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Semi-Infinite Weil Complex and the Virasoro Algebra

Boris Feigin[1] **and Edward Frenkel**[2]

[1] Landau Institute for Theoretical Physics, Moscow, USSR
[2] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA

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Abstract. We define a semi-infinite analogue of the Weil algebra associated an infinite-dimensional Lie algebra. It can be used for the definition of semi-infinite characteristic classes by analogy with the Chern-Weil construction. The second term of a spectral sequence of this Weil complex consists of the semi-infinite cohomology of the Lie algebra with coefficients in its "adjoint semi-infinite symmetric powers." We compute this cohomology for the Virasoro algebra. This is just the BRST cohomology of the bosonic $\beta\gamma$-system with central charge 26. We give a complete description of the Fock representations of this bosonic system as modules over the Virasoro algebra, using Friedan-Martinec-Shenker bosonization. We derive a combinatorial identity from this result.

1. Introduction

It is well-known that the Weil algebra, associated to a finite-dimensional Lie algebra is very useful in geometry and topology.

Let us recall its definition.

Let G be a finite-dimensional Lie group, g – its Lie algebra. Denote by $\wedge^*(g')$ and $S^*(g')$ exterior and symmetric algebras of the dual space to g, correspondingly. Put $W(g)=\bigwedge^*(g')\otimes S^*(g')$. Introduce grading on $W(g)$:

$$W(g)=\bigoplus_{k\geqq 0} W^k(g),$$

where

$$W^k(g)=\bigoplus_{r+2p=k}\bigwedge^r(g')\otimes S^p(g').$$

Let X_i be basic elements of g. Denote by c_i and γ_i the images of X_i' in $\wedge^*(g')$ and $S^*(g')$, correspondingly. They are the generators of $W(g)$. Define the differential d in

$W(g)$ as follows:

$$d\cdot c_{i_1}\dots c_{i_r}\otimes\gamma_{j_1}\dots\gamma_{j_p}=\sum_k(-1)^{k+1}c_{i_1}\dots\hat{c}_{i_k}\dots c_{i_r}\otimes\gamma_{i_k}\gamma_{j_1}\dots\gamma_{j_p}$$
$$+\frac{1}{2}\sum_{k,l,m}(-1)^{m+1}f_{kl}^{i_m}c_kc_lc_{i_1}\dots\hat{c}_{i_m}\dots c_{i_r}\otimes\gamma_{j_1}\dots\gamma_{j_p}$$
$$+\sum_{k,l,m}f_{kl}^{j_m}c_kc_{i_1}\dots c_{i_r}\otimes\gamma_l\gamma_{j_1}\dots\hat{\gamma}_{j_m}\dots\gamma_{j_p},$$

where f_{jk}^i are the structural constants of g.

This equips $W(g)$ with the structure of a differential graded algebra. This algebra is called the Weil algebra.

The differential d is a sum of the Koszul differential d_k and the differential d_c of the standard cohomology complex of the Lie algebra g with coefficients in the module $S^*(g')$ of coadjoint symmetric powers. This decomposition gives us two spectral sequences.

In one of them the 0^{th} differential is the Koszul differential d_k and we conclude that $W(g)$ is an acyclic complex: $H^0(W(g))=\mathbf{C}$, $H^i(W(g))=0$, $i\neq 0$.

In another spectral sequence the 0^{th} differential is d_c and the first differential is trivial. Therefore the second term of this spectral sequence is

$$E_2^{i,j}=\begin{cases}0, & \text{if } i \text{ is odd}\\ H^j(g,S^{i/2}(g')) & \text{if } i \text{ is even}\end{cases}.$$

If G is compact, then the Weil algebra is homotopically equivalent to the algebra of invariant differential forms on the universal bundle of G and this spectral sequence coincides with the Leray spectral sequence of the universal bundle, beginning from the second term.

Let us define, following [40] the universal connection and curvature over the Weil algebra:

$$\theta=\sum_i c_iX_i\in W^1(g)\otimes g,\qquad \Omega=\sum_i\gamma_iX_i\in W^2(g)\otimes g.$$

One has:

$$d\cdot\theta=\Omega-\tfrac{1}{2}[\theta,\theta],\qquad d\cdot\Omega=[\Omega,\theta].$$

Now let $P\to M$ be a principal G-bundle. Denote by $\Omega(P)$ the complex of the differential forms on P. Suppose we are given a connection θ' and a curvature Ω' on P, $\theta'\in\Omega^1(P)$, $\theta'\in\Omega^2(P)$. They determine the maps

$$g'\to\Omega^1(P),\qquad g'\to\Omega^2(P),$$

which can be uniquely extended to the homomorphism of graded algebras

$$W(g)\to\Omega(P),$$

which carries the universal connection and curvature over the Weil algebra to the connection and the curvature on P.

We also have a map from the spectral sequence of $W(g)$ to the Leray spectral sequence of the bundle P. In particular, we have a map

$$H^q(g,S^p(g'))\to H^{2p}(M,H^q(G)).$$

If G is compact, then $H^q(g,S^p(g'))=H^q(g)\otimes H^0(g,S^p(g'))$ and this map is the Chern-Weil homomorphism

$$H^0(g,S^p(g'))\to H^{2p}(M).$$

It gives the construction of the characteristic classes of P in terms of the space $H^0(g, S^*(g'))$ of invariant polynomials on g.

Thus, one can define the characteristic classes, using the Weil algebra. The Weil algebra is also used for definition of the equivariant differential forms [3, 5, 9, 40] universal Thom class [40], for computation of Gelfand-Fuchs cohomology [28, 29], and in the proof of the local Riemann-Roch theorem [21]. In the works [4, 35, 47] the Weil algebra is shown to be useful in an algebraic definition of the BRST procedure in topological quantum field theory.

In this work we define a semi-infinite analogue of the Weil algebra, associated to an infinite-dimensional Lie algebra g. We call it the semi-infinite Weil complex. This complex has two spectral sequences, as well as the standard Weil algebra. The first of these spectral sequences ensures that our complex is acyclic. The second term of the second spectral sequence consists of the semi-infinite cohomology of g with coefficients in a g-module $S^{\infty/2+*}(g)$, which can be called "(co)adjoint semi-infinite symmetric powers of g." In this work we compute this cohomology in the case when g is the Virasoro algebra.

Semi-infinite cohomology theory was formulated in standard mathematical terms in [14]. In the past few years it was studied extensively in physics literature as the BRST cohomology [8, 45] (see [35a] for a detailed exposition of the BRST construction). The main benefit of [14] and subsequent works [25, 39, 48, 49] is that the powerful homological algebra technique, like spectral sequences, Hodge decomposition, etc., can be used in this cohomology theory as well. In particular, these methods were applied to give a new simple proof of the famous "no ghost" theorem and its generalizations [22, 25, 38], and to carry out the quantum Drinfeld-Sokolov reduction [18].

Semi-infinite cohomology theory is particularly suitable for the representations with highest weight. For example, in [16] we proved a semi-infinite analogue of the Borel-Weil-Bott-Kostant theorem on the semi-infinite cohomology of the current algebra of a nilpotent Lie algebra [1]. This result follows from the "two-sided" Bernstein-Gelfand-Gelfand resolution and it is closely related to the geometry of the semi-infinite flag manifold [16].

Thus, we see that many of the classical results on the finite-dimensional Lie algebras have "semi-infinite analogues."

There are also many signs of the existence of a semi-infinite geometry: elliptic genus [41], [8a] Floer cohomology [2, 23], semi-infinite flag manifolds and sheaves [16]. One can expect that in this geometry the semi-infinite Weil complex plays a role, similar to the role of the Weil algebra in the finite-dimensional geometry, namely, gives a natural construction of "semi-infinite characteristic classes." More precisely, one can expect that there exist characteristic homomorphisms:

$$H^{\infty/2+p}(g, S^{\infty/2+q}(g)) \to H^{\infty/2+2q}(M, H^{\infty/2+p}(G))$$

for a principal G-bundle $P \to M$, where $H^{\infty/2+*}$ denotes Lie algebra cohomology in the left-hand side, and cohomology of the space (e.g. Floer cohomology) – in the right-hand side. The idea is again to use algebraic methods for the definition of geometric invariants. Hopefully, these characteristic classes are suitable for the formulation of the index theorem on the loop spaces.

Usually in the definition of the semi-infinite cohomology one has obstructions to the nilpotency of the differential [14], thus, in the case of the Virasoro algebra the differential is nilpotent only if the central charge is equal to 26. On the contrary,

[1] Note that it was also proved by other means and connected with $N=2$ coset models in [30, 37]

the differential of the Weil complex is always nilpotent due to the "cancellation of anomalies." In particular, in the Virasoro case it is equivalent to the fact, that the central charge of its adjoint symmetric powers modules is equal to 26.

These modules are interesting on their own right. They are the Fock representations of the bosonic $\beta\gamma$-system with conformal dimensions 2 and -1.

