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Separation of Phases at Low Temperatures in a One-Dimensional Continuous Gas

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Abstract. We show the existence of a phase separation at low temperatures in a one-dimensional one-component classical gas in the canonical ensemble with interaction *hard core* $-1/r^{\alpha}$, $1<\alpha\leqq 2$. This implies that for sufficiently low temperatures there are values of the chemical potential at which the pressure is not differentiable as a function of the chemical potential.

0. Introduction

Most of the results on phase transitions in continuous models are for phase separation in mixtures and, to the author's knowledge, there are no results on the existence of a phase transition in a one-component classical continuous gas, see however Israel [1]. Extending ideas developed in Johansson [2] we will prove that a one-dimensional continuous gas in the canonical ensemble with attractive pair-interaction $1/r^{\alpha}$, $1<\alpha\leqq 2$, and a hard core has a phase transition at sufficiently low temperatures.

In the proof we rewrite the partition function for the continuous model as an integral of partition functions for discrete models. These discrete models are similar to a one-dimensional lattice gas in the canonical ensemble.

In the first section we define the model and state our results. The second section contains the representation of the continuous model as an integral of discrete models, the definition of blocks, partitions, and the rearrangement procedure and the main steps in the energy-entropy argument. In Sect. 3 and 4 we prove the basic entropy and energy estimates.

Many arguments in this paper are similar to the corresponding arguments in Johansson [2], which we will refer to as [I].

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1. Preliminaries and Results

Consider N particles at positions $x_1, \ldots, x_N$ in $\Omega = [0, L]$, where $L \in Z^+$. The particles interact via the potential

$$V(r) = \begin{cases} +\infty & \text{for } 0 < r < 1 \\ -1/r^\alpha & \text{for } r \geqq 1, \end{cases}$$

where $\alpha > 1$. Without loss of generality we have put the hard-core radius equal to 1. As boundary conditions we let (L, ∞) be empty and we put fixed particles at $x_k = -(k+1)$, $k = 0, 1, \ldots$ in $(-\infty, 0)$. The total interaction energy is then

$$H(\underline{x}) = \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=-\infty, j \neq i}^{N} V(|x_i - x_j|),$$

where $\underline{x} = (x_1, \ldots, x_N) \in \Omega^N$. Since $H(\underline{x})$ is symmetric with respect to permutations of $x_1, \ldots, x_N$ we can restrict our attention to ordered configurations. Let

$$X = \{\underline{x} \in \Omega^N;\ x_1 > 0,\ x_N < L,\ x_{k+1} - x_k > 1,\ k = 1, \ldots, N-1\}$$

and define for $A \subseteqq X$,

$$Z(A) = \int_A e^{-\beta H(\underline{x})} dx_1 \ldots dx_N .$$

The configurational canonical probability measure for the ordered configurations is

$$P(A) = Z(A)/Z(X), \qquad A \subseteqq X. \tag{1.1}$$

The density, $d(\tau_1, \tau_2)(\underline{x})$, of the configuration $\underline{x}$ in the interval $[[\tau_1 L], [\tau_2 L] + 1)$, $0 \leqq \tau_1 < \tau_2 \leqq 1$, is the number of particles in $\underline{x}$ in this interval divided by the length of the interval. Here $[\cdot]$ denotes integer part. Let the asymptotic average density ϱ, $0 \leqq \varrho < 1$, be given and write $\Omega \to R^+$ for the thermodynamic limit $N, L \to \infty$, $N/L \to \varrho$. We can now define what it means for the gas to have a uniform/non-uniform density in the thermodynamic limit exactly as in [I].

For a given small $\delta > 0$ and given $\varrho > \delta$ we put

$$d_1 = (1-\delta)^{-1}(\varrho - \delta), \qquad d_2 = (1/2 - \delta)^{-1} \varrho . \tag{1.2}$$

The main theorem of this paper is

Theorem 1.3. *Assume that $1 < \alpha \leqq 2$ and $0 < \varrho < 1/2$. There exist positive constants K, ξ, β_0 depending only on α and ϱ, such that if $\beta > \beta_0$ and $\delta = K \exp(-\xi\beta)$, then for each $\varepsilon > 0$,*

$$\lim_{\Omega \to R^+} P\{\underline{x} \in X;\ d(\tau_1, \tau_2)(\underline{x}) \geqq 1/2 - 2\delta\} = 1$$

for any fixed τ_1, τ_2, $0 \leqq \tau_1 < \tau_2 \leqq d_1 - \varepsilon$ and

$$\lim_{\Omega \to R^+} P\{\underline{x} \in X;\ d(\tau_1, \tau_2)(\underline{x}) \leqq 2\delta\} = 1$$

for any fixed τ_1, τ_2, $d_2 + \varepsilon \leqq \tau_1 < \tau_2 \leqq 1$.

The constants in the theorem are such that $\delta < 1/16$ and $0 < d_1 < d_2 < 1$ when $\beta \geqq \beta_0$. This means that we have a non-uniform density in the thermodynamic limit. By an argument analogous to the corresponding one in [I] this implies

Corollary 1.4. *Let $1<\alpha\leqq 2$. Then if $\beta\geqq\beta_0$ there is a value of the chemical potential μ for which the pressure $p(\mu,\beta)$ is not differentiable as a function of μ.*

2. Proof of the Main Theorem

2.1. The Discrete Model

Let $\Lambda=\{0,\ldots,L-1\}$ and let K denote the set of all $\underline{n}\in\{0,1\}^Z$ such that $n_k=1$ if $k\leqq -1$, $n_k=0$ if $k\geqq L$ and

$$\sum_{i=0}^{L-1} n_i=N.$$

Given $\underline{n}\in K$ we define $\underline{p}(\underline{n})\in\Lambda^N$ by $n_{p_k(\underline{n})}=1, 0\leqq p_1(\underline{n})<\ldots<p_N(\underline{n})\leqq L-1$. For every $\underline{x}\in X$ we define $\underline{n}=\underline{n}(\underline{x})\in K$ by $n_{[x_k]}=1, k=1,\ldots,N, n_i=1$ if $i\leqq -1$ and all other n_i's are $=0$. We also define $\underline{s}=\underline{s}(\underline{x})\in[0,1)^Z$ by $s_{[x_k]}=x_k-[x_k]$, $k=1,\ldots,N$, and $s_i=0$ otherwise. Given $\underline{n}$ and $\underline{s}, \underline{x}$ is uniquely determined since

$$x_k=p_k(\underline{n})+s_{p_k(\underline{n})},\qquad k=1,\ldots,N,$$

and consequently the map $F:X\to K\times[0,1)^Z$ defined by $F(\underline{x})=(\underline{n}(\underline{x}),\underline{s}(\underline{x}))$ is injective. Let

$$T=\{\underline{t}\in[0,1]^N;\ t_1<\ldots<t_N\}$$

and $f_\sigma(\underline{t})=(t_{\sigma(1)},\ldots,t_{\sigma(N)})$ for $\sigma\in S_N$, the permutation group on $\{1,\ldots,N\}$.

Note that for each $\underline{x}\in X$ there are unique $\underline{t}=\underline{t}(\underline{x})\in T$ and $\sigma=\sigma(\underline{x})\in S_N$ such that $x_k-[x_k]=t_{\sigma(k)}$, $k=1,\ldots,N$. Given a subset $A\subseteqq X$ and a $\underline{t}\in T$ we write

$$A(\underline{t})=\{\underline{x}\in A;\ \underline{t}(\underline{x})=\underline{t}\}$$

and

$$Q(\underline{t},A)=F(A(\underline{t}))\subseteqq K\times[0,1)^Z.$$

If $F(\underline{x})=(\underline{n},\underline{s})$ then

$$H(\underline{x})=H(\underline{n},\underline{s})=-\frac{1}{2}\sum_{\substack{k\in\Lambda,\, l\in Z\\ k\neq l}}\frac{n_k n_l}{|k-l+s_k-s_l|^\alpha}.$$

Define

$$Z(\underline{t},A)=\sum_{(\underline{n},\underline{s})\in Q(\underline{t},A)} e^{-\beta H(\underline{n},\underline{s})}.$$

This defines our discrete model for a given $\underline{t}$. $Q(\underline{t},X)$ is always non-empty since $\underline{p}(\underline{n})+\underline{t}\in A(\underline{t})$ for any $\underline{t}\in T$ and any $\underline{n}\in K$. $Z(\underline{t},X)$ is the partition function for our discrete model. The next lemma says that our continuous model is an integral over these discrete models.

Lemma 2.1. *For each $A\subseteqq X$,*

$$Z(A)=\int_T Z(\underline{t},A)d^N t.$$

Proof. For $\underline{m}\in K$ we let

$$J(\underline{m})=\{\underline{x}-p(\underline{m});\ \underline{n}(\underline{x})=\underline{m} \text{ and } \underline{x}\in A\}.$$

Then $J(\underline{m})+p(\underline{m})$, $\underline{m}\in K$ are disjoint with union A. The sets $f_\sigma(T)$, $\sigma\in S_N$ are also disjoint with union $[0,1]^N$ apart from a set of measure zero. Put $I(\underline{m},\sigma)=f_\sigma^{-1}(J(\underline{m})\cap f_\sigma(T))$ a subset of T. Then

$$\begin{aligned}\int_A e^{-\beta H(\underline{x})}d^N x &= \sum_{\underline{m}\in K}\sum_{\sigma\in S_N}\int_{J(\underline{m})\cap f_\sigma(T)} e^{-\beta H(\underline{\tau}+p(\underline{m}))}d^N\tau \\ &=\int_T \sum_{\underline{m}\in K}\sum_{\sigma\in S_N} 1_{I(\underline{m},\sigma)}(\underline{t})e^{-\beta H(f_\sigma(\underline{t})+p(\underline{m}))}d^N t.\end{aligned} \tag{2.1}$$

For a given $\underline{t}\in T$ we define $G: A(\underline{t})\to K\times S_N$ by $\underline{x}\to(\underline{n}(\underline{x}),\sigma(\underline{x}))$. G is injective since $\underline{x}=p(\underline{n})+f_\sigma(\underline{t})$. Now

$$\begin{aligned}G(A(\underline{t}))&=\{(\underline{m},\sigma);\ p(\underline{m})+f_\sigma(\underline{t})\in A\}\\ &=\{(\underline{m},\sigma);\ \underline{t}\in I(\underline{m},\sigma)\}.\end{aligned}$$

Thus the integrand in (2.1) can be written as

$$\sum_{(\underline{m},\sigma)\in G(A(\underline{t}))} e^{-\beta H(p(\underline{m})+f_\sigma(\underline{t}))}=\sum_{\underline{x}\in A(\underline{t})} e^{-\beta H(\underline{x})}=Z(\underline{t},A). \quad \square$$

2.2. Definition of Blocks and Partitions

We now fix $\underline{t}\in T$ and take $Q=Q(\underline{t},X)$ as our configuration space. Let $0\leqq a<a'\leqq L$ be two integers and $(\underline{n},\underline{s})\in Q$ a configuration. Then

$$A=\langle a,a'\rangle=\{(n_a,n_{a+1},\ldots,n_{a'-1}),(s_a,s_{a+1},\ldots,s_{a'-1})\}$$

is called a *block* in $(\underline{n},\underline{s})$. A is an *o-block* if $n_a=1$, $n_{a'-2}=1$, and $n_{a'-1}=0$, and an *e-block* if $n_1=n_{a'-1}=0$. We also define $\langle-\infty,a\rangle$ and $\langle a,\infty\rangle$ in the obvious way. They are always an o-respectively an e-block. Two o-(e-)blocks $A=\langle a,a'\rangle$ and $B=\langle a',a''\rangle$ can be joined to a new o-(e-)block $AB=\langle a,a''\rangle$.

