdot)V_{ij}(x^{(ij)})$ on the left-hand side by $\bar{j}_a(\tau)\chi(\tau)$, $\bar{j}_a(\tau)\chi(\tau)\omega\left(\frac{x}{v\tau}\right)$ or by $\bar{j}_a(\tau)\chi(\tau)\omega\left(\frac{x}{v\tau}\right)\frac{x}{2\tau}$, hold.

Using (3.9), (3.10) and the statements above we conclude that the right-hand side of (3.8) is given as a finite sum of terms of the form

$$\{h_1 I_a(\tau) h_2 \cdots\} O_1(p_a) I_a(\tau) h_m O_1(p), \tag{3.11}$$

where $O_1(p_a)$ and $O_1(p)$ are first order polynomials in components of p_a and p, respectively, and with constant coefficients, and $h_j = h_j(x, \tau)$ are smooth and satisfy

$$|\partial^\alpha_{(\tau,x)} h(x,\tau)| \leqq C_\alpha \tau^{-|\alpha|}. \tag{3.12}$$

The form of $A(\tau)$ is a finite sum of terms of the form

$$\tau h_1 I_a(\tau) h_2 O_1(p_a) I_a(\tau) h_3, \text{ with } h_j \text{ and } O_1(p_a) \text{ as above.} \tag{3.13}$$

As for the time derivative of $A(\tau)$, it is a finite sum of terms of the form

$$\{h_1 I_a(\tau) h_2 \cdots\} O_1(p_a) I_a(\tau) h_m, h_j(x,\tau) \text{ as above.} \tag{3.14}$$

In order to verify Assumption 2.2(1)–(5) (the other part $q(\beta_0, \alpha_0, \delta)$ of Corollary 2.6 will be discussed afterwards) we must examine the commutators of operators of the form (3.13) with some of the form (3.11) and (3.14). For that it is convenient to take a more general point of view by introducing operators B_a of the form

$$B_a = (C_a \tau^{2\kappa_0} + O_1(p_a)\tau^{\kappa_0} + O_2(p_a))(\tau^{2\kappa_0} + (p_a)^2)^{-1}$$

with $O_1(p_a)$ as above and $O_2(p_a)$ second order polynomials similarly defined. Given such operator B_a and $h(x,\tau)$ smooth and obeying (3.12), we get by an elementary computation the following (convenient) identity:

$$\begin{aligned}[B_a, h] = \text{finite sum of terms of form } \tau^{-(1+\kappa_0)} \tilde{B}_a \tilde{h} \tilde{\tilde{B}}_a,\\ \text{with } \tilde{B}_a, \tilde{\tilde{B}}_a \text{ and } \tilde{h} \text{ given similarly.}\end{aligned} \tag{3.15}$$

By the statements associated with (3.11), (3.13) and (3.14) $A(\tau)$ is a finite sum of terms of the form

$$\tau^{1+\kappa_0} h_1 B_1 h_2 B_2 h_3, \tag{3.16}$$

$[H, A(\tau)]$ of terms of the form

$$\tau^{\kappa_0} h_1 B_1 h_2 \cdots B_{m-1} h_m O_1(p), \tag{3.17}$$

and $d_\tau A(\tau)$ of terms of the form

$$\tau^{\kappa_0} h_1 B_1 h_2 \cdots B_{m-1} h_m. \tag{3.18}$$

In all cases each B_j is given by some B_a as above. Moreover among these factors there will always exist at least one of the specific form

$$B_j = O_1(p_a)\tau^{\kappa_0}(\tau^{2\kappa_0} + (p_a)^2)^{-1}. \tag{3.19}$$

Now using (3.15) and the statement associated with (3.16) repeatedly together with the ones associated with (3.17)–(3.19) we obtain that for any n, $\mathrm{ad}^n_{A(\tau)}(H)$ is given by terms either of the form (3.17) and with one B_j of the form (3.19), or of the form $\tau^{2\kappa_0} h_1 B_1 \cdots B_{m-1} h_m$ and with *two* of the B_js of the form (3.19). In particular

$$\mathrm{ad}^n_{A(\tau)}(H) = \sum h_1 B_1 \cdots B_{m-1} h_m O_2(p). \tag{3.20}$$

Similarly $\mathrm{ad}^n_{A(\tau)}(d_\tau A(\tau))$ is given by terms of the form (3.18) and with one B_j of the form (3.19). Hence

$$\mathrm{ad}^{n-1}_{A(\tau)}(d_\tau A(\tau)) = \tau^{\kappa_0} \sum h_1 B_1 \cdots B_{m-1} h_m, \tag{3.21}$$

and also

$$\mathrm{ad}^{n-1}_{A(\tau)}(d_\tau A(\tau)) = \sum h_1 B_1 \cdots B_{m-1} h_m O_1(p). \tag{3.22}$$

It follows from (3.20)–(3.22) that Assumption 2.2(1)–(5) hold.

For future applications (in particular some in Sect. 4) we state some estimates: With $h(x,\tau)$ smooth and satisfying (3.12), and for any $m, n_1 \in \mathbf{N}$,

$$\mathrm{ad}^m_{A(\tau)}(h(x,\tau)) = \sum_{|\alpha| \leqq n_1 + m - 1} O(\tau^{|\alpha|})(\partial^\alpha_x h)(x,\tau) + O(\tau^{-n_1(1+\kappa_0)}) \quad \text{for} \quad \tau \to \infty. \tag{3.23}$$

In particular

$$\mathrm{ad}^m_{A(\tau)}(h(x,\tau)) = O(1). \tag{3.24}$$

For any $h \in C_0^\infty(\mathbf{R}^n)$ and $n \in \mathbf{N}$

$$\mathrm{ad}^n_{A(\tau)}(h(p)) = O(1). \tag{3.25}$$

It is remarked that (3.23) follows by a closer examination of the right-hand side of (3.15), while (3.25) follows by an application of the calculus (for example) of the Ps.D.Op.s. introduced in Sect. 4. The details are omitted.

