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Topological Couplings and Contact Terms in 2d Field Theory

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Abstract. In string theory and in topological quantum field theory one encounters operators whose effect in correlation functions is simply to measure the topology of 2d spacetime. In particular these "dilaton"-type operators count the number of other operators via contact terms with the latter. While contact terms in general have a reputation for being convention-dependent, the ones considered here are well-defined by virtue of their simple geometrical meaning: they reflect the geometry of the stable-curve compactification. We give an unambiguous prescription for their evaluation which involves no analytic continuation in momenta.

1. Introduction

Usually the various n-point correlation functions of a quantum field theory are related to each other in complicated, indirect ways. One obtains such relations, for example, from unitarity of the scattering matrix. In certain limiting cases one obtains very simple results, for example the famous low-energy theorems relating amplitudes with and without a zero-momentum pion. String theory reflects such relations on scattering amplitudes via affiliated relations between an N-point amplitude and the *integral* of an $(N+1)$-point amplitude over the location of the last point. More generally, the latter integral can be regarded as the change of the n-point functions of some conformal field theory as we deform the theory slightly. The statement then becomes that certain deformations modify amplitudes in very simple ways.

The most famous low-energy theorem in string theory is the statement that the zero-momentum mode of the dilaton is the string *coupling constant*. Thus deforming any amplitude by this operator merely *multiplies* it by $(1+\varepsilon)^M$, where ε is the strength of the perturbation and M is the order in which the given n-point function enters in string perturbation theory. In other words, the $(N+1)$-point function involving a zero-momentum dilaton, integrated over 2d spacetime, must equal M

times the corresponding N-point answer; M in turn is proportional to $2g-2+N$, where g describes the topology of spacetime. But other 2d theories can have additional relations of this sort. Topological gravity [1–3] is an example par excellence: according to [4], *all* amplitudes are determined from just a couple of basic ones by application of a family of recursion formulas generalizing the one just mentioned.

Suppose that $N=0$, so that we are considering the dilaton 1-point function. How can the answer be a topological invariant, independent of the shape of the spacetime Σ? The basic observation is that the dilaton corresponds to a field $c\partial^2 c - \bar{c}\bar{\partial}^2\bar{c}$ in the 2d theory which fails to satisfy the usual physical state condition, and so its insertion depends on a choice of normal ordering. Similar states show up in the heterotic string [5] and superstring. If we normal-order with the help of a metric on Σ ("Weyl normal ordering"), we get a curvature factor whose integral is the Euler number [6]. If we normal-order by choosing local coordinate families we directly recover the Cech definition of the Euler number [7, 5].

It remains to understand $N>0$. At first it may seem that we again get $2g-2$, since the obstruction to finding a smooth family of coordinates seems to depend on g, regardless of other insertions. Similarly, the choice of a smooth metric again leads to the integral of the curvature. We could propose to use a metric with singularities at the N "other" points [4], but this prescription raises various troubling issues. It seems to depend on which vertex we choose to integrate first; apparently we can choose any kind of singularity we like. Nor is this an unfamiliar lament. To count the number of other punctures we want the dilaton vertex to have "contact terms," δ-functions in its correlations with anything else. Such terms do not seem to appear in the original framework of CFT [8]. They have in other contexts been interpreted as convention-dependent, essentially describing a connection over the space of CFT's [9] whose magnitude can be adjusted by some kind of "gauge transformation." And yet in the present context we claim that they are well defined and moreover universal, the same regardless of what operator the dilaton hits.[1] This suggests an interpretation for them purely in terms of the geometry of moduli space.

We will give such an interpretation for the bosonic string dilaton, leaving topological gravity for the future. We will review the ideas of [10, 11], as well as the insertion prescription of [5, 7]. Then we will develop the notion of a degenerating family of punctured surfaces with coordinates.[2] Our construction has both a clear physical and mathematical meaning, and it gives both the contact term and the dilaton–dilaton pole we need. Presumably in topological gravity the pole is absent, since there is no tachyon.

Other approaches to 2d contact terms in string theory require the use of 2*d* auxiliary fields [18], analytic continuation in momentum, or even relaxing conservation of momentum [19]. In more general 2d theories it is not clear what

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[1] There is an exception when *two* dilatons collide. In the bosonic string two dilatons at zero momentum can fuse to form a tachyon and hence a quadratic divergence. This pole too is hard to obtain if one simply considers one dilaton as measuring curvature induced by the other, but we will see it emerging automatically from the geometry.

[2] Sonoda and Zwiebach have addressed this issue in the framework of string field theory [12, 13]. For discussions of the stable-curve compactification see e.g. [14, 15], and in the present context [16, 17]

these mean; for example the bosonic string has no auxiliary fields, whie topological gravity has no momentum! Still other calculations write the amplitude as the integral of a total derivative [20]; here the delicate issue is evaluating the boundary contribution. The present approach has none of these technical issues. We simply get a smoothed-out δ-function when we interpolate between normal ordering appropriate to a singular surface and that appropriate to a non-singular one. The answer does not depend on how we do the interpolation, except of course for the divergent two-dilaton case.

Finally, the reader may wonder at all the formalism lavished on just one field, $c\partial^2 c - \bar{c}\bar{\partial}^2\bar{c}$. Can we not just omit it? The answer is no. This state appears in the propagator; amplitudes factorize on it as is crucial for consistency of loop corrections [5, 6]. Moreover, as mentioned, in topological gravity *all* states are generalizations of this one; all fail the usual physical state condition, either in the formalism of [4] or in that of [2]. This is of course precisely why they can all measure topological properties of the line bundles $\mathscr{L}_i$ described in [1]. Nor is the need for local coordinates at insertions some sort of pathology of the operator formalism. Any method of defining amplitudes with fields of this sort will have to face the same problem (see e.g. [21]).

2. Operator Formalism

2.1. The Measure. We recall how one builds a measure in the operator formalism, following [6, 7, 11]. The basic observation is that to insert an arbitrary state $|\psi\rangle$ into a Riemann surface Σ we need to choose not a point but a finite parametrized *hole* on Σ. Equivalently one can choose (the germ of) a local complex coordinate z, and consider the loop $\{z = e^{i\theta}, 0 < \theta < 2\pi\}$. We can then remove the disk inside the loop and put boundary conditions on the worldsheet fields corresponding to $|\psi\rangle$. The resulting path integral may be regarded as the pairing of $|\psi\rangle$ with a state $\langle\Sigma, z|$. Conformal invariance implies that this state depends only on the isomorphism class of the pair (Σ, z). Accordingly we introduce the space $\mathscr{P}_{g,1}$ of such isomorphism classes. This is an infinite-dimensional space. Clearly we can map $\mathscr{P}_{g,1} \xrightarrow{\pi} \mathscr{M}_{g,1}$ by forgetting everything about z except the location of $P = \{z = 0\}$. When we have several punctures we similarly construct $\mathscr{P}_{g,N} \xrightarrow{\pi} \mathscr{M}_{g,N}$.