A set of integers $\gamma=\{a_1,\ldots,a_r\}$, $0\leqq a_1<\ldots<a_r\leqq L$ defines a *partition* of $(\underline{n},\underline{s})$ into blocks $\langle a_k,a_{k+1}\rangle$, $k=0,\ldots,r$, where $a_0=-\infty$ and $a_{r+1}=\infty$. We will say that $\langle a_k,a_{k+1}\rangle$ is a block in $(\underline{n},\underline{s},\gamma)$. Our partitions will depend only on $\underline{n}$ and not on $\underline{s}$ and we will write $\gamma=\gamma(\underline{n})$ to indicate this dependence.

For $x,y\in Z$ and $(\underline{u},\underline{s})\in Q$ we define $N(x,y)(\underline{n})$ as in [I], (1.5). Fix a $\beta\geqq\beta_0$ and let δ be as in Theorem 1.3. The constants K, ξ, and β_0 will be defined in Sect. 2.4.

Definition 2.2. Let γ be a partition. We will say that $(\underline{n},\underline{s},\gamma)$ has *the density property* if the blocks in $(\underline{n},\underline{s},\gamma)$ alternate between o- and e-blocks and for each o-(e-)block $A=\langle a,a'\rangle$ in $(\underline{n},\underline{s},\gamma)$

(i) $N(a,x)(\underline{n})\geqq(1/2-\delta)(x-a-1)$ $(\leqq\delta(x-a))$.

(ii) $N(x,a'-1)(\underline{n})\geqq(1/2-\delta)(a'-x-1)$ $(\leqq\delta(a'-1-x))$ if $a\leqq x<a'$.

We will now define a partition $\gamma_1(\underline{n})$ for every configuration $(\underline{n},\underline{s})\in Q_1:=Q(\underline{t},X)$. Put

$$u_k=2^{k-1},\quad v_k=\frac{2}{\delta}4^{k-1},$$

$k=1,2,\ldots$. Define

$$\gamma^{(0)}(\underline{n})=\{i\in Z;\ (n_{i-2},n_{i-1},n_i)=(1,0,0) \text{ or } =(0,0,1)\}.$$

The blocks in $(\underline{n}, \underline{s}, \gamma^{(0)}(n))$ will alternate between o- and e-blocks. A 1 followed by a double 0, $(1,0,0)$, means going from an o- to an e-block and at the next 1, $(0,0,1)$, a new o-block starts. In the same way as in [I] we successively define $\gamma^{(k)}(\underline{n})$, $k=1,2,\dots$. We let

$$\gamma_1(\underline{n})=\gamma^{(\nu k_N)}(\underline{n}), \tag{2.2}$$

where

$$k_N=[\omega \log N] \tag{2.3}$$

and ν, ω are constants depending only on α. They are given by (3.7) and (3.16) respectively.

2.3. The Rearrangement Procedure

Let $(\underline{n}, \underline{s})$ be a configuration and $\gamma=\{a_1, \dots, a_{2r-1}\}$ a partition into $2r$ blocks such that the density property is satisfied. The operation $S_{k,k+1}(\underline{n}, \underline{s}, \gamma, \delta)=(\underline{n}', \underline{s}', \gamma', \delta')$ defined by letting block number k change place with block number $k+1$ is defined exactly as in [I]. Recall that δ is a set whose elements are old partition points removed during the rearrangement procedure.

Lemma 2.3. *If* $(\underline{n}, \underline{s}) \in Q(\underline{t}, X)$, *then* $S_{k,k+1}(\underline{n}, \underline{s}) \in Q(\underline{t}, X)$, $\underline{t} \in T$.

Proof. From the definition of $Q(\underline{t}, X)$ it follows that $(\underline{n}, \underline{s}) \in Q(\underline{t}, X)$ if and only if $s_{p_k(\underline{n})}=t_{\sigma(k)}$, $k=1, \dots, N$, for some $\sigma \in S_N$, all other $s_j=0$, and

$$\underline{x}=F^{-1}(\underline{n}, \underline{s})=(p_k(\underline{n})+s_{p_k(\underline{n})})_{k=1}^N \in X .$$

Recall that $\underline{x} \in X$ if $\underline{x}$ satisfies the hard-core condition $x_{k+1}-x_k>1$, $k=1, \dots, N-1$, and $x_1>0$, $x_N<L$. Write $(\underline{n}', \underline{s}')=S_{k,k+1}(\underline{n}, \underline{s})$ and $\underline{x}'=F^{-1}(\underline{n}', \underline{s}')$. Since we get $\underline{s}'$ from $\underline{s}$ by a permutation of the elements of $\underline{s}$ it is clear that $s'_{p_k(\underline{n}')}=t_{\sigma'(k)}$, $k=1, \dots, N$, for some $\sigma' \in S_N$. If $\langle a_{k-1}, a_k\rangle$, $k=1, \dots, 2r$, are the blocks in $(\underline{n}, \underline{s}, \gamma)$, then from the definition of o- and e-blocks we have that $n_{a_k-1}=0$, $k=1, \dots, 2r-1$ and hence also $s_{a_k-1}=0$. Thus the hard-core conditions place no restriction on the order of the blocks $\langle a_{k-1}, a_k\rangle$. Consequently $\underline{x}'$ also satisfies the hard core conditions and $(\underline{n}', \underline{s}') \in Q(\underline{t}, X)$. □

Write $A_k=\langle a_{2(k-1)}, a_{2k-1}\rangle$, $B_k=\langle a_{2k-1}, a_{2k}\rangle$, $k=1, \dots, r$, so that $A_1, \dots, A_r$ are o-blocks and $B_1, \dots, B_r$ are e-blocks. λ denotes a constant, depending only on α, which will be specified in Sect. 4. We now turn to the definition of *the elementary rearrangement operation S*. Assume first that an o-block A_k is the shortest block. If one of its neighbouring e-blocks has length $\geqq \lambda$ times the length of the other, we let A_k change place with the shortest of its neighbours. Otherwise we let A_k change place with that neighbouring block which gives the lowest energy for the resulting configuration. If an e-block, B_j, is shortest we take the shortest of its neighbouring o-blocks and apply the procedure just described to this o-block.

More formally we consider the shortest block among $A_1, B_1, \dots, A_r, B_r$ or the leftmost if it is not unique.

(a) Assume that A_k is the shortest block:

(i) if $|B_k| \geqq \lambda |B_{k-1}|$, then $S=S_{2k-2,2k-1}$,

(ii) if $|B_{k-1}| \geqq \lambda |B_k|$, then $S=S_{2k-1,2k}$,

(iii) if $\lambda^{-1}|B_k|<|B_{k-1}|<\lambda|B_k|$, then $S=S_{2k-2,2k-1}$ in case

$$H(S_{2k-2,2k-1}(\underline{n},\underline{s}))\leqq H(S_{2k-1,2k}(\underline{n},\underline{s}))$$

and $S=S_{2k-1,2k}$ otherwise.

(b) Assume that B_j is the shortest block and $|A_j|\leqq|A_{j+1}|$. Then S is defined as in (a) with $k=j$. If $|A_j|>|A_{j+1}|$, then S is defined as in (a) with $k=j+1$.

We can now define Q_j and partitions $\gamma_j(\underline{n})$ for each $(\underline{n},\underline{s})\in Q_j$ precisely as in [I], and the rearranged configuration we get starting from $(\underline{m},\underline{s},\gamma_{j-1}(\underline{m}))$ is denoted by

$$(R(\underline{m}),R(\underline{s}),R\gamma(\underline{m}),R\delta(\underline{m}))\,.$$

It follows from Lemma 2.3 that $Q_j\subseteqq Q=Q_1$.

Lemma 2.4. *$(\underline{m},\underline{s})\in Q_{j-1}$, $j\geqq 2$, is uniquely determined by $(R(\underline{m}),R(\underline{s}),R\gamma(\underline{m}),R\delta(\underline{m}))$.*

This is proved exactly as Lemma 2.4 in [I]. The proof of the next lemma is, apart from minor changes, the same as the proof of Lemmas 2.3 and 2.5 in [I]. The necessary modifications will be outlined in Sect. 3.2.

Lemma 2.5. *For every $(\underline{n},\underline{s})\in Q_j$, $1\leqq j\leqq k_N$, $(\underline{n},\underline{s},\gamma_j(\underline{n}))$ has the density property and all blocks in $(\underline{n},\underline{s},\gamma_j(\underline{n}))$ have length $\geqq u_j$.*

2.4. The Energy-Entropy Argument

We now turn to the proof of Theorem 1.3. Fix $0\leqq\tau_1<\tau_2\leqq d_1-\varepsilon$, where $\varepsilon>0$ is small and d_1 is given by (1.2) with δ as in the theorem. Define

$$A_{\tau_1,\tau_2}=\{\underline{x}\in X;\ d(\tau_1,\tau_2)(\underline{n})<1/2-2\delta\}\,.$$

If $d_2+\varepsilon\leqq\tau_1<\tau_2\leqq 1$ we define instead

$$A_{\tau_1,\tau_2}=\{\underline{x}\in X;\ d(\tau_1,\tau_2)(\underline{n})>2\delta\}\,.$$

We will show that there is a constant C independent of N and $\underline{t}\in T$ such that

$$\frac{Z(\underline{t},A_{\tau_1,\tau_2})}{Z(\underline{t},X)}\leqq\frac{C}{N}\,. \tag{2.4}$$

Lemma 2.1, (1.1) and (2.4) imply that $P(A_{\tau_1,\tau_2})\to 0$ as $\Omega\to R^+$ and Theorem 1.3 follows.