We are left with verifying $q(\beta_0, \alpha_0, \delta)$, $\delta = \min\{\varepsilon_0, 2\kappa_0\}$:

Formally $\left(\text{with } D = i[H, \cdot] + \dfrac{d}{dt}\right)$,

$$D\tilde{A}(\tau) = \frac{2}{v}\left(p - \frac{x}{2\tau}\right)\omega_*\left(\frac{x}{v\tau}\right)\left(p - \frac{x}{2\tau}\right)$$
$$- \tau \sum_{(ij)} \omega\left(\frac{x}{v\tau}\right) \cdot \nabla V_{ij} - \frac{1}{2v(v\tau)^2}(\Delta \nabla \cdot \omega)\left(\frac{x}{v\tau}\right).$$

By Lemma 3.5(1) and (2) the first term is non-negative and the (ij)-indexed term in the summation is supported in the region where $|x^{(ij)}| \geqq r_1 v\tau$ ($> R_0$ by (3.4)). Here V_{ij} is smooth and the estimate $\left|\tau\omega\left(\frac{x}{v\tau}\right) \cdot \nabla V_{ij}\right| \leqq C\tau^{-\varepsilon_0}\left|\omega\left(\frac{x}{v\tau}\right)\right|$ holds (by Assumption 3.1(2)).

Thus

$$\sum_a f_2(H)\chi(\tau)B_a(\tau)\chi(\tau)(D\tilde{A}(\tau))\chi(\tau)B_a(\tau)\chi(\tau)f_2(H) \geqq O(\tau^{-\varepsilon_0}). \tag{3.26}$$

As for the factor $\chi(\tau)$:

$$D\chi(\tau) = \chi'(\cdot)\frac{x}{2\tau^2}p + p\frac{x}{2\tau^2}\chi'(\cdot) - \chi'(\cdot)\frac{x^2}{2\tau^3}, \quad \chi'(\cdot) = \frac{d}{dy}\chi(y < -1)_{|y = (x^2/4\tau^2) - E''}.$$

To treat the contributions to $f_2(H)DA(\tau)f_2(H)$ from terms containing such factors we note that $\chi'(\cdot) = \chi_1(\cdot)^2\chi'(\cdot)$, where $\chi_1(\cdot) = \chi\left(E' - \dfrac{x^2}{4\tau^2} < -1\right)$, $E' = E'' - 4$. We pull one factor $\chi_1(\cdot)$ to the left and the other factor to the right, and obtain (using Lemma 2.11(1)) that the form of such terms is

$$\chi_1(\cdot)O(1)\chi_1(\cdot)+O(\tau^{-1}). \tag{3.27}$$

Clearly the latter satisfies (2.5). The former (2.6) since cf. Lemma 2.7(1) we can write

$$\chi_1(\cdot)\tilde{g}(A(\tau),\tau)=\sum_{m=0}^{n-1}\tilde{g}^{(m)}(A(\tau),\tau)(m!)^{-1}\,\mathrm{ad}^m_{A(\tau)}(\chi_1(\cdot))$$
$$+\text{ remainder},\quad \tilde{g}=(g^{(1)})^{1/2}.$$

For n large enough the right-hand side is of the form

$$O(\tau^{(\alpha-1)/2(1+\kappa_0)})\chi\left(E'-\frac{x^2}{4\tau^2}<-\frac{1}{2}\right)+O(\tau^{-1})$$

by an application of (3.23) with $h=\chi_1(\cdot)$. (We use that $\|\tilde{g}^{(m)}(A(\tau),\tau)\|\leqq C\tau^{(\alpha-1)/2(1+\kappa_0)}$ which in turn follows by (3.7).) But by Theorem 3.4 (with B as introduced in the beginning of Example 4, for E' large enough and f supported in a small neighbourhood of E)

$$\tau^{(\alpha-1)/2(1+\kappa_0)}\chi\left(E'-\frac{x^2}{4\tau^2}<-\frac{1}{2}\right)e^{-itH}f(H)B^{-\alpha/2}=O(\tau^{-(1/2)(1+\kappa_0)}). \tag{3.28}$$

This completes the discussion of terms of the form (3.27).

It remains to consider

$$R:=f_2(H)\left\{\sum_a\chi(\tau)(DB_a(\tau))\chi(\tau)\tilde{A}(\tau)\chi(\tau)B_a(\tau)\chi(\tau)\right.$$
$$\left.+\sum_a\chi(\tau)B_a(\tau)\chi(\tau)\tilde{A}(\tau)\chi(\tau)(DB_a(\tau))\chi(\tau)\right\}f_2(H).$$

We use (3.9) and (3.10) in computing

$$R=f_2(H)\sum_a\chi(\tau)R_a(\tau)\chi(\tau)\tilde{A}(\tau)\chi(\tau)^2f_2(H)+O(\tau^{-\varepsilon_0}),$$

$$R_a(\tau)=I_a(\tau)^2D\bar{j}_a(\tau)^4+\bar{j}_a(\tau)^4\frac{d}{dt}I_a(\tau)^2$$

$$=D\bar{j}_a(\tau)^4-\{I-I_a(\tau)^2\}D\bar{j}_a(\tau)^4+\bar{j}_a(\tau)^4\frac{d}{dt}\{I_a(\tau)^2-I\}. \tag{3.29}$$

Terms of the first type do not contribute to the summation on the right-hand side of (3.29), since

$$\sum_a D\bar{j}_a(\tau)^4=D\sum_a\bar{j}_a(\tau)^4=0.$$

Because $I-I_a(\tau)^2=\{I+I_a(\tau)\}(p_a)^2(\tau^{2\kappa_0}+(p_a)^2)^{-1}$ the remaining terms contribute with an operator which is $O(\tau^{-2\kappa_0})$. Putting together

$$R=O(\tau^{-2\kappa_0})+O(\tau^{-\varepsilon_0}). \tag{3.30}$$

By (3.26), (3.30) and the statements following (3.27) $q(\beta_0, \alpha_0, \delta)$, $\delta = \min\{\varepsilon_0, 2\kappa_0\}$, holds.

We conclude that the conclusion of Theorem 2.4 holds.

