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Continuity and Relative Hamiltonians

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Abstract. Let $(\omega_n)_{n\geqq 1}$ be a norm convergent sequence of normal states on a von Neumann algebra $\mathscr{A}$ with $\omega_n \to \omega$. Let $(k_n)_{n\geqq 1}$ be a strongly convergent sequence of self-adjoint elements of $\mathscr{A}$ with $k_n \to k$. It is shown that the sequence $(\omega_n^{k_n})_{n\geqq 1}$ of perturbed states converges in norm to ω^k. A related result holds for C^*-algebras. A counter-example is provided to show that it is not sufficient to assume weak convergence of $(\omega_n)_{n\geqq 1}$ even when $k_n = k$ for all n. However, conditions are given which, together with weak convergence, are sufficient. Relative entropy methods are used, and a relative entropy inequality is proved.

1. Continuity of ω^k in ω and in k

Given a faithful state ω on a von Neumann algebra $\mathscr{A}$ and a self-adjoint element $k \in \mathscr{A}$, Araki [1] defined, using perturbation theory and modular theory, a state denoted by ω^k. The motivation for this definition came from quantum statistical mechanics. If ω represents the equilibrium state of a physical system, then ω^k will represent the equilibrium state of the perturbed system in which the energy of each state σ has been increased by $\sigma(k)$. Araki's definition has proved useful for the analysis of stability properties for equilibrium states and in demonstrating the invariance of such states under given symmetry groups.

In [7], I have used the equivalent, but more direct, definition, that ω^k is the unique state maximizing the function $\sigma \mapsto \mathrm{ent}_{\mathscr{A}}(\sigma|\omega) - \sigma(k)$, where $\mathrm{ent}_{\mathscr{A}}(\sigma|\omega)$ is the relative entropy of σ with respect to ω. With this definition, it is possible to use relative entropy techniques to give alternative – and, in my view, simpler – proofs of the results in [1]. Also, the definition and many of the results can be extended to the case in which k is a lower-bounded self-adjoint operator affiliated with $\mathscr{A}$. In this paper, relative entropy techniques are used to prove a powerful continuity result for ω^k with k bounded. Background and full elucidation of the notation are given in [7]. I have used [7] wherever possible in this paper as a unified source of results about ω^k, but, of course, many of the results quoted were originally proved by Araki. [7] gives the detailed references. This paper focuses on von Neumann algebra results, but there is an immediate extension to the C^*-algebra case. This is given as Theorem 1.9.

Notation. $\mathscr{A}$ is a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. It may be assumed, by results explained in [7], that $\mathscr{A}$ has a faithful normal state and is in standard form. $\Sigma_*(\mathscr{A})$ is the set of normal states on $\mathscr{A}$. For $\omega \in \Sigma_*(\mathscr{A})$ and $k=k^* \in \mathscr{A}$, $s(\omega)$ is the support projection of ω, and we set

$$c(\omega, k) = \mathrm{ent}_{\mathscr{A}}(\omega^k|\omega) - \omega^k(k) = \sup\{\mathrm{ent}_{\mathscr{A}}(\sigma|\omega) - \sigma(k) : \sigma \in \Sigma_*(\mathscr{A})\}.$$

Note that, as in [7], $\mathrm{ent}_{\mathscr{A}}(\sigma|\omega)$ is defined with the convention of [3] so that, for example, $\mathrm{ent}_{\mathscr{B}(\mathscr{H})}(\sigma|\omega\} = \mathrm{tr}(-\sigma\log\sigma + \sigma\log\omega) \leqq 0$. Also following [3, 7], ω^k is defined to be normalized. $c(\omega, k)$ is a difference of free energies [5] and is equivalent to the function denoted by $\log\omega^k(1)$ in several recent papers (e.g. [10, 11]) which, following [1, 2], use a non-normalized version of ω^k.

For a given natural positive cone $\mathscr{P}$, $\xi(\omega)$ will be the unique vector in $\mathscr{P}$ such that $\omega(A) = (\xi(\omega), A\xi(\omega))$ for all $A \in \mathscr{A}$, J will denote the modular conjugation.