The rearrangement procedure defines a map $\mathscr{R}:Q\to Q_{k_N}$ by $(\underline{n},\underline{s})\to R^{k_N-1}(\underline{n},\underline{s})=(\mathscr{R}(\underline{n}),\mathscr{R}(\underline{s}))$. Let

$$H_1(\underline{t})=\{(\underline{n},\underline{s})\in Q(\underline{t},X);\ |\gamma_{k_N}(\mathscr{R}(\underline{n}))|\geqq 3\}$$

and $H_j(\underline{t})=R(H_{j-1}(\underline{t}))$, $2\leqq j\leqq k_N$, for every $\underline{t}\in T$.

Lemma 2.6. *Let d_1 and d_2 be given by (1.2). If $0\leqq\tau_1<\tau_2\leqq d_1-\varepsilon$ or $d_2+\varepsilon\leqq\tau_1<\tau_2\leqq 1$, then*

$$Q(\underline{t},A_{\tau_1,\tau_2})\subseteqq H_1(\underline{t})$$

for all $\underline{t}\in T$.

We postpone the proof of this lemma to the end of Sect. 3.2.

By Lemma 2.6, (2.4) will follow if we can prove

$$\frac{1}{Z(\underline{t}, X)} \sum_{(\underline{n}, \underline{s}) \in H_1(\underline{t})} e^{-\beta H(\underline{n}, \underline{s})} \leqq \frac{C}{N} \tag{2.5}$$

with C independent of $\underline{t}$ and N. The proof of (2.5) is an energy-entropy argument which is completely analogous to the corresponding energy-entropy argument in [I]. In Sect. 3 we will prove that Lemma 3.2 in [I] is true also in the present case if we let $C_1 = \log(C_1'/\delta)$, where C_1' depends only on α.

The constants in Theorem 1.3 are defined as follows. Let

$$\beta_0 = (1 + 4\log(C_1'/\delta_0))/\kappa ,$$

where

$$\delta_0 = \min\{\varrho, 1/4 - \varrho L, 1/16, c_7\}$$

and c_7, which depends only on α, is given by (4.12) below. For given $\beta \geqq \beta_0$, δ is defined by $\beta = (1 + 4\log(C_1'/\delta))/\kappa$ so that $\beta \geqq \beta_0$ implies $\delta \leqq \delta_0$. This gives $\delta = K\exp(-\xi\beta)$ with $\xi = \kappa/4$.

If all blocks in $(\underline{n}, \underline{s}, \gamma)$ have length $\geqq u_j$ and $(\underline{n}, \underline{s}, \gamma)$ satisfies the density property, then

$$H((\underline{n}, \underline{s})) - H(S((\underline{n}, \underline{s}))) \geqq 2\kappa j , \tag{2.6}$$

where κ is a constant that depends only on α. This energy estimate will be proved in Sect. 4. It follows from repeated use of (2.6) that

$$H(\underline{n}, \underline{s}) - H(R(\underline{n}, \underline{s})) \geqq \kappa |R\delta(\underline{n})| j \tag{2.7}$$

for every $(\underline{n}, \underline{s}) \in Q_{j-1}$. Using the entropy estimate (3.1) in [I], (2.7) and $\kappa\beta - C_1 \geqq 1$ we can do exactly the same computation as in Sect. 3.3 in [I] to show that

$$\sum_{(\underline{n}, \underline{s}) \in H_1(\underline{t})} e^{-\beta H(\underline{n}, \underline{s})} \leqq \eta \sum_{(\underline{n}, \underline{s}) \in H_{k_N}(\underline{t})} e^{-\beta H(\underline{n}, \underline{s}) + \log(2k_N) |\gamma_{k_N}(\underline{n})|} , \tag{2.8}$$

where η is a numerical constant. From the definition of $H_1(\underline{t})$ we know that $|\gamma_{k_N}(\underline{n})| \geqq 3$ if $(\underline{n}, \underline{s}) \in H_1(\underline{t})$. We can now estimate the right-hand side of (2.8) using a final global rearrangement in exactly the same way as in Sect. 3.4 in [I]. This gives

$$\frac{1}{Z(\underline{t}, X)} \sum_{(\underline{n}, \underline{s}) \in H_1(\underline{t})} e^{-\beta H(\underline{n}, \underline{s})} \leqq CL^3 (\log N)^3 N^{-2\kappa\omega\beta} , \tag{2.9}$$

where ω is the constant in (2.3). At the end of Sect. 3 we will see that if $\beta \geqq \beta_0$ then

$$2\kappa\omega\beta \geqq 4 . \tag{2.10}$$

Thus (2.5) follows from (2.9) and we have proved Theorem 1.3.

3. Proof of Some Lemmas

3.1. The Entropy Estimate

The proof of Lemma 3.2 in [I] is based on the following lemma. Let

$$w_{j,k} = \zeta^{j-1} v_k , \qquad w_j = w_{j,j+1} , \tag{3.1}$$

where ζ will be specified at the end of this section. Recall that λ is the constant in the definition of an elementary rearrangement. The constants λ and ζ depend only on α.

Lemma 3.1. *There is a constant C_λ that only depends on λ, such that for all $(\underline{n}, \underline{s}) \in Q_{j-1}$ the distance from an element in $R\delta(\underline{n})$ to the closest element in $R\gamma(\underline{n})$ is $\leqq C_\lambda w_{j-1}$, $j=2, \ldots, k_N$.*

Proof. Denote the assertion in the lemma for a given j by $(a)_j$. The proof is similar to the proof of Lemma 3.1 in [I] but is more involved due to the more complicated definition of an elementary rearrangement. Let $(b)_j$, $1 \leqq j < k_N$ denote the following assertion

> Consider two o-(e-)blocks of length $\geqq w_{j,k}$ in $(\underline{n}, \underline{s}, \gamma_j(\underline{n}))$, $(\underline{n}, \underline{s}) \in Q_j$. Then the total length of the o-(e-)blocks between them is $\geqq u_k$, $j \leqq k \leqq v(k_N - j + 1) + k_N$.

Here v is the constant in (2.2). That $(b)_1$ is true follows from the definition of $\gamma_1(\underline{n})$ in the same way as in [I]. We will prove Lemma 3.1 inductively by showing that $(b)_{j-1}$ implies $(a)_j$ and that $(b)_{j-1}$ and $(a)_j$ together imply $(b)_j$, $2 \leqq j < k_N$.

Assume that $(b)_{j-1}$ is true. Below A and B with some index will always denote an o-block and e-block respectively. Consider first the elements of $R\delta(\underline{n})$ inside an o-block A in $(R(\underline{n}), R(\underline{s}), R\gamma(\underline{n}))$. A is built up from o-blocks $A_1, \ldots, A_r$, $r \geqq 1$ in $(\underline{n}, \underline{s}, \gamma_{j-1}(\underline{n}))$:

$$(\underline{n}, \underline{s}) = \ldots B_0 A_1 B_1 A_2 \ldots B_{r-1} A_r B_r \ldots,$$
$$(R(\underline{n}), R(\underline{s})) = \ldots B_0 \ldots B_{s-1} A_1 \ldots A_r B_s \ldots B_r \ldots,$$
$$= \ldots BAB' \ldots,$$

where $B_0, \ldots, B_r$ are e-blocks in $(\underline{n}, \underline{s}, \gamma_{j-1}(\underline{n}))$.

We will prove that there is a t, $1 \leqq t \leqq r$, such that

$$\max\{|A_1 \ldots A_{t-1}|, |A_{t+1} \ldots A_r|\} \leqq C_\lambda w_{j-1}. \tag{3.2}$$

The left-hand side gives an upper bound on the distance from an element in $R\delta(\underline{n})$ inside A to the closest element in $R\gamma(\underline{n})$.

Claim 1. *Suppose that at some step in the rearrangement procedure from $(\underline{n}, \underline{s}, \gamma_{j-1}(\underline{n}))$ to $(R(\underline{n}), R(\underline{s}), R\gamma(\underline{n}))$ the elementary rearrangement*

$$(\underline{m}, \underline{r}) = \ldots A^0 B^0 A^1 B^1 A^2 B^2 \ldots,$$
$$(S(\underline{m}), S(\underline{r})) = \ldots A^0 B^0 B^1 A^1 A^2 B^2 \ldots,$$

was done. Then one of (i)–(iii) *below must hold*

(i) $|A^1| < u_j$, $|A^1| \leqq |A^2|$, *and* $|B^1| < \lambda |B^0|$,
(ii) $|A^1| \leqq |A^2|$, *and* $|B^1| < u_j$,
(iii) $|A^0| > |A^1|$, $|B^0| < u_j$, *and* $|B^1| < \lambda |B^0|$.

To see this we use the definition of the S-operation in Sect. 2.3. If A^1 is the shortest block, then $|A^1| < u_j$ and $|A^1| \leqq |A^2|$. In case $|B^1| \geqq \lambda |B^0|$, A^1 and B^0 would have changed place instead. Thus (i) holds. If A^1 is not shortest block, either B^0 or B^1 must have been shortest. If B^1 is shortest, $|B^1| < u_j$ and since A^1 and B^1 changed places, $|A^1| \leqq |A^2|$. Thus (ii) holds. If B^0 is shortest, $|B^0| < u_j$, $|A^0| > |A^1|$ since otherwise A^1 would not have been involved. Furthermore if $|B^1| \geqq |B^0|$, A^1 and B^0

would have changed place. Consequently, $|B^1| < \lambda |B^0|$ and (iii) holds. This establishes Claim 1.