In particular (to be used in Sect. 4) for any $E' \geqq E$ sufficiently large and f supported in a small neighbourhood of E (so that (3.28) holds), and for any $s \geqq 0$ and $\varepsilon > 0$,

$$\left(\frac{-A(\tau)}{\tau}\right)^{1/2} \chi\left(\frac{A(\tau)}{\tau} < -\varepsilon\right) e^{-itH} f(H) \langle x \rangle^{-s(1+\kappa_0)} = O(\tau^{-s}) \quad \text{for} \quad \tau \to \infty. \quad (3.31)$$

4. Free Channel Propagation Estimates

In this section we shall apply the results of Sect. 3 to obtain two propagation estimates (stated as Theorems 4.4 and 4.5) for the free channel, by which we mean certain estimates involving decoupling operators χ_{fr} defined as follows:

Consider $C^\infty(X)$-functions $\chi_{fr}(\cdot)$, homogeneous of degree zero outside the unitsphere in X and satisfying the support condition

$$\operatorname{supp} \chi_{fr}(\cdot) \cap \bigcup_{\substack{a \neq \\ (1)\cdots(N)}} X_a \neq \emptyset. \quad (4.1)$$

Clearly (with dist = distance)

$$\delta := \operatorname{dist}\left\{\operatorname{supp} \chi_{fr}(\cdot), \bigcup_{\substack{a \neq \\ (1)\cdots(N)}} X_a\right\} > 0. \quad (4.2)$$

With R_0 given in Assumption 3.1(2) and $\chi_{fr}(\cdot)$ and δ as above, χ_{fr} and $\tilde{\chi}_{fr}$ are the operators given by multiplication by $\chi_{fr}(x)$ respectively

$$\tilde{\chi}_{fr}(x) = \chi\left(-|x| + \frac{R_0}{\delta} < -1\right) \chi_{fr}(x). \quad (4.3)$$

We notice that (by Assumption 3.1(2)) $V(x)\tilde{\chi}_{fr}(x)$ is smooth and satisfies

$$\partial_x^\alpha \{V(x)\tilde{\chi}_{fr}(x)\} = O(|x|^{-|\alpha|-\varepsilon_0}), \quad \forall \alpha. \quad (4.4)$$

We introduce the following class of pseudodifferential operators (Ps.D.Op.s): Let S_l^m be the symbol class of $C^\infty(X \times X)$-functions $p(x, \xi)$ with

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leqq C_{\alpha,\beta} \langle x \rangle^{l-|\alpha|} \langle \xi \rangle^m, \quad \forall x, \xi \in X \quad \forall \alpha, \beta; m, l \in \mathbf{R}.$$

The corresponding class of Ps.D.Op.s defined by

$$(P(X, D)\psi)(x) = (2\pi)^{-v(N-1)} \iint e^{i(x-y)\cdot\xi} p(x, \xi) \psi(y)\, dy\, d\xi$$

will be denoted by $\bar{S}_l^m$.

If $p(x, \xi) \in S_l^m$ is supported away from an interval $\Delta \subset \mathbf{R}^+$ in the sense that $p(x, \xi) = 0$ if $\xi^2 \in \Delta$, then by convention $p(x, \xi) \in S_l^m(\Delta^c)$ and $P(X, D) \in \bar{S}_l^m(\Delta^c)$. Moreover $\bar{S}_\infty^\infty(\Delta^c) := \bigcup_l \bigcup_m \bar{S}_l^m(\Delta^c)$.

As for the calculus of the above Ps.D.Op.s we refer to [H–S1] and [K]. It will be used without further references in the following.

We will need the following extension of [H–S1, Lemma 3.3].

Lemma 4.1. *Let $\Delta\subset\mathbf{R}^+$ a compact interval, $f\in C_0^\infty(\dot{\Delta})$ ($\dot{\Delta}=$ the interior of Δ), $s\in\mathbf{R}$, $P(X,D)\in\bar{S}_\infty^\infty(\Delta^c)$ and $\tilde{\chi}_{fr}(x)$ of the form (4.3) be given.*
Then

$$P(X,D)\tilde{\chi}_{fr}f(H)\langle x\rangle^s\in\mathscr{B}(\mathscr{H}).$$

Idea of proof. We will only sketch the proof since it goes like the one of [H–S1, Lemma 3.3].

For a suitable closed curve Γ in $\mathbf{C}$ around $\operatorname{supp} f$ and intersecting $\mathbf{R}$ in $\dot{\Delta}$ we write

$$P(X,D)\tilde{\chi}_{fr}f(H)=\frac{1}{2\pi i}\int_\Gamma dz\,P(X,D)\{R_0(z)\tilde{\chi}_{fr}-\tilde{\chi}_{fr}R(z)\}f(H),$$

where $H_0=p^2$, $R_0(z)=(H_0-z)^{-1}$ and $R(z)=(H-z)^{-1}$.
But

$$\begin{aligned}R_0(z)\tilde{\chi}_{fr}-\tilde{\chi}_{fr}R(z)&=R_0(z)\{\tilde{\chi}_{fr}H-H_0\tilde{\chi}_{fr}\}R(z)\\&=R_0(z)\{i(p\cdot(\nabla\tilde{\chi}_{fr})(\cdot)+(\nabla\tilde{\chi}_{fr})(\cdot)\cdot p)+V\tilde{\chi}_{fr}\}R(z).\end{aligned}$$

Using the above facts, the calculus and (4.4) we obtain that

$$P(X,D)\tilde{\chi}_{fr}f(H)=\int_\Gamma dz\{G_1(X,D,z)+\tilde{G}_1(X,D,z)\}\tilde{\chi}_{fr,1}R(z)f(H),$$

where $\tilde{\chi}_{fr,1}(x)=1$ on $\operatorname{supp}\tilde{\chi}_{fr}(\cdot)$ and given with only slightly larger support, $\tilde{G}_1(X,D,z)\in\bar{S}^m_{-s}$ for some m and $G_1(X,D,z)\in\bar{S}_\infty^\infty(\Delta^c)$ with the x-decay of the symbol improved by a factor $\langle x\rangle^{-\varepsilon_0}$.

Now we iterate the arguments to obtain the total x-decay to the power $-s$. □

Lemma 4.2. *With the conditions of Lemma 4.1 and for any $0<s<s'$,*

$$P(X,D)\tilde{\chi}_{fr}e^{-itH}f(H)\langle x\rangle^{-s'}=O(t^{-s})\quad\textit{for}\quad t\to+\infty.$$

This result is a consequence of Lemma 4.1 and the following lemma, which in turn follows from Theorem 3.3 and a covering argument.

Lemma 4.3. *With f as above and for any $0<s<s'$,*

$$\langle x\rangle^{-s}e^{-itH}f(H)\langle x\rangle^{-s'}=O(t^{-s})\quad\textit{for}\quad t\to+\infty.$$

Theorem 4.4. *Suppose $P_-(X,D)\in\bar{S}_0^0$ and that $\operatorname{supp}p_-\subset\{(x,\xi)|x\cdot\xi<(1-\varepsilon_1)|x||\xi|\}$ for some $\varepsilon_1>0$. Then for any decoupling operator χ_{fr}, $f\in C_0^\infty(\mathbf{R}^+)$ and $0<s<s'$,*

$$P_-(X,D)\chi_{fr}e^{-itH}f(H)\langle x\rangle^{-s'}=O(t^{-s})\quad\textit{for}\quad t\to+\infty.$$

Proof. Let $\varepsilon_1>0$, χ_{fr}, $0<s<s'$, $0<E$ and $\xi_0\in X$ with $|\xi_0|^2=E$ be given.