Theorem 1.1. *Let $(\omega_n)_{n\geq 1}$ be a sequence of normal states on $\mathscr{A}$ with $\omega_n \xrightarrow{n} \omega$ and let $(k_n)_{n\geq 1}$ be a sequence of self-adjoint elements of $\mathscr{A}$ with $k_n \xrightarrow{s} k$. Then $\omega_n^{k_n} \xrightarrow{n} \omega^k$, $\mathrm{ent}_{\mathscr{A}}(\omega_n^{k_n}|\omega_n) \to \mathrm{ent}_{\mathscr{A}}(\omega^k|\omega)$, and $c(\omega_n, k_n) \to c(\omega, k)$.*

This result is a strengthening of [1, Proposition 4.1], reproved in [7, Proposition A.1], which dealt with the case that $\omega_n = \omega$ for all n. The proof is along the lines of [7, Propositions 3.15 and A.1], but is more difficult. There are two strands. One, (Lemmas 1.6 and 1.7), involves what amounts to a compactness property for $\{\omega_n^{k_n} : n \geqq 1\}$, proved by a relative entropy inequality (Proposition 2.2) which may be more widely useful. The other strand, to which we turn first, involves the construction of a sequence $(\varrho_n)_{n\geqq 1}$ such that $\mathrm{ent}_{\mathscr{A}}(\varrho_n|\omega_n) - \varrho_n(k_n) \to c(\omega, k)$ (cf. Proposition 3.5). In fact, we construct a double sequence $(\varrho_{n,m})_{n,m\geqq 1}$. First we approximate ω^k by $(\omega^{a_m})_{m\geqq 1}$, where the $(a_m)_{m\geqq 1}$ are analytic approximations to k. Then, for each m, we approximate ω^{a_m} by a sequence to which we can apply the following lemma – proved by Araki in [2, Theorem 3.7 (2)].

Lemma 1.2. *Let $(\sigma_n)_{n\geqq 1}$ and $(\omega_n)_{n\geqq 1} \subset \Sigma_*(\mathscr{A})$ with $\sigma_n \xrightarrow{n} \sigma$ and $\omega_n \xrightarrow{n} \omega$. Suppose that there exists $X \in \mathbb{R}$ such that $\sigma_n \leqq X\omega_n$ for all $n \geqq 1$.*

Then $\mathrm{ent}_{\mathscr{A}}(\sigma_n|\omega_n) \to \mathrm{ent}_{\mathscr{A}}(\sigma|\omega)$.

Lemma 1.3. *There exists $(a_m)_{m\geqq 1} \subset \mathscr{A}$ and $(T_m)_{m\geqq 1} \subset \mathscr{A}'$ such that $a_m = a_m^*$, $\omega^{a_m} \xrightarrow{n} \omega^k$, $\mathrm{ent}_{\mathscr{A}}(\omega^{a_m}|\omega) \to \mathrm{ent}_{\mathscr{A}}(\omega^k|\omega)$, $T_m \geqq 0$, $\omega^{a_m}(A) = (\xi(\omega), T_m A\xi(\omega))$ for all $A \in \mathscr{A}$, and $\omega^{a_m} \leqq \|T_m\|\,\omega$.*

Proof. Assume first that ω is faithful. Take $(a_m)_{m\geqq 1}$ to be the sequence of analytic approximations to k defined in the appendix to [7] and denoted there by $(k_n)_{n\geqq 1}$ [7, Eq. (A.2)]. From that appendix, $a_m = a_m^*$, $\omega^{a_m} \xrightarrow{n} \omega^k$, and $\mathrm{ent}_{\mathscr{A}}(\omega^{a_m}|\omega) \to \mathrm{ent}_{\mathscr{A}}(\omega^k|\omega)$. That appendix also defines an operator $\Gamma_m(\frac{1}{2}i) \in \mathscr{A}$ and $c_m \in \mathbb{R}$ such that $\xi(\omega^{a_m}) = \Gamma_m(\frac{1}{2}i)e^{-\frac{1}{2}c_m}\xi(\omega)$. But $\xi(\omega^{a_m}) = J\xi(\omega^{a_m}) = J\Gamma_m(\frac{1}{2}i)e^{-\frac{1}{2}c_m}J\xi(\omega)$, so, for $A \in \mathscr{A}$,

$\omega^{a_m}(A)=(\xi(\omega^{a_m}), A\xi(\omega^{a_m}))=(\xi(\omega), T_mA\xi(\omega))$, where

$$T_m=J\Gamma_m(\tfrac{1}{2}i)^*\Gamma_m(\tfrac{1}{2}i)e^{-c_m}J\in\mathscr{A}'_+ .$$

That $\omega^{a_m}\leqq\|T_m\|\,\omega$ is then elementary.