The Virasoro algebra and its conjugate act on $\mathscr{P}_{g,1}$. Replacing z by $\varphi \circ z = z - \varepsilon z^{n+1}$, $n \geqq 0$ makes an infinitesimal motion in $\mathscr{P}_{g,1}$ which we associate to the real vector[3] $\varepsilon\ell_n + \bar{\varepsilon}\bar{\ell}_n \in \mathrm{Vect} \otimes \overline{\mathrm{Vect}}$. Here Vect is the Lie algebra of meromorphic vector fields on $\mathbb{C}$ generated by $\ell_n = -z^{n+1}\dfrac{\partial}{\partial z}$. For $n < 0$ ℓ_n again acts, in general to change (Σ, P) as well as z. We cut Σ into $D = \{|z| < 1\}$ and $\Sigma \backslash D$, then reglue these pieces with the help of $z \to \varphi(z)$. We will refer to this operation as "Schiffer variation of (Σ, z) by φ at P." For example $\varepsilon\ell_{-1} + \bar{\varepsilon}\bar{\ell}_{-1}$ takes (Σ, z) to $(\Sigma, z - \varepsilon)$ and so moves P. For $n \leqq -1$, however, the action is trivial. Converting the vector field

[3] The appearance of the conjugate term may seem surprising. Bear in mind that (Σ, z) and $(\Sigma, \varphi \circ z)$ are both points of the manifold $\mathscr{P}_{g,1}$ and as such must be connected by a *real* tangent vector. But $\ell_{-1}, \bar{\ell}_{-1}$ separately generate *complex* tangents in $T^{1,0}\mathscr{P}$ and $T^{0,1}\mathscr{P}$, respectively

$z^{n+1}\frac{\partial}{\partial z}$ on $\mathbb{C}$ to a holomorphic vector field v and Σ near P, if v extends holomorphically to to $\Sigma\backslash\{P\}$ then the regluing just mentioned can be undone by an analytic map of $\Sigma\backslash D$ to itself. Thus we introduce the vector space $B(\Sigma, z)$ of vector fields on $\mathbb{C}$ whose images v on Σ near P extend to $\Sigma\backslash P$, and we find $T^{1,0}\mathscr{P}_{g,1}|_{(\Sigma,z)} \cong \mathrm{Vect}/B(\Sigma, z)$. Similarly one finds

$$T^{1,0}\mathscr{P}_{g,N}|_{(\Sigma,z_1,\ldots)} = [\mathrm{Vect} \oplus \cdots \oplus \mathrm{Vect}]/B(\Sigma, z_1, \ldots, z_N). \tag{2.1}$$

This time the "Borel subalgebra" $B(\Sigma, z_1, \ldots, z_N)$ consists of N-tuples of vector fields in Vect, $(v_1, \ldots, v_N)$. Taking each v_i to a vector field on Σ near $P_i = \{z_i = 0\}$ using z_i, all must be the restrictions of a single v holomorphic on $\Sigma\backslash\{P_1, \ldots, P_n\}$.

If $\vec{v} \equiv (v_1, \ldots, v_N)$ does *not* lie in $B(\Sigma, z_1, \ldots)$ then it generates a nonzero tangent vector $\tilde{V}_{\vec{v}}$ to $\mathscr{P}_{g,N}$ at $(\Sigma, z_1, \ldots)$. We want to show that given any set of states $|\psi_1\rangle, \ldots, |\psi_N\rangle$ we get a differential form on $\mathscr{P}_{g,N}$ of degree $3g-3+N$. Let

$$b[\vec{v}] = \sum_{i=1}^{N} \oint b_{zz}^{(i)}(z) v_i^z(z) dz,$$

an operator on $\mathscr{H}^{\otimes N}$. Here $b_{zz}^{(i)}(z)$ is the ghost field, an operator-valued form on the z-plane acting on the i-th copy of $\mathscr{H}$. Then we let

$$\begin{aligned}\Omega(\tilde{V}_1, \ldots, \tilde{V}_{3g-3+N}, \bar{\tilde{V}}_1, \ldots, \bar{\tilde{V}}_{3g-3+N})|_{(\Sigma,z_1,\ldots)} \\ \equiv \langle \Sigma, z_1, \ldots | b[\vec{v}_1] \cdots b[\vec{v}_{3g-3+N}] \bar{b}[\vec{v}_1] \cdots \bar{b}[\vec{v}_{3g-3+N}] |\psi_1\rangle_{P_1} \otimes \cdots \otimes |\psi_N\rangle_{P_N}.\end{aligned} \tag{2.2}$$

Here $\tilde{V}_i$ are in $T^{1,0}\mathscr{P}_{g,N}$ at $(\Sigma, z_1, \ldots)$, $\bar{\tilde{V}}_i$ are their conjugates, and $\vec{v}_i$ is any N-tuple representing $\tilde{V}_i$. The subscript on $|\psi_i\rangle_{P_i}$ means it is to be paired with the i-th slot of the vector $\langle \Sigma, z_1, \ldots|$.

Expression (2.2) can be generalized in the obvious way to arbitrary complex vectors $\tilde{V}_i \in T_{\mathbb{C}}\mathscr{P}$, with representatives $\vec{v}_i = (v_{i1}, \bar{v}_{i1}, \ldots) \in \mathrm{Vect} \otimes \overline{\mathrm{Vect}} \oplus \cdots$ and

$$b[\vec{v}_i] = \sum_{j=1}^{N} [\oint b_{zz}^{(j)}(z) v_{ij}^z(z) dz + \oint \bar{b}_{\bar{z}\bar{z}}^{(j)}(\bar{z}) \bar{v}_{ij}^{\bar{z}}(\bar{z}) d\bar{z}].$$

Then $\tilde{\Omega}$ is certainly antisymmetric in the $\tilde{V}$'s. It is also well defined. For, suppose $\vec{v}_i \in B(\Sigma, z_1, \ldots)$. Then $\langle \Sigma, z_1, \ldots | b[\vec{v}_i] = 0$ by the operator formalism construction of $\langle \Sigma, z_1, \ldots |$(see Ref. [11]). In path integral language we can take one of the terms $\oint b^j v_{ij} dz$ and deform the contour to the other insertion points, where it cancels the other terms.

Of course we do not really want a form $\tilde{\Omega}$ on $\mathscr{P}_{g,N}$. We instead want a form Ω on $\mathscr{M}_{g,N}$, so that we can integrate it to get answers. In fact, however, one sometimes finds that $\tilde{\Omega}$ is simply the lift of the desired form:

$$\tilde{\Omega} = \pi^* \Omega. \tag{2.3}$$

This happens whenever the $|\psi_i\rangle$ all satisfy [11],

$$\text{SPSC:} \quad L_n|\psi\rangle = b_n|\psi\rangle = \bar{L}_n|\psi\rangle = \bar{b}_n|\psi\rangle = 0 \quad n \geqq 0. \tag{2.4}$$

We will call (2.4) the "strong physical state condition." As is well known, states $|\psi\rangle$ of ghost number 2 obeying the SPSC correspond to transverse, on-shell particles in string theory.

2.2. Weak Physical States. Another option was given in [7]. Even if $|\psi_i\rangle$ fail the SPSC we can imagine choosing a section $\sigma:\mathscr{M}_{g,N}\to\mathscr{P}_{g,N}$ and taking

$$\Omega=\sigma^*\tilde{\Omega}. \tag{2.5}$$

Thus very roughly speaking, given $(\Sigma,P_1,\dots,P_N)$ we plug the $(\Sigma,z_1,\dots,z_N)$ given by σ into (2.2). Of course now the measure Ω depends on the choice of σ, but this is only what one expects from off-shell states. It may happen, however, that $\tilde{\Omega}$ is a *closed* differential form. In this case if we change σ to a nearby σ', the difference can be written as the action of a vertical vector field $\tilde{V}$ on $\mathscr{P}$. Then for $\varepsilon\ll 1$,

$$\Omega'=(\sigma')^*\tilde{\Omega}=\sigma^*(\tilde{\Omega}+\varepsilon\mathscr{L}_{\tilde{V}}\tilde{\Omega})=\Omega+\varepsilon d[\sigma^*(\iota_{\tilde{V}}\tilde{\Omega})], \tag{2.6}$$

since $d\tilde{\Omega}=0$. Thus the measure is a well-defined *cohomology* class. If we specify the behavior of σ at the relevant boundaries of $\mathscr{M}_{g,N}$ then we get a well-defined *integral* for Ω.

