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Scaling Limit for Interacting Ornstein-Uhlenbeck Processes *

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Abstract. The problem of describing the bulk behavior of an interacting system consisting of a large number of particles comes up in different contexts. See for example [1] for a recent exposition. In [4] one of the authors considered the case of interacting diffusions on a circle and proved that the density of particles evolves according to a nonlinear diffusion equation. The interacting particles evolved according to a generator that was symmetric in equilibrium. In this article we consider interacting Ornstein-Uhlenbeck processes. Here the diffusion generator is not symmetric relative to the equilibrium and the earlier methods have to be modified considerably. We use some ideas that were employed in [3] to extend the central limit theorem from the symmetric to nonsymmetric cases.

1. The Model and Its Macroscopic Equation

Let S be the circle of circumference 1. For each positive integer N we consider a system of N interacting particles with positions on S and velocities in R. The system is described by the following stochastic differential equations in phase space $(\underset{\sim}{x}, \underset{\sim}{v}) = \{(x_1, v_1), (x_2, v_2), \ldots, (x_N, v_N)\}$,

$$\begin{aligned} dx_i(t) &= N v_i(t)\,dt \\ dv_i(t) &= -N^2 \sum_{j \neq i} 2V'(N(x_i(t)-x_j(t)))\,dt \\ &\quad - \frac{N^2}{2} v_i(t)\,dt + N\,dw_i(t) \end{aligned} \tag{1.1}$$

for $i=1,2,\ldots,N$; $0 \leqq t \leqq T$. Here $\{w_i(t), i=1,2,\ldots,N\}$ are N independent Wiener processes and V is an even function on R with compact support describing a pair interaction.

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We make the following assumptions on V:

(i) $V \geqq 0$, $V(0) > 0$ and V has compact support (super stability).
(ii) V is once continuously differentiable.
(iii) $\psi(z) = -zV'(z) \geqq 0$ (Repulsive Interaction).

The generator of the Markov Process $(\underline{x}(t), \underline{v}(t))$ on $(S \times R)^N$ is given by

$$L_N = S_N + J_N,$$

where

$$\begin{aligned} S_N &= \frac{N^2}{2} \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial v_i^2} - v_i \frac{\partial}{\partial v_i} \right), \\ J_N &= N \sum_{i=1}^{N} \left\{ v_i \frac{\partial}{\partial x_i} - N \sum_{j \neq i} 2V'(N(x_i - x_j)) \frac{\partial}{\partial v_i} \right\}. \end{aligned} \tag{1.2}$$

The "equilibrium" measure on $(S \times R)^N$ for L_N is given by

$$d\mu_N(\underline{x}, \underline{v}) = \frac{1}{Z_N} e^{-\frac{1}{2} \sum_{i=1}^{N} v^2 - \sum_{i \neq j} V(N(x_i - x_j))} d^N \underline{x} \, d^N \underline{v}, \tag{1.3}$$

where Z_N is the normalizing constant. We note that S_N and J_N are respectively the symmetric and antisymmetric part of L_N with respect to μ_N.

We will assume that the initial distribution of the process has a density $f_N^0(\underline{x}, \underline{v})$ with respect to μ_N and it satisfies the following entropy bound:

$$H_N(f_N^0) = \int f_N^0(\underline{x}, \underline{v}) \log f_N^0(\underline{x}, \underline{v}) \, d\mu_N(\underline{x}, \underline{v}) \leqq CN \tag{1.4}$$

for some suitable constant C.

The empirical distribution of the process at time t is the probability measure on S defined by

$$\alpha_N(t, d\theta) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t)}(d\theta). \tag{1.5}$$

We view α_N as an element of $C[[0, T], M_1(S)]$, the space of continuous functions on $[0, T]$ with values in the space $M_1(S)$ of probability measures on S, endowed with the weak topology. We denote by Q_N the distribution of α_N on $C[[0, T], M_1(S)]$. The aim of this article is to study the limiting behavior of Q_N. The main result is that as $N \to \infty$, Q_N will concentrate around a single trajectory which is the solution of a nonlinear diffusion equation.