For a general value of conformal dimension λ these representations can be constructed in the following way. Denote by $\mathscr{L}$ the Lie algebra of vector fields on the circle. Let $\mathscr{F}_{\lambda,\mu}$ be the $\mathscr{L}$-module of λ-differentials on the circle with monodromy $e^{2\pi i\mu}$. It consists of elements of the form $f(z)z^{\mu}dz^{\lambda}$. There is a bilinear non-degenerate $\mathscr{L}$-invariant pairing between $\mathscr{F}_{\lambda,\mu}$ and $\mathscr{F}_{1-\lambda,-1-\mu}$:

$$(f(z)z^{\mu}dz^{\lambda}, g(z)z^{-1-\mu}dz^{1-\lambda}) \to \int_{S^1} f(z)g(z)\frac{dz}{z}$$

(z is a Fourier coordinate on the circle S^1). In a standard way this defines a non-degenerate anti-symmetric scalar product on $V_{\lambda,\mu}=\mathscr{F}_{\lambda,\mu}\oplus\mathscr{F}_{1-\lambda,-1-\mu}$. The Heisenberg algebra $H_{\lambda,\mu}$ associated to $V_{\lambda,\mu}$ is just the bosonic $\beta\gamma$-system with conformal dimensions λ and $1-\lambda$ [27].

Let $p\in\mathbf{Z}$ and $W_{\lambda,\mu}(p)$ be a subspace of $\mathscr{F}_{\lambda,\mu}$, spanned by elements $z^{n+\mu}dz^{\lambda}$, $n\geqq p$. Denote by $W'_{1-\lambda,-1-\mu}(p)$ the subspace of $\mathscr{F}_{1-\lambda,-1-\mu}$, spanned by elements $z^{n-\mu}dz^{1-\lambda}$, $n>-p$. $W_{\lambda,\mu}\oplus W'_{1-\lambda,-1-\mu}$ is a maximal isotropic subspace in $V_{\lambda,\mu}$. It defines the irreducible Fock representation $M^{-}_{\lambda,\mu}(p)$ of the Heisenberg algebra $H_{\lambda,\mu}$. There is the (projective) Weil representation of the Lie algebra of symplectic transformations of $V_{\lambda,\mu}$ (preserving our scalar product) in $M^{-}_{\lambda,\mu}(p)$. $\mathscr{L}$ preserves the scalar product, therefore the Virasoro algebra $\hat{\mathscr{L}}$, which is the central extension of $\mathscr{L}$, acts on $M^{-}_{\lambda,\mu}(p)$ with central charge $c=2(6\lambda^2-6\lambda+1)$. This is a representation with highest weight.

There is a fermionic counterpart of this representation. One can introduce on $V_{\lambda,\mu}$ a symmetric scalar product and associate to it the Clifford algebra $C_{\lambda,\mu}$, or, in other words, fermionic bc-system. Let $M^{+}_{\lambda,\mu}(p)$ be the irreducible Fock representation of $C_{\lambda,\mu}$, associated with the maximal isotropic subspace $W_{\lambda,\mu}\oplus W'_{1-\lambda,-1-\mu}$ of $V_{\lambda,\mu}$ [35a]. The Virasoro algebra acts on $M^{+}_{\lambda,\mu}(p)$ as well, with central charge $c=-2(6\lambda^2-6\lambda+1)$. This is also a representation with highest weight.

It is natural to ask about the structure of $M^{+}_{\lambda,\mu}$ and $M^{-}_{\lambda,\mu}$ as modules over the Virasoro algebra.

The structure of $M^{+}_{\lambda,\mu}(p)$ was found in [19], using bosonization of fermions [10, 24, 27] (for bosonization on higher genus Riemann surfaces see [1]). Bosonization means that we express all operators, acting on $M^{+}_{\lambda,\mu}(p)$, in terms of a Heisenberg algebra A. This algebra is the central extension of the commutative Lie algebra $\mathscr{F}_{0,0}$ of functions on the circle by means of the 2-cocycle:

$$(f(z), g(z)) \to \int_{S^1} f(z)dg(z).$$

Note that this algebra is different from $H_{\lambda,\mu}$. In physics literature it is called the free scalar field.

This bosonization is very important in representation theory, conformal field theory, and integrable systems (KP hierarchy [10]).

After bosonization the module $M^{+}_{\lambda,\mu}(p)$ becomes the direct sum of standard irreducible Fock representations of A, and the action of the Virasoro algebra is given by a quadratic expression on the generators of A. This simplifies the study of the structure of these representations and enables to give explicit formulas for intertwining operators between them. The structure of these representations then can be determined completely, using standard representation theoretical arguments [19].

In this work we solve this problem for bosonic modules $M^-_{\lambda,\mu}(p)$. To do this we use the bosonization, which was found by Friedan, Martinec, and Shenker [27] (see [36] and references therein for an account of this bosonization on Riemann surfaces). A similar construction was also given by Kac and van de Leur and used by them for a construction of a super KP hierarchy [31, 32]. It is possible to express operators, acting on $M^-_{\lambda,\mu}(p)$, in terms of the algebra $A \oplus A$ of two scalar fields. However our representation $M^-_{\lambda,\mu}(p)$ is not isomorphic to the direct sum of Fock representations of $A \oplus A$, but is its submodule. Actually, this submodule is the kernel of a vertex operator, acting on this bosonic Fock space. We can determine the structure of this kernel, using the results of [19]. This enables us to determine the structure of $M^-_{\lambda,\mu}(p)$ as a module over the Virasoro algebra. Using this result we reprove a very interesting combinatorial identity, which was proved by Kac and Peterson in [34] by other means and which goes back to 1898 [42] (see also [6, 7]). Note that the famous Jacobi triple product identity can be proved, using the bosonization of fermions.

The modules $M^-_{2,-2}(p)$ are just the modules of adjoint semi-infinite powers of the Virasoro algebra. Using the information on the structure of these modules we can compute their cohomology. This gives us a description of the second term of the spectral sequence of the semi-infinite Weil complex of the Virasoro algebra. In fact, more careful analysis shows that the natural map

$$E_2^{p,q}(\hat{\mathscr{L}}) \to E_2^{p,q}(h),$$

where h is the Cartan subalgebra of the Virasoro algebra is an isomorphism. So "semi-infinite symmetric polynomials" on the Virasoro algebra coincide with the symmetric polynomials on its Cartan subalgebra. This can be viewed as an analogue of the Chevalle theorem.

We want to mention that in recent works [11, 12, 46] the bosonic representations $M^-_{2,-2}(p)$ are shown to be relevant to $2D$ gravity. Cohomology, which we compute in this paper, is an important ingredient there. We hope to discuss these questions in a separate paper.

Similar problems are still open for affine Kac-Moody algebras. Work along these lines is in progress.

The paper is arranged as follows. In Sect. 2 we give the definition of the semi-infinite cohomology, symmetric and exterior powers, Weil complex, and modules $M^-_{\lambda,\mu}(p)$ and $M^+_{\lambda,\mu}(p)$. We state the main results at the end of Sect. 2. In Sect. 3 we describe the structure of the modules $M^-_{\lambda,\mu}(p)$, using the bosonization procedure. We also prove a combinatorial identity. In Sect. 4 we give the proof of the main result on the semi-infinite cohomology, using standard homological technique.

The main results of this paper were announced in [17].

2. Preliminaries and Statement of the Main Results

2.1. Semi-Infinite Coadjoint Symmetric and Exterior Powers

Let g be a Lie algebra. Linear space $V(g) = g \oplus g'$ carries non-degenerate exterior and symmetric forms: $(,)$ and $\{,\}$. If we denote by $\langle , \rangle$ the standard pairing between g and g', then

$$\begin{aligned} -(a,c)=\{a,c\}=\langle a,c\rangle, &\quad \text{if} \quad a\in g, c\in g', \\ (a,c)=\{a,c\}=\langle a,c\rangle, &\quad \text{if} \quad a\in g', c\in g, \\ (a,c)=\{a,c\}=0, &\quad \text{if both} \quad a, c\in g \quad \text{or} \quad g'. \end{aligned}$$

This defines two algebras: the Heisenberg algebra $H(g)$ and the Clifford algebra $C(g)$. The former is the quotient of the tensor algebra $T^*(g)$ of g by the ideal, generated by the elements of the form $[a,c]=ac-ca=(a,c)$, where a, $c\in V(g)$, and the latter is the quotient of $T^*(g)$ by the ideal, generated by the elements of the form $[a,c]_+=ac+ca=\{a,c\}$, where a, $c\in V(g)$. Introduce grading on these algebras by putting $\deg a=-1$, if $a\in g$, $\deg a=1$, if $a\in g'$.

Suppose we are given a decomposition of g into the sum of two linear subspaces: $g=b\oplus n$, which satisfies the following condition. Linear subspaces d of g, such that $\dim b/b\cap d<\infty$ and $\dim d/b\cap d<\infty$, generate some topology on g. The adjoint action of any $x\in g$ is continuous in this topology.