As is well known, $\tilde{\Omega}$ is indeed closed whenever the $|\psi_i\rangle$ obey a weaker condition than (2.4), namely

$$(Q_B+\bar{Q}_B)|\psi\rangle=0, \tag{2.7}$$

where Q_B is the BRST operator. Thus we admit states which are not transverse, though still on shell.

Actually (2.7) is too weak. It is not possible to find a global smooth section σ. Consider for example $\mathscr{M}_{g,1}$. A choice of σ means that for every $P\in\Sigma$ we have $z_P(\cdot)$ depending parametrically on P and centered at P i.e., $z_P(P)=0$. Thus $dz_P|_P$ is a nonzero 1-form at P. Varying P we get a smooth, nowhere-zero 1-form field on Σ, which is impossible if $g\neq 1$. However, there is a way out. We *can* always find a section everywhere smooth and defined up to constant phases. Thus as P varies we allow $z_P(Q)$ to jump to $e^{i\alpha(P)}z_P(Q)$, where α is a real function independent of Q. If $\tilde{\Omega}$ does not "feel" such jumps then we again get a good Ω from (2.5).[4] The conditions for this to happen are a subset of (2.4) [7]:

$$(Q_B+\bar{Q}_B)|\psi\rangle=(L_0-\bar{L}_0)|\psi\rangle=(b_0-\bar{b}_0)|\psi\rangle. \tag{2.8}$$

At this point it is convenient to note that the dilaton state, as well as its cousins in topological gravity, all satisfy a slightly stronger condition than (2.8). While not strictly necessary, this stronger condition will save us a bit of algebra. Accordingly we introduce the "weak physical state condition"

$$\text{WPSC:}\quad Q_B|\psi\rangle=L_0|\psi\rangle=b_0|\psi\rangle=\bar{Q}_B|\psi\rangle=\bar{L}_0|\psi\rangle=\bar{b}_0|\psi\rangle=0. \tag{2.9}$$

For states obeying the WPSC both the overall scale and phase of the family $z_P(\cdot)$ are irrelevant. Higher Taylor coefficients *will* in general matter, however. For example the state

$$|D\rangle=(c_1c_{-1}-\bar{c}_1\bar{c}_{-1})|\mathbf{1}\rangle$$

can tell the difference between z_P and $z_P+(z_P)^2$. Here $|\mathbf{1}\rangle$ is the $SL(2)$-invariant state.

We should pause to comment on the generality of our prescription. When

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[4] Recently an explicit slice σ, defined up to phase, has been constructed using string field theory methods by Zwiebach [13]. It would be very interesting to apply this slice to topological gravity computations

constructing $\tilde{\Omega}$ any holomorphic anticommuting (2, 0) field $\mathscr{B}$ would have worked in place of b, since all we required was that $\oint \mathscr{B} v dz$ be invariant and that contour deformation worked. When constructing Ω we could have used any operator $\mathscr{Q}$ with $\mathscr{Q}^2 = 0$ and $\{\mathscr{Q}, \mathscr{B}\} = T$, $\mathscr{Q} = \oint \mathscr{J}$ for some holomorphic current $\mathscr{J}$. For, then $\sum_{i=1}^{N} \langle \Sigma, z | \mathscr{Q}^{(i)} = 0$ by contour deformation and the appropriately modified (2.7) guarantees that $\tilde{\Omega}$ is closed. More generally we map $\mathscr{Q}$-cohomology to ordinary deRham cohomology of moduli space [22]. Similar remarks apply to super geometry [5, 23]. This generality means that the methods of this paper carry over to topological gravity, where $\mathscr{Q} = Q_B + Q_s$ and $\mathscr{B}$ can be b, G, or some combination. Here Q_s is the scalar supersymmetry charge $\oint b\gamma$ and G is the super partner of the stress tensor T. As mentioned earlier all the observables of topological gravity fail the appropriately modified version of the SPSC (2.4).

2.3. A Warmup. To fix notation we briefly recall the story of the one-point function [5–7]. The state $|D\rangle = \frac{1}{2}(Q_B + \bar{Q}_B)(c_0 - \bar{c}_0)|\mathbf{1}\rangle$ is BRST-exact, yet does not decouple. This is because while its 1-point measure $\tilde{\Omega} = d\tilde{\alpha}$, still $\tilde{\alpha}$ is *not* insensitive to phase changes of slice because $(c_0 - \bar{c}_0)|\mathbf{1}\rangle$ fails even the WPSC, (2.9). Cutting Σ into patches we find [5] that across patch boundaries the *change* in $\alpha = \sigma^* \tilde{\alpha}$ is itself a total derivative, and thus we build up a Cech 2-class, the Euler class of Σ, times the partition function.

Let us do the calculation directly on S^2, without the total-derivative trick just used. This will illustrate how we compute the insertions $b[\sigma_*(V)]$. Let ζ be the usual, fixed, coordinate on the plane. Throughout most of the Riemann sphere we can take a slice $\zeta_Q(\cdot) = \zeta(\cdot) - \zeta(Q)$, the "conformal normal ordering (CNO) slice" [21]. At $\zeta = \infty$ this clearly breaks down. We can instead try $w_Q(\cdot) = \zeta(\cdot)^{-1} - \zeta(Q)^{-1}$, but near the equator this choice does not agree with $\zeta_Q(\cdot)$ even modulo $U(1)$. Indeed one finds

$$w_Q = -\zeta_Q \zeta(Q)^{-2} + (\zeta_Q)^2 \zeta(Q)^{-3} + \cdots.$$

Hence one must interpolate. This is good news. If we just use ζ_Q then moving Q makes a constant shift in ζ_Q, which is the action of ℓ_{-1}. This means that the modulus associated to Q gives $b[\sigma_*(\partial/\partial\zeta(Q))] = b_{-1}$, so we insert $b_{-1} b_{-1}|D\rangle \equiv 0$. Thus whenever we use ζ_Q (or w_Q) we get zero; any contribution to the 1-point function must come from the interpolation region.

Let $\zeta = e^{y+i\theta}$. For $0 < y < \varepsilon$ consider the slice

$$z_Q = \zeta_Q - \varepsilon^{-1} y(Q) \zeta(Q)^{-1} (\zeta_Q)^2 + \cdots.$$

The ellipsis denotes higher powers of ζ_Q; since $b_n|D\rangle = 0$, $n > 1$, we need not bother with such terms. Thus for $y < 0$, $y > \varepsilon$ the slice z_Q matches ζ_Q and w_Q respectively, up to a (Q-dependent) constant. Also we will work to lowest nontrivial order in y, which will be seen to justified as $\varepsilon \to 0$. Recall that z_Q, ζ_Q are functions while $\zeta(Q)$ is just a number. We now vary Q:

$$\begin{aligned} \frac{\partial z_Q}{\partial \zeta(Q)} &= -1 + \frac{2y}{\varepsilon \zeta(Q)} z_Q + \mathcal{O}(z_Q^2), \\ \frac{\partial z_Q}{\partial \bar{\zeta}(Q)} &= \frac{-1}{2\varepsilon |\zeta(Q)|^2} (z_Q)^2 + \mathcal{O}(z_Q^3). \end{aligned} \tag{2.10}$$