Claim 2. *Suppose that at some step in the rearrangement procedure from* $(\underline{n}, \underline{s}, \gamma_{j-1}(\underline{n}))$ *to* $(R(\underline{n}), R(\underline{s}), R\gamma(\underline{n}))$ *the elementary rearrangement*

$$\dots A_{k-(j_2+1)} B_{k-j_4} \dots B_{k-(j_3+1)} A_{k-j_2} \dots A_{k-(j_1+1)} B_{k-j_3} \dots$$
$$\dots B_{k-1} A_{k-j_1} \dots A_d B_k \dots$$

to

$$\dots A_{k-(j_2+1)} B_{k-j_4} \dots B_{k-1} A_{k-j_2} \dots A_{k-j_1} \dots A_d B_k \dots$$

was performed. Assume furthermore that $j_2 > 2[2\lambda]+1$, $|A_{k-j_2} \dots A_{k-(j_1+1)}| \geqq u_j$, *and* $j_3 > j_1 \geqq 0$. *Then* $j_3 = 1$, $|B_{k-1}| < u_j$, *and*

$$|A_{k-j_2} \dots A_{k-(j_1+1)}| \leqq |A_{k-j_1} \dots A_d| . \tag{3.3}$$

It is clear that $j_2 - (j_1+1) \leqq j_4$ and thus $j_2 \leqq j_4 + j_1 + 1$. Since $|A_{k-j_2} \dots A_{k-(j_1+1)}| \geqq u_j$ it follows from Claim 1 that either $|B_{k-j_3} \dots B_{k-1}| < u_j$ and (3.3) holds, or

$$|B_{k-j_4} \dots B_{k-(j_3+1)}| < u_j \quad \text{and} \quad |B_{k-j_3} \dots B_{k-1}| < \lambda u_j .$$

In the first case we get $j_3 = 1$ and $|B_{k-1}| < u_j$. In the second case we get $j_4 = j_3 + 1$ and $j_3 \leqq [2\lambda]$, since all B_i's have length $\geqq u_{j-1}$. Thus $j_4 \leqq [2\lambda]+1$ and $j_1 \leqq j_3 - 1 \leqq [2\lambda]-1$. This gives $j_2 \leqq j_4 + j_1 \leqq 2[2\lambda]+1$, which contradicts the assumption $j_2 > 2[2\lambda]+1$, and the claim is proved.

If B_{s-1} ends up to the left of A and B_s to the right of A, then A_s must have been fixed throughout the rearrangement procedure, since if A_s has been moved B_{s-1} and B_s would have been joined. We now define two integers q_1 and q_2 as follows. If $s=1$ or if $|A_i| < u_j$ for $1 \leqq i \leqq s-1$ we put $q_1 = 0$. Otherwise

$$q_1 = \max\{i; |A_{s-i}| \geqq u_j, \ 1 \leqq i \leqq s-1\} .$$

If $s \leqq 3$ we put $q_2 = 1$. If $s \geqq 4$ we define, for $1 \leqq i \leqq s-3$, $v_i = 0$ if the first time A_i is joined with another o-block, this o-block contains A_{s-2}, otherwise $v_i = 1$. If $v_i = 0$ for all i, $1 \leqq i \leqq s-3$ we put $q_2 = 2$, otherwise

$$q_2 = \max\{i; v_{s-i} = 1, \ 3 \leqq i \leqq s-1\} .$$

Let $q = \max\{q_1, q_2\}$. Then the following claim is true.

Claim 3. $|A_1 \dots A_{s-q-1}| \leqq 3\lambda^2 w_{j-1}$.

Since $q \geqq q_1$, $A_1, \dots, A_{s-q-1}$ all have length $< u_j$ and we can assume that $s-q-1 > 3$, otherwise the bound follows trivially. Also $q \geqq q_2$ and the definition of q_2 gives that $A_{s-q-1}, \dots, A_1$ were joined successively with an o-block containing A_{s-2}. According to Claim 1, when A_2 is joined with the o-block containing A_3 we must have

$$\lambda |B_1| \geqq |B_2 \dots B_{s-q-1}| .$$

If $|B_2| \geqq \lambda w_{j-1}$ then $|B_1| \geqq w_{j-1}$ and by $(b)_{j-1}$, $|A_2| \geqq u_j$ and we get a contradiction. Consequently $|B_2| < \lambda w_{j-1}$. Similarly we must have

$$\lambda |B_2| \geqq |B_3 \dots B_{s-q-1}| ,$$

and hence $|B_3 \dots B_{s-q-1}| < \lambda^2 w_{j-1}$ which implies $s-q-3 \leqq \lambda^2 w_{j-1}/u_{j-1}$. Thus

$$|A_1 \dots A_{s-q-1}| \leqq (s-q-1)u_j \leqq 3\lambda^2 w_{j-1}$$

and the claim is proved.

Claim 4. $|B_{s-i}| < \lambda u_j$ *if* $1 \leqq i \leqq q_1$.

If $q_1 = 0$ there is nothing to prove, so assume that $q_1 \geqq 1$ and $|B_{s-i}| \geqq \lambda u_j$ for some i, $1 \leqq i \leqq q_1$. Since B_{s-1} ends up in B the same holds for B_{s-i}. At some step in the rearrangement procedure an e-block of length $\geqq \lambda u_j$ containing B_{s-i} must have changed place with an o-block containing A_{s-q_1} of length $\geqq u_j$. By Claim 1 this is not possible. Thus $|B_{s-i}| < \lambda u_j$.

It will be convenient to write

$$\chi = 2[2\lambda] + 3.$$

Claim 5. *At least one of the following two assertions is true:*
(i) $|A_1 \dots A_{s-2}| \leqq (3\lambda^2 + 2)w_{j-1}$,
(ii) $|A_1 \dots A_{s-q_1-1}| \leqq (3\lambda^2 + \chi)w_{j-1}$ *and* $q_1 \leqq \chi$.

Assume that $q > \chi$. At some step in the rearrangement procedure an o-block containing A_{s-q} must have been joined with an o-block containing A_s. The o-block containing A_{s-q} must then have length $\geqq u_j$, because if $q = q_1$, $|A_{s-q}| \geqq u_j$ and if $q = q_2$, A_{s-q} is joined with some other o-block before it is joined with A_{s-2}, and hence before it is joined with A_s. Thus we perform a rearrangement of the type given in Claim 2 with $k = s$, $d \geqq s$, and $j_2 \geqq q \geqq j_1 + 1$ since the o-block closest to the left of A_s is always B_{s-1}. We see that $|A_{s-j_2} \dots A_{s-(j_1+1)}| \geqq u_j$ and $j_2 > 2[2\lambda] + 1$. Furthermore, $j_3 > j_1$ since the blocks $B_{s-(j_1+1)}, \dots, B_{s-1}$ must all lie in $B_{s-j_3} \dots B_{s-1}$. Claim 2 now gives $j_3 = 1$, $j_1 = 0$, and $|B_{s-1}| < u_j$.

Consequently at some previous step in the rearrangement procedure an o-block containing A_{s-q} must have been joined with an o-block containing A_{s-1}. Again we have a rearrangement of the type given in Claim 2, this time with $k = s = 1$, $d = s-1$, $j_3 > j_1$, and $s-1-j_2 \leqq s-q$, i.e. $j_2 \geqq q-1 > 2[2\lambda]+2$. Claim 2 gives $j_3 = 1$, $|B_{s-2}| < u_j$ and

$$|A_{s-q} \dots A_{s-2}| \leqq |A_{s-1}|. \tag{3.4}$$

We see that at some previous step in the rearrangement procedure an o-block containing A_{s-q} must have been joined with an o-block containing A_{s-2}. The same argument as above using Claim 2 now gives

$$|A_{s-q} \dots A_{s-3}| \leqq |A_{s-2}|. \tag{3.5}$$

Suppose that $|A_{s-2}| \geqq w_{j-1}$. It follows from (3.4) that $|A_{s-1}| \geqq w_{j-1}$, and hence $(b)_{j-1}$ gives $|B_{s-2}| \geqq u_j$. This contradicts $|B_{s-2}| < u_j$ and consequently $|A_{s-2}| < w_{j-1}$ and $|A_{s-q} \dots A_{s-2}| \leqq 2w_{j-1}$ by (3.5). Combining this with Claim 3 we see that (i) holds.

Assume now that $q \leqq \chi$. If $q = q_1$, $|A_1 \dots A_{s-q_1-1}| \leqq 3\lambda^2 w_{j-1}$ by Claim 3 and (ii) holds. In case $q = q_2$, $|A_1 \dots A_{s-q_2-1}| \leqq 3\lambda^2 w_{j-1}$ by Claim 3 and

$$|A_{s-q_2} \dots A_{s-q_1-1}| \leqq (q_2 - q_1)u_j \leqq \chi w_{j-1}$$

and (ii) follows. This establishes Claim 5.

By symmetry we can apply the same argument to blocks to the right of A_s. We introduce integers p_1, p_2 analogous to q_1, q_2 and prove the next claim.

Claim 6. $|B_{s+i}|<\lambda u_j$ *if* $0\leqq i<p_1$ *and at least one of the following assertions hold* (iii) $|A_{s+2}\dots A_r|\leqq(3\lambda^2+2)w_{j-1}$, (iv) $|A_{s+p_1+1}\dots A_r|\leqq(3\lambda^2+\chi)w_{j-1}$ *and* $p_1\leqq\chi$.

We are now in position to prove (3.2). Let $1\leqq\mu_1,\ \mu_2\leqq\chi$, let A_t be the longest block among $A_{s-\mu_1},\dots,A_{s+\mu_2}$ and A_d the second longest. If $|A_d|<u_j$, then

$$\max\{|A_{s-\mu_1}\dots A_{t-1}|,\ |A_{t+1}\dots A_{s+\mu_2}\}\leqq 2\chi u_j. \tag{3.6}$$

If $|A_d|\geqq u_j$ it follows from Claims 4 and 6 that the e-blocks between A_d and A_t have length $\leqq|d-t|\lambda u_j\leqq(\mu_1+\mu_2)\lambda u_j\leqq 2\lambda\chi u_j$. Now define the constant v in (2.2) and in $(b)_j$ by

$$v=[\log_2(2\lambda\chi)]+1. \tag{3.7}$$

If $|A_d|\geqq w_{j-1,j+v}$ and $j+v\leqq v(k_N-j+2)+k_N$, i.e. $j\leqq v(k_N-j+1)+k_N$, then $(b)_{j-1}$ gives that the total length of the e-blocks between A_d and A_t is $\geqq u_{j+v}>2\lambda\chi u_j$. Thus we get a contradiction and conclude that $|A_d|<w_{j-1,j+v}\leqq(2\lambda\chi)^2w_{j-1}$. Hence

$$\max\{|A_{s-\mu_1}\dots A_{t-1}|,\ |A_{t+1}\dots A_{s+\mu_2}\}\leqq 2\chi(2\lambda\chi)^2w_{j-1}. \tag{3.8}$$

There are four possible combinations of the assertions in Claims 5 and 6. For the combinations (i) and (iii), (i) and (iv), (ii) and (iii), (ii) and (iv) we choose respectively $\mu_1=\mu_2=1$, $\mu_1=1$, $\mu_2=p_1$, $\mu_1=q_1$, $\mu_2=1$ and $\mu_1=p_1$, $\mu_2=p_1$. In all cases we obtain (3.2) by combining the assertions in the claims with (3.6) or (3.8).