Theorem 1.1 (Hydrodynamic Limit). *Assume that $H_N(f_N^0) \leqq CN$ and that there exists a function $m_0(\theta)$ such that $m_0(\theta) \geqq 0$, $\int m_0(\theta) d\theta = 1$ and for every $J: S \to R$ which is smooth and every $\delta > 0$,*

$$\lim_{N \to \infty} \int_{A_{N,\delta,J}} f_N^0(\underline{x}, \underline{v}) \, d\mu_N(\underline{x}, \underline{v}) = 0,$$

where

$$A_{N,\delta,J} = \left\{ (\underline{x}, \underline{v}) \in (S \times R)^N : \left| \frac{1}{N} \sum_{i=1}^{N} J(x_i) - \int J(\theta) m_0(\theta) d\theta \right| \geqq \delta \right\},$$

then $Q = \lim_{N\to\infty} Q_N$ *exists and it is concentrated on the trajectory of the form* $\mu(t, d\theta)$ *in* $C[[0,T], M_1(S)]$ *satisfying*

(i) $\mu(t,d\theta)=m(t,\theta)d\theta$ *for all* t.

(ii) $\int_0^T dt \int_S d\theta [m(t,\theta)]^3 < \infty$.

(iii) $\int_0^T dt \int_S \frac{1}{m(t,\theta)} \left[\frac{\partial}{\partial\theta} P(m(t,\theta))\right]^2 d\theta < \infty$.

(iv) $m(t,\theta)$ *is a weak solution of the equation*

$$\frac{\partial m}{\partial t} = 2\frac{\partial^2 P(m(\theta,t))}{\partial\theta^2} \tag{1.6}$$

with initial condition $m(t,\theta)$ *at* $t=0$ *given by* $m_0(\theta)$. *Furthermore the weak solution* (1.6) *satisfying* (ii) *and* (iii) *above is unique. The function* $P(\cdot)$ *is the thermodynamic function "pressure" which is defined in Sect. 2.*

This theorem will be proved in Sect. 5 using the basic results of Sects. 3 and 4. The main idea is the following.

Let $f_N^t(\underset{\sim}{x}, \underset{\sim}{v})$ be the solution of the forward equation

$$\frac{\partial f_N^t}{\partial t} = L_N^* f_N^t \tag{1.7}$$

with initial condition f_N^0, where L_N^* is adjoint of L_N with respect to the measure μ_N. The core of the problem is to prove certain local ergodic properties of the time average

$$\bar{f}_N(\underset{\sim}{x}, \underset{\sim}{v}) = \frac{1}{T}\int_0^T f_N^t(\underset{\sim}{x}, \underset{\sim}{v})dt. \tag{1.8}$$

These local ergodic properties will be established in Sect. 4 and are based on certain bounds on entropy and its rate of change. An elementary computation yields

$$\frac{d}{dt} H_N(f_N^t) = -\frac{N^2}{2}\int \frac{1}{f_N^t}\sum_{i=1}^N \left(\frac{\partial f_N^t}{\partial v_i}\right)^2 d\mu_N \leqq 0. \tag{1.9}$$

This implies immediately that $H_N(f_N^t) \leqq CN$ for all $t \geqq 0$. Defining for any density f relative to μ_N

$$I_N(f) = \frac{1}{2}\int \frac{1}{f}\sum_{i=1}^N \left(\frac{\partial f}{\partial v_i}\right)^2 d\mu_N,$$

we have

$$\int_0^T I_N(f_N^t)dt = -\frac{1}{N^2}(H_N(f_N^T) - H_N(f_N^0)) \leqq \frac{C}{N}.$$

Since both $I_N(f)$ and $H_N(f)$ are convex functionals of f we have

$$I_N(\bar{f}_N) \leqq \frac{C}{TN}, \tag{1.10}$$

$$H_N(\bar{f}_N) \leqq CN. \tag{1.11}$$

2. Canonical and Grand Canonical Gibbs Measures

Let $\mathscr{X}$ denote the set of all locally finite configurations of particles on the real line R. By a locally finite configuration we mean a countable subset of R with no accumulation points in R. For each value of a real parameter λ called the chemical activity the Grand Canonical Partition function in a finite interval Λ is defined as

$$Z(\lambda, \Lambda) = \sum_{n \geqq 0} \frac{e^{n\lambda}}{n!} \int_{\Lambda^n} \exp\left[- \sum_{i \neq j} V(q_i - q_j) \right] dq_1 \dots dq_n . \tag{2.1}$$