By covering arguments it suffices to find neighbourhoods N_E of E and N_{ξ_0} of ξ_0, respectively, such that the estimate holds for any $f\in C_0^\infty(\mathbf{R}^+)$ and $P_-(X,D)\in\bar{S}_0^0$ with the properties: $\operatorname{supp}f\subset N_E$, $\operatorname{supp}p_-\subset\{(x,\xi)|x\cdot\xi<(1-\varepsilon_1)|x||\xi|\}$ and $p_-(x,\xi)=0$ for $\xi\notin N_{\xi_0}$. Here we use Lemmas 4.2 and 4.3.

For that it is sufficient (by the calculus and Lemma 4.3) to prove the estimate for $P_-(X,D)$ having a certain product form to be specified below.

Let $\chi_1 = \chi_1(X)$ be a decoupling operator as χ_{fr} but with the additional property

$$\operatorname{supp} \chi_1(\cdot) \subset \left\{ x \,|\, x \cdot \xi_0 < \left(1 - \frac{\varepsilon_1}{2}\right) |x| \, |\xi_0| \right\}, \tag{4.5}$$

and let δ be the corresponding positive number given by (4.2).

We introduce $r_1, r_2 > 0, \omega$ and $\{\tilde{j}_a\}$ in accordance with Lemma 3.2 with $k = 2$.

Put $v = (2r_2)^{-1} \delta E^{1/2}$, and let t_0 be chosen in accordance with (3.4), and $\kappa_0 > 0$ with $s(1 + \kappa_0) = s'$.

Corresponding to the inputs E given above and $\varepsilon = \frac{1}{3}$ in Theorem 3.4 we can find a neighbourhood N_E^1 of E and $E' \geqq E$, such that the estimates of the theorem hold for E' and for any f with $\operatorname{supp} f \subset N_E^1$.

With this E' (and the other quantities introduced above) we define $A(\tau)$ and $\tilde{A}(\tau)$ by (3.5). Clearly (3.28) and (3.31) hold for any f with $\operatorname{supp} f \subset N_E^1$.

We choose $8^{-1}E > \varepsilon > 0$ so small that with $d = (4v)^{-1} E \varepsilon_1 \left(1 - \frac{\varepsilon_1}{4}\right)$,

$$\frac{\varepsilon_1}{8} E < (E - 6\varepsilon)\left(1 - \frac{\varepsilon_1}{4}\right) - (E - 6\varepsilon)^{1/2} \left(1 - \frac{\varepsilon_1}{2}\right) E^{1/2}, \tag{4.6}$$

and

$$\sup_{|x| < 2v^{-1}(E' + 2)^{1/2}} |\omega(x)| \, 3\varepsilon < \frac{d}{6}. \tag{4.7}$$

Corresponding to the inputs E and ε (above) in Theorem 3.3 we can find a neighbourhood N_E^2 of E, such that the estimates of the theorem hold for any f with $\operatorname{supp} f \subset N_E^2$. Put $N_E = N_E^1 \cap N_E^2$.

With $s, s', \chi_{fr}, \chi_1, \varepsilon, \xi_0$ and N_E as above we shall prove the estimate of Theorem 4.4 with $P_-(X, D) = \chi_1(X)\chi_2(D)$, where $\chi_2(\xi)$ is any smooth function with

$$\operatorname{supp} \chi_2(\cdot) \subset B_\varepsilon(\xi_0) = \{\xi \,\|\, \xi - \xi_0| \leqq \varepsilon\},$$

and for any f with $\operatorname{supp} f \subset N_E$.

Explicitly we shall prove for $\phi \in \mathscr{H}$ and with $\psi(t) = e^{-itH} f(H) \langle x \rangle^{-s'} \phi$, $\chi_2 = \chi_2(D)$ and $\psi_1(t) = \chi_1 \chi_2 \chi_{fr} \psi(t)$ that

$$\|\psi_1(t)\| \leqq C t^{-s} \|\phi\|. \tag{4.8}$$

For that let

$$B(t) = \left(I - \chi\left(\frac{x^2}{4t^2} - E < -3\varepsilon\right)\right)\left(I - \chi\left(E' - \frac{x^2}{4t^2} < -1\right)\right).$$

We claim that

$$\|\psi_1(t)\|^2 \leqq 2\|B(t)\psi_1(t)\|^2 + C t^{-2s} \|\phi\|^2. \tag{4.9}$$

This estimate is verified by writing $(I - B(t))\chi_2$ as a sum of three terms. The contributions from two of these can be handled by using the identity

$$\chi\left(\frac{x^2}{4t^2} - E < -3\varepsilon\right)\chi_2 = \chi_2 \chi\left(\frac{x^2}{4t^2} - E < -3\varepsilon\right)$$

$$+\left[\chi\left(\frac{x^2}{4t^2}-E<-3\varepsilon\right),\chi_2\right]\chi\left(\frac{x^2}{4t^2}-E<-\varepsilon\right)$$
$$+\left[\chi\left(\frac{x^2}{4t^2}-E<-3\varepsilon\right),\chi_2\right]\left(I-\chi\left(\frac{x^2}{4t^2}-E<-\varepsilon\right)\right).$$

As for the ones from the first two terms on the right-hand side we use our assumptions (cf. Theorem 3.3), and as for the contribution from the last term we notice that the operator is $O(t^{-s})$ (by the calculus of Ps.D.Op.s). A similar argument works for the third term $\chi\left(E'-\frac{x^2}{4t^2}<-1\right)\chi_2$.