It remains to consider elements of $R\delta(\underline{n})$ inside an e-block B in $(R(\underline{n}), R(\underline{s}), R\gamma(\underline{n}))$. B has been built up from e-blocks in $(\underline{n},\underline{s},\gamma_{j-1}(\underline{n}))$, and one of these e-blocks must have remained fixed during the rearrangement procedure, say that B_0 was fixed. Then

$$(\underline{n},\underline{s})=\dots A_{-u}B_{-u}A_{-u+1}\dots A_0B_0A_1B_1\dots B_{s-1}A_sB_{s+1}\dots A_rB_r\dots,$$

$$(R(\underline{n}),R(\underline{s}))=\dots A_{-u}\dots A_0B_{-u}\dots B_0\dots B_{s-1}A_1\dots A_s\dots A_rB_{s+1},$$

Claim 7. (i) *If* $|B_i|<\lambda u_j$ *for* $t\leqq i<s$, *where* $t\geqq 2$, *then*

$$|B_t\dots B_{s-1}|\leqq(6\lambda^3+3\lambda\chi)w_{j-1}.$$

(ii) *If* A_{t-1} *and* A_t *are joined before* A_t *and* A_{t+1} *are joined then* $|B_i|<\lambda u_j$ *for* $t\leqq i<s$.

Let q_1 and χ be defined as above. If $q_1>\chi$ then by Claim 5, $|A_1\dots A_{s-2}|\leqq(3\lambda^2+2)w_{j-1}$ and since all blocks have length $\geqq u_{j-1}$ this implies

$$|B_t\dots B_{s-1}|\leqq(s-2)\lambda u_j\leqq(3\lambda^2+2)w_{j-1}\lambda u_j/u_{j-1}=(3\lambda^3+2\lambda)w_{j-1}.$$

Assume that $q_1\leqq\chi$. By Claim 5 it follows that we always have $s-q_1-1\leqq(3\lambda^2+\chi)w_{j-1}/u_{j-1}$ and we get

$$|B_t\dots B_{s-q_1-1}|\leqq(s-q_1-1)\lambda u_j\leqq(6\lambda^3+2\chi)w_{j-1}$$

and

$$|B_{s-q_1}\dots B_{s-1}|\leqq q_1\lambda u_j\leqq\chi\lambda w_{j-1}.$$

This proves (i).

If A_t and A_{t+1} have not been joined, B_t lies to the right of A_t. When A_{t-1} and A_t have been joined B_t will lie to the right of the o-block containing $A_{t-1}A_t$. If $|B_i|\geqq\lambda u_j$

for some $i, t \leqq i < s$, then at some step in the rearrangement procedure an e-block of length $\geqq \lambda u_j$ containing B_i must change place with an o-block of length $\geqq u_j$ containing $A_{t-1}A_t$, but this is impossible by Claim 1.

Claim 8. *There exists a* $p \leqq \chi$ *such that*

$$|B_p \dots B_{s-1}| \leqq (6\lambda^3 + 3\lambda\chi + 2\lambda^2) w_{j-1} \tag{3.9}$$

and $|A_i| < u_j$ *when* $1 \leqq i < p$.

By Claim 4, $|B_{s-i}| < \lambda u_j$ when $1 \leqq i \leqq q$ and hence Claim 7 (i) gives

$$|B_{s-q_1} \dots B_{s-1}| \leqq (6\lambda^3 + 3\lambda\chi) w_{j-1} .$$

If $s - q_1 \leqq \chi$ we can take $p = s - q_1$. Note that by the definition of q_1, $|A_i| < u_j$ when $1 \leqq i < s - q_1$. Assume that $s - q_1 > \chi$. If A_{t-1} and A_t are joined before A_t and A_{t+1} are joined for some t, $2 \leqq t \leqq \chi$, then Claim 7 shows that (3.9) holds with $p = t$. Suppose that this is not the case. Consider the step when A_χ and $A_{\chi+1}$ are joined. Then, by our assumption, $A_{\chi-1}$ and A_χ have not been joined and consequently nor have $A_{\chi-2}$ and $A_{\chi-1}$ or $A_{\chi-3}$ and $A_{\chi-2}$. The following elementary rearrangement is done:

$$\dots A_{\chi-2} B_{\chi-2} A_{\chi-1} B_{\chi-1} A_\chi B_\chi \dots B_{\chi+j_1} A_{\chi+1} \dots$$

to

$$\dots A_{\chi-2} B_{\chi-2} A_{\chi-1} B_{\chi-1} B_\chi \dots B_{\chi+j_1} A_\chi A_{\chi+1} \dots .$$

By Claim 1, $|B_\chi \dots B_{\chi+j_1}| < 2\lambda |B_{\chi-1}|$. As in Claim 7 it follows that $|B_i| < \lambda u_j$ when $i > \chi + j_1$ and Claim 7 gives

$$|B_{\chi+j_1+1} \dots B_{s-1}| \leqq (6\lambda^3 + 3\lambda\chi) w_{j-1} . \tag{3.10}$$

At some later step $A_{\chi-1}$ and A_χ will be joined and the same argument gives

$$|B_{\chi-1} \dots B_{\chi+j_2}| \leqq 2\lambda |B_{\chi-2}| ,$$

where $j_2 \geqq j_1$. If $|B_{\chi-1}| \geqq 2\lambda w_{j-1}$ then $|B_{\chi-2}| \geqq w_{j-1}$ and since $|A_{\chi-1}| < u_j$ this contradicts $(b)_{j-1}$. Hence $|B_{\chi-1}| < 2\lambda w_{j-1}$ and we get $|B_\chi \dots B_{\chi+j_1}| < 4\lambda^2 w_{j-1}$. Together with (3.10) this proves (3.9) with $p = \chi$.

We can now prove the following claim by a completely analogous argument.

Claim 9. *There is a* $q \leqq \chi$ *such that*

$$|B_{-u} \dots B_{-q}| \leqq (6\lambda^3 + 3\lambda\chi + 2\lambda^2) w_{j-1}$$

and $|A_{-i}| < u_j$ *if* $0 \leqq i < q - 1$.

Now let B_t be the longest block among $B_{-q+1}, \dots, B_{p-1}$. Using Claim 8, Claim 9 and $(b)_{j-1}$ we can apply the same argument as that after Claim 6 to prove that

$$\max\{|B_{-u} \dots B_{t-1}|, |B_{t+1} \dots B_{s-1}|\} \leqq C_\lambda w_{j-1}$$

with a suitable C_λ. This completes the proof of $(a)_j$.

We now turn to the proof of $(b)_j$ given that $(b)_{j-1}$ and $(a)_j$ are true.

Claim 10. *Suppose that we have two o-(e-)blocks* C *and* C' *of length* $\geqq w_{j,k}$ *in* $(\underline{n}, \underline{s}, \gamma_j(\underline{n}))$, $(\underline{n}, \underline{s}) \in Q_j$ *for some* $k, j \leqq k \leqq v(k_N - j + 1) + k_N$ *such that the length of the* *e-(o-)blocks between them is* $<u_k$. *Let* $v'=[\log_2(3+2\lambda)]+1$. *Then there is a configuration* $(\underline{m},\underline{r})\in Q_{j-1}$, $R(\underline{m},\underline{r})=(\underline{n},\underline{s})$ *with the following property. In* $(\underline{m},\underline{r},\gamma_{j-1}(\underline{m}))$ *there are two o-(e-)blocks* C_2 *and* C_2', *of length* $\geqq w_{j-1,k+v'}$, *such that the length of the e-(o-)blocks between them is* $<3u_k+2\lambda u_j$.

Since $v'\leqq v$, $k\leqq v(k_N-j+1)+k_N$ implies that $k+v'\leqq v(k_N-(j-1)+1)+k_N$. Now $3u_k+2\lambda u_j<u_{k+v'}$, so Claim 10 contradicts $(b)_{j-1}$. Hence the assumption in Claim 10 must be wrong and $(b)_j$ follows.

To prove Claim 10 we first show that there is an $(\underline{m},\underline{r})\in Q_{j-1}$ with $R(\underline{m},\underline{r})=(\underline{n},\underline{s})$ and blocks C_1 and C_1' in $(R(\underline{m}),R(\underline{r}),R\gamma(\underline{m}))$ of length $\geqq w_{j,k}$, such that the length of the e-(o-)blocks between them is $<3u_k$. Assume first that C and C' are e-blocks and let $A_1,\dots,A_p$ and $B_1,\dots,B_{p-1}$ be, respectively, the o- and e-blocks between C and C' in $(\underline{n},\underline{s},\gamma_j(\underline{n}))$. We write $C=\langle b_0,a_1\rangle$, $C'=\langle b_p,a_{p+1}\rangle$, $A_i=\langle a_i,b_i\rangle$, and $B_i=\langle b_i,a_{i+1}\rangle$. There is a $(\underline{m},\underline{r})\in Q_{j-1}$ with $R(\underline{m},\underline{r})=(\underline{n},\underline{s})$ such that $a_1\in R\gamma(\underline{m})$. At least one of b_i, $1\leqq i\leqq p$, must belong to $R\gamma(\underline{m})$ since otherwise we would have an o-block $\langle a_1,b\rangle$, in $(R(\underline{m}),R(\underline{r}),R\gamma(\underline{m}))$ with $b\geqq a_{p+1}$. Since $|A_1|+\dots+|A_p|<u_k$ and $|C'|\geqq w_{j,k}$ this would contradict the density property of $(R(\underline{m}),R(\underline{r}),R\gamma(\underline{m}))$. Let b_q be the largest among b_i, $1\leqq i\leqq p$, that belongs to $R\gamma(\underline{m})$. In $(R(\underline{m}),R(\underline{r}),R\gamma(\underline{m}))$ we have two e-blocks $C_1=\langle c,a_1\rangle$ and $C_1'=\langle b_q,c'\rangle$, where $c\leqq b_0$ and $c'\geqq a_{p+1}$. Clearly C_1 and C_1' both have length $\geqq w_{j,k}$. Consider an o-block $A=\langle a_{i_1},b_{i_2}\rangle$, $1\leqq i_1\leqq i_2\leqq q$, between C_1 and C_1' in $(R(\underline{m}),R(\underline{r}),R\gamma(\underline{m}))$. If $i_1=i_2$, $|A|=|A_{i_1}|$. Suppose that $i_1<i_2$ so that $A=A_{i_1}B_{i_1}\dots B_{i_2-1}A_{i_2}$. Using the density property of $(R(\underline{m}),R(\underline{r}),R\gamma(\underline{m}))$ we get that the number of occupied positions in A is

$$\geqq(1/2-\delta)|A|\geqq(1/2-\delta)(|A_{i_1}|+|B_{i_1}|+\dots+|B_{i_2-1}|+|A_{i_2}|)\,.$$

On the other hand, using the density property of $(\underline{n},\underline{s},\gamma_j(\underline{n}))$, we see that the number of occupied positions in A is

$$\leqq|A_{i_1}|+\dots+|A_{i_2}|+\delta(|B_{i_1}|+\dots+|B_{i_2-1}|)\,.$$

This gives

$$|A|\leqq\left(1+\frac{1/2+\delta}{1/2-2\delta}\right)(|A_{i_1}|+\dots+|A_s|)\leqq3(|A_{i_1}|+\dots+|A_{i_2}|)$$

since $\delta\leqq1/16$. It follows that the total length of the o-blocks in $(R(\underline{m}),R(\underline{r}),R\gamma(\underline{m}))$ between C_1 and C_1' is $<3u_k$. The case when C_1 and C_1' are o-blocks is analogous.