If ω is not faithful, then we use the canonical reduction $\lambda_1:\mathscr{A}\to\mathscr{A}_{s(\omega)}$ defined by $\lambda_1(A)=s(\omega)A|_{s(\omega)\mathscr{H}}$ to map the von Neumann algebra $(\mathscr{A},\mathscr{H})$ to the von Neumann algebra $(\mathscr{A}_{s(\omega)}=s(\omega)\mathscr{A}s(\omega),\ s(\omega)\mathscr{H})$. Denote the inverse embedding by λ_2. $\omega\circ\lambda_2$ defined by $\omega\circ\lambda_2(s(\omega)As(\omega))=\omega(A)$ is then faithful. Applying the result just proved gives $(a_m)_{m\geqq1}\subset\mathscr{A}_{s(\omega)}$ and $(X_m)_{m\geqq1}\subset\mathbb{R}$ such that $a_m=a_m^*$, $(\omega\circ\lambda_2)^{a_m}\xrightarrow{n}(\omega\circ\lambda_2)^{\lambda_1(k)}$, $\mathrm{ent}_{\mathscr{A}_{s(\omega)}}((\omega\circ\lambda_2)^{a_m}|\omega\circ\lambda_2)\to\mathrm{ent}_{\mathscr{A}_{s(\omega)}}((\omega\circ\lambda_2)^{\lambda_1(k)}|\omega\circ\lambda_2)$, and $(\omega\circ\lambda_2)^{a_m}\leqq X_m\omega\circ\lambda_2$. By [7, Lemma 2.3F], λ_2 defines an isomorphism $(\sigma\mapsto\sigma\circ\lambda_2)$ between $\{\sigma\in\Sigma_*(\mathscr{A}):s(\sigma)\leqq s(\omega)\}$ and $\Sigma_*(\mathscr{A}_{s(\omega)})$ satisfying

$$\mathrm{ent}_{\mathscr{A}}(\sigma|\omega)-\sigma(s(\omega)ks(\omega))=\mathrm{ent}_{\mathscr{A}_{s(\omega)}}(\sigma\circ\lambda_2|\omega\circ\lambda_2)-(\sigma\circ\lambda_2)(\lambda_1(k)) .$$

Invoking [7, Theorem 3.1A and C, and Lemma 2.3D], yields $\omega^k=\omega^{s(\omega)ks(\omega)}$, $\omega^k\circ\lambda_2=(\omega\circ\lambda_2)^{\lambda_1(k)}$, $\mathrm{ent}_{\mathscr{A}_{s(\omega)}}((\omega\circ\lambda_2)^{\lambda_1(k)}|\omega\circ\lambda_2)=\mathrm{ent}_{\mathscr{A}}(\omega^k|\omega)$, and similarly, $(\omega\circ\lambda_2)^{a_m}=\omega^{\lambda_2(a_m)}\circ\lambda_2$ and $\mathrm{ent}_{\mathscr{A}_{s(\omega)}}((\omega\circ\lambda_2)^{a_m}|\omega\circ\lambda_2)=\mathrm{ent}_{\mathscr{A}}(\omega^{\lambda_2(a_m)}|\omega)$. Also $\omega^{\lambda_2(a_m)}\xrightarrow{n}\omega^k$. All that remains is to deduce the existence of T_m from $\omega^{\lambda_2(a_m)}\leqq X_m\omega$. This is standard (e.g. [12, Lemma 5.19]). □

Lemma 1.4. *For $n,m\geqq1$, define $\varrho_{n,m}\in\Sigma_*(\mathscr{A})$ by $\varrho_{n,m}=\omega_n$ if $(\xi(\omega_n), T_m\xi(\omega_n))\leqq\frac{1}{2}$ and, otherwise, by*

$$\varrho_{n,m}(A)=\frac{(\xi(\omega_n), T_mA\xi(\omega_n))}{(\xi(\omega_n), T_m\xi(\omega_n))} .$$

Then $\varrho_{n,m}\leqq2\|T_m\|\,\omega_n$, and, as $n\to\infty$, $\varrho_{n,m}\xrightarrow{n}\omega^{a_m}$ and

$$\mathrm{ent}_{\mathscr{A}}(\varrho_{n,m}|\omega_n)\to\mathrm{ent}_{\mathscr{A}}(\omega^{a_m}|\omega) .$$

Proof. Since $\xi(\omega_n)\xrightarrow{n}\xi(\omega)$ (e.g. [3, Theorem 2.5.31(b)]) and since $1=\omega^{a_m}(1)=(\xi(\omega), T_m\xi(\omega))$, we have that, for each fixed m, $\varrho_{n,m}\xrightarrow{n}\omega^{a_m}$. That $\varrho_{n,m}\leqq2\|T_m\|\,\omega_n$ is elementary and allows Lemma 1.2 to be applied to give $\mathrm{ent}_{\mathscr{A}}(\varrho_{n,m}|\omega_n)\to\mathrm{ent}_{\mathscr{A}}(\omega^{a_m}|\omega)$. □

Set $\kappa=1+\sup_{n\geqq1}\|k_n\|$.