The free energy as a function $F(\lambda)$ of λ is defined by

$$F(\lambda) = \lim_{\Lambda \uparrow R} \frac{1}{|\Lambda|} \log Z(\lambda, \Lambda). \tag{2.2}$$

By the assumptions made on the interaction V, the limit (2.2) exists and defines a convex function $F(\lambda)$ of λ. In our case since the dimension is 1, the function $F(\lambda)$ is continuously differentiable with a derivative

$$\varrho(\lambda) = \frac{dF(\lambda)}{d\lambda}. \tag{2.3}$$

The function $\varrho(\lambda)$ is continuous and strictly increasing with $\varrho(\lambda) \to 0$ as $\lambda \to -\infty$ and $\varrho(\lambda) \to +\infty$ as $\lambda \to +\infty$. The inverse $\lambda = \lambda(\varrho)$ exists as a continuous increasing function and

$$P(\varrho) = F(\lambda(\varrho)) \tag{2.4}$$

defines the pressure $P(\varrho)$ as a continuous, strictly increasing function of ϱ.

The canonical Gibbs measure in a finite interval Λ with particle number n and external boundary condition ω is a probability measure $\mu^{\omega}_{n,\Lambda}$ on configurations in Λ with n points or equivalently, a symmetric probability measure on Λ^n. The external boundary condition is a configuration on Λ^c. We define

$$d\mu^{\omega}_{n,\Lambda} = \frac{1}{n!} [Z^{\omega}_{n,\Lambda}]^{-1} \exp[-\sum V(q_i - q_j) - 2\sum V(q_i - y_\alpha)] dq_1 \dots dq_n, \tag{2.5}$$

where $Z^{\omega}_{n,\Lambda}$ is the normalization constant,

$$Z^{\omega}_{n,\Lambda} = \frac{1}{n!} \int_{\Lambda^n} \exp[-\sum V(q_i - q_j) - 2\sum V(q_i - y_\alpha)] dq_1 \dots dq_N . \tag{2.6}$$

The Grand Canonical Gibbs measure on a finite interval Λ with activity λ is a convex combination of $\mu^{\omega}_{n,\Lambda}$ with special weights. We define

$$\hat{\mu}^{\omega}_{\lambda,\Lambda} = [\hat{Z}^{\omega}_{\lambda,\Lambda}]^{-1} \sum_{n=0}^{\infty} e^{\lambda n} Z^{\omega}_{n,\Lambda} \mu^{\omega}_{n,\Lambda},$$

where

$$\hat{Z}^{\omega}_{\lambda,\Lambda} = \sum_{n=0}^{\infty} e^{\lambda n} Z^{\omega}_{n,\lambda} .$$

The Grand Canonical Gibbs measure on the infinite line with activity λ is a point process μ_λ on the configuration space $\mathscr{X}$ such that for every finite interval Λ the conditional distribution of the configuration in Λ given the configuration ω in the exterior Λ^c, is given by $\hat{\mu}^{\omega}_{\lambda,\Lambda}$ for almost all ω.

One of the important aspects of the absence of phase transition in our one dimensional system is the following theorem. It is stated here without proof. A sketch of the proof can be found in the appendix of [4].