If we let ζ in (3.1) be given by

$$\zeta=4(3+2\lambda)^2+(2+\lambda)C_\lambda\,, \tag{3.11}$$

it is easily shown that

$$\begin{aligned}&w_{j,k}-2C_\lambda w_{j-1}\geqq w_{k-1,k+v'}\,,\\&\lambda^{-1}(w_{j,k}-2C_\lambda w_{j-1})>C_\lambda w_{j-1}\,.\end{aligned} \tag{3.12}$$

The o-(e-)blocks C_1 and C_1' have been built up from o-(e-)blocks in $(\underline{m},\underline{r},\gamma_{j-1}(\underline{m}))$. It follows from $(a)_j$ that there exists o-(e-)blocks C_2 and C_2' in $(\underline{m},\underline{r},\gamma_{j-1}(m))$ contained in C_1 respectively C_1', such that C_2 and C_2' have length $w_{j,k}-2C_\lambda w_{j-1}\geqq w_{k-1,k+v'}$. Assume first that C_2 and C_2' are e-blocks. Let $A_1,\dots,A_p$ be the o-blocks between C_2 and C_2' in $(\underline{m},\underline{r},\gamma_{j-1}(\underline{m}))$. If an o-block containing one or several of $A_1,\dots,A_p$ changes place with an e-block containing one of C_2 or C_2', it follows that the length of this e-block increases by at least $\lambda^{-1}|C_2|$ or $\lambda^{-1}|C_2'|$ respectively, i.e. using (3.12) by at least $C_\lambda w_{j-1}$. But this contradicts $(a)_j$. Thus all $A_1, \ldots, A_p$ are included in o-blocks between C_1 and C'_1 in $(R(\underline{m}), R(\underline{r}), R\gamma(\underline{m}))$ and it follows that $|A_1|+\ldots+|A_p| < 3u_k$.

Assume now that C_2 and C'_2 are o-blocks and let $B_1, \ldots, B_p$ be the e-blocks between C_2 and C'_2 in $(\underline{m}, \underline{r}, \gamma_{j-1}(\underline{m}))$. Suppose that an e-block B^1 containing e-blocks among $B_1, \ldots, B_p$ changes place with an o-block A^1 containing C_2:

$$\ldots A^0 B^0 A^1 B^1 A^2 \ldots \to \ldots A^0 B^0 B^1 A^1 A^2 \ldots$$

By Claim 1 either $|A^1| \leqq |A^2|$, which will contradict $(a)_j$ in the same way as above, or $|B^0| < u_j$ and $|B^1| < \lambda |B^0|$ and consequently $|B^1| < \lambda u_j$. Since $|B^0 B^1| \geqq u_j$ this second case cannot be repeated. The same argument can be applied with C'_2 instead of C_2. It follows that the length of the e-blocks between C_1 and C'_1 is at least $|B_1|+\ldots+|B_p|-2\lambda u_j$. Hence $|B_1|+\ldots+|B_p|-2\lambda u_j < 3u_k$. This establishes the claim and completes the proof of Lemma 3.1. □

The proof of the entropy estimate using Lemma 3.1 is now exactly as the proof of Lemma 3.2 in [I], except that $16w_{j-1}$ is replaced by $2C_\lambda w_{j-1}$ and ζ in (3.1) is not $=9$ but is given by (3.11). This gives $C_1 = \log(C'_1/\delta)$, where C'_1 can be taken to be $=8\zeta$.

3.2. Proof of the Density Property for the Partitions

The proof of Lemma 2.5 is very similar to the proof of the Lemmas 2.3 and 2.6 in [I]. The proof that $(\underline{n}, \underline{s}, \gamma_1(\underline{n}))$ satisfies the density property for all $(\underline{n}, \underline{s}) \in Q_1$ is the same as the proof of Lemma 2.3 in [I]. The only difference is that $\gamma^{(0)}(\underline{n})$ is defined differently. If $\langle a, a' \rangle$ is an o-block in $(\underline{n}, \underline{s}, \gamma^{(0)}(\underline{n}))$ then $n_a = 1, n_{a'-2} = 1, n_{a'-1} = 0$ and we do not have two consecutive zeros in the sequence $n_a, \ldots, n_{a'-1}$. This means that $1-\delta_k$ has to be replaced by $1/2-\delta_k$ everywhere. The proof, by induction on j, that $(\underline{n}, \underline{s}, \gamma_j(\underline{n}))$ has the density property for every $(\underline{n}, \underline{s}) \in Q_j$ is the same as the proof of Lemma 2.6 in [I] except that $1-\delta$ is replaced by $1/2-\delta$.

We will now prove that all blocks in $(\underline{n}, \underline{s}, \gamma_j(\underline{n}))$ have length $\geqq u_j$. Let $A = \langle a, a' \rangle$ be the shortest block in $(\underline{n}, \underline{s}, \gamma_j(\underline{n}))$ and let $B = \langle a', a'' \rangle$ be its right neighbour, $|B| \geqq |A|$. There is a $(\underline{m}, \underline{r}) \in Q_{j-1}$ such that $R(\underline{m}, \underline{r}) = (\underline{n}, \underline{s})$ and $a \in R\gamma(\underline{m})$. If $|B| < u_j$, then $a' \notin R\gamma(\underline{m})$ since all blocks in $(\underline{n}, \underline{s}, R\gamma(\underline{m}))$ have length $\geqq u_j$. The next point, b, to the right of a in $R\gamma(\underline{m})$ is $\geqq a''$. If A is an e-block then B is an o-block and $\langle a, b \rangle$ must be an e-block in $(\underline{n}, \underline{s}, \gamma(\underline{m}))$. The density property gives

$$N(a, a''-1)(\underline{n}) \leqq \delta(a''-a). \tag{3.13}$$

On the other hand the density property for $(\underline{n}, \underline{s}, \gamma_j(\underline{n}))$ gives

$$\begin{aligned} N(a, a''-1)(\underline{n}) &= N(a, a'-1)(\underline{n}) + N(a', a''-1)(\underline{n}) \\ &\geqq 0 + (1/2-\delta)(a''-a') \geqq (1/2-\delta)\tfrac{1}{2}(a''-a), \end{aligned} \tag{3.14}$$

since $|B| \geqq |A|$. Now (3.13) and (3.14) are contradictory if $\delta \leqq 1/16$ so we must have $|A| \geqq u_j$.

Assume now that A is an o-block and hence B is an e-block. Recall that $a < a' < a'' \leqq b$ and $a''-a \geqq a'-a$. $\langle a, b \rangle$ must be an o-block in $(\underline{n}, \underline{s}, R\gamma(\underline{m}))$. We will prove the following property for the o-block $\langle a, b \rangle$:

$$\text{If} \quad a \leqq x < x+s < b \quad \text{and} \quad N(x, x+s)(\underline{n}) \leqq \delta d, \quad \text{then} \quad x-a \geqq 2s. \tag{3.15}$$

Thus $a'-a \geqq 2(a''-a-1)$ and we get a contradiction. Hence if we can prove (3.15) we are finished.

If every o-block $\langle a,b\rangle$ in $(\underline{n},\underline{s},\gamma)$ satisfies (3.15), then so does every o-block in $S(\underline{n},\underline{s},\gamma)$. To see this suppose that the o-blocks A_1 and A_2 have been joined to $A_1A_2 = \langle a,b\rangle$. Let a', $a<a'<b$, be the position of the old partition point. If $a'\in(x,x+s)$, then since $N(x,x+s)(\underline{n})\leqq\delta s$ we must have either $N(x,a'-1)(\underline{n})\leqq\delta(a'-x)$ or $N(a',x+s)(\underline{n})\leqq(x+s-a')$, which both are impossible by the density property. Hence $\langle x,x+s\rangle$ must be completely within A_1 or A_2 and we are done.

Thus if every o-block in $(\underline{m},\underline{r},\gamma_{j-1}(\underline{n}))$, $(\underline{m},\underline{r})\in Q_{j-1}$ satisfies (3.15), then so does every o-block in $(R(\underline{m}),R(\underline{r}),R\gamma(\underline{m}))$ and since o-blocks in $(\underline{n},\underline{s},\gamma_j(\underline{n}))$ are parts of o-blocks in $(R(\underline{m}),R(\underline{r}),R\gamma(\underline{m}))$ for some $(\underline{m},\underline{r})\in Q_{j-1}$, $R(\underline{m},\underline{r})=(\underline{n},\underline{s})$, we see that (3.15) holds for o-blocks in $(\underline{n},\underline{s},\gamma_j(\underline{n}))$. Hence it suffices to show that (3.15) holds for every o-block $\langle a,b\rangle$ in $(\underline{n},\underline{s},\gamma_1(\underline{n}))$ for each $(\underline{n},\underline{s})\in Q_1$. This is done inductively by showing that (3.15) holds for o-blocks in $(\underline{n},\underline{s},\gamma^{(k)}(\underline{n}))$, $k=0,\ldots,vk_N$. That (3.15) is true for $k=0$ is trivial since $N(x,x+s)(\underline{n})\leqq\delta s$ is impossible. The argument is now very similar to the proof of Lemma 4.1 in [I]. Assume that (3.15) is true for o-blocks in $(\underline{n},\underline{s},\gamma^{(k-1)}(\underline{n}))$ and let $A=\langle a,b\rangle$ be an o-block in $(\underline{n},\underline{s},\gamma^{(k)}(\underline{n}))$. If y is the length of the e-blocks in $(\underline{n},\underline{s},\gamma^{(k-1)}(\underline{n}))$ that wholly or partly lie in $\langle x,x+s\rangle$, then just as in the proof of Lemma 4.1 in [I] we get $x-a\geqq v_k-(s-y)$, $y\leqq u_k$, and

$$N(x,x+s)(\underline{n})\geqq(1/2-\delta)(s-y).$$

Together with $N(x,x+s)(\underline{n})\leqq\delta s$ and $\delta\leqq 1/16$ these estimates show that (3.15) holds.