Lemma 1.5. $\liminf_{n\to\infty} c(\omega_n,k_n)\geqq c(\omega,k)$.

Proof. Suppose not. Then, by passing to a subsequence, we may assume that, for some $\varepsilon>0$, $c(\omega_n,k_n)<c(\omega,k)-\varepsilon$ for all n.

Using Lemma 1.3, choose and fix M such that $|\mathrm{ent}_{\mathscr{A}}(\omega^{a_M}|\omega)-\mathrm{ent}_{\mathscr{A}}(\omega^k|\omega)|<\varepsilon/8$ and $\|\omega^{a_M}-\omega^k\|(1+\|k\|)<\varepsilon/8$.

Using Lemma 1.4, find N (depending on M) such that $n\geqq N\Rightarrow|\mathrm{ent}_{\mathscr{A}}(\varrho_{n,M}|\omega_n)-\mathrm{ent}_{\mathscr{A}}(\omega^{a_M}|\omega)|<\varepsilon/8$, $\|\varrho_{n,M}-\omega^{a_M}\|\,\kappa<\varepsilon/16$, and $|\omega^{a_M}(k_n-k)|<\varepsilon/16$.

Then $n\geqq N\Rightarrow|\mathrm{ent}_{\mathscr{A}}(\varrho_{n,M}|\omega_n)-\mathrm{ent}_{\mathscr{A}}(\omega^k|\omega)|<\varepsilon/4$ and

$$|\varrho_{n,M}(k_n)-\omega^k(k)|\leqq|\varrho_{n,M}(k_n)-\omega^{a_M}(k_n)|+|\omega^{a_M}(k_n)-\omega^{a_M}(k)|+|\omega^{a_M}(k)-\omega^k(k)|<\varepsilon/4 .$$

But, from the definition of c, for all n and m, $c(\omega_n, k_n) \geqq \mathrm{ent}_{\mathscr{A}}(\varrho_{n,m}|\omega_n) - \varrho_{n,m}(k_n)$.
Thus $n \geqq N \Rightarrow c(\omega_n, k_n) \geqq \mathrm{ent}_{\mathscr{A}}(\omega^k|\omega) - \omega^k(k) - \varepsilon/2 = c(\omega, k) - \varepsilon/2$. This is a contradiction. □

Lemma 1.6. *$\omega_n^{k_n}(k_n - k) \to 0$ as $n \to \infty$.*

Proof. By the Peierls-Bogoliubov inequality – [7, Lemma 3.8] –

$$\mathrm{ent}_{\mathscr{A}}(\omega_n^{k_n}|\omega_n) \geqq \omega_n^{k_n}(k_n) - \omega_n(k_n) \geqq -2\kappa .$$

Set $A_n = (k_n - k)/2\kappa$. Then Proposition 2.2 yields

$$\begin{aligned} |\omega_n^{k_n}(A_n)|^2 &\leqq \omega_n^{k_n}(A_n^2) \leqq -(1 + 1/e - \mathrm{ent}_{\mathscr{A}}(\omega_n^{k_n}|\omega_n))/\log \omega_n(A_n^2) \\ &\leqq -(1 + 1/e + 2\kappa)/\log \omega_n(A_n^2) \\ &= -(1 + 1/e + 2\kappa)/\log(\omega(A_n^2) + (\omega_n - \omega)(A_n^2)) . \end{aligned}$$

$\omega(A_n^2) \to 0$ because $k_n \xrightarrow{s} k$ while $|(\omega_n - \omega)(A_n^2)| \leqq \|\omega_n - \omega\| \to 0$ because $\omega_n \xrightarrow{n} \omega$. □

Lemma 1.7. *Let $(\omega_\alpha^{k_\alpha})_{\alpha \in I}$ be a w*-convergent subnet of $(\omega_n^{k_n})_{n \geqq 1}$ with $\omega_\alpha^{k_\alpha} \to \omega''$. Then $\omega_\alpha^{k_\alpha}(k_\alpha) \to \omega''(k)$.*

Proof. Choose $\varepsilon > 0$. $\exists \alpha_0$ such that $\alpha > \alpha_0 \Rightarrow |(\omega'' - \omega_\alpha^{k_\alpha})(k)| < \varepsilon/2$ (by w*-convergence) and $|\omega_\alpha^{k_\alpha}(k_\alpha - k)| < \varepsilon/2$ (by Lemma 1.6).
Then $\alpha > \alpha_0 \Rightarrow |(\omega''(k) - \omega_\alpha^{k_\alpha}(k_\alpha)| < \varepsilon$. □