The linear subspace $W=b\oplus n'$ is a maximal isotropic (or Lagrangian) subspace of $V(g)$ with respect to the bilinear forms $(,)$ and $\{,\}$. Symmetric algebra $S^*(W)$ is embedded into $H(g)$, and exterior algebra $\wedge^*(W)$ is embedded into $C(g)$. Let M^- and M^+ be the representations of $H(g)$ and $C(g)$, induced from the trivial representations of $S^*(W)$ and $\wedge^*(W)$, correspondingly. These modules inherit grading from $H(g)$ and $C(g)$: $M^\pm=\bigoplus_{l\in\mathbf{Z}} M_l^\pm$. Let us define an action of the Lie algebra g on them.

We will call the elements of $V(g)$, which belong to W, annihilation operators, and the elements of $V(g)$, which belong to W', creation operators.

Let $a_1\dots a_l$ be a monomial in $T^*(V)$, where a_i belong to either W or W'. The monomial $a_{i_1}\dots a_{i_l}$, where

$$\sigma=\begin{pmatrix} 1\dots l \\ i_1\dots i_l\end{pmatrix}$$

is a permutation, is called normally ordered, if for some m operators $a_{i_1},\dots,a_{i_m}$ are creation operators, while $a_{i_{m+1}},\dots,a_{i_l}$ are annihilation operators.

Let $a_{i_1}\dots a_{i_l}$ be any normally ordered monomial. We denote by $:a_1\dots a_l:$ the element $\pi^-(a_{i_1}\dots a_{i_l})$ of $H(g)$ and the element $\operatorname{sgn}(\sigma)\pi^+(a_{i_1}\dots a_{i_l})$ of $C(g)$, where π^-, π^+ are the projections of $T^*(V)$ onto $H(g)$ and $C(g)$, correspondingly. This definition is well-defined due to the isotropy of W and W'. By linearity we extend it to the whole $T^*(V)$.

Let x be an element of the Lie algebra g. Adjoint action of g defines the element $\bar{x}$ of $\operatorname{Hom}(g,g)=g'\otimes g\subset T^*(V)$. This gives the projective representations $\varrho^\pm$ of the Lie algebra g in $M^\pm:\varrho^\pm(x)=:\bar{x}:$. Note, that $:\bar{x}:$ is well-defined on $M^\pm$, and it belongs to some completions of $H(g)$ and $C(g)$. If $b=g$ (or $b=0$). then $M^\pm$ are just the modules of (co)adjoint symmetric and exterior powers of g. In the case when g is infinite-dimensional and both b and n are infinite-dimensional we get some semi-infinite analogues of the symmetric and exterior power modules, which we will also denote by $S^{\infty/2+*}(g)$ and $\wedge^{\infty/2+*}(g)$. In this case, in general, the action of g on $M^\pm$ is projective, so that it defines a central extension of g and therefore cohomology classes $\omega_\pm$ from $H^2(g)$. Let us compute them.

Let g_i, $i\geqq 0$ be a basis of b, and g_i, $i<0$ be a basis of n. Denote by g_i', $i\in\mathbf{Z}$ the dual basis of g'. The Heisenberg algebra $H(g)$ is generated by β_i, γ_i, $i\in\mathbf{Z}$ (images of g_i, g_i') with the commutation relations:

$$[\gamma_i,\beta_j]=\delta_{i,j}, \qquad [\beta_i,\beta_j]=[\gamma_i,\gamma_j]=0.$$

Module M^- is the irreducible representation of $H(g)$ with vacuum vector, annihilated by β_i, $i \geqq 0$, γ_i, $i<0$.

Analogously, the Clifford algebra $C(g)$ is generated by b_i, c_i, $i \in \mathbf{Z}$ (images of g_i, g_i') with the anticommutation relations:

$$[c_i, b_j]_+ = \delta_{i,j}, \qquad [b_i, b_j]_+ = [c_i, c_j]_+ = 0.$$

Module M^+ is the irreducible representation of $C(g)$ with vacuum vector, annihilated by b_i, $i \geqq 0$, c_i, $i<0$.

Let f^i_{jk} be the structural constants of g:

$$[g_j, g_k] = \sum_i f^i_{jk} g_i.$$

Then

$$\varrho^-(g_j) = \sum_{i,k} f^i_{jk} :\beta_i \gamma_k:, \qquad \varrho^+(g_j) = \sum_{i,k} f^i_{jk} :b_i c_k:. \tag{1}$$

Let U be an infinite-dimensional linear space with the basis u_i, $i \in \mathbf{Z}$. The subspaces U_j of U, spanned by u_i, $i<j$, generates some topology on U. Denote by gl the Lie algebra of continuous endomorphisms of U. It consists of infinite matrices (a_{ij}), with finitely many non-zero entries in the quadrant $i>n, j<n$ for any n. We can define representations $\nu^\pm$ of gl in $M^\pm$ by putting $\nu^-(E_{ij}) = :\beta_i \gamma_j:$, $\nu^+(E_{ij}) = :b_i c_j:$, where E_{ij} is the standard matrix from gl. As is well-known [10, 33, 43] this representation of gl is projective and it defines elements $\tilde{\omega}^\pm$ from $H^2(gl) = \mathbf{C}$ [20]. Wick theorem says, that $\tilde{\omega}^- = -\tilde{\omega}^+$.

Formulas (1) define embeddings of the central extension of g into the central extension of gl, and $\omega^\pm \in H^2(g)$ are the restrictions of $\tilde{\omega}^\pm \in H^2(gl)$. So $\omega^- = -\omega^+$.

Under the action of the central extension of the Lie algebra g the modules $M^\pm$ are decomposed into the direct sum of submodules: $M^\pm = \bigoplus_{l \in \mathbf{Z}} M_l^\pm$.

2.2. Semi-Infinite Cohomology

Now let N be a representation of g, which is locally finite with respect to b (that is b-submodule, generated by any element of N is finite-dimensional). Define an operator d_N of degree 1, acting on $C(g, N) = N \otimes M^+$. Let d_1 be the canonical element in $\mathrm{Hom}(g, g) = g' \otimes g \subset U(g) \otimes C(g)$, corresponding to the identity, and d_2 be the canonical element from $\mathrm{Hom}(g \otimes g, g) = g' \otimes g' \otimes g \subset T(g)$, defined by the commutation in the Lie algebra g. Note that $:\bar{d}_2:$ is well defined on M^+. We put

$$d_N = d_1 = \mathbf{1} \otimes :\bar{d}_2:.$$

This operator is well-defined on $C(g, N)$. Let us compute d_N^2. In order to do that we have to write an explicit formula for d_N, using the basis, introduced in Sect. 2.1. We have:

$$d_N = \sum_i \varrho_N(g_i) \otimes c_i - 1/2 \sum_{i,j,k} f^i_{jk} :b_i c_j c_k:,$$

where ϱ_N denotes the action of g on N.

One can easily check that

$$d_N^2 = \sum_{i,j} \omega^+_{ij} c_i c_j,$$

where $\omega^+_{ij} = \omega^+(g_i, g_j)$.

The cocycle $-\omega^+$ defines a central extension of the Lie algebra g

$$0 \to g \to \hat{g} \to \mathbf{C} \to 0.$$

An element of the Lie algebra $\hat{g}$ is a pair (x, c), $x \in g$, $c \in \mathbf{C}$. The commutation relations are:

$$[(x,c),(y,c')] = ([x,y], -\omega^+(x,y)).$$

Now let N be a $\hat{g}$-module, where any element c from the center acts by multiplication by c. Then obviously, $d_N^2 = 0$, and so d_N equips $C(g,N)$ with the structure of complex. If $b=0$ (or $b=g$), then $\omega^- = 0$ and this complex coincides with the standard complex of (co)homologies of the Lie algebra g. If both b and n are infinite-dimensional, then this complex is the complex of semi-infinite cohomologies of the Lie algebra g with respect to the decomposition $g = n \oplus b$ [14]. Its i^{th} cohomology group is denoted by $H^{\infty/2+i}(g,N)$.

If $H^2(g)$ is not equal to 0, then semi-infinite cohomology is not defined, unless the module N is a projective representation of g, corresponding to the cocycle $-\omega^+$ ($d_N^2 = 0$ only on this case).

In a standard way one can define relative semi-infinite cohomology. Let h be a subalgebra of g. Then $h = h_n \oplus h_b$, where $h_n = h \cap n$, $h_b = h \cap b$. Let N satisfies the conditions above. We define a subcomplex $C(g,h,N)$ of $C(g,N)$, such that
1) $\mathbf{1} \otimes (h_n \oplus h_b') \subset U(g) \otimes C(g)$ acts on $C(g,h,N)$ by 0;
2) for any $x \in h$, $x \otimes \mathbf{1} + \mathbf{1} \otimes \varrho^-(x)$ acts on $C(g,h,N)$ by 0.

Note, that if $\tilde{g}$ is the universal central extension of g

$$0 \to g \to \tilde{g} \to K \to 0,$$

then $H^2(g) = 0$ and the semi-infinite cohomology complex $C(\tilde{g}, N)$ is well-defined for any $\tilde{g}$-module N. Cocycle $-\omega^+ \in H^2(g)$ defines a character χ of the center K. It is clear, that $C(\tilde{g}, K, N) = C(g, N)$, if any $k \in K$ acts on N by multiplication by $\chi(k)$, and $C(\tilde{g}, K, N) = 0$, otherwise.