Theorem 2.1. *For each λ, there is exactly one Grand Canonical Gibbs measure μ_λ on $\mathscr{X}$ corresponding to the activity λ. It is a stationary point process with density $\varrho=\varrho(\lambda)$ given by* (2.3). *If $\mu^{\omega}_{n,\Lambda}$ are canonical Gibbs measures such that $\Lambda\uparrow R$ and $\frac{n}{|\Lambda|}\to\varrho$, then for any continuous bounded local function $H(\omega')$ on the configuration space,*

$$\lim_{\substack{n/|\Lambda|\to\varrho\\ \Lambda\uparrow R}}\ \sup_{\omega}\ \mu^{\omega}_{n,\Lambda}\left[\left|\frac{1}{|\Lambda|}\int_{\Lambda} H(\omega'_x)dx-\hat{H}(\varrho)\right|\geqq\delta\right]=0. \tag{2.7}$$

Here $\hat{H}(\varrho)=E^{\mu_\lambda}[H(\omega')]$ with λ chosen so that $\varrho(\lambda)=\varrho$. The configuration ω' translated in space by x is denoted by ω'_x.

The correlation measures $R^{(k)}_\lambda(dz_1,\ldots,dz_k)$ are defined so that

$$E^{\mu_\lambda}\left[\sum_{(z_1,\ldots,z_k)} f(z_1,\ldots,z_k)\right]=\int_{R^k} f(z_1,\ldots,z_k)R^{(k)}_\lambda(dz_1,\ldots,dz_k).$$

On the left-hand side the summation is over all k-tuples $(z_1,\ldots,z_k)$ that belong to the configuration. $f(z_1,\ldots,z_k)$ is assumed to be a bounded continuous function with compact support. $R^{(k)}_\lambda(dz_1,\ldots,dz_k)$ is a σ-finite measure on R^k invariant with respect to diagonal translations $z_1,\ldots,z_k\to z_1+a,\ldots,z_k+a$. In particular

$$R^{(1)}_\lambda(dz)=\varrho(\lambda)dz.$$

We have the identity proved in the appendix of [4].

Theorem 2.2. *For every λ and ϱ related by $\varrho=\varrho(\lambda)$,*

$$\varrho+\iint\varphi(x-y)h(y)R^{(2)}_\lambda(dx,dy)=P(\varrho),$$

where $h(y)$ is any function which has compact support with $\int_{-\infty}^{\infty}h(y)dy=1$.

3. Some Estimates Based on Entropy and the Dirichlet Form

In this section our aim is to obtain some elementary bounds based on the inequalities

$$H_N(f_N)\leqq AN \tag{3.1}$$

and

$$I_N(f_N)\leqq\frac{C}{N} \tag{3.2}$$

for an arbitrary density $f_N(\underline{x},\underline{v})$ relative to μ_N.

Theorem 3.1. *Let f_N satisfy* (3.1). *Then*

(i)
$$\int|\underline{v}|^2 f_N d\mu_N\leqq N(\log\sqrt{2}+A). \tag{3.3}$$

(ii) *For any function ϕ on $C(R)$ with compact support there exists a constant C' such that for all N and $\lambda \leqq 1$,*

$$\int \lambda \sum_{i,j} \phi(\lambda N(x_i - x_j)) f_N d\mu_N \leqq C'N. \tag{3.4}$$

Proof.

$$\text{(i)}\ \tfrac{1}{4}\int \left[\sum_{i=1}^{N} v_i^2\right] f_N d\mu_N \leqq \log \int \exp[\tfrac{1}{4}\sum v_i^2] d\mu_N + H_N(f_N)$$
$$= N(\log\sqrt{2} + A).$$

(ii) The proof can be found in Lemma 4.2 of [4].

In the following we will denote the Gaussian product distribution on R^N by

$$G_N(d\underline{v}) = \frac{1}{(2\pi)^{N/2}} e^{-(1/2)|\underline{v}|^2} d^N \underline{v}, \tag{3.5}$$

and the equilibrium distribution on S^N by

$$d\mu_N^*(\underline{x}) = \frac{1}{Z_N^*} \exp\left(-\sum_{i \neq j} V(N(x_i - x_j))\right) d^N \underline{x} \tag{3.6}$$

so that $d\mu_N(\underline{x}, \underline{v}) = dG_N(\underline{v}) \cdot \mu_N^*(\underline{x})$. Given a probability density $f_N(x, v)$ relative to μ_N, we denote by $f_N(\underline{v}|\underline{x})$ the density of the conditional distribution of $\underline{v}$ given $\underline{x}$, relative to $G_N(d\underline{v})$. By the logarithmic Sobolev inequality (see [1] for a proof) for any configuration $\underline{x}$ we have

$$\int f_N(\underline{v}|\underline{x}) \log f_N(\underline{v}|\underline{x}) G_N(d\underline{v}) \leqq 2 \int \frac{1}{f_N(\underline{v}|\underline{x})} |\nabla_v f_N(\underline{v}|\underline{x})|^2 G_N(d\underline{v}).$$