The fact that z_Q is *not* holomorphic in Q is crucial in order for either term of $|D\rangle$ to contribute. Near $y=0$ we have $|\zeta(Q)|\sim 1$ so

$$b\left[\sigma_*\left(\frac{\partial}{\partial\zeta(Q)}\right)\right]b\left[\sigma_*\left(\frac{\partial}{\partial\bar{\zeta}(Q)}\right)\right]|D\rangle$$
$$=\left(b_{-1}+\frac{1}{2\varepsilon}\bar{b}_1+(b_n \text{ terms}, n\geqq 0)+(\bar{b}_n \text{ terms}, n>1)\right)(\text{c.c.})(c_1c_{-1}-\bar{c}_1\bar{c}_{-1})|\mathbf{1}\rangle$$
$$-\frac{1}{\varepsilon}|\mathbf{1}\rangle. \tag{2.11}$$

The extra terms beyond $b_{-1}\bar{b}_{-1}$ amount to the $\hat{b}$ prescription of Polchinski [6, 7]. Note that the push-forward of a holomorphic vector like $\frac{\partial}{\partial z(Q)}$ is not holomorphic because σ itself is not.

Recalling that $d\zeta\wedge d\bar{\zeta}=-2i dy\wedge d\theta+\mathcal{O}(y)$, the contribution from the strip $0<y<\varepsilon$ is $-2i(2\pi)\int_0^\varepsilon(-\varepsilon^{-1})dy\cdot Z=4\pi i Z=-2\pi i\chi Z$. Here Z is the 1-point function of $|\mathbf{1}\rangle$, i.e. the partition function, and $\chi=-2$ is the Euler number of S^2. Outside the equatorial strip we get zero. Moreover generally on any Σ we also get [6]

$$\langle\langle D\rangle\rangle=-2\pi i\chi Z,$$

where $\langle\langle\cdots\rangle\rangle$ means the string integral.

3. Degenerating Curves with Coordinates

We have seen that the correlation functions of WPSC states (like the dilaton) in general depend on the behavior of the slice $\sigma:\mathcal{M}_{g,N}\to\mathcal{P}_{g,N}$ near the boundary of moduli space. However, there is a notion of a "good" slice near the boundary, which essentially fixes this ambiguity.

Given a completely pinched surface Σ, we can build a family of surfaces degenerating to Σ by separating Σ into Σ_L, Σ_R and rejoining the two halves with the universal "plumbing fixture" (see e.g. [14]). Thus we choose coordinates z_L and z_R near the attachment points P_L, P_R and identify the punctured neighborhoods via

$$z_L\sim q/z_R. \tag{3.1}$$

The resulting q-dependent family of curves includes the original pinched curve at $q=0$.

This construction of course generalizes to the case when $\Sigma_{L,R}$ have additional marked points. We join them via the plumbing fixture (3.1) to obtain Σ with marked points. Following [10], we denote this joining operation by

$$(\Sigma_L,Q_1,\ldots,Q_{N_L},z_L)\infty(\Sigma_R,z_R,Q_{N_L+1},\ldots,Q_N)=(\Sigma,Q_1,\ldots,Q_N).$$

We stress that the only q-dependence that we allow is in the plumbing fixture, and that the punctured surfaces Σ_L,Σ_R are completely q-independent (though they may depend on the other moduli). This description of the moduli space is only good in the neighborhood of $q=0$.

Now, in this construction, the neighborhood of the node on Σ is modeled on two disks, with the centers of the disks (the points P_L, P_R) identified. In the family of degenerating surfaces, the double point is replaced by a narrow neck joining the two disks, whose size shrinks to zero as $q \to 0$. As is familiar to string theorists, there is a conformally equivalent picture, in which rather than being joined by a narrow neck, the two sides of the surface are joined by a long thin tube, whose length goes to infinity as $q \to 0$. In discussing the *physics* associated to degenerating surfaces, this is perhaps the more appropriate picture to use.

Let us, therefore, describe an alternative form of the plumbing fixture construction which is more appropriate to the "long thin tube" picture. Let u_L, u_R be coordinates centered at the attachment points P_L, P_R. Now we glue together the neighborhoods of the attachment points via the identification

$$1 - e^{u_L} \sim q/(1 - e^{u_R}). \tag{3.2}$$

This is the *same* surface we obtained by old plumbing fixture construction (3.1), provided the new local coordinates are related to the old ones by $u_{L,R} = \log(1 - z_{L,R})$. But now notice the crucial change in perspective. Whereas the boundaries of the disks which we joined, $|z| = 1$, were a finite distance from the attachment points in the z-plane, they are now an *infinite* distance away in the u-plane. This logarithmic change of coordinates is simply the change of coordinates appropriate to describing the cylinder, rather than the disk. We will denote this modified plumbing fixture construction, using (3.2), by

$$(\Sigma_L, Q_1, \ldots, Q_{N_L}, u_L) \# (\Sigma_R, u_R, Q_{N_L+1}, \ldots, Q_N) = (\Sigma, Q_1, \ldots, Q_N).$$

The "long thin tube" picture is also appropriate to describe the neighborhood of a puncture, where the vertex operator insertion is thought of as creating an infinitely long thin tube. Thus, when vertex operators approach each other we should also use the logarithmic coordinates described above. In the standard plumbing fixture construction, the "collision" of two vertex operators is described by sewing a 3-punctured sphere onto the rest of the Riemann surface (Fig. 1): $(\Sigma, P, Q) = (\Sigma, z) \infty (\mathbb{P}^1, \xi^{-1}, P, Q)$. Here ξ^{-1} is a local coordinate near infinity on the Riemann sphere. As $q \to 0$, rather than the two vertex operators approaching each other, the sphere pinches off. Just as there is a universal plumbing fixture to describe all degenerations, there is a universal 3-punctured sphere with local coordinates at the puctures. By $SL(2, \mathbb{C})$ invariance, we can locate the punctures at $\xi = \{\infty, 1, 0\}$. An appropriate set of "long thin tube" local coordinates is

$$(\mathbb{P}^1, \log(1 - \xi^{-1}), \log(\xi), \log(1 - \xi)). \tag{3.3}$$

Note that each of the three punctures is "infinitely far" from the others in these local coordinates, and that the local coordinates are invariant (up to sign) under the mœbius transformations that permute the three punctures.[5] We now glue this

[5] This last condition, of mœbius invariance, was the motivating criterion of Sonoda and Zwiebach [12], who were looking for the most general 3-string vertex in closed string field theory. They found an infinite class of 3-punctured spheres with local coordinates, invariant under mœbius transformations which permuted the punctures. For any of the solutions found in [12], one can find a compatible sewing prescription analogous to (3.2). The resulting dilaton contact terms will be exactly the same as the ones we calculate here

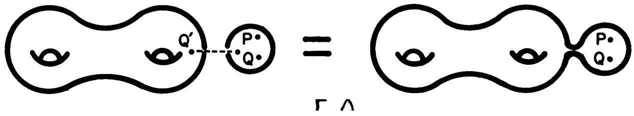

Fig. 1. Sewing to get two colliding points

3-punctured sphere with local coordinates onto the rest of the Riemann surface using (3.2) (see Fig. 1):

$$\begin{aligned}(\Sigma, \log(1-\zeta_{Q'}))\#(\mathbb{P}^1, \log(1-\xi^{-1}), \log(\xi), \log(1-\xi))\\ =(\Sigma, \log(\zeta_Q/q), \log(1-\zeta_Q/q)).\end{aligned} \tag{3.4}$$

The pinching parameter q and the location of the attachment point Q' are the two moduli in (3.4). Here $\zeta_{Q'}(\cdot)$ is some family of local coordinates at the attachment point Q', which depends on the other moduli, but is *independent of* q. Similarly, if there are other punctures on Σ, the local coordinates at those punctures are also chosen to be independent of q. The only dependence on q is in the plumbing fixture.