In order to estimate the first term on the right-hand side of (4.9) it is noticed that on the support of $\left(1-\chi\left(\frac{x^2}{4t^2}-E<-3\varepsilon\right)\right)\chi_1(x)$, $\frac{x^2}{4t^2}-E\geqq-6\varepsilon$, and hence $\frac{|x|}{t}\geqq 2(E-6\varepsilon)^{1/2}>E^{1/2}$. Thus on this support, for any $t>E^{-1/2}$ and $a\neq(1)\cdots(N)$, $|x|>1$ and

$$\left|\frac{x^a}{vt}\right|=\left|\frac{x^a}{|x|}\right|\frac{|x|}{vt}>\frac{\delta}{v}E^{1/2}.$$

In particular by our choice of v for all x and a as above and $t>\max\{E^{-1/2},t_0\}$ (and with $\tau=t+t_0$)

$$\left|\frac{x^a}{v\tau}\right|\geqq\frac{t}{\tau}2r_2>r_2. \tag{4.10}$$

By Lemma 3.2(5) and (4.10) we obtain that for $t>\max\{E^{-1/2},t_0\}$

$$\left(\omega\left(\frac{x}{v\tau}\right)-\frac{x}{v\tau}\right)\chi_1(x)B(t)=0. \tag{4.11}$$

Due to (4.5), (4.6), (4.9) and (4.11) the following estimate holds for all $t>\max\left\{E^{-1/2},t_0\frac{4}{\varepsilon_1}\left(1-\frac{\varepsilon_1}{4}\right)\right\}$ (and with d as defined above)

$$\|\psi_1(t)\|^2\leqq\frac{2}{d}\left\langle B(t)\psi_1(t),-\omega\left(\frac{x}{v\tau}\right)\cdot\left(\xi_0-\frac{x}{2\tau}\right)B(t)\psi_1(t)\right\rangle+Ct^{-2s}\|\phi\|^2. \tag{4.12}$$

$\left(\text{Notice that the constraint on } t \text{ implies that } \frac{t}{\tau}>1-\frac{\varepsilon_1}{4}.\right)$

The next step is to replace ξ_0 on the right-hand side of (4.12) by p. For that pick $\chi_3(\xi)\in C^\infty(X)$ such that $\operatorname{supp}\chi_3\subset B_{3\varepsilon}(\xi_0)$ and $\chi_3(\xi)=1$ on $B_{2\varepsilon}(\xi_0)$. Then

$$(I-\chi_3(D))B(t)\chi_1\chi_2=O(t^{-2s}). \tag{4.13}$$

By applying (4.13) twice

$$\|\psi_1(t)\|^2\leqq-\frac{2}{d}\left\langle B(t)\psi_1(t),\frac{\tilde{A}(\tau)}{\tau}B(t)\psi_1(t)\right\rangle$$
$$+\frac{2}{d}\left\|B(t)\omega\left(\frac{x}{v\tau}\right)\cdot(p-\xi_0)\chi_3(D)B(t)\right\|\|\psi_1(t)\|^2+Ct^{-2s}\|\phi\|^2. \tag{4.14}$$

But by (4.7)

$$\left\| B(t)\omega\left(\frac{x}{v\tau}\right)\cdot(p-\xi_0)\chi_3(D)B(t)\right\| \leqq \sup_{x^2/4t^2<E'+2}\left|\omega\left(\frac{x}{v\tau}\right)\right| \|(p-\xi_0)\chi_3(D)\| \leqq \frac{d}{6}. \tag{4.15}$$

As for the first term on the right-hand side of (4.14), we notice that (by using (4.13) again)

$$-\left\langle B\psi_1(t), \frac{\tilde{A}(\tau)}{\tau}B\psi_1(t)\right\rangle \leqq -\left\langle B\psi_1(t), \frac{A(\tau)}{\tau}B\psi_1(t)\right\rangle + C_1 t^{-2s}\|\phi\|^2 + C_2 t^{-\min\{2\kappa_0,1\}}\|\psi_1(t)\|^2. \tag{4.16}$$

We choose $t'>0$ such that

$$C_2(t')^{-\min\{2\kappa_0,1\}} < \frac{d}{6}. \tag{4.17}$$

We obtain by (4.14)–(4.17) and a subtraction that for

$$t > \max\left\{E^{-1/2}, t_0\frac{4}{\varepsilon_1}\left(1-\frac{\varepsilon_1}{4}\right), t'\right\}$$

$$\frac{1}{3}\|\psi_1(t)\|^2 \leqq -\frac{2}{d}\left\langle B(t)\psi_1(t), \frac{A(\tau)}{\tau}B(t)\psi_1(t)\right\rangle + Ct^{-2s}\|\phi\|^2. \tag{4.18}$$

Putting $\varepsilon' = \dfrac{d}{24}$ we get from (4.18) and another subtraction, that for all $t>0$,

$$\frac{1}{6}\|\psi_1(t)\|^2 \leqq -\frac{2}{d}\left\langle B(t)\psi_1(t), \frac{A(\tau)}{\tau}\chi^2\left(\frac{A(\tau)}{\tau} < -\varepsilon'\right)B(t)\psi_1(t)\right\rangle + Ct^{-2s}\|\phi\|^2. \tag{4.19}$$

Next we decompose

$$B(t)\psi_1(t) = B(t)\chi_1\chi_2\chi\left(\frac{x^2}{4t^2}-E<-\varepsilon\right)\chi_{fr}\psi(t) + B(t)\chi_1\chi_2\left(1-\chi\left(\frac{x^2}{4t^2}-E<-\varepsilon\right)\right)\chi_{fr}\psi(t).$$

The norm of the first term is bounded by $Ct^{-s}\|\phi\|$, as is the case when multiplied by p.

Using these facts together with (4.19) we obtain the estimate

$$\|\psi_1(t)\|^2 \leqq C_1\|g_\tau(A(\tau))P(t)\psi(t)\|^2 + C_2 t^{-2s}\|\phi\|^2, \tag{4.20}$$

where $g_\tau(\cdot) = \left(-\dfrac{\cdot}{\tau}\right)^{1/2}\chi\left(\dfrac{\cdot}{\tau} < -\varepsilon'\right)$ and $P(t) = h_1\chi_2 h_2$ with h_1 and h_2 smooth functions in x and τ and satisfying (3.12).

By an extension of Lemma 2.7(2) for any $n\in\mathbf{N}$,

$$g_\tau(A(\tau))P(t) = \sum_{m=0}^{n-1}\mathrm{ad}^m_{A(\tau)}(P(t))\frac{(-1)^m}{m!}g_\tau^{(m)}(A(\tau)) + \text{remainder}. \tag{4.21}$$

Since by (3.24) and (3.25) $\mathrm{ad}^m_{A(\tau)}(P(t)) = O(1)$ for $\tau \to \infty$ (for any m), we finally conclude from (3.31), (4.20) and (4.21) that (4.8) holds.