Proof of Theorem 1.1. By Lemma 1.7, w* upper semicontinuity of ent [4, 9], and Lemma 1.5

$$\begin{aligned} \mathrm{ent}_{\mathscr{A}}(\omega''|\omega) - \omega''(k) &\geqq \limsup_{\alpha \in I} (\mathrm{ent}_{\mathscr{A}}(\omega_\alpha^{k_\alpha}|\omega_\alpha) - \omega_\alpha^{k_\alpha}(k_\alpha)) \\ &\geqq c(\omega, k) . \end{aligned} \tag{1.8}$$

It follows from the definition and uniqueness of ω^k that $\omega'' = \omega^k$. Then, as the subnet $(\omega_\alpha^{k_\alpha})_{\alpha \in I}$ was arbitrary, $\omega_n^{k_n} \xrightarrow{w} \omega^k$.
(1.8) and Lemma 1.5 yield $c(\omega_n, k_n) \to c(\omega, k)$. Lemma 1.7 yields $\omega_n^{k_n}(k_n) \to \omega^k(k)$. It follows that $\mathrm{ent}_{\mathscr{A}}(\omega_n^{k_n}|\omega_n) \to \mathrm{ent}_{\mathscr{A}}(\omega^k|\omega)$.

All that remains is to show that $\omega_n^{k_n} \xrightarrow{n} \omega^k$. Suppose given $\delta > 0$. Find $\varepsilon > 0$ such that $\sqrt{2\varepsilon} + 3\varepsilon/16 < \delta$. As in the proof of Lemma 1.5, we can find M and N such that $n \geqq N \Rightarrow$

$$\|\omega^{a_M} - \omega^k\| < \varepsilon/8 , \qquad \|\varrho_{n,M} - \omega^{a_M}\| < \varepsilon/16 ,$$

and

$$\mathrm{ent}_{\mathscr{A}}(\varrho_{n,M}|\omega_n) - \varrho_{n,M}(k_n) \geqq c(\omega, k) - \varepsilon/2 .$$

Choose $N_1 \geqq N$ such that $n \geqq N_1 \Rightarrow |c(\omega_n, k_n) - c(\omega, k)| < \varepsilon/2$.
But, by [7, Eq. 3.4],

$$\mathrm{ent}_{\mathscr{A}}(\varrho_{n,M}|\omega_n) - \varrho_{n,M}(k_n) = c(\omega_n, k_n) + \mathrm{ent}_{\mathscr{A}}(\varrho_{n,M}|\omega_n^{k_n})$$

so that $n \geqq N_1 \Rightarrow |\mathrm{ent}_{\mathscr{A}}(\varrho_{n,M}|\omega_n^{k_n})| < \varepsilon$.
[8, Theorem 3.1] gives $\frac{1}{2}\|\varrho_{n,M} - \omega_n^{k_n}\|^2 \leqq -\mathrm{ent}_{\mathscr{A}}(\varrho_{n,M}|\omega_n^{k_n})$, so that

$$n \geqq N_1 \Rightarrow \|\varrho_{n,M} - \omega_n^{k_n}\| < \sqrt{2\varepsilon} \quad \text{and} \quad \|\omega_n^{k_n} - \omega^k\| < \sqrt{2\varepsilon} + \varepsilon/16 + \varepsilon/8 < \delta . \quad \square$$

Theorem 1.9. *Let $(\omega_n)_{n\geqq 1}$ be a sequence of states on a C^*-algebra $\mathscr{C}$ with $\omega_n \xrightarrow{n} \omega$ and let $(k_n)_{n\geqq 1}$ be a sequence of self-adjoint elements of $\mathscr{C}$ with $k_n \xrightarrow{n} k$. Then $\omega_n^{k_n} \xrightarrow{n} \omega^k$, $\mathrm{ent}_{\mathscr{C}}(\omega_n^{k_n}|\omega_n) \to \mathrm{ent}_{\mathscr{C}}(\omega^k|\omega)$, and $c(\omega_n, k_n) \to c(\omega, k)$.*

Proof. If σ is a state on $\mathscr{C}$ then we denote by $\theta(\sigma)$ the unique normal extension of σ to the universal enveloping algebra $\mathscr{C}^{**}$. It was noted in [7] that, whether or not the GNS vector representative of σ is separating, σ^k can be defined by $\theta(\sigma^k) = \theta(\sigma)^k$ and that

$$\mathrm{ent}_{\mathscr{C}}(\sigma|\omega) = \mathrm{ent}_{\mathscr{C}^{**}}(\theta(\sigma)|\theta(\omega))$$