Because of the mœbius invariance of (3.3), it doesn't matter which of the three points we glue onto the rest of the surface. Indeed, there are higher codimension components of the boundary of moduli space, such as when a puncture approaches an already existing pinch on the Riemann surface, in which case two of the three points on the $\mathbb{P}^1$ are glued onto the rest of the Riemann surface. This too is described by the universal construction (3.2), (3.3), and though it is not relevant to the bosonic string, it is important for topological gravity.

The modified sewing prescription (3.2) thus has both a conceptual advantage over (3.1) and a mathematical one. Conceptually, it corresponds to the desired physical picture of long thin tubes. Mathematically, it allows us to take the manifestly mœbius-invariant "universal 3-punctured sphere with local coordinates" (3.3) and sew it onto the rest of the Riemann surface in a uniform way. Of course, any punctured Riemann surface with coordinates constructed using (3.2) could also be constructed using (3.1) provided we choose appropriate coordinates at the attachment points. In so doing, however, we would break the manifest mœbius invariance of the 3-punctured sphere (3.3).

Long thin tubes are very nice, but, away from $q=0$, another coordinate family is much easier to work with. Instead of (3.4), we take

$$(\Sigma, \zeta_Q - q, \zeta_Q). \tag{3.5}$$

Here ζ_Q is a local coordinate centered at Q and our two moduli are again the location of Q and $q=\zeta_Q(P)$. As we will discuss more fully below, (3.5) vastly simplifies the task of integrating over the location of P. Furthermore, as we saw in Sect. (2.3), we can, almost everywhere, choose ζ_Q to be conformal normal ordered (CNO) [21], and hence the dilaton measure vanishes in those regions.

Since we have argued that we must use (3.4) near $q=0$, what we really want is a slice which interpolates between (3.4) and (3.5). Such a slice can be gotten as

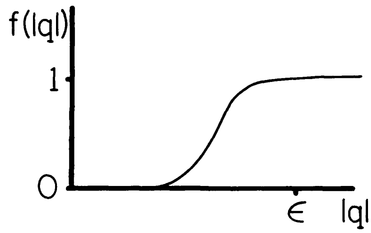

Fig. 2. The interpolating function f

follows. Let $f(|q|)$ be some smooth function which rises from zero near $|q|=0$ to one for $|q|>\varepsilon$ (see Fig. 2). Then let

$$z_P(\cdot)=\frac{1}{f}|q|^f((\zeta_Q(\cdot)/q)^f-1),$$

$$z_Q(\cdot)=\frac{1}{f}|q|^f((1-\zeta_Q(\cdot)/q)^f-1). \tag{3.6}$$

For small $|q|$, f goes to zero, and $z_P=\log(\zeta_Q/q)$, $z_Q=\log(1-\zeta_Q/q)$, which agrees with (3.4). For $|q|>\varepsilon$, $f=1$ and $z_P=(|q|/q)(\zeta_Q-q)$, $z_Q=-(|q|/q)\zeta_Q$, which agree with (3.5) up to a $U(1)$ phase (which is good enough, since the dilaton obeys the WPSC (2.9)). So, for *any* choice of the smooth function $f(|q|)$, we have a slice which interpolates between CNO coordinates and the degenerating "long thin tube" coordinates.

We will show in the next section that the slice (3.6) allows us, in a precisely controlled way, to smear out the "contact term" when a dilaton approaches a state obeying the SPSC (2.4). The lesson to be learned is that nonanalytic terms, like δ-functions, arise not from some failure of conformal field theory itself, but from the fact that the degenerating slice (3.4) used to insert states cannot in general be analytically continued to all of moduli space. It can, however, be *smoothly* interpolated, and it is the interpolation (3.6) which introduces the nonanalytic behavior.

4. Contact Terms

4.1. Dilaton–Strong Physical State. Now we turn to the calculation of the contact term which results when a dilaton at P approaches a state $|\Psi\rangle$ obeying the SPSC at Q. For q not too small, we can use the coordinate family (3.5) and, locally, we can take ζ_Q to be a conformal normal ordered (CNO) coordinate $\zeta_Q=\zeta-\zeta(Q)$ [21], in which case, as we saw in Sect. 2, the dilaton amplitude vanishes identically. However, for small q, we must use the slice (3.6), which is not holomorphic. The contribution from this region looks like

$$\int dq\wedge d\bar{q}\, b[\sigma_*(\partial/\partial q)]b[\sigma_*(\partial/\partial\bar{q})]|D\rangle_P\otimes|\Psi\rangle_Q. \tag{4.1}$$

Our first task is to evaluate the push-forward of $\partial/\partial q$. Since the coordinates at other punctures on the surface are chosen in a q-independent way, we push-forward only has support on the circles around P and Q. Moreover, the vector $\partial/\partial q$ moves the point P, but *by construction* it does not move the point Q. The push-forward certainly has positive frequency contributions, ℓ_n, $n \geqq 0$, at Q (after all, the coordinate z_Q *does* depend on q). But it has no ℓ_{-1} piece at Q. Since the state $|\Psi\rangle$ is annihilated by all the positive frequency modes of b, we can safely ignore the contribution to the push-forward from the circle around Q. The contribution at P must be calculated, but fortunately, that's not too hard. By an argument similar to (2.10), (2.11) (see also [5]),

$$\begin{aligned}\sigma_{P*}\left(\frac{\partial}{\partial q}\right) &= \frac{\partial z_P}{\partial q}\frac{\partial}{\partial z_P} + \frac{\partial \bar{z}_P}{\partial q}\frac{\partial}{\partial \bar{z}_P} \\ &= \frac{|q|^f}{q}\ell_{-1} + \frac{f}{2q}(\ell_0 - \bar{\ell}_0) - \frac{|q|}{2q}f'(|q|)\log|q|(\ell_0 + \bar{\ell}_0) \\ &\quad - \frac{|q|}{4q}f'(|q|)|q|^{-f}(\ell_1 + \bar{\ell}_1) + \cdots, \end{aligned} \tag{4.2}$$

where $\ell_n = -z_P^{n+1}\partial/\partial z_P$. This means that the b insertions are

$$b[\sigma_*(\partial/\partial q)]b[\sigma_*(\partial/\partial \bar{q})] = \frac{1}{4|q|}f'(|q|)(b_1^{(P)}b_{-1}^{(P)} - \bar{b}_1^{(P)}\bar{b}_{-1}^{(P)}) + \cdots,$$

where the b oscillators act at P, and the dots indicate terms which annihilate the state. The dilaton state, again, is $|D\rangle = (c_1c_{-1} - \bar{c}_1\bar{c}_{-1})|\mathbf{1}\rangle$. Plugging this into (4.1), we get

$$\int \frac{dq \wedge d\bar{q}}{2|q|} f'(|q|)|\mathbf{1}\rangle_P \otimes |\Psi\rangle_Q .$$