Theorem 4.5 (*maximal velocity estimate*). *Let $E, \varepsilon > 0$. Then for any $f \in C_0^\infty(\mathbf{R})$ supported in a small neighbourhood of E, decoupling operator χ_{fr} and $s' > s \geqq l \geqq 0$,*

$$\langle x \rangle^l \chi\left(E - \frac{x^2}{4t^2} < -\varepsilon\right) \chi_{fr} e^{-itH} f(H) \langle x \rangle^{-s'} = O(t^{-s+l}) \quad \textit{for} \quad t \to +\infty.$$

Idea of proof. We will only give a brief outline of the proof since it is very similar to the one of Theorem 4.4.

By Theorem 3.4 it can be assumed that $l = 0$.

As in the proof of Theorem 4.4 we define κ_0 by $s(1+\kappa_0) = s'$, and introduce cutoff functions $\chi_1(x)$ and $\chi_2(\xi)$ with similar properties except for (4.5), and the operator $A(\tau)$ of Example 4 in Sect. 3. Then we derive an estimate of the type (4.18) but with

$$B(t) = \left(I - \chi\left(E' - \frac{x^2}{4t^2} < -1\right)\right) \quad \text{and} \quad \psi_1(t) = \chi\left(E - \frac{x^2}{4t^2} < -\varepsilon\right)\chi_1(X)\chi_2(D)\chi_{fr}\psi(t),$$

and apply it together with (3.31) (as in the proof of Theorem 4.4). □

Appendix A

We shall prove Lemma 3.2(9).

By the construction of $\omega(x)$ in [G],

$$\partial_x^\alpha(\omega(x) - x) = \sum_a \int_{\bigcap_{i=0}^{m^a} \Omega_i^a} (\partial_x^\alpha \psi)(x-y) y^a \, dy,$$

where $\psi \in C_0^\infty(X)$, for some positive numbers $c_0^a, c_1^a, \ldots, c_{m^a}^a$ and orthogonal projections $P_1^a, \ldots, P_{m^a}^a$, $\Omega_0^a = \{y \,|\, |\Pi^a y| \leqq c_0^a\}$ and Ω_i^a, $1 \leqq i \leqq m^a$, are given by either $\{y \,|\, |P_i^a y| \leqq c_i^a\}$ or $\{y \,|\, |P_i^a y| > c_i^a\}$.

We are thus led to proving

Lemma A. *Let $\phi \in C_0^\infty(X)$, $F \in C^\infty(X, X)$, $P_0, \ldots, P_m$ be orthogonal projections on X, $c_0, \ldots, c_m$ be positive numbers, $\Omega_i = \{y \,|\, |P_i y| \leqq c_i\}$ for $i = 0, \ldots, m$, and $\Omega = \bigcap_{i=0}^m \Omega_i$. Then for any $n \in \mathbf{N} \cup \{0\}$,*

$$\sup_{x \in X} \left| \int_\Omega F(P_0 y)(x \cdot \nabla_y)^n \phi(x-y) \, dy \right| < \infty.$$

Proof. Let $P = P_0 \vee \cdots \vee P_m$, the orthogonal projection onto the span of vectors in the ranges of P_i, and $P^\perp = I - P$. Then since Ω is invariant with respect to translation with vectors in the ranges of $P^\perp$, a change of variables shows that

$$\int_\Omega F(P_0 y)(-x \cdot \nabla_y)^n \phi(x-y) \, dy = \int_\Omega \left(\frac{d}{dt}\right)^n \{F(P_0(tP^\perp x + y))\phi(x + tPx - y)\}_{|t=0} \, dy.$$

The norm of the right-hand side is uniformly bounded since $P_0 P^\perp = 0$ and the integral is zero for Px outside a bounded set. The latter statement follows from the fact that $P\Omega$ is bounded, which in turn follows by a repeated application of a formula in [H] (expressing for instance $P_0 \vee P_1$ analytically in terms of P_0 and P_1). $\square$

Appendix B

We shall prove Mourre-type estimates for the operator $A_R := \dfrac{R\omega(x/R)p + p\omega(x/R)R}{2}$, where ω is given by setting $k=2$ in Lemma 3.2 and $R>0$ is large (cf. Example 1 in Sect. 3).

Lemma B1. *For any cluster decomposition $b \neq (1)\cdots(N)$ there exist $r_1, r_2 > 0$ and $C^\infty(X^b)$-partition of unity $\{\tilde{j}^b_{b_1}\}_{b_1 \subset b}$ such that*

(1) $|x^{b_1}| > 2r_1$ *on* $\operatorname{supp} \tilde{j}^b_{b_2}$ *if* $b_1 \not\subset b_2$,
(2) $|x^{b_1}| < r_2$ *on* $\operatorname{supp} \tilde{j}^b_{b_1}$,
(3) *For any* $\alpha \exists C > 0 : |\partial^\alpha_x \tilde{j}^b_{b_1}(x)| \leq C$.

Proof. If $b = (1\cdots N)$ the result is obtained from Lemma 3.2 by putting $\tilde{j}^b_{b_1} = \tilde{j}_{b_1}$. Otherwise we copy the proof (i.e. we introduce certain characteristic functions as in [G] however now involving only cluster decompositions contained in b, and then we smooth out by convolution).

Corresponding to the partition of unity $\{\tilde{j}_a\}$ of Lemma 3.2 let (cf. [G])

$$j_a = \tilde{j}_a \left(\sum_b \tilde{j}_b^2 \right)^{-1/2}.$$

For any cluster decompositions a and $b, b \subset a$ and $a \neq (1)\cdots(N)$, let

$$H^a_b = -\Delta + \sum_{(ij) \subset b} V_{ij}(\Pi^{(ij)} \cdot) \quad \text{on} \quad L^2(X^a).$$

We abbreviate $H^{(1\cdots N)}_b = H_b$ and $H^a_a = H^a$. By convention for $a = (1)\cdots(N)$ $H^a = 0$ on $L^2(X^a) = \mathbf{C}$.

Clearly for any a,

$$H_a = (p_a)^2 \otimes I + I \otimes H^a \quad \text{on} \quad L^2(X) = L^2(X_a) \otimes L^2(X^a).$$

As is well-known the continuous spectrum of H is given by

$$\sigma_c(H) = \left[\min_{a \neq (1\cdots N)} \sigma(H^a), \infty \right).$$

The set of thresholds is defined by

$$\mathcal{F} = \bigcup_{a \neq (1\cdots N)} \{\text{eigenvalues of } H^a\}.$$

It is known that $\mathcal{F} \cup \{\text{eigenvalues of } H\}$ is closed and countable, see [F–H1] (a fact which also follows directly and quite easy from the proof given below).