2.3. Semi-Infinite Weil Complex

Now let $g = b \oplus n$ be a decomposition into the direct sum of two subalgebras. Let $W(g) = M^- \otimes M^+$. Introduce grading on $W(g)$ by putting $W^i(g) = \bigoplus_{j+2k=1} M_k^- \otimes M_j^+$. Let us introduce an operator d of degree 1 on $W(g)$. Let d_0 be the canonical element in $g' \otimes g \subset H(g) \otimes C(g)$, corresponding to the identity operator on g. Put

$$d = d_0 + d_{M^-}.$$

Proposition 1. $d_0^2 = d_0 d_{M^-} + d_{M^-} d_0 = d_{M^-}^2 = 0$.

Proof is clear, the last equality follows from the fact, that $\omega^- = -\omega^+$ (see Sect. 2.1). $\square$

Definition 1. The complex $(W(g), d)$ is called the semi-infinite Weil complex of the Lie algebra g with respect to the decomposition into the direct sum of two subalgebras $g = b \oplus n$.

If $b=0$, then this complex is just the Weil algebra, which was defined in the Introduction.

We can easily write the explicit formula for the differential d in terms of the basis, introduced in Sect. 2.1:

$$d=\sum_i \gamma_i \otimes b_i - 1/2 \sum_{i,j,k} f_{jk}^i :b_i c_j c_k: + \sum_{i,j,k} f_{jk}^i :\beta_i \gamma_j c_k:.$$

As in the standard case we can define two filtrations on $W(g)$: F_1^* and F_2^*,

$$F_1^p = \bigoplus_{2j \geqq p} M_j^- \otimes M_i^+, \qquad F_2^p = \bigoplus_{i+j \geqq p} M_j^- \otimes M_i^+.$$

The following proposition is the precise analogue of the key property of the usual Weil algebra (see the Introduction).

Proposition 2. 1) *$W(g)$ is an acyclic complex: $H^0(W(g))=\mathbf{C}$, $H^i(W(g))=0$, $i \neq 0$,* 2) *in the spectral sequence, associated with the filtration F_1^*, the second term is*

$$E_2^{p,q}(g) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ H^{\infty/2+q}(g, S^{\infty/2+p/2}(g)) & \text{if } p \text{ is even} \end{cases}.$$

Proof is the same, as in the standard case. The convergence of the spectral sequences, associated to our filtrations, is ensured by the fact, that both b and n are subalgebras. □

2.4. Virasoro Algebra

Virasoro algebra $\hat{\mathscr{L}}$ is the central extension of the Lie algebra $\mathscr{L}$ of vector fields on the circle. If z is a Fourier coordinate on the circle, then we can choose the basis $T(i) = z^{-i+1} d/dz$, $i \in \mathbf{Z}$, C in $\hat{\mathscr{L}}$, where C is the central element and the following commutation relations hold:

$$[T(n), T(m)] = (n-m)T(n+m) + 1/12 \ (n^3-n)\delta_{n,-m} C.$$

For any $p \in \mathbf{Z}$ let b_p be the subalgebra of $\hat{\mathscr{L}}$, spanned by $T(m)$, $m > -p$, and n_p be the subalgebra of $\hat{\mathscr{L}}$, spanned by $T(m)$, $m \leqq -p$. Denote $b=b_1$, $n=n_1$.

Let S_p^* and $\wedge_p^*$ be the modules of semi-infinite adjoint symmetric and exterior powers of the Virasoro algebra with respect to the decomposition $\mathscr{L} = b_p \oplus n_p$.

These modules appear to be the particular cases of the families of the Fock representations of bosonic and fermionic ghost systems [27] (corresponding to dimensions 2 and -1).

2.5. Bosonic and Fermionic Ghost Systems

Let $\lambda, \mu \in \mathbf{C}$ and $\beta(n)$, $\gamma(n)$ be the generators of the Heisenberg algebra with the commutation relations:

$$[\gamma(n), \beta(m)] = \delta_{n,-m}. \tag{2}$$

We can introduce fields $\beta(z)$, $\gamma(z)$ with the operator product

$$\beta(z)\gamma(w)\sim\frac{1}{w-z}\left(\frac{w}{z}\right)^{\mu+2\lambda}, \tag{3}$$

$$\gamma(z)\beta(w)\sim\frac{1}{z-w}\left(\frac{z}{w}\right)^{\mu+2\lambda}, \tag{4}$$

and mode expansion

$$\beta(z)=\sum_{n\in\mathbf{Z}}\beta(n)z^{-n-\mu-2\lambda}, \tag{5}$$

$$\gamma(z)=\sum_{n\in\mathbf{Z}}\gamma(n)z^{-n+\mu+2\lambda-1}. \tag{6}$$

For $p\in\mathbf{Z}$ let $M(p)$ be the irreducible representation of this Heisenberg algebra with the vacuum vector $|p\rangle$, satisfying the following conditions:

$$\gamma(n)|p\rangle=0, \quad n\geqq p, \quad \beta(n)|p\rangle=0, \quad n>-p. \tag{7}$$

Our definitions are slightly different from the standard ones [27], because we allow λ, μ to be arbitrary complex numbers.

Put

$$\tilde{j}(z)=-:\beta(z)\gamma(z):=\sum_{n\in\mathbf{Z}}\tilde{j}(n)z^{-n-1}. \tag{8}$$

Here and further we use the "universal" normal ordering, defined with respect to the vector $|0\rangle$.

We have:

$$\begin{aligned}&[\tilde{j}(n),\beta(m)]=-\beta(n+m),\\ &[\tilde{j}(n),\gamma(m)]=\gamma(n+m),\\ &[\tilde{j}(n),\tilde{j}(m)]=n\delta_{n,-m}.\end{aligned} \tag{9}$$

We have a grading on $M^-(p)$ with respect to the action of the operator $\tilde{j}(0)$: $M^-(p)=\bigoplus_{l\in\mathbf{Z}}M^-(p)_l$, so that $|p\rangle\in M^-(p)_p$.

Now let us define the action of the Virasoro algebra on $M^-(p)_l$. Introduce the field $T(z)=\sum_{n\in\mathbf{Z}}T(n)z^{-n-2}$ and put

$$T(z)=-\lambda:\beta(z)\partial_z\gamma(z):+(1-\lambda):\partial_z\beta(z)\gamma(z):-\tfrac{1}{2}(\mu+1)(\mu+2\lambda)z^{-2}. \tag{10}$$

The operators $T(n)$ generate the Virasoro algebra with central charge $2(6\lambda^2-6\lambda+1)$, and the following relations hold:

$$[T(n),\gamma(m)]=(-m+\mu-(n-1)\lambda)\gamma(n+m), \tag{11}$$

$$[T(n),\beta(m)]=(-m+(-1-\mu)-(n-1)(1-\lambda))\beta(n+m). \tag{12}$$

Thus, the operators $T(n)=z^{-n+1}d/dz$ act on $\gamma(m)$ as on λ-differentials $z^{-m+\mu}dz^{\lambda}$ and on $\beta(m)$ as on $(1-\lambda)$-differentials $z^{-m-1-\mu}dz^{1-\lambda}$. So $M^-(p)$ is just the Weil representation $M^-_{\lambda,\mu}(p)$ of the Virasoro algebra, which was defined in the Introduction.

In particular, $\mathscr{L}=\mathscr{F}_{-1}$ and $\mathscr{L}'=\mathscr{F}_2$ due to the pairing, defined in the Introduction. Therefore $M^-_{2,-2}(p)_l$ is nothing but S^l_p.

The fermionic counterpart of this construction is basically the same. We have the generators $b(n)$, $c(n)$, $n\in\mathbf{Z}$ of the Clifford algebra with the anti-commutation relations:

$$[b(n), c(n)]_{+} = \delta_{n,-m}. \tag{13}$$

We define the fields $b(z)$, $c(z)$ with the operator product

$$b(z)c(w) \sim \frac{1}{z-w}\left(\frac{w}{z}\right)^{\mu+2\lambda}, \tag{14}$$

$$c(z)b(w) \sim \frac{1}{z-w}\left(\frac{z}{w}\right)^{\mu+2\lambda}, \tag{15}$$

and mode expansion as in (5), (6).

To construct the fermionic Fock representation $M^{+}(p)$ we just have to replace the bosonic generators in the formulas (7), (8), (9) by the fermionic generators. We have a grading on $M^{+}(p)$ by eigenvalues of the operator $\tilde{j}(0): M^{+}(p) = \bigoplus_{l \in \mathbf{Z}} M^{+}(p)_l$, such that $|p\rangle \in M^{+}(p)_{-p}$. The formula

$$T(z) = -\lambda : b(z)\partial_z c(z): + (1-\lambda) : \partial_z b(z) c(z): + \tfrac{1}{2}(\mu+1)(\mu+2\lambda)z^{-2} \tag{16}$$

defines the fermionic representations $M^{+}_{\lambda,\mu}(p)$ of the Virasoro algebra (semi-infinite forms modules), with central charge $-2(6\lambda^2 - 6\lambda + 1)$. If we replace $\beta(n)$, $\gamma(n)$ by $b(n)$, $c(n)$, then the formulas (11), (12) remain true. This shows that $M^{+}_{\lambda,\mu}(p)$ is the spinor representation of the Virasoro algebra, associated with the linear space $V_{\lambda,\mu}$, equipped with the symmetric scalar product (see Introduction). It is clear that the module $M^{+}_{2,-2}(p)_l$ coincides with $\wedge^{l}_{p}$.