Now, the unit state at P can simply be erased, for it corresponds to the identity operator acting on the Hilbert space at Q. If we write the q-integral in polar coordinates: $dq \wedge d\bar{q} = -2i|q|d|q| \wedge d\theta$, the angular integral can be done immediately, and we get

$$2\pi i \int d|q| f'(|q|)|\Psi\rangle_Q = -2\pi i|\Psi\rangle_Q . \tag{4.3}$$

Compare this with the contribution to the dilaton from the surface, which we calculated in Sect. 2.3. There we found that integrating the dilaton over the surface gave us a factor of $-2\pi i\chi$, where $\chi = 2g-2$ is the Euler characteristic of the Riemann surface. We now see that we get an extra contribution from the neighborhood of each SPSC puncture equal to $-2\pi i$. So the full result of integrating over the location of the dilaton is $-2\pi i(2g-2+N)$, where N is the number of other punctures. As expected, the result of integrating out the dilaton is proportional to the Euler characteristic of the punctured Riemann surface.[6]

Note that we never had to integrate all the way down to $q=0$. Indeed, we could cut off the integral at some small, but finite, value of $|q|$ for which $f = f' = 0$ and obtain the whole value of the "contact term." The entire contribution to the

[6] To make it precisely *equal* to the Euler characteristic of the punctured surface, we could multiply the dilaton state by the conventional factor of $i/2\pi$

integral came from the interpolation region where f is turning on from 0 to 1, i.e. from the "shoulder" region, where the "long thin tube" is attached to the rest of the surface. Note further that the details of the shape of f completely drop out of (4.3). Any function which interpolates from zero to one gives the same answer. This is as expected – the integrand in (4.3) is a total derivative, df, on that patch of moduli space. So the integral just measures the jump in f across the patch.

This calculation suggests an alternative derivation of the results of Sect. 2.3. Consider the coordinate family

$$z_Q(\cdot) = \frac{1}{f}|\zeta(Q)|^f\left(\left(\frac{\zeta(\cdot)}{\zeta(Q)}\right)^f - 1\right)\frac{\zeta(Q)}{|\zeta(Q)|},$$

where $f(|\zeta(Q)|)$ is some smooth function which goes from $+1$ when $\zeta(Q)=0$ to -1 when $\zeta(Q)$ lies on the equator. This family interpolates from $z_Q(\cdot)=\zeta(\cdot)-\zeta(Q)$ near the origin to $z_Q(\cdot) = -(\zeta(Q)/|\zeta(Q)|)^2(\zeta(\cdot)^{-1} - \zeta(Q)^{-1})$ near the equator. This is exactly the sort of interpolating family we sought in Sect. 2.3. But we have just calculated the resulting dilaton measure. The result is

$$\begin{aligned}\langle\langle D\rangle\rangle &= -2\pi i \int d|\zeta(Q)| f'(|\zeta(Q)|)Z \\ &= -2\pi i(-2)Z.\end{aligned}$$

4.2. Strong Physical State–Dilaton. In the previous subsection, we found that if we integrate over the location of the dilaton (P), there is a "contact term" when P approaches the location of a strong physical state (Q). We expect that there should also be a contact term if we put the SPSC state at P, and the dilaton at Q. But our treatment of the two points was rather asymmetrical, so we should check whether this is indeed the case.

The best way to approach the problem is to be slightly more general and to consider integrating both over q, the relative positions of P and Q, and over $r \equiv \zeta(Q)$, the position of Q. To this end, let us write our coordinate family (3.6) explicitly as

$$\begin{aligned}z_P(\cdot) &= \frac{1}{f}|q|^f\left(\left(\frac{\zeta(\cdot)-r}{q}\right)^f - 1\right), \\ z_Q(\cdot) &= \frac{1}{f}|q|^f\left(\left(1-\frac{\zeta(\cdot)-r}{q}\right)^f - 1\right),\end{aligned}$$

and calculate the state

$$\begin{aligned}&\int dr \wedge d\bar{r} \wedge dq \wedge d\bar{q}\left(\frac{q}{|q|^f}\right)^{L_0^{(P)}+L_0^{(Q)}}\left(\frac{\bar{q}}{|q|^f}\right)^{\bar{L}_0^{(P)}+\bar{L}_0^{(Q)}} \\ &\cdot b\left[\sigma_*\frac{\partial}{\partial r}\right] b\left[\sigma_*\frac{\partial}{\partial \bar{r}}\right] b\left[\sigma_*\frac{\partial}{\partial q}\right] b\left[\sigma_*\frac{\partial}{\partial \bar{q}}\right] |\Psi\rangle_P \otimes |D\rangle_Q.\end{aligned} \tag{4.5}$$

The factor of $(q|q|^{-f})^{L_0}(\bar{q}|q|^{-f})^{\bar{L}_0}$ deserves comment. The states at P, Q are inserted with the local coordinates (4.4). However, we would like to compare this with the insertion of a state onto Σ using the "standard" CNO coordinate $\zeta - \zeta(Q)$. In evaluating (4.1) in the previous subsection, the state which resulted after all the dust had settled obeyed the SPSC, and so it didn't matter what coordinate we used to insert it. When the dust settles here, we will have to insert the state $b_{-1}\bar{b}_{-1}|\Psi\rangle$, which, though primary, has $L_0 = \bar{L}_0 = 1$, and so does depend on the coordinate used to insert it. But it doesn't depend very much. Only the scale factor in the change of coordinates matters, and the effect of this scale factor is taken into account precisely by the above factor.

So now, we need to evaluate the push-forward of the vector fields $\partial/\partial r$ and $\partial/\partial q$ using the coordinate family (4.4), and fold the result into the b-insertions in (4.5). Keeping only those terms which do not annihilate either an SPSC state or a dilation, we get

$$
\begin{aligned}
& b\left[\sigma_* \frac{\partial}{\partial r}\right] b\left[\sigma_* \frac{\partial}{\partial \bar{r}}\right] b\left[\sigma_* \frac{\partial}{\partial q}\right] b\left[\sigma_* \frac{\partial}{\partial \bar{q}}\right] \\
&\quad = |q|^{4(f-1)} b_{-1}^{(P)} \bar{b}_{-1}^{(P)} b_{-1}^{(Q)} \bar{b}_{-1}^{(Q)} \\
&\qquad + |q|^{2(f-1)} \frac{f'}{4|q|} [b_{-1}^{(P)} \bar{b}_{-1}^{(P)} (b_1^{(Q)} b_{-1}^{(Q)} - \bar{b}_1^{(Q)} \bar{b}_{-1}^{(Q)}) + b_{-1}^{(Q)} \bar{b}_{-1}^{(Q)} (b_1^{(P)} b_{-1}^{(P)} - \bar{b}_1^{(P)} \bar{b}_{-1}^{(P)})] \\
&\qquad + \frac{1}{4|q|^4} (1-f)(1-f+|q| f')(b_1^{(P)} b_{-1}^{(P)} \bar{b}_1^{(Q)} \bar{b}_{-1}^{(Q)} + \bar{b}_1^{(P)} \bar{b}_{-1}^{(P)} b_1^{(Q)} b_{-1}^{(Q)}).
\end{aligned} \tag{4.6}
$$

Note that each term comes with the appropriate power of $|q|^{(f-1)}$ to cancel the explicit power of $|q|^{(f-1)}$ in (4.5). The first term takes two SPSC states and gives the traditional answer for the integrated form of the vertex operators. The second term has two pieces, one of which is responsible for the contact term we calculated in the previous subsection for $|D\rangle_P \otimes |\Psi\rangle_Q$. The other piece gives the contact term for $|\Psi\rangle_P \otimes |D\rangle_Q$. Plugging (4.6) into (4.5) and doing the q-integral, we get

$$\int dr \wedge d\bar{r} |\mathbf{1}\rangle \otimes b_{-1}\bar{b}_{-1}|\Psi\rangle \sim \int dr \wedge d\bar{r} b_{-1}\bar{b}_{-1}|\Psi\rangle. \tag{4.7}$$

This is very satisfying. Despite the gross asymmetry in the treatment of the points P, Q in the slice (4.4), the resulting integration measure is completely symmetrical. We should have expected this. After all, (3.3) is completely symmetrical, and the detailed form of the interpolating slice should not matter.