(cf. [2, Sect. 5]). The result follows immediately. □

2. A Relative Entropy Inequality

Lemma 2.1. *Let $\sigma, \varrho \in \Sigma_*(\mathscr{A})$ and $(A_n)_{n=1}^N \subset \mathscr{A}$, with N finite or infinite, $A_n \geqq 0$ for each n, and $\sum_{n=1}^N A_n = 1$. Then*

$$\mathrm{ent}_{\mathscr{A}}(\sigma|\varrho) \leqq -\sum_{n=1}^N \sigma(A_n)\log(\sigma(A_n)/\varrho(A_n)).$$

Proof. It is sufficient to prove this result for finite N, since

$$\begin{aligned} &-\sum_{n=1}^N \sigma(A_n)\log(\sigma(A_n)/\varrho(A_n)) \\ &= \sum_{n=1}^N (-\sigma(A_n)\log(\sigma(A_n)/\varrho(A_n)) + \sigma(A_n) - \varrho(A_n)) \end{aligned}$$

and the right-hand side is a sum of negative terms.

For N finite and $\mathscr{A} = \mathscr{B}(\mathscr{H})$ – the algebra of all bounded operators on some Hilbert space $\mathscr{H}$ – the result is proved as Proposition 8.7 of [4]. This proof extends without significant change to the present context. Indeed, if $\mathscr{A}$ is represented on a Hilbert space $\mathscr{H}$, then, by a theorem of Naimark, one can construct a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$, with $e \in \mathscr{B}(\mathscr{K})$ the orthogonal projection onto $\mathscr{H}$, and mutually orthogonal projections $(P_n)_{n=1}^N \subset \mathscr{B}(\mathscr{K})$ such that $A_n = eP_n|_{\mathscr{B}(e\mathscr{K})}$. Then, let $\mathscr{A}_1$ be the von Neumann algebra on $\mathscr{K}$ generated by $(P_n)_{n=1}^N \cup \{eAe : A \in \mathscr{A}\}$. To extend the argument of [4, Proposition 8.7] it is only necessary to remark that $(\mathscr{A}_1)_e = \mathscr{A}$.

One should note that in this paper, as in [7], I an using Araki's version of ent – which is defined only for algebras. This was denoted by ent^S in [4]. Nevertheless, I still view the extension of relative entropy to non-algebras given in [4] as being of possible fundamental physical significance [6, Eq. (5.6)]. □

Proposition 2.2. *Let $\sigma, \varrho \in \Sigma_*(\mathscr{A})$ and $A \in \mathscr{A}$ with $1 \geqq A \geqq 0$. Then*

$$\sigma(A) \leqq -(1 + 1/e - \mathrm{ent}_{\mathscr{A}}(\sigma|\varrho))/\log\varrho(A).$$

(This applies even if $\varrho(A) = 0$, but it is vacuous when $\mathrm{ent}_{\mathscr{A}}(\sigma|\varrho) = -\infty$.)

Proof. Set $s = \sigma(A)$ and $r = \varrho(A)$.

Lemma 2.1 gives $-s\log(s/r) - (1-s)\log((1-s)/(1-r)) \geqq \mathrm{ent}_{\mathscr{A}}(\sigma|\varrho)$. But, $-(1-s)\log((1-s)/(1-r)) \leqq s - r \leqq 1$ and $-s\log s \leqq 1/e$, so

$$s\log r \geqq \mathrm{ent}_{\mathscr{A}}(\sigma|\varrho) - 1 - 1/e. \quad \square$$

Comparing this with [13, Theorem III.5.4] and with [7, Lemma 2.3C and Proposition A.1], we see that it gives a precise measure of the weak compactness of $\{\sigma : \mathrm{ent}_{\mathscr{A}}(\sigma|\varrho) \geqq \alpha\}$ for $\alpha > -\infty$.

3. Weak Convergence – A Counter-Example and A Sufficient Condition

In this section a counter-example (Example 3.2) is provided demonstrating that Theorem 1.1 does not, in general, extend to the case in which ω_n converges weakly to ω, even if $k_n = k$ for all n. However, sufficient conditions (Proposition 3.5) are given for the extension of Theorem 1.1. It is also shown that Lemma 1.2 does not extend to cover weak convergence (Example 3.3) and that Theorem 1.1 does not, in general, extend to the case in which k_n converges σ-weakly to k, even if $\omega_n = \omega$ for all n (Example 3.4).