2.6. Statement of the Main Results

The main result of our paper is the computation of the semi-infinite cohomologies of the Virasoro algebra with coefficients in its adjoint semi-infinite symmetric power modules. The cohomology will be computed with respect to the decomposition $\mathscr{L} = b \oplus n$.

First we compute the relative cohomologies with respect to "Cartan subalgebra" of the Virasoro algebra. This subalgebra h is spanned by $T(0)$. Let us denote

$$h_p^{j,l} = \dim H^{\infty/2+j}(\mathscr{L}, h, S_p^l).$$

j ↑

									l →
. . .	m+1	m+1	m+1	0 m	m−1	. . .	2	1	
. . .	−2 m	−1 m	0 m	1 −1 m−1	2 m−2	. . .	m−1 1	m 0	

Diagram 1. Values of $h_p^{j,l}$ for $p = -2m$, $m \geq 0$

. . .	m+1	m+1	m+1	1 m	m−1	. . .	2	1
. . .	2m+1	2m+1	2m+1	0 2m−1	2m−3	. . .	3	1
. . .	−2 m	−1 m	0 m	1 −1 m−1	2 m−2	. . .	m−1 1	m 0

Diagram 2. Values of $d_p^{j,l}$ for $p=-2m$, $m \geqq 0$

Theorem 1. 1) *If* $p=2m$, $m \leqq 0$, *then*

$$h_p^{0,l}=m+1\,, \qquad h_p^{0,l}=m+1-l\,,$$
$$h_p^{-1,l}=m\,, \qquad l \leqq 0; \qquad h_p^{-1,l}=m-l\,, \qquad 0<l \leqq m\,,$$
$$h_p^{j,l}=0\,, \qquad l>m;$$

2) *If* $p=2m-1$, $m \leqq 0$, *then* $h_p^{j,l}=h_p^{j-1,l}$;
3) *if* $p>0$, *then* $h_p^{j,l}=h_{1-p}^{-j,-l}$ *(Poincaré duality).*

This theorem will be proved in Sect. 4.

Note, that "Poincaré duality" follows from the existence of the anti-involution ν on the Virasoro algebra: $\nu(T(n))=T(-n)$, $\nu(C)=C$. Obviously, $\nu(n_p)=b_{1-p}$, $\nu(b_p)=n_{1-p}$, that it why the modules S_p^l and $\bigwedge_p^j$ are contragradient to the modules S_{1-p}^{-l} and $\bigwedge_{1-p}^{-j}$, correspondingly.

From this theorem one can deduce the result on the absolute cohomologies $H^{\infty/2+j}(\mathscr{L}, S_p^l)$. Let us denote the dimension of this space by $d_p^{j,l}$.

We can use the so-called Serre-Hochshild spectral sequence [28], associated with the Virasoro algebra and its subalgebra h. Its second term is equal to

$$E_2^{i,j}=H^i(h,\mathbf{C})\otimes H^{\infty/2+j}(\mathscr{L},h,S_p^l)\,.$$

We know, that $H^0(h,\mathbf{C})=\mathbf{C}$, $H^1(h,\mathbf{C})=\mathbf{C}$, and all other cohomologies vanish. From the Theorem 1 we see that this spectral sequence collapses in the second term. So

$$d_p^{j,l}=h_p^{j,l}+h_p^{j-1,l}\,.$$

This gives the following result.

Theorem 2. 1) *If* $p=-2m$, $m \geqq 0$, *then*

$$d_p^{1,l}=m+1\,, \qquad d_p^{1,l}=m+1-l\,,$$
$$d_p^{0,l}=2m+1\,, \qquad d_p^{0,l}=2(m-l)+1\,,$$
$$d_p^{-1,l}=m\,, \qquad l \leqq 0; \qquad d_p^{-1,l}=m-l\,, \qquad 0<l \leqq m;$$
$$d_p^{j,l}=0\,, \qquad l>m;$$

2) *If* $p=-2m-1$, $m \geqq 0$, *then* $d_p^{j,l}=d_p^{j-1,l}$;
3) *If* $p>0$, *then* $d_p^{j,l}=d_{1-p}^{1-j,-l}$.

Now let us consider the semi-infinite Weil complex of the Virasoro algebra, associated with the decomposition $\mathscr{L}=b\oplus n$.

If we put $p=1$, Theorem 2 will give us the description of the second term of the spectral sequence, associated to the filtration F_2^* (see Sect. 2.3) of this Weil complex. Actually, from the explicit construction of the cohomology classes, which is implied in the proof of the Theorem 1 (see Sect. 4), we have a stronger result.

Theorem 3. *In the spectral sequence, associated with the filtration F_2 of the semi-infinite Weil complex of the Virasoro algebra the second term is given by*

$$E_2^{j,l}(\mathscr{L})=\begin{cases}\mathbf{C}, & \text{if } j=0,1 \text{ and } l\geqq 0\\ 0, & \text{otherwise}.\end{cases}$$

The map $E_2^{j,l}(\mathscr{L})\to E_2^{j,l}(h)$, induced by the embedding $h\to\mathscr{L}$ is an isomorphism.

Our plans are the following: in the next section we study in detail the structure of $M^{\pm}_{\lambda,\mu}(p)$ as the modules over the Virasoro algebra by means of Friedan-Martinec-Shenker bosonization. We then use this information in order to deduce some combinatorial identities and prove our theorems.

3. Bosonization and the Structure of Representations

In this section we apply the Friedan-Martinec-Shenker bosonization to the Fock representations $M^{-}_{\lambda,\mu}(p)_l$ of the bosonic ghosts to determine their structure as modules over the Virasoro algebra. But first let us recall how bosonization works in the fermionic case.

3.1. Bosonization of Fermions

Let A be a Heisenberg algebra with generators $j(n)$, $n\in\mathbf{Z}$ and commutation relations

$$[j(n),j(m)]=n\delta_{n,-m}.$$

We introduce fields $j(z)=\sum_{n\in\mathbf{Z}} j(n)z^{-n-1}$ and $\phi(z)=q+j(0)\log z-\sum_{n\neq 0} j(n)z^{-n}$, where the operator q is the conjugate to $j(0):[q,j(0)]=\delta_{n,0}$. It is clear, that $j(z)=\partial_z\phi(z)$. The field $\phi(z)$ is called the free bosonic scalar field. We have the operator product

$$\phi(z)\phi(w)\sim\log(z-w). \tag{17}$$

Let $\alpha\in\mathbf{C}$ and $\mathscr{H}_\alpha$ be the irreducible representation of A with the vacuum vector v_α, satisfying the following conditions:

$$j(n)v_\alpha=\alpha\delta_{n,0}v_\alpha,\qquad n\geqq 0.$$

We see that $\exp(\eta q)(v_\alpha)=v_{\alpha+\eta}$.

For any $\eta\in\mathbf{C}$, $\eta\neq 0$ introduce the so-called vertex operator

$$V(\eta,z)=\sum_{n\in\mathbf{Z}}\exp(\eta q)z^{\alpha\eta}V(\eta,n)z^{-n},$$

acting from $\mathscr{H}_\alpha$ to $\mathscr{H}_{\alpha+\eta}$:

$$V(\eta,z)=:\exp(\eta\phi(z)):=\exp(\eta q)z^{\alpha\eta}\times\exp\Big(\eta\sum_{n>0} j(-n)z^n\Big/n\Big)\exp\Big(\eta\sum_{n<0} j(-n)z^n\Big/n\Big).$$

We have the well-known operator product

$$j(z)V(\eta,w)\sim\frac{\eta V(\eta,w)}{z-w}+\frac{1}{\eta}\partial_w V(\eta,w),$$

which means that

$$[j(n), V(\eta, m)] = \eta V(\eta, n+m), \qquad :j(z)V(\eta, z): = \frac{1}{\eta}\partial_z V(\eta, z).$$

Also

$$V(\eta, z)V(v, w) = (z-w)^{\eta v} : V(\eta, z)V(v, w):. \tag{18}$$

Let us define the action of the generators $b(n)$, $c(n)$ of the Clifford algebra on the module $\bigoplus_{l \in \mathbf{Z}} \mathscr{H}_{\alpha+l}$ by the formulas

$$b(z) = \sum_{n \in \mathbf{Z}} b(n) z^{-n-\alpha+1} = V(-1, z), \tag{19}$$

$$c(z) = \sum_{n \in \mathbf{Z}} c(n) z^{-n+\alpha} = V(1, z), \tag{20}$$

where $\alpha = \mu + 2\lambda$.