If we denote the marginal of $f_N(\underline{x}, \underline{v})$ by

$$f_N^*(\underline{x}) = \int f_N(\underline{x}, \underline{v}) G_N(d\underline{v}),$$

then by a simple calculation

$$\int_{S^N} f_N^*(\underline{x}) d\mu_N^*(\underline{x}) \int_{R^N} f_N(\underline{v}|\underline{x}) \log f_N(\underline{v}|\underline{x}) G_N(d\underline{v})$$
$$\leqq 2 \int \frac{1}{f_N(\underline{x}, \underline{v})} |\nabla_v f_N(\underline{x}, \underline{v})|^2 d\mu_N(\underline{x}, \underline{v}) = 4 I_N(f_N). \tag{3.7}$$

We have therefore proved

Lemma 3.2. *Let f_N be such that $I_N(f_N) \leqq C/N$. Then the conditional density $f_N(\underline{v}|\underline{x})$ satisfies*

$$\int f_N^*(\underline{x}) d\mu_N^*(\underline{x}) \int G_N(d\underline{v}) f_N(\underline{v}|\underline{x}) \log f_N(\underline{v}|\underline{x}) \leqq \frac{4C}{N}. \tag{3.8}$$

4. Local Equilibrium Distributions

We would like to show that, relative to the probability distribution $\bar{f}_N d\mu_N$, "average microscopic quantities" like $\frac{1}{N} \sum \psi(N(x_i - x_j)) J(x_i)$ are close, for large N, to the expression involving only the macroscopic density function.

The approach we use here is similar to the one used in [4], but the information we have on $\bar{f}_N$ here are different. In [4] two basic estimates available on $\bar{f}_N$ are used,

$$\int \bar{f}_N \log \bar{f}_N d\mu_N \leqq BN \tag{4.1}$$

and

$$\frac{1}{2}\int \frac{1}{\bar{f}_N} \sum_{i=1}^{N} \left(\frac{\partial \bar{f}_N}{\partial x_i}\right)^2 d\mu_N \leqq DN. \tag{4.2}$$

In our case we do not have (4.2) but instead have

$$\frac{1}{2}\int \frac{1}{\bar{f}_N} \sum_{i=1}^{N} \left(\frac{\partial \bar{f}_N}{\partial v_i}\right)^2 d\mu_N \leqq \frac{D}{N}. \tag{4.3}$$

As we saw in the previous section, by the logarithmic Sobolev inequality this will tell us that the distribution of the velocities conditioned on the positions are all very close to the equilibrium. But this says nothing about the distribution of positions. To infer from this that the local position distributions are close to equilibrium distributions and that the macroscopic density has no unnecessary fluctuations, we have to actually use information that $\bar{f}_N$ are time averages of the solution to the forward equation.

Consider a function $\Phi(\omega)$ on the configuration of points in R which is bounded, continuous and localized in some finite interval $[-l, l]$. For any given $x \in S$ and any $\underline{x} \in S^N$ we can consider the configuration ω_N^x on the line

$$\omega_N^x = \{N(x_i - x) : |x_i - x| < \tfrac{1}{4}\}.$$

If N is sufficiently large $\Phi(\omega_N^x)$ makes sense and we are interested in the quantity

$$\xi_N(\underline{x}) = \int_S \Phi(\omega_N^x) J(x) dx,$$

let $h(x)$ be a nonnegative smooth function with $\int h(x)dx = 1$, supported on $[-\frac{1}{2}, \frac{1}{2}]$. For $\lambda > 1$, we define

$$\begin{aligned}\varrho_\lambda(x) &= \frac{\lambda}{N} \sum_{i=1}^{N} h(\lambda(x_i - x)) \\ &= (h_