Let $\mathscr{A} = L^\infty([0,1]) \otimes \mathscr{B}(\mathbb{C}^2)$ acting on $\mathscr{H} = L^2([0,1]) \otimes \mathbb{C}^2$. $A \in \mathscr{A}$ takes the form $A = (A_{ij})_{i,j=1,2}$ with $A_{ij} \in L^\infty([0,1])$, while $f \in \mathscr{A}_*$ takes the form $f = (f_{ij})_{i,j=1,2}$ with $f_{ij} \in L^1([0,1])$, and $f(A) = \sum_{i,j=1}^{2} f_{ij}(A_{ji}) = \sum_{i,j=1}^{2} \int_0^1 f_{ij}(x) A_{ji}(x)\,dx$.

$\tau = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is a faithful trace on $\mathscr{A}$. Define $\tau_n = (t^n_{ij}) \in \Sigma_*(\mathscr{A})$ by

$$t^n_{12} = t^n_{21} = 0,$$

$$\begin{aligned} t^n_{11}(x) = 1 - t^n_{22}(x) &= 1 \quad \text{for} \quad x \in \bigcup_{m=0}^{2^{n-1}-1} \left[\frac{2m}{2^n}, \frac{2m+1}{2^n}\right) \\ &= 0 \quad \text{for} \quad x \in \bigcup_{m=0}^{2^{n-1}-1} \left[\frac{2m+1}{2^n}, \frac{2m+2}{2^n}\right). \end{aligned}$$

Lemma 3.1. $\tau_n \xrightarrow{w} \tau$.

Proof. $s(\tau_n) = (t^n_{ij})$ and $\tau_n(A) = 2\tau(s(\tau_n)A)$ for $A \in \mathscr{A}$, so it is sufficient to prove that $s(\tau_n) \to \frac{1}{2}$ σ-weakly. Indeed, it is sufficient to prove that t^n_{11} converges σ-weakly to t in $L^\infty([0,1])$, where $t(x) = \frac{1}{2}$ for all $x \in [0,1]$.

Choose $\varepsilon > 0$ and $f \in L^1([0,1])$.

$$\exists g \in C([0,1]) \text{ such that } \int_0^1 |f(x) - g(x)|\,dx < \varepsilon/2\,.$$

$$\exists N \text{ such that } |x-y| \leqq 1/2^N \Rightarrow |g(x) - g(y)| < \varepsilon\,.$$

Then $n \geqq N$

$$\Rightarrow \left| \int_{\frac{2m+1}{2^n}}^{\frac{2m+2}{2^n}} g(x)\,dx - \int_{\frac{2m}{2^n}}^{\frac{2m+1}{2^n}} g(x)\,dx \right| < \varepsilon/2^n \quad \text{for} \quad m = 0, 1, \ldots, 2^{n-1} - 1$$

$$\Rightarrow \left| g(t) - g(t^n_{11}) \right| \leqq \tfrac{1}{2} \sum_{m=0}^{2^{n-1}-1} \left| \int_{\frac{2m+1}{2^n}}^{\frac{2m+2}{2^n}} g(x)\,dx - \int_{\frac{2m}{2^n}}^{\frac{2m+1}{2^n}} g(x)\,dx \right| < \varepsilon/4$$

and

$$|f(t)-f(t_{11}^n)|\leqq|(f-g)(t)|+|g(t)-g(t_{11}^n)|+|(g-f)(t_{11}^n)|$$
$$\leqq\tfrac{1}{2}\int_0^1|f(x)-g(x)|dx+\varepsilon/4+\int_0^1|g(x)-f(x)|dx<\varepsilon. \quad \square$$

Example 3.2. Set $k=\begin{pmatrix}0&1\\1&0\end{pmatrix}\in\mathscr{A}$. τ_n was constructed precisely so that $s(\tau_n)ks(\tau_n)=0$. By [7, Theorem 3.1C], $\tau_n^k=\tau_n^{s(\tau_n)ks(\tau_n)}=\tau_n$. However, by [7, Corollary 3.14], $\tau^k\neq\tau$. It follows that $\tau_n^k\not\to\tau^k$.

This example can easily be extended to produce a sequence $(\varrho_n)_{n\geqq 1}$ of faithful states such that $\varrho_n\xrightarrow{w}\tau$ but $\varrho_n^k\not\to\tau^k$. To do this, note that, as $x\to 0$, $(1-x)\tau_n+x\tau\xrightarrow{n}\tau_n$, and use Theorem 1.1 to choose x_n such that $\|\varrho_n^k-\tau_n^k\|\leqq\frac{1}{2}\|\tau^k-\tau\|$, where $\varrho_n=(1-x_n)\tau_n+x_n\tau$. Now weak convergence of ϱ_n^k to τ^k would contradict the fact that $\tau_n^k=\tau_n\xrightarrow{w}\tau$ and the weak lower semicontinuity of the norm. $\square$