From (18) it follows that $b(z)$, $c(z)$ have the operator product, given by (14), (15), and therefore $b(n)$, $c(n)$ obey the standard anti-commutation relations (13). The vector $v_{\alpha-p}$ satisfies the conditions (6), imposed on the vector $|p\rangle$. So here is the unique homomorphism $\varepsilon^+ : M^+_{\lambda,\mu}(p) \to \bigoplus_{l \in \mathbf{Z}} \mathscr{H}_{\alpha+l}$ over the Clifford algebra, such that $\varepsilon^+(|p\rangle) = v_{\alpha-p}$.

On the other hand, we can define the action of the algebra A on $M^+_{\lambda,\mu}(p)$ by the operators $\tilde{j}(n) + \alpha z^{-1}$, which obey the same commutation relations as $j(n)$. This gives us a homomorphisms $\mathscr{H}_{\alpha-p+l} \to M^+_{\lambda,\mu}(p)_l$ over the Heisenberg algebra A, which is in fact an isomorphism, because $\mathscr{H}_{\alpha-p+l}$ is irreducible over A. So we have established an isomorphism between $M^+_{\lambda,\mu}(p)$ and $\bigoplus_{l \in \mathbf{Z}} \mathscr{H}_{\alpha+l}$ over both Heisenberg and Clifford algebras. In particular, we see that $M^+_{\lambda,\mu}(p)_l$ is isomorphic to $M^+_{\lambda,\mu+l-p}(0)_0 = M^+_{\lambda,\mu-p+l}$.

Let us rewrite the action (16) of the Virasoro algebra on $M^+_{\lambda,\mu}$, given in terms of fermions, as an action of this algebra on $\mathscr{H}_\alpha$ in terms of bosons. We have:

$$T(z) = \tfrac{1}{2} :j(z)^2: + \alpha_0 \partial_z j(z), \tag{21}$$

where $\alpha_0 = \lambda - 1/2$. Denote this representation of the Virasoro algebra by $\mathscr{H}_{\alpha_0,\alpha}$. This is a representation with highest weight $h = \alpha(\alpha - 2\alpha_0)/2$ and central charge $c = 1 - 12\alpha_0^2$.

In [19, 44] (see [15] for a brief account of the results) intertwining operators between modules $\mathscr{H}_{\alpha_0,\alpha}$ were constructed, using vertex operators (so-called, screening operators). This gave the possibility to determine completely the structure of these representations (see [19]). We want to note also that these results gave a method for computation of the correlation functions in the two-dimensional minimal models (the integral representation) [13], the method which has also proved to be very useful in other models of conformal field theory.

3.2. Bosonization of Bosons

Denote $N(p) = \bigoplus_{l \in \mathbf{Z}} M^+_{0,l} \otimes \mathscr{H}_{i(-\alpha+p+l)}$, where $\alpha = \mu + 2\lambda$ $(i = \sqrt{-1})$ (see Fig. 1).

Let us define an action of the generatory $\beta(n)$, $\gamma(n)$ of the Heisenberg algebra on $N(p)$, putting

$$\beta(z) = \sum_{n \in \mathbf{Z}} \beta(n) z^{-n-\alpha} = \partial_z b(z) V(-i, z), \tag{22}$$

$$\gamma(z) = \sum_{n \in \mathbf{Z}} \gamma(n) z^{-n+\alpha-1} = c(z) V(i, z). \tag{23}$$

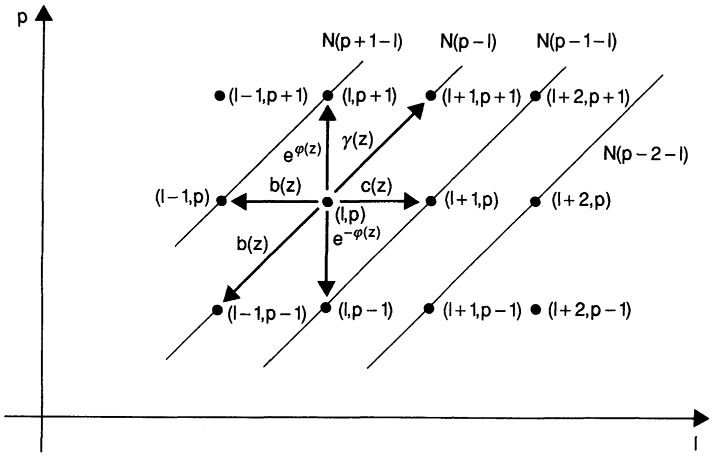

Fig. 1. The bosonization of bosons. The dot (l, p) denotes the module $M_{0,l}^{+} \otimes \mathscr{H}_{i(-\alpha+p)}$

It is easy to check that these operators satisfy the operator products (3), (4), and therefore the operators $\beta(z)$, $\gamma(z)$ satisfy the standard commutation relations (2).

The vector $|0\rangle \otimes v_{i(-\alpha+p)}$ satisfies the conditions (6), imposed on the vacuum vector $|p\rangle$ of $M_{\lambda,\mu}^{-}(p)$. This allows to define the homomorphism $\varepsilon^{-}: M_{\lambda,\mu}^{-}(p) \to N(p)$ over the Heisenberg algebra, generated by $\beta(n)$, $\gamma(n)$, putting $\varepsilon^{-}(|p\rangle) = |0\rangle \otimes v_{i(-\alpha+p)}$.

Because of the irreducibility of $M_{\lambda,\mu}^{-}(p)$ this homomorphism is an embedding. We would like to characterize the image of this embedding.

Proposition 3. *The image of the homomorphism ε^{-} coincides with the kernel of the operator $c(0)$, acting from $N(p)$ to $N(p-1)$.*

Proof. The operators $\beta(n)$, $\gamma(n)$, given by formulas (22), (23) do not depend on $b(0)$ and therefore commute with $c(0)$. So $\varepsilon^{-}(M_{\lambda,\mu}^{-}(p))$ is contained in the kernel of the operator $c(0)$. To prove that it coincides with the kernel, one should define an action of the operators $j(n)$, $b(n)$, $n \in \mathbf{Z}$ and $c(m)$, $m \neq 0$ on $M_{\lambda,\mu}^{-}(p)$, which corresponds to the action of this operator on $N(p)$ under the homomorphism ε^{-}. It can be easily done [26],

$$b(z) = \partial_z \beta(z) V(i, z), \qquad \partial_z c(z) = \partial_z \gamma(z) V(-i, z),$$

and the action of $j(z) = \partial_z \phi(z)$ is given by the operators $i(\tilde{j}(z) - \alpha z^{-1})$. $\square$

Now let us rewrite the action (10) of the Virasoro algebra on the module $M_{\lambda,\mu}^{-}(p)$ in terms of the operator $j(n)$, $b(z)$, $c(n)$. We have

$$T(z) = :\partial_z b(z) c(z): + \tfrac{1}{2} :j(z)^2: + i\alpha_0 \partial_z j(z), \tag{24}$$

where $\alpha_0 = -\lambda + 1/2$. Thus $T(z)$ is a sum of two Virasoro algebras with central charges $c = -2$ and $c = 1 + 12\alpha_0^2$.

In order to make total bosonization we can bosonize the bc-ghosts by another free scalar field $\psi(z)$ according to (19), (20). Denote $j'(z) = \partial_z \psi(z)$. Then we have

$$\beta(z) = -:\partial_z \psi(z) \exp(-\psi(z) - i\phi(z)):,$$
$$\gamma(z) = :\exp(\psi(z) + i\phi(z)):$$

This gives the isomorphism $M^+_{0,l} \cong \mathscr{H}_{-\frac{1}{2},l}$, and, according to (21),

$$T(z) = \tfrac{1}{2} :j'(z)^2: - \tfrac{1}{2}\partial_z j'(z) + \tfrac{1}{2} :j(z)^2: + i\alpha_0 \partial_z j(z)$$

is the corresponding action of the Virasoro algebra on $N(p)_l = \mathscr{H}_{-\frac{1}{2},l} \otimes \mathscr{H}_{i\alpha_0, i(-\alpha+p+l)}$, $l \in \mathbf{Z}$.

Denote by $\bar{\mathscr{H}}_l$ the kernel of the operator $Q = \int :\exp(\psi(z)): dz$, acting from $\mathscr{H}_{-\frac{1}{2},l}$ to $\mathscr{H}_{-\frac{1}{2},l+1}$. Note, that Q is just the intertwining operator, constructed in [19, 44], in this particular case. From Proposition 4 it follows that

$$M^-_{\lambda,\mu}(p)_l \cong \bar{\mathscr{H}}_l \otimes \mathscr{H}_{i\alpha_0, i(-\alpha+p+l)}.$$

So in order to describe the structure of $M^+_{\lambda,\mu}(p)_l$ as a module over the Virasoro algebra we have to determine the structure of the module $\bar{\mathscr{H}}_l$.

Denote by L_m the irreducible representation of the Virasoro algebra with highest weight $h_m = m(m+1)/2$ and central charge $c = -2$.

Proposition 4. $\bar{\mathscr{H}}_l \cong \bigoplus_{k \geqq 0} L_{|l|+2k}$.