4.3. Dilaton–Dilaton. Flush with success, having reproduced the expected contact terms between a dilaton and an SPSC state, we plunge on to consider two dilatons approaching each other. What do we expect? There ought to be a contact term, but at first glance, it may be hard to see. We know that the dilaton measure vanishes wherever the dilaton is inserted with local coordinate depending holomorphically on the moduli. Since we assumed that the "background" coordinate ζ_Q is CNO, this suggests that the resulting contact term will vanish. That is indeed correct, but it is also easy to fix. Instead of the slice (4.4), let us take the more general slice

$$
\begin{aligned}
z_P(\cdot) &= \frac{1}{f}|q|^f\left(\left(\frac{\zeta_Q(\cdot)}{q}\right)^f - 1\right), \\
z_Q(\cdot) &= \frac{1}{f}|q|^f\left(\left(1 - \frac{\zeta_Q(\cdot)}{q}\right)^f - 1\right),
\end{aligned} \tag{4.8}
$$

where

$$\zeta_Q(\cdot) = \zeta(\cdot) - r + \bar{r}\beta(\zeta(\cdot) - r)^2 + \cdots. \tag{4.9}$$

This family interpolates to the non-CNO coordinates $\zeta_Q - q$ and ζ_Q centered at P and Q, respectively. Here β is a real constant which one can think of as measuring the background curvature of Σ at Q' (see Fig. 1) [6, 7]. To first order in β, the effect of considering this more general family is to add a term to (4.6),

$$\begin{aligned} b\left[\sigma_* \frac{\partial}{\partial r}\right] b\left[\sigma_* \frac{\partial}{\partial \bar{r}}\right] b\left[\sigma_* \frac{\partial}{\partial q}\right]\left[\sigma_* \frac{\partial}{\partial \bar{q}}\right] \\ = \cdots + \frac{\beta}{8|q|}[2f' \cdot (b_1^{(P)} b_{-1}^{(P)} - \bar{b}_1^{(P)} \bar{b}_{-1}^{(P)})(b_1^{(Q)} b_{-1}^{(Q)} - \bar{b}_1^{(Q)} \bar{b}_{-1}^{(Q)}) \\ + (f^2)' \cdot (b_1^{(P)} b_{-1}^{(P)} b_1^{(Q)} b_{-1}^{(Q)} + \bar{b}_1^{(P)} \bar{b}_{-1}^{(P)} \bar{b}_1^{(Q)} \bar{b}_{-1}^{(Q)})] \end{aligned}$$

and indeed, from this term we get the expected contact term between two dilatons which is proportional to β.

Somewhat surprisingly, *even* in the flat-space limit, when $\beta = 0$, the two-dilaton measure does not vanish. The third term in (4.6) still contributes to the measure, and the contribution is quadratically divergent! Using the slice (4.4),

$$\begin{aligned} \int dr \wedge d\bar{r} \wedge dq \wedge d\bar{q}\, b\left[\sigma_* \frac{\partial}{\partial r}\right] b\left[\sigma_* \frac{\partial}{\partial \bar{r}}\right] b\left[\sigma_* \frac{\partial}{\partial q}\right] b\left[\sigma_* \frac{\partial}{\partial \bar{q}}\right] |D\rangle_P \otimes |D\rangle_Q \\ = \int dr \wedge d\bar{r} \wedge dq \wedge d\bar{q} \frac{1}{2|q|^4}(1-f)(1-f+|q|f')|\mathbf{1}\rangle_P \otimes |\mathbf{1}\rangle_Q. \end{aligned} \tag{4.10}$$

The origin of this divergence is very simple. In the bosonic string, two zero momentum dilatons can fuse into a zero momentum tachyon (the state $c_1\bar{c}_1|\mathbf{1}\rangle$). The zero momentum tachyon, being off-shell, contributes to the quadratic divergence of the string measure. This could have been expected from the start, but is rather hard to see in other approaches to the dilaton.

5. Some Comments

The surface Σ can pitch in many different ways. Instead of $P \to Q$ with everything else fixed, we can have several punctures all coming together, etc. In fact so far we have neglected any other punctures altogether. Let us see why this is justified.

We have made extensive use of the fact that $b_1|D\rangle \neq 0$. Actually, however, this is the *only* positive mode of b which violates the SPSC, Eq. (2.4). While this situation is not generic in topological gravity (the higher $\mathcal{O}_n$ do not have it), it is very convenient for our purposes, as it implies

$$\text{MPSC:} \quad b_{n_1} b_{n_2} b_{n_3} |D\rangle = 0, \quad \text{all} \quad n_i \geqq -1 \tag{5.1}$$

together with similar relations with some or all b's replaced by $\bar{b}$'s. We will refer to these conditions plus (2.9) as the "medium physical state condition." Note that any state satisfying the SPSC (2.4) automatically satisfies (5.1) as well, since $b_n b_n = 0$. Again the MPSC (5.1) is *not* strictly necessary. Let us see what it buys us when satisfied.

The usefulness of (5.1) lies in the following observation. Equations (2.2), (2.5) certainly define a volume form on $\mathcal{M}_{g,N}$. Moreover, we have a natural projection $\mathcal{M}_{g,N+1} \xrightarrow{p_k} \mathcal{M}_{g,N}$ which forgets the k-th puncture. Our plan is to integrate Ω along the fibers of p_k and compare it to the N-point measure times $2g-2+N$. Things will certainly be simple if the latter two forms are equal, but from the discussion of Sect. 2.2 it is clear that in general they can disagree by a total derivative.

Let us recall what the formula (2.5) for the measure means. Suppose we are given $(\Sigma, Q_1, \ldots, Q_{N+1})$ and tangents $V_1, \ldots, V_{3g-2+N}$ to $\mathcal{M}_{g,N+1}$ (we will suppress the conjugate quantities). We are supposed to find a number, $\Omega(V_1, \ldots, V_{3g-2+N})$. To this end we evaluate the right-hand side of (2.2) at the point $\sigma(\Sigma, Q_1, \ldots)$, substituting $\sigma_*(V_i)$ for $\tilde{V}_i$. The ensuing expression is in general complicated; to get a given moduli deformation we in general must insert $b[\vec{v}]$ where $\vec{v}$ has nonzero bits at *every* puncture! This is very inconvenient for our plan. We would prefer to separate the V's into two sets: V_0, which moves the first puncture keeping everything else fixed, and $W_1, \ldots, W_{3g-3+N}$ corresponding to the moduli of $\mathcal{M}_{g,N}$. We would like V_0 to lead to a b insertion just at $P \equiv Q_0$ while the W's lead to insertions just at $Q_i, i>0$. Indeed we never even mentioned the W's in Sect. 4. Was this justified?