Example 3.3. Let $\sigma_n=\frac{1}{2}\tau_n+\frac{1}{2}\tau$. Clearly, $\frac{1}{2}\tau\leqq\sigma_n\leqq 3/2\tau$ and $\sigma_n\xrightarrow{w}\tau$. By monotonicity of ent – or even by Lemma 2.1 –

$$\begin{aligned}\operatorname{ent}_{\mathscr{A}}(\sigma_n|\tau)&\leqq-\sigma_n(s(\tau_n))\log(\sigma_n(s(\tau_n))/\tau(s(\tau_n)))\\&\quad-\sigma_n(1-s(\tau_n))\log(\sigma_n(1-s(\tau_n))/\tau(1-s(\tau_n)))\\&=-3/4\log 3/2-\tfrac{1}{4}\log\tfrac{1}{2}<0\end{aligned}$$

so that $\operatorname{ent}_{\mathscr{A}}(\sigma_n|\tau)\not\to\operatorname{ent}_{\mathscr{A}}(\tau|\tau)=0$, and Lemma 1.2 does not extend. $\square$

Example 3.4. Let $P=\begin{pmatrix}1&0\\0&0\end{pmatrix}\in\mathscr{A}$ and set $a_n=Ps(\tau_n)P$. The proof of Lemma 3.1 shows that a_n converges σ-weakly to $\frac{1}{2}P$. By using [10, Theorem 4] to restrict to the abelian subalgebra $(PL^\infty([0,1])P)\oplus((1-P)L^\infty([0,1])(1-P))$ of $\mathscr{A}$, it can be seen that, for $A\in\mathscr{A}$, $\tau^{a_n}(A)=\tau(e^{-a_n}A)/\tau(e^{-a_n})$ and $\tau^{\frac{1}{2}P}(A)=\tau(e^{-\frac{1}{2}P}A)/\tau(e^{-\frac{1}{2}P})$. But then $\tau^{a_n}(A)\to(\frac{1}{2}(e^{-1}+1)\tau(PA)+\tau((1-P)A))/(\frac{1}{4}e^{-1}+3/4)$ so that $\tau^{a_n}\not\to\tau^{\frac{1}{2}P}$. $\square$

Proposition 3.5. *Let $(\omega_\alpha)_{\alpha\in I}$ be a net of normal states on $\mathscr{A}$ with $\omega_\alpha\xrightarrow{w}\omega\in\Sigma_*(\mathscr{A})$.*
A) *Let $(k_\alpha)_{\alpha\in I}$ be a net of self-adjoint elements of $\mathscr{A}$ with $k_\alpha\xrightarrow{n}k$. If $\liminf_{\alpha\in I}c(\omega_\alpha,k_\alpha)\geqq c(\omega,k)$ then $\omega_\alpha^{k_\alpha}\xrightarrow{w}\omega^k$ and $c(\omega_\alpha,k_\alpha)\to c(\omega,k)$.*
B) *Let h be an extended-valued lower-bounded operator affiliated with $\mathscr{A}$ in the sense of* [7, *Definition 2.12*]. *If $\liminf_{\alpha\in I}c(\omega_\alpha,h)\geqq c(\omega,h)>-\infty$ then $\omega_\alpha^h\xrightarrow{w}\omega^h$ and $c(\omega_\alpha,h)\to c(\omega,h)$.*

Proof. A) We use techniques already explained:

Let $(\omega_\beta^{k_\beta})_{\beta\in J}$ be a w^*-convergent subnet of $(\omega_\alpha^{k_\alpha})_{\alpha\in I}$ with $\omega_\beta^{k_\beta}\to\omega''$. Then

$$\begin{aligned}|(\omega''(k)-\omega_\beta^{k_\beta}(k_\beta)|&\leqq|(\omega''-\omega_\beta^{k_\beta})(k)|+|\omega_\beta^{k_\beta}(k-k_\beta)|\\&\leqq|(\omega''-\omega_\beta^{k_\beta})(k)|+\|k-k_\beta\|\to 0\ \text{so that}\end{aligned}$$

$$\begin{aligned}\operatorname{ent}_{\mathscr{A}}(\omega''|\omega)-\omega''(k)&\geqq\limsup_{\beta\in J}(\operatorname{ent}_{\mathscr{A}}(\omega_\beta^{k_\beta}|\omega_\beta)-\omega_\beta^{k_\beta}(k_\beta))\\&\geqq c(\omega,k).\end{aligned}$$

Hence $\omega''=\omega^k$ and $\limsup_{\beta\in J} c(\omega_\beta, k_\beta)=c(\omega, k)$.

The proof of B is similar, but uses [7, Lemma 2.3A] and the weak lower semicontinuity of h [7, Definition 2.12]. □

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