Proof. Consider the complex C^* of modules over the Virasoro algebra, $C^l = M^+_{0,l} = \mathscr{H}_{-\frac{1}{2},l}$, with the differential Q. Obviously, $Q^2 = 0$, because $Q = c(0)$. Moreover, this complex is acyclic, because the kernel $\operatorname{Ker} Q$ of the operator Q is the subspace of $\bigoplus_{l \in \mathbf{Z}} M^+_{0,l}$, generated by $b(n)$, $c(n)$, $n < 0$, and the image coincides with $b(0) \operatorname{Ker} Q$, so that Q is the isomorphism between its image and kernel.

The composition structure of the modules $\mathscr{H}_{-\frac{1}{2},l}$ over the Virasoro algebra is shown on the Fig. 2 [19]. The dots denote the irreducibles quotients L_m of $\mathscr{H}_{-\frac{1}{2},l}$ and the arrows show the interrelations between them. In particular, we see that

$$0 \to \bigoplus_{k \geqq 0} L_{|l|+2k} \to \mathscr{H}_{-\frac{1}{2},l} \to \bigoplus_{k \geqq 0} L_{|l|+2k+1} \to 0.$$

Note also that $\mathscr{H}_{-\frac{1}{2},-1-l}$ is contragradient to $\mathscr{H}_{-\frac{1}{2},l}$. We have to prove that $\bigoplus_{k \geqq 0} L_{|l|+2k}$ coincides with the kernel of the operator Q.

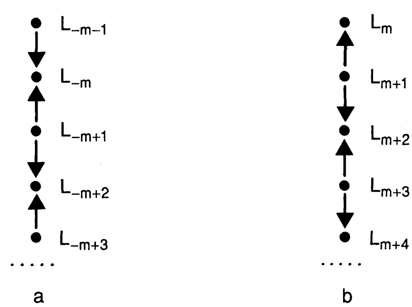

Fig. 2. The structure of the modules $\mathscr{H}_{-1/2,m}$. (a) $m < 0$, (b) $m \geqq 0$

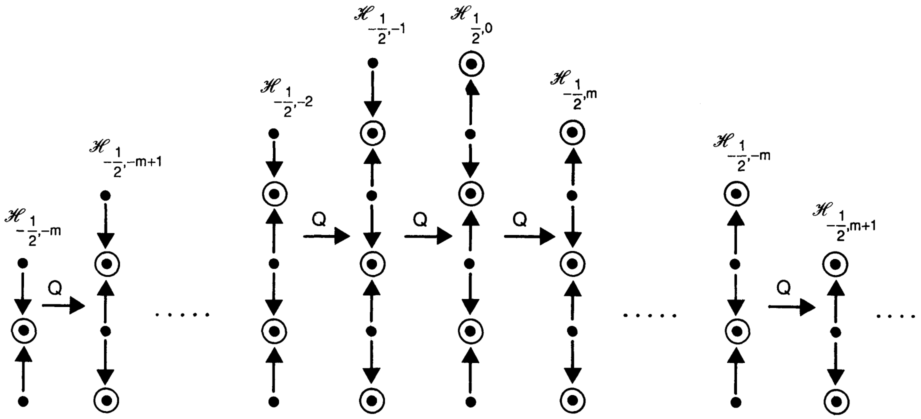

Fig. 3. The action of Q on $\bigoplus_{l\in\mathbf{Z}} M_{0,l}^{+}$. Encircled are the factors, which belong to Ker$Q$

Suppose that for some k, $L_{|l|+2k}$ does not belong to the kernel. Then it should map onto a submodule $L_{|l|+2k}$ of $\mathscr{H}_{-\frac{1}{2},l+1}$, but there are no such submodules in there, see Fig. 3.

Otherwise, suppose, that for some k, $L_{|l|+2k+1}$ is a factor of the kernel. Our complex C^* is acyclic, therefore there should be a quotient $L_{|l|+2k+1}$ of $\mathscr{H}_{-\frac{1}{2},l-1}$, mapping onto. But there is no such quotients in there. $\square$

As a corollary we get a complete description of the structure of the modules $M_{\lambda,\mu}^{-}(p)_l$.

Theorem 4.

$$M_{\lambda,\mu}^{-}(p)_l \cong \bigoplus_{k\geqq 0} L_{|l|+2k} \otimes \mathscr{H}_{i\alpha_0,\, i(-\alpha+p+l)},$$

where $\alpha_0 = -\lambda + 1/2$, $\alpha = \mu + 2\lambda$.

In particular, if we put $\lambda = 2$, $\mu = -2$, then, as we know (see Sect. 2.5) $M_{2,-2}^{-}(p)_l = S_p^l$ is the module of semi-infinite symmetric powers of the Virasoro algebra. In this case we have the following.

Corollary 1.

$$S_p^l \cong \bigoplus_{k\geqq 0} L_{|l|+2k} \otimes \mathscr{H}_{-\frac{3}{2}i,\, i(-2+p+l)}.$$

In the next subsection we will deduce from this result an amusing combinatorial identity, which can be viewed as a bosonic counterpart of the Jacobi triple-product identity. Then, in Sect. 4, we will prove Theorem 1.

In the conclusion of this subsection, we would like to mention a simple consequence of our results: the extension

$$0 \to M^{-}(p) \to N(p) \to M^{-}(p-1) \to 0$$

does not split over the $\beta\gamma$-Heisenberg algebra. We also want to note that $M^{-}(p)$ coincides with the 0^{th} cohomology space of the complex

$$0 \to N(p) \to N(p-1) \to \dots,$$

which makes it possible to consider Q as BRST-like operator.

3.3. Combinatorial Identities

In this subsection we will prove a character identity, using Theorem 3. But first let us deduce a fermionic counterpart of this identity.

Let us take the module $M^+ = M^+_{0,0}(0)$ and compute its character

$$\mathrm{ch}^+(u,q) = \mathrm{tr}_{M^+} u^{-j(0)} q^{T(0)}.$$

The module M^+ is generated by $b(n)$, $n \leqq 0$ and $c(n)$, $n < 0$. Therefore from the formulas (8), (12) it is clear that

$$\mathrm{ch}^+(u,q) = \prod_{n \geqq 0} (1+uq^n)(1+u^{-1}q^{n+1}).$$

On the other hand, $M^+ \cong \bigoplus_{l \in \mathbf{Z}} \mathscr{H}_{-\frac{1}{2},l}$. The module $\mathscr{H}_{-\frac{1}{2},l}$ is generated by $j(n)$, $n<0$, and it is easy to see that $[T(0), j(n)] = -nj(n)$, $n \neq 0$. So its character is equal to $q^{l(l+1)/2}\phi(q)$, where $\phi(q)$ stands for the Euler partition function $\phi(q) = \prod_{n>0}(1-q^n)^{-1}$. Therefore $\mathrm{ch}^+(u,q) = \phi(q) \sum_{l \in \mathbf{Z}} u^{-l} q^{l(l+1)/2}$, and we get the famous Jacobi triple product identity

$$\prod_{n \geqq 0} (1+uq^n)(1+u^{-1}q^{n+1}) = \phi(q) \sum_{l \in \mathbf{Z}} u^{-l} q^{l(l+1)/2}.$$

Now let us take the bosonic module $M^- = M^-_{0,0}(0)$ and compute its character $\mathrm{ch}^-(u,q) = \mathrm{tr}_{M^-} u^{-j(0)} q^{T(0)}$. This module is generated by $\beta(n)$, $n \leqq 0$ and $\gamma(n)$, $n<0$. Again, using (8), (12), we have

$$\mathrm{ch}^-(u,q) = \prod_{n \geqq 0} (1-uq^n)^{-1}(1-u^{-1}q^{n+1})^{-1}.$$

On the other hand, Theorem 4 says that

$$M^- \cong \bigoplus_{l \in \mathbf{Z}} \bigoplus_{k \geqq 0} L_{|l|+2k} \otimes \mathscr{H}_{\frac{i}{2}, il}.$$

The character of L_m, $m \geqq 0$ is equal to $(q^{m(m+1)/2} - q^{(m+1)(m+2)/2})\phi(q)$, due to the exact sequence

$$0 \to L_m \to \mathscr{H}_{-\frac{1}{2},m} \to \mathscr{H}_{-\frac{1}{2},m+1} \to 0.$$

Therefore $\mathrm{ch}^-(u,q)$ is equal to

$$\phi(q)^2 \sum_{l \in \mathbf{Z}} u^l f_l(q),$$

$f_l(q) = \sum_{j \geqq |l|} (-1)^{j-l} q^{h_{l,j}}$, where $h_{l,j}$ is the highest weight of the module $\mathscr{H}_{-\frac{1}{2},j} \otimes \mathscr{H}_{-\frac{i}{2},-il}$, $j \geqq |l|$. It is clear that $h_{l,j} = (j-l+1)(j+l+2)/2$. It is instructive to look at Fig. 4, where $h_{l,j}$ are marked on the plane (x = power of u, y = power of q). It is clear from this picture that any of them lie on a line of the form $y = |l|(|l|+1)/2 + lx$, $y \geqq 0$. Therefore

$$\sum_{l \in \mathbf{Z}} u^l f_l(q) = \sum_{l \in \mathbf{Z}} (-1)^l q^{l(l+1)/2} (1-uq^l)^{-1},$$

where, certainly,

$$(1-uq^l)^{-1} = \begin{cases} \sum_{m \geqq 0} u^m q^{lm}, & \text{if } l \geqq 0 \\ -\sum_{m<0} u^m q^{lm}, & \text{if } l<0 \end{cases}.$$