We will now discuss the proof of Lemma 2.6. The proof is almost exactly the same as that of Lemma 3.4 in [I]. Fix $\underline{t}\in T$. If $F(\underline{x})=(\underline{n},\underline{s})$, $\underline{x}\in X$, then

$$d(\tau_1,\tau_2)(\underline{x})=\frac{1}{L(\tau_2-\tau_1)}N([\tau_1L],[\tau_2L])(\underline{n}).$$

Hence if $0\leqq\tau_1<\tau_2\leqq d_1-\varepsilon$, then

$$Q(\underline{t},A_{\tau_1,\tau_2})=\left\{(\underline{n},\underline{s})\in Q(\underline{t},X);\ \frac{1}{L(\tau-\tau_1)}N([\tau_1L],[\tau_2L])(\underline{n})\leqq\frac{1}{2}-2\delta\right\}$$

and similarly for $d_2+\varepsilon\leqq\tau_1<\tau_e\leqq 1$. Thus we can copy the proof of Lemma 3.4 in [I] almost verbatim, except that $1-2\delta$ and $1-\delta$ must be replaced by $1/2-2\delta$ respectively $1/2-\delta$. The only other modification is that $8w_{j-1}$ in [I] is replaced by $C_\lambda w_{j-1}$ and formula (4.4) in [I] changes to

$$\lambda-v_1\leqq C_\lambda\sum_{j=2}^{k_N}w_{j-1}\leqq C'_\lambda N^\gamma$$

with $\gamma=\omega\log 4\zeta$, where ω comes from (2.3) and ζ from (3.1). Put

$$\omega=\frac{1}{2\log 4\zeta} \tag{3.16}$$

so that $\gamma=1/2$; ω depends only on α.

We can now verify (2.10). We have

$$2\kappa\beta\geqq 8\log(C'_1/\delta_0)\geqq 8\log(16\cdot 8\zeta)\geqq 8\log 4\zeta$$

since $\delta\leqq\delta_0\leqq 1/16$. Hence $2\kappa\beta\omega\geqq 4$.

4. Proof of the Energy Estimate

Let $(\underline{n}, \underline{s})$ be a configuration and γ a partition such that $(\underline{n}, \underline{s}, \gamma)$ has the density property. Denote by $A_1, B_1, \ldots, A_r, B_r$ the blocks in $(\underline{n}, \underline{s}, \gamma)$. By assumption all the blocks have length $\geqq u_j$. An elementary rearrangement is always of the form that an o-block, A_k say, changes place with one of its neighbouring e-blocks, B_{k-1} or B_k. Recall that these operations are denoted by $S_{2k-2,2k-1}$ respectively $S_{2k-1,2k}$. Let

$$\Delta E_1 = H(S_{2k-2,2k-1}(\underline{n}, \underline{s})) - H(\underline{n}, \underline{s}),$$
$$\Delta E_2 = H(S_{2k-1,2k}(\underline{n}, \underline{s})) - H(\underline{n}, \underline{s}).$$

We want to show that:

(i) If $|B_k| \geqq \lambda |B_{k-1}|$ and $|A_{k-1}| \geqq |A_k|$, then $\Delta E_1 \geqq 2\kappa j$.

(ii) If $|B_{k-1}| \geqq \lambda |B_k|$ and $|A_k| \geqq |A_{k+1}|$, then $\Delta E_2 \geqq 2\kappa j$.

(iii) If $\lambda^{-1}|B_{k-1}| \leqq |B_k| \leqq \lambda |B_{k-1}|$, then $\max\{\Delta E_1, \Delta E_2\} \geqq 2\kappa j$.

Here κ is a constant that only depends on α. Write $A_{k-1} = \langle a_1, b_1 \rangle$, $B_{k-1} = \langle b_1, a_2 \rangle$, $A_k = \langle a_2, b_2 \rangle$, $B_k = \langle b_2, a_3 \rangle$, and $A_{k+1} = \langle a_3, b_3 \rangle$. The lengths of A_{k-1}, B_{k-1}, A_k, B_k, A_{k+1} are respectively x_1, y_1, x_2, y_2, and x_3.

We write $\Delta E_1 = \Delta E_1^0 - \Delta E_1^1$ and $\Delta E_2 = \Delta E_2^0 - \Delta E_2^1$, where ΔE_1^0 and ΔE_2^0 are the changes in energy which we would have if the e-blocks B_{k-1} and B_k were empty, and ΔE_1^1 and ΔE_2^1 are the changes in energy due to the particles in B_{k-1} and B_k. Then

$$\Delta E_1^0 = \sum_{\substack{i < b_1 \\ a_2 \leqq j < b_2}} n_i n_j ((j - i + s_j - s_i - y_1)^{-\alpha} - (j - i + s_j - s_i)^{-\alpha})$$
$$- = \sum_{\substack{a_2 \leqq j < b_2 \\ k \geqq a_3}} n_j n_k ((k - j + s_k - s_j)^{-\alpha} - (k - j + s_k - s_j + y_1)^{-\alpha})$$

and

$$\Delta E_2^0 = \sum_{\substack{a_2 \leqq j < b_2 \\ k \geqq a_3}} n_j n_k ((k - j + s_k - s_j - y_2)^{-\alpha} - (k - j + s_k - s_j)^{-\alpha})$$
$$= \sum_{\substack{i < b_1 \\ a_2 \leqq j < b_2}} n_i n_j ((j - i + s_j - s_i)^{-\alpha} - (j - i + s_j - s_i + y_2)^{-\alpha}).$$

If we write $\sigma_{ij} = s_{a_2+j} - s_{b_1-i}$ and $\tau_{ik} = s_{a_3+k} - s_{b_2-j}$ these formulas can be rewritten as

$$\Delta E_1^0 = \sum_{i=1}^{\infty} \sum_{j=0}^{x_2-1} n_{b_1-i} n_{a_2+j} ((j + i + \sigma_{ij})^{-\alpha} - (j + i + \sigma_{ij} + y_1)^{-\alpha})$$
$$- \sum_{j=1}^{x_2} \sum_{k=0}^{\infty} n_{b_2-j} n_{a_3+k} ((k + j + \tau_{jk} + y_2)^{-\alpha} - (k + j + \tau_{jk} + y_2 + y_1)^{-\alpha})$$

and

$$\Delta E_2^0 = \sum_{j=1}^{x_2} \sum_{k=0}^{\infty} n_{b_2-j} n_{a_3+k} ((k + j + \tau_{jk})^{-\alpha} - (k + j + \tau_{jk} + y_2)^{-\alpha})$$
$$- \sum_{i=1}^{\infty} \sum_{j=0}^{x_2-1} n_{b_1-i} n_{a_2+j} ((j + i + \sigma_{ij} + y_1)^{-\alpha} - (j + i + \sigma_{ij} + y_1 + y_2)^{-\alpha}).$$

We will use the following facts, the proofs of which will be sketched at the end of the section.

(a) If $1 \leqq x \leqq 2z$, then

$$x^{-\alpha}-(x+z)^{-\alpha} \geqq (1-(2/3)^{\alpha})x^{-\alpha}.$$

(b) For x, y_1, y_2 define

$$f(x, y_1, y_2)=\frac{y_2}{y_1}(x^{-\alpha}-(x+y_1)^{-\alpha})-((x+y_1)^{-\alpha}-(x+y_1+y_2)^{-\alpha}).$$

Then $f(x, y_1, y_2)>0$ and $f(x, y_1, y_2)$ is a decreasing function of x for fixed y_1, y_2. Furthermore there are constants c_1 and c_2, depending only on α, such that, if $1 \leqq x \leqq c_1 y_1$ and $y_2/y_1 \geqq 1/\lambda$, then

$$f(x, y_1, y_2) \geqq c_2/x^{\alpha}. \tag{4.1}$$

We assume to begin with that $1<\alpha<2$. Consider first the case (i). Then $y_2 \geqq \lambda y_1$ and $x_1 \geqq x_2 \geqq 2$. Let us prove a lower bound on ΔE_1^0.

From the definition of o-blocks we know that $n_{b_1-1}=0$ and the density property gives

$$\begin{aligned} &\sum_{i=1}^{p} n_{b_1-i} \geqq (1/2-\delta)(p-1), \quad 1 \leqq p \leqq x_1, \\ &\sum_{j=0}^{p} n_{a_2+j} \geqq (1/2-\delta)(p+1), \quad 0 \leqq p \leqq x_2-1. \end{aligned} \tag{4.2}$$

If we use $0 \leqq n_i \leqq 1$ and $-1 \leqq \sigma_{ij}, \tau_{jk} \leqq 1$ we obtain

$$\begin{aligned} \Delta E_1^0 \geqq &\sum_{i=1}^{x_1} \sum_{j=0}^{x_2-1} n_{b_1-i} n_{a_2+j}((j+i+1)^{-\alpha}-(j+i+1+y_1)^{-\alpha}) \\ &-\sum_{j=1}^{x_2} \sum_{k=0}^{\infty}((j+k-1+y_2)^{-\alpha}-(j+k-1+y_2+y_1)^{-\alpha}). \end{aligned}$$

A summation by parts using (4.2) gives

$$\begin{aligned} \Delta E_1^0 \geqq (1/2-\delta)^2 &\sum_{i=2}^{x_1} \sum_{j=0}^{x_2-1}((j+i+1)^{-\alpha}-(j+i+1+y_1)^{-\alpha}) \\ &-\sum_{j=1}^{x_2} \sum_{k=0}^{\infty}((j+k-1+y_2)^{-\alpha}-(j+k-1+y_2+y_1)^{-\alpha}). \end{aligned} \tag{4.3}$$