Theorem B2. *Let $E \in \sigma_c(H) \backslash \mathcal{F} \cup \{\text{eigenvalues of } H\}$ and $d(E)$ be the distance from*

E to $\{E' \in \mathscr{F} \mid E' < E\}$. Then for any given $\varepsilon > 0$ and (sufficiently) large R, the following estimate holds for any $f \in C_0^\infty(\mathbf{R})$ supported in a small neighbourhood of E:

$$f(H)i[H, A_R]f(H) \geqq (2d(E) - \varepsilon)f(H)^2.$$

Proof. For purely notational convenience we assume that $E > 0$. Let $\varepsilon > 0$ be given.

By combining Lemma 3.2(1) and (6) we obtain (cf. [G])

$$\omega_*\left(\frac{x}{R}\right) \geqq \sum_a \Pi_a j_a^2\left(\frac{x}{2R}\right).$$

Using this estimate together with Lemma 3.2(2) and (9) and a straightforward computation leads to

$$f(H)i[H, A_R]f(H) \geqq f(H)\left\{\sum_a j_a\left(\frac{x}{2R}\right)2(p_a)^2 j_a\left(\frac{x}{2R}\right) + O(R^{-\varepsilon_0})\right\}f(H),$$

valid for any $f \in C_0^\infty(\mathbf{R})$.

Introducing the abbreviation

$$B_{a,R} = j_a\left(\frac{x}{2R}\right)F\left((p_a)^2 < E - \frac{\varepsilon}{3}\right)j_a\left(\frac{x}{2R}\right),$$

where the second operator on the right-hand side is the spectral operator associated with $(p_a)^2$ and the characteristic function of the interval $\left(-\infty, E - \frac{\varepsilon}{3}\right)$, we conclude that for R large enough

$$f(H)i[H, A_R]f(H) \geqq (2E - \varepsilon)f(H)^2 + \frac{\varepsilon}{4}f(H)^2 - 2E\sum_a f(H)B_{a,R}f(H).$$

Hence the theorem follows if for any given $C > 0$, cluster decomposition a and large R we can prove that for any $f \in C_0^\infty(\mathbf{R})$ supported in a small neighbourhood of E

$$C^{-1}f(H)^2 \geqq f(H)B_{a,R}f(H). \tag{B.1}$$

So let a cluster decomposition a and $C > 0$ be given. Then there exists a $g \in C_0^\infty(\mathbf{R})$ supported in the region $x > \frac{\varepsilon}{9}$ such that for all $b \supset a$ and $f_1 \in C_0^\infty(\mathbf{R})$ supported in the interval $B_{\varepsilon/9}(E)$ we have that

$$f_1(H_b)B_{a,R}f(H) = f_1(H_b)\{g(H^b)B_{a,R} + O(R^{-1})\}f(H), \tag{B.2}$$

where we suppress a tensor symbol on the right-hand side (as also will be done in the following). Notice that (by the Fourier transform) for suitable g as above $f_1(H_b)(I - g(H^b))F((p_b)^2 < E - \frac{3}{10}\varepsilon) = 0$ and that $(I - F((p_b)^2 < E - \frac{3}{10}\varepsilon))B_{a,R} = O(R^{-1})$.

We also remark (for another future application) that (suppressing the Fourier transform)

$$f_1(H_b)g(H^b) = \int \oplus\, d\xi_b f_1((\xi_b)^2 + H^b)g(H^b). \tag{B.3}$$

We will prove (B.1) by first expanding the right-hand side into a sum of terms indexed by decreasing strings of cluster decompositions: $(1\cdots N)=b_0 \supsetneqq b_1 \supsetneqq \cdots \supset a$. In doing so we go through a finite number of very similar steps (to be explained).

The expansion involves families of $C_0^\infty(\mathbf{R})$-functions $\{f_b\}$ indexed by $b\supset a$ and with the following properties:

$$|f_b|\leqq 1,\quad \exists 0<\varepsilon_1<\varepsilon_2<\cdots<\varepsilon_{\#a}<\frac{\varepsilon}{9},\quad 5\varepsilon_{j-1}<\varepsilon_j:$$

$$\operatorname{supp} f_b \subset B_{\varepsilon_{\#b}}(E),\quad f_b(x)=1 \text{ on } B_{1/2\varepsilon_{\#b}}(E).$$

With this order of scale in mind we perform the first step in expanding the right-hand side of (B.1): Pick $0<\delta<1$. Then for any f supported in $B_{1/2\varepsilon_1}(E)$,

$$f(H)B_{a,R}f(H)=f(H)f_{b_0}(H)Kf(H) + f(H)f_{b_0}(H)\sum_{b_0 \supsetneqq b_1 \supset a} \tilde{j}_{b_1}\left(\frac{x}{|x|^\delta}\right) f_{b_1}(H_{b_1})B_{a,R}f(H), \tag{B.4}$$

where K is compact and independent of f_{b_0}. Here we use that $\tilde{j}_b\left(\frac{x}{|x|^\delta}\right)j_a\left(\frac{x}{2R}\right)$ is compactly supported if either $b=b_0$ or $b\not\supset a$ (by Lemma B1), and the fact that for a suitable curve Γ independent of f_{b_0} (but not of f_{b_1})

$$f_{b_0}(H)\tilde{j}_{b_1}\left(\frac{x}{|x|^\delta}\right)(I-f_{b_1}(H_{b_1}))$$

$$=\frac{1}{2\pi i}\int_\Gamma dz f_{b_0}(H)\{\cdots\}(I-f_{b_1}(H_{b_1})),$$

$$\text{with}\quad \{\cdots\}=\left\{(H-z)^{-1}\tilde{j}_{b_1}\left(\frac{x}{|x|^\delta}\right)-\tilde{j}_{b_1}\left(\frac{x}{|x|^\delta}\right)(H_{b_1}-z)^{-1}\right\}. \tag{B.5}$$

The form is (by Lemma B1 and a computation similar to the one in the proof of Lemma 4.1) $f_{b_0}(H)K$, where K is as above.