In fact our procedure works when the MPSC (5.1) holds. We begin with $(\Sigma, Q_1, \ldots, Q_N) \in \mathcal{M}_{g,N}$, which we suppose at first to be nonsingular. In particular none of the Q_i may coincide. We want to introduce a zeroth point P, insert a dilaton there, and integrate "holding everything else fixed." To do this choose a basis $W_1, \ldots, W_{3g-3+N}$ tangent to $\mathcal{M}_{g,N}$. Choose any additional point $P \in \Sigma$ and let V_0 be a vertical tangent to $\mathcal{M}_{g,N+1}$ which moves only P. Let $V_1, \ldots, V_N$ be any tangents projecting to $W_1, \ldots, W_N$ under $(p_0)_*$. The measure $\Omega(V_0, V_1, \ldots, V_N)$ should depend only on V_0 and the W's, not on the choice of V's, since any two choices for V_i differ by a multiple of V_0. Everything so far makes sense even if P coincides with one of the Q_i.

Next we choose a nice slice. Let $\hat{\sigma}: \mathcal{M}_{g,N} \to \mathcal{P}_{g,N}$ be some slice defined near $(\Sigma, Q_1, \ldots, Q_N)$. If P avoids the Q_i then we extend $\hat{\sigma}$ to $\sigma: \mathcal{M}_{g,N+1} \to \mathcal{P}_{g,N+1}$ simply by choosing a family of local coordinates at P. Thus σ has the key property that the local coordinate at each Q_i, while in general depending on all of the $\{Q_j\}$, is independent of P. We have already seen in Sect. 3 that as $P \to Q_i$ we have to relax this stipulation, with interesting consequences. But away from $q=0$ the slice (3.5) did have this property, and even at $P=Q_i$ it was independent of the Q_j, $j \neq i$.

Consider $\sigma_*(V_0)$. Since V_0 moves P leaving $(\Sigma, Q_1, \ldots, Q_N)$ unchanged, and since the coordinates at Q_i are independent of the location of P, we see that $\sigma_*(V_0)$ is just a Schiffer variation at P. The corresponding ghost insertion $b[\sigma_*(V_0)]$ then lives entirely on the first copy of $\mathcal{H}$. Reinstating the conjugate quantities we get $b^{(0)}[\sigma_*(V_0)]b^{(0)}[\sigma_*(\bar{V}_0)]$. Then if we insert a state like $|D\rangle$ (i.e., obeying the MPSC) at P, we find by (5.1) that *no more* b_n insertions, $n \geqq -1$, are allowed at P.

Next consider $\sigma_*(V_i)$ (a vector on $\mathcal{P}_{g,N+1}$) versus $\hat{\sigma}_*(W_i)$ (a vector on $\mathcal{P}_{g,N}$). Suppose that the latter is represented as a Schiffer variation by $(v_{i1}, \ldots, v_{iN})$.[7] Then the former can be represented by $(v_{i0}, v_{i1}, \ldots, v_{iN})$, where the last N entries are the

[7] Recall our convention that v refers to a vector field on Σ, V, W refer to tangents to $\mathcal{M}$, and $\tilde{V}$ is tangent to $\mathcal{P}$

same as before (and hence independent of P), while v_0 involves only the generators $\ell_n, n \geqq -1$. This follows since σ at Q_i is independent of P, while V_i differs from W_i only by a term which moves P. But we have just argued that $\oint b^{(0)} v_{i0}$ can be neglected in view of (5.1). In other words, the utility of the MPSC is that, with the nice choice of σ made above, it ensures that P's moduli insertions live entirely at P and are independent of the others, while the others live entirely away from P and are exactly the same for the $(N+1)$- and N-point functions. This makes it easy to integrate P holding everything else fixed.

We can thus imagine trying to prove

$$(2g-2+N)\Omega_N(W_1,\ldots,W_N) = (\text{const})\cdot \int_{P\in\Sigma} dz_P d\bar{z}_P \Omega_{N+1}(V_0, V_1,\ldots,V_N), \tag{5.2}$$

where V_i correspond to W_i as before and z_P is a coordinate near P normalized so that its derivative along V_0 equals one. Equation (5.2) may seem a bit mysterious. If the insertion of $|D\rangle$ is independent of all the other moduli, how can the right-hand side of (5.2) "know" how many other punctures are present? But again we saw in Sect. 3 that when P collides with one of the Q_i then we cannot continue to ask that σ be "good," i.e., that σ at Q_i be independent of P. Instead we invented a slice σ which smoothly interpolated between the required behavior for $P \sim Q_i$ and convenient behavior elsewhere. Taking the results of Sect. 4 and introducing the W_j, $j \neq i$ as spectators we do indeed obtain (5.2); the interpolation builds up a smoothed δ-function which contributes to the factor of N on the left-hand side. Equation (5.2) at once implies the "dilaton equation" of [4] when integrated over $\mathscr{M}_{g,N}$.

The structure of (5.2) deserves comment. Since the multiplicative factor is a topological invariant, as the original $(\Sigma, Q_1,\ldots,Q_N)$ approaches a singular surface nothing happens to this prefactor. We therefore get no surprises at the boundary of moduli space other than the one alluded to as $P \to Q_i$. This fact can be related to the MPSC as follows.

Suppose a pinch separates Σ into parts of genus g_L, g_R with N_L+1, N_R+1 punctures, $N_L + N_R = N$, and suppose $g_L, g_R > 0$. We want to see how our answers depend on a change of slice σ, to see whether δ-function contributions are also possible at this pinch, as we just claimed they were for $P \to Q_i$. We therefore must evaluate the second term of (2.6). But the vector $\tilde{V}$ effecting the change is some positive ℓ_n acting at one of the punctures; $\iota_{\tilde{V}}$ thus puts an extra b_n $n \geqq 0$ at one puncture [22]. By the MPSC this kills the whole term, and there is no change. How do we avoid this conclusion when $P \to Q_i$? In this case one component is a *sphere*, $g_R = 0$, with three punctures. The sphere has conformal Killing vectors, and so some b insertions are absent and we do *not* kill the pinching term.

We learn two things. First, when inserted states obey the MPSC (5.1) then extra pinching terms will not appear in recursion formulas, as implied by (5.2). But when they do *not* obey it we *do* expect such terms; this is exactly what is claimed in topological gravity for the $\mathcal{O}_n$ $n > 1$ [1,4].

6. Conclusion

It is well known that not just any degeneration in a Riemann surface should be allowed in the string integral, and more generally in the deformation of any CFT.

Bad degenerations include those with anything worse than a double point. Good degenerations correspond to the physical picture of long thin tubes; they lead us to the mathematical notion of a stable family of curves, originally developed for other reasons. What we have done in this paper is in a sense to extend this notion to "stable" families of curves with coordinates. We have not specified the actual coordinates at colliding points; clearly there is no natural choice. Instead we specified the *relation* between the coordinates at P and Q using the universal fixture (3.3) and (3.2). Again we were led to this construction by the physical picture of long thin tubes. We think that the precise mathematical elaboration of this notion, and its link to [13], would be quite interesting.

What we found was a conflict between families of curves with coordinates which are good in the above sense near the pinched locus and families which are good elsewhere. More precisely one cannot find a global holomorphic family of such curves. This is quite different from the situation with pointed curves without coordinates: there the only obstruction to finding a complex universal curve was modular identifications. With coordinates, however, one is inexorably led to introduce *nonanalytic* families, as we did in (3.6). This is the origin of nonanalytic behavior, like contact terms, not some failure of the axioms of CFT itself.

We have pointed out that the presence of pinching terms in topological recursion formulas like those of [1, 4] is to be expected for generic states satisfying the weak physical state condition (2.9). For dilatons the story is simpler, however, and the desired dilaton equation follows quite simply. In particular the contact terms arise without ambiguity. It would of course be quite interesting to extend this analysis to the rest of the hierarchy of states in topological gravity.

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Note added in proof: The analysis of this paper can be applied to 2d topological gravity as well; details appear in refs. 24.

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