Let $z=\min\{x_2, y_1\}$ and introduce the function

$$g_\alpha(z)=(2-\alpha)^{-1}(z^{2-\alpha}-1)+1.$$

We want to show that if we choose λ sufficiently large, depending on α, then $\Delta E_1^0 \geqq c_3 g_\alpha(z)$ for some constant $c_3>0$ that only depends on α. Consider the first double sum in (4.3) and assume that $z \geqq 2$. Using property (a) above and estimating sums by integrals obtain

$$\begin{aligned} &\sum_{i=2}^{x_1} \sum_{j=0}^{x_2-1}((j+i+1)^{-\alpha}-(j+i+1+y_1)^{-\alpha}) \\ &\geqq \sum_{i=2}^{z} \sum_{j=1}^{z}((j+i)^{-\alpha}-(j+i+y_1)^{-\alpha}) \\ &\geqq (1-(2/3)^{\alpha}) \sum_{i=2}^{z} \sum_{j=1}^{z}(j+i)^{-\alpha} \geqq (1-(2/3)^{\alpha}) c g_\alpha(z), \end{aligned} \tag{4.4}$$

where c only depends on α. This is easily checked to hold also for $z=1$. Now consider the second sum in (4.3). Cancellation between terms and estimation of sums by integrals gives

$$\begin{aligned}
&\sum_{j=1}^{x_2}\sum_{k=0}^{\infty}((j+k-1+y_2)^{-\alpha}-(j+k-1+y_2+y_1)^{-\alpha})\\
&=\sum_{j=1}^{x_2}\sum_{k=0}^{\infty}(j+k-1+y_2)^{-\alpha}\leqq\sum_{j=1}^{z}\sum_{k=0}^{\infty}(j+k+y_2-1)^{-\alpha}\\
&\leqq(1-\alpha)^{-1}(2-\alpha)^{-1}[(z+y_2-2)^{2-\alpha}-(y_2-2)^{2-\alpha}]\\
&\leqq c'[(\lambda-1)^{2-\alpha}-(\lambda-2)^{2-\alpha}]g_\alpha(z), \qquad (4.5)
\end{aligned}$$

where c' only depends on α. We have used the fact that $y_2\geqq\lambda y_1\geqq\lambda z$. If we use $\delta\leqq 1/16$ we get $(1/2-\delta)^2\geqq 1/6$, and combining (4.4) and (4.5) we see that by choosing λ sufficiently large, depending on α, we get $\Delta E_1^0\geqq c_3 g_\alpha(z)$.

We must also estimate the effect on energy changes, ΔE_1^1, of particles in B_{k-1} and B_k. If we only consider energy losses and not energy gains, there are three quantities to be estimated: the change in interaction energy between B_{k-1} and everything to the left of B_{k-1}, between B_{k-1} and A_k, and between A_k and B_k. These quantities are all estimated in a similar way and we only treat the first one. The density property gives

$$\sum_{j=0}^{p} n_{b_1+j}\leqq\delta p,\qquad 0\leqq p\leqq y_1-1.$$

Using this in a summation by parts, $0\leqq n_i\leqq 1$ and cancellation between terms we see that the change in interaction energy between B_{k-1} and everything to the left of B_{k-1} is

$$\begin{aligned}
&\sum_{\substack{i<b_1\\ b_1\leqq j<a_2}} n_i n_j[(j+s_j-i-s_i)^{-\alpha}-(j+s_j-i-s_i+x_2)^{-\alpha}]\\
&\leqq\sum_{i=1}^{\infty}\sum_{j=1}^{y_1-1} n_{b_1+j}[(j+i-1)^{-\alpha}-(j+i-1+x_2)^{-\alpha}]\\
&\leqq\sum_{i=0}^{x_2-1}\sum_{j=1}^{y_1-1} n_{b_1+j}(i+j)^{-\alpha}\leqq\delta\sum_{i=0}^{x_2-1}\sum_{j=1}^{y_1-1}(i+j)^{-\alpha}\\
&\leqq\delta c_4 g_\alpha(z),
\end{aligned}$$

where $c_4>0$ only depends on α. The estimates for the other quantities are the same and we get

$$\Delta E_1^1\leqq 3\delta c_4 g_\alpha(z). \qquad (4.6)$$

Thus

$$\Delta E_1\geqq(c_3-3\delta c_4)g_\alpha(z)\geqq 2\kappa(\log z+1)\geqq 2\kappa j,$$

if $\delta\leqq c_3/6c_4$ and $\kappa\leqq\frac{1}{4}c_3\log 2$. The second inequality follows from the fact that as $\alpha\nearrow 2$, $g_\alpha(z)\searrow 1+\log z$. It can be checked that c_3 and c_4 remain positive as $\alpha\nearrow 2$, so the same estimate holds for $\alpha=2$. The last inequality comes from $z\geqq 2^{j-1}$.

Claim (ii) is handled in exactly the same way and one proves that $\Delta E_2^0\geqq c_3 g_\alpha(z')$ and

$$\Delta E_2^1\leqq 3\delta c_4 g_\alpha(z'), \qquad (4.7)$$

where $z'=\min\{x_2, y_2\}$.

It remains to treat Claim (iii). We thus assume that $\lambda^{-1} \leqq y_2/y_1 \leqq \lambda$ and we will prove that

$$\frac{y_2}{y_1} \Delta E_1^0 + \Delta E_2^0 \geqq c_5 g_\alpha(\zeta), \tag{4.8}$$

where $\zeta = \min\{x_1, y_1, x_2, y_2, x_3\}$. From this it follows immediately that

$$\max\{\Delta E_1^0, \Delta E_2^0\} \geqq \frac{c_5}{\lambda+1} g_\alpha(\zeta). \tag{4.9}$$

Using (4.6), (4.7), and (4.9) we want to conclude that

$$\max\{\Delta E_1, \Delta E_2\} \geqq c_6 g_\alpha(\zeta) \geqq 2\kappa j, \tag{4.10}$$

where $c_6 > 0$ only depends on α, and $\kappa \leqq \frac{1}{2} c_6 \log 2$. There are three possibilities. Either A_k is the shortest block and $z = z' = \zeta$, or B_{k-1} is shortest, $z = \zeta$ and $z' \leqq \lambda\zeta$ since $y_2 \leqq \lambda y_1$, or B_k is shortest, $z' = \zeta$ and $z \leqq \lambda\zeta$ since $y_1 \leqq \lambda y_2$. Thus we always have $z, z' \leqq \lambda\zeta$ and (4.6) and (4.7) give, after some computation, that

$$\max\{\Delta E_1^1, \Delta E_2^1\} \leqq \delta\lambda c_4 g_\alpha(\zeta). \tag{4.11}$$

Equation (4.10) follows from (4.9) and (4.11) if we assume that $\delta \leqq c_5(2\lambda(\lambda+1)c_4)^{-1}$. Hence we know that (i)–(iii) hold with $\kappa = \min\{\frac{1}{4} c_3 \log 2, \frac{1}{2} c_6 \log 2\}$ if

$$\delta_0 \leqq \min\{c_3/6c_4, c_5(2\lambda(\lambda+1)c_4)^{-1}\} = c_7. \tag{4.12}$$

We still have to prove (4.9). If f is defined as in (b), then

$$\frac{y_2}{y_1} \Delta E_1^0 + \Delta E_2^0 = \sum_{i=1}^{\infty} \sum_{j=0}^{x_2-1} n_{b_1-i} n_{a_2+j} f(i+j+\sigma_{ij}, y_1, y_2) + \sum_{j=1}^{x_2} \sum_{k=0}^{\infty} n_{b_2-j} n_{a_3+k} f(k+j+\tau_{jk}, y_1, y_2).$$

Using the properties (b) of f and the density property we can sum by parts and get

$$\frac{y_2}{y_1} \Delta E_1^0 + \Delta E_2^0 \geqq \left(\frac{1}{2} - \delta\right)^2 \left[\sum_{i=2}^{x_1} \sum_{j=0}^{x_2-1} f(i+j+1, y_1, y_2) + \sum_{j=2}^{x_2} \sum_{k=0}^{x_3-1} f(k+j+1, y_1, y_2) \right]. \tag{4.13}$$

Let c_1 be the constant in (b). If $c_1\zeta/2 < 2$ we estimate (4.13) by keeping only the first term in the sums.

$$\frac{y_2}{y_1} \Delta E_1^0 + \Delta E_2^0 \geqq \frac{2}{6} f(3, y_1, y_2) \geqq c_8 \geqq \frac{c_8}{\zeta}(1 + \zeta - 1) \geqq \frac{4c_8}{c_1} g_\alpha(\zeta).$$

Here we have used the fact that $f(3, y_1, y_2) \geqq 3c_8$ if $y_1, y_2 \geqq 1$, where $c_8 > 0$ only depends on α. To get this estimate we can argue as follows. If $c_1 y_1 < 3$, there are only finitely many possibilities for y_1, y_2 and we can take $3c_8$ less than the smallest of the possible values of $f(3, y_1, y_2)$, which are all positive. If $c_1 y_1 \geqq 3$ we can use (4.1).

If $c_1\zeta/2 \geqq 2$ we use (4.1) to get

$$\frac{y_2}{y_1}\Delta E_1^0 + \Delta E_2^0 \geqq \frac{c_2}{3}\sum_{i=2}^{[c_1\zeta/2]}\sum_{j=1}^{[c_1\zeta/2]}(i+j)^{-\alpha} \geqq c_9 g_\alpha(\zeta),$$

where c_9 only depends on α. Equation (4.8) now follows with $c_5 = \min\{c_9, 4c_8/c_1\}$.

We will now sketch the proofs of (a) and (b) above. (a) is obtained as follows:

$$x^{-\alpha} - (x+z)^{-\alpha} = x^{-\alpha}\left(1 - \left(\frac{x}{x+z}\right)^\alpha\right) \geqq x^{-\alpha}\left(1 - \left(\frac{2}{3}\right)^\alpha\right)$$

if $1 \leqq x \leqq 2z$. That $f > 0$ follows immediately from the strict convexity of $x \to 1/x^\alpha$, and that f is decreasing as a function of x follows from $\partial f/\partial x < 0$, which is a consequence of the strict convexity of $x \to 1/x^{\alpha+1}$. The inequality (4.1) is obtained as follows:

$$\begin{aligned} f(x, y_1, y_2) &\geqq \frac{1}{\lambda}\left(\frac{1}{x^\alpha} - \frac{1}{(x+y_1)^\alpha}\right) - \frac{1}{(x+y_1)^\alpha} \\ &= \frac{1}{x^\alpha}\left(\frac{1}{\lambda} - \left(1 + \frac{1}{\lambda}\right)\left(\frac{x}{x+y_1}\right)^\alpha\right) \\ &\geqq \frac{1}{x^\alpha}\left(\frac{1}{\lambda} - \left(1 + \frac{1}{\lambda}\right)\left(\frac{c_1}{c_1+y_1}\right)^\alpha\right) \geqq \frac{c_2}{x^\alpha}, \end{aligned}$$

where $c_2 > 0$ if c_1 is chosen sufficiently small. This completes the proof of the energy estimate (2.6).

References

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