By combining (B.2) and (B.4) we arrive at the statement

$$f(H)B_{a,R}f(H)=f(H)\{f_{b_0}(H)K+O(R^{-1})\}f(H) + f(H)\sum_{b_0 \supsetneqq b_1 \supset a} B_{b_1}f_{b_1}(H_{b_1})g(H^{b_1})B_{a,R}f(H),$$

with B_{b_1} bounded, uniformly with respect to f_b, $b\supset a$, K compact and independent of f_{b_0}, and finally the estimate $O(R^{-1})$ uniform with respect to f_b, $b\supset a$.

Now we repeat this procedure by letting $f_{b_1}(H_{b_1})g(H^{b_1})$ play the role of $f_{b_0}(H)$ in writing as the first part of the second step,

$$f_{b_1}(H_{b_1})g(H^{b_1})B_{a,R}f(H)=f_{b_1}(H_{b_1})g(H^{b_1})K_{b_1}\tilde{B}_{b_1}f(H) + f_{b_1}(H_{b_1})g(H^{b_1})\sum_{b_1 \supsetneqq b_2 \supset a} \tilde{j}^{b_1}_{b_2}\left(\frac{x^{b_1}}{|x^{b_1}|^\delta}\right) f_{b_2}(H_{b_2})B_{a,R}f(H),$$

with $\tilde{B}_{b_1}$ bounded uniformly with respect to $\{f_b | \#b\leqq \#b_1\}$, and $K_{b_1}=I_{L^2(X_{b_1})}\otimes K^{b_1}$, where the second factor on the right-hand side is compact on $L^2(X^{b_1})$ and independent of $\{f_b | \#b \leqq \#b_1\}$. (This is true with $K^{b_1} = \tilde{g}(H^{b_1})\langle x^{b_1}\rangle^{-\delta\varepsilon_0}$, $\tilde{g} \in C_0^\infty(\mathbf{R})$ and $\tilde{g}g = g$.) Here we use Lemma B1 and an analogue to (B.5). The second part consists in invoking (B.2) (as before).

By continuing in this way (or rather by a simple induction) we obtain the following expansion valid for any f supported in $B_{1/2\varepsilon_1}(E)$:

$$\begin{aligned} &f(H)B_{a,R}f(H) \\ &\quad = f(H)\{f_{b_0}(H)K + O(R^{-1})\}f(H) \\ &\quad + f(H) \sum_{1 \leqq j \leqq \#a-1} \sum_{b_0 \supsetneqq b_1 \supsetneqq \cdots \supsetneqq b_j \supset a} B_{b_1,\ldots,b_j} f_{b_j}(H_{b_j}) g(H^{b_j}) K_{b_1,\ldots,b_j} \tilde{B}_{b_1,\ldots,b_j} f(H), \end{aligned} \tag{B.6}$$

where $B_{b_1,\ldots,b_j}$ and $\tilde{B}_{b_1,\ldots,b_j}$ are bounded, the former uniformly with respect to f_b, $b \supset a$, and the latter uniformly with respect to $\{f_b | \#b \leqq \#b_j\}$, $K_{b_1,\ldots,b_j} = I_{L^2(X_{b_j})} \otimes K^{b_1,\ldots,b_j}$, where the second factor on the right-hand side is compact on $L^2(X^{b_j})$ and independent of $\{f_b | \#b \leqq \#b_j\}$, K is compact and independent of f_{b_0}, and finally the estimate $O(R^{-1})$ is uniform with respect to f_b, $b \supset a$.

When multiplied (from the right) by a compact operator on $L^2(X^b)$ each fiber on the right-hand side of (B.3) goes to zero (in uniform topology) when the support of the function f_1 shrinks. This convergence is uniform with respect to ξ_b, which means that the integral goes to zero. (See [M1, p. 295] for a similar argument.) Hence the statement (B.1) follows from (B.6) by first choosing R large, then f_a with $\varepsilon_{\#a}$ small, then f_b with $\#b = \#a - 1$ and $\varepsilon_{\#b} \ll \varepsilon_{\#a}$, and so on, at last f_{b_0} with $\varepsilon_1 \ll \varepsilon_2$. □

Remark. One can also prove Theorem B2 along the lines of the proof of the usual Mourre estimate as given in [F–H1]. This was communicated to us by G. M. Graf. The crucial point of that proof is (as in the proof of the usual estimate) a formula relating the vector field to vector fields of subsystems, so that an induction argument applies. As for the proof of the usual estimate an analysis at thresholds is required. This was not the case for the proof given above.

References

[F–H1] Froese, R. G., Herbst, I.: A new proof of the Mourre estimate. Duke Math. J. **49** (4), 1075–1085 (1982)

[F–H2] Froese, R. G., Herbst, I.: Exponential bounds and absence of positive eigenvalues for N-body Schroedinger operators. Commun. Math. Phys. **87**, 429–447 (1982)

[G] Graf, G. M.: Asymptotic completeness for N-body short-range quantum systems: A new proof. ETH-preprint 89-50. Commun. Math. Phys. **132**, 73–101 (1990)

[H] Hill, R. N.: Proof that the H^- ion has only one bound state. Details and extension to finite nuclear mass. J. Math. Phys. **18** (12), 2316–2330 (1977)

[H–S1] Herbst, I., Skibsted, E.: Time-dependent approach to radiation conditions. Duke Math. J. **64** (1), (1991)

[H–S2] Herbst, I., Skibsted, E.: in preparation

[I] Isozaki, H.: Differentiability of generalized Fourier transforms associated with Schroedinger operators. J. Math. Kyoto Univ. **25** (4), 789–806 (1985)

[J] Jensen, A.: Propagation estimates for Schroedinger-type operators. Trans. Am. Math. Soc. **291** (1), (1985) [J–M–P] Jensen, A., Mourre, E., Perry, P.: Multiple commutator estimates and resolvent smoothness in quantum scattering theory. Ann. Inst. Henri Poincaré **41** (2), 207–225 (1984)

[K] Kitada, H.: Fourier integral operators with weighted symbols and micro-local resolvent estimates. J. Math. Soc. Jpn **39** (3), 455–476 (1987)

[M1] Mourre, E.: Operateurs conjugues et proprietes de propagation. Commun. Math. Phys. **91**, 279–300 (1983)

[M2] Mourre, E.: Absence of singular continuous spectrum for certain selfadjoint operators. Commun. Math. Phys. **78**, 391–408 (1981)

[R–S] Reed, M., Simon, B.: Methods of modern mathematical physics II. New York: Academic Press 1975

[S–S] Sigal, I. M., Soffer, A.: Local decay and propagation estimates for time-dependent and time-independent Hamiltonians. University of Princeton preprint 88

Communicated by B. Simon