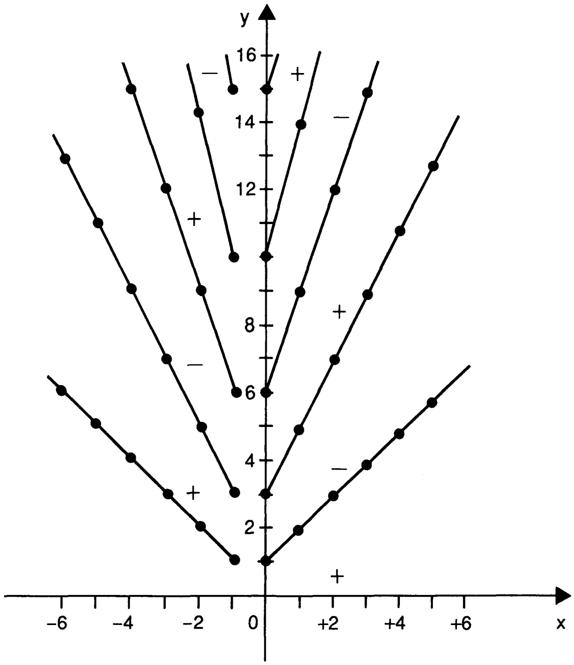

Fig. 4. Powers of u and q in the right-hand side of the identity

Finally, we obtain the identity in the most beautiful form, which is due to [34],

$$\prod_{n\geqq 0}(1-uq^n)^{-1}(1-u^{-1}q^{n+1})^{-1}=\phi(q)^2\sum_{l\in\mathbf{Z}}(-1)^l q^{l(l+1)/2}(1-uq^l)^{-1}.$$

This identity was also proved in [6, 7, 34], ..., and it goes back to 1898 [42]. Note that multiplying both sides by $\phi(q)$ we obtain an expression of the Kostant partition function of the affine Kac-Moody algebra $A_1^{(1)}$ in terms of the Euler partition function [34].

4. Proof of the Main Theorem

In this section we prove Theorem 1, which was formulated in Sect. 2, using the results of Sect. 3.

Let us recall some notations. Suppose, χ is an element of $\hat{h}'=h\oplus\mathbf{C}C$-dual space to the Cartan subalgebra $\hat{h}$ of the Virasoro algebra. So χ is a pair $(h=\chi(T(0))$, $c=\chi(C))$. Denote by $\mathbf{C}_\chi$ the one-dimensional representation of $\hat{h}$, corresponding to χ. It can be trivially extended to the Borel subalgebra $\hat{b}=b\oplus\mathbf{C}C$.

Introduce the Verma module $M_\chi=M_{h,c}$ with highest weight χ over the Virasoro algebra $\hat{\mathscr{L}}$, $M_\chi=U(\hat{\mathscr{L}})\underset{U(\hat{b})}{\otimes}\mathbf{C}_\chi$. Denote by M_χ^* the contragradient module to M_χ. For $\chi=(h,c)$ let $\chi'=(1-h,26-c)$. For any $\mathscr{L}$-module N with central charge 26 put $h^j(N)=\dim H^{\infty/2+j}(\mathscr{L},h,N)$ (with respect to the decomposition $\mathscr{L}=b\oplus n$).

Proposition 5 [14].

$$h^j(M_\chi \otimes M_\nu^*) = \begin{cases} 1 & \textit{if} \quad \nu = \chi',\ j=0 \\ 0 & \textit{otherwise}. \end{cases}$$

Proof follows from the semi-infinite analogue of the Shapiro lemma (see also [25]). □

Suppose M_χ is a Verma module of the type II or III^0 in the classification of [19]. It means that there is only one "chain" of singular vectors in M_χ, or, by other words, there is a weight $\chi_1 \in \hat{h}'$, such that

$$0 \to M_{\chi_1} \to M_\chi \to L_\chi \to 0,$$

is exact, where L_χ denotes the irreducible module with highest weight χ.

Introduce the weights χ_m inductively by putting $\chi_{m+1} = (\chi_m)_1$. There is a duality in the representations of the Virasoro algebra [14], which gives $\chi'_m = (\chi')_{-m}$.

Proposition 6.

$$h^0(L_\chi \otimes L_{\chi'}) = 1, \qquad h^0(L_\chi \otimes L_{\chi'_2}) = 1,$$
$$h^1(L_\chi \otimes L_{\chi'_1}) = 1, \qquad h^{-1}(L_\chi \otimes L_{\chi'_1}) = 1,$$

in all other cases $h^j(L_\chi \otimes L_\nu) = 0$.

Proof. We can replace the module L_χ by the complex $M_{\chi_1} \to M_\chi$ and the module L_ν by the complex $M_\nu^* \to M_{\nu_1}^*$. It is easy to compute the cohomology of the tensor product of these complexes, using Proposition 5. This gives the result, if we observe that $\chi'_1 = \nu_1$ implies $\nu = \chi'_2$. □

We will compute $h_p^{j,l} = h^j(S_p^l)$ (see Sect. 2.6), using this result. We confine ourselves with the case $p \leqq 0$. The case $p > 0$ can be pursued in the same way.

From the Corollary to the Theorem 4 we know that

$$S_p^l \cong \bigoplus_{k \geqq 0} L_{|l|+2k} \otimes \mathscr{H}_{-\frac{3}{2}i,\, i(-2+p+l)}.$$

The composition structure of the modules $\mathscr{H}_p = \mathscr{H}_{-\frac{3}{2}i,\, i(-2+p)}$ is shown on Fig. 5.

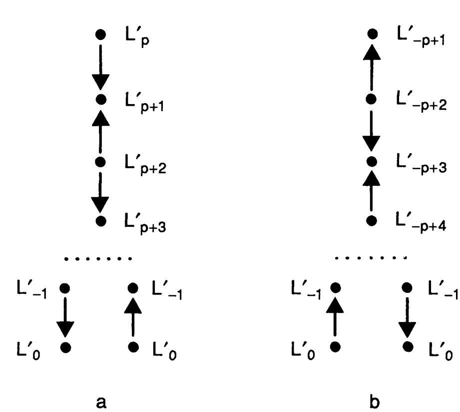

Fig. 5. The structure of the modules $\mathscr{H}_p$. (a) $p \leqq 0$, (b) $p > 0$

In particular, the highest weights of the composition quotients of $\mathscr{H}_p$ are of the form $-(m-2)(m+1)/2 = h'_{m-1}$, where $h_m = m(m+1)/2$.

Denote $L'_{-k} \doteq L_{h'_k}$, $k \geqq 0$. If $m \leqq 0$, then we have the following exact sequences:

$$0 \to \bigoplus_{m+1 \leqq j \leqq 0} L'_{2j-1} \to \mathscr{H}_{2m} \to \bigoplus_{m \leqq j \leqq 0} L'_{2j} \to 0, \tag{25}$$

$$0 \to \bigoplus_{m \leqq j \leqq 0} L'_{2j} \to \mathscr{H}_{2m-1} \to \bigoplus_{m \leqq j \leqq 0} L'_{2j-1} \to 0. \tag{26}$$

Note, that $\mathscr{H}_p$ is contragradient to $\mathscr{H}_{1-p}$.

Proposition 7. 1) *Let $m \geqq 0$, $n \leqq 0$ be of equal parity. Then*

$$h^j(L_m \otimes \mathscr{H}_n) = \begin{cases} 0, & m > -n \\ \delta_{j,0}, & m = -n \\ \delta_{j,0} + \delta_{j,-1}, & 0 \leqq m < -n. \end{cases}$$

2) *Let $m \geqq 0$, $n \leqq 0$ be of different parity. Then*

$$h^j(L_m \otimes \mathscr{H}_n) = \begin{cases} 0, & m > -n-1 \\ \delta_{j,1}, & m = -n-1 \\ \delta_{j,0} + \delta_{j,1}, & 0 \leqq m < -n-1. \end{cases}$$

In other words, $h^j(L_m \otimes \mathscr{H}_n) = h^{j-1}(L_m \otimes \mathscr{H}_{n+1})$.
3) *Let $m, n \geqq 0$. If $m < n$, then $h^j(L_m \otimes \mathscr{H}_n) = 0$.*

Proof is based on the straightforward computation of the long exact sequences, associated to the short exact sequences (25), (26), or their dual. □

Proof of Theorem 1. Let $p = -2m$, $m \geqq 0$. By the Theorem 1, $h_p^{j,l} = \sum_{k \geqq 0} h^j(L_{|l|+2k} \otimes \mathscr{H}_{p+l})$. By the Proposition 7, it is equal to $(m+1)\delta_{j,0} + m\delta_{j,-1}$, if $l \leqq 0$, $(m+1-l)\delta_{j,0} + (m-l)\delta_{j,-1}$, if $0 < l \leqq m$, and 0, if $l > 0$.

In all other cases the proof is the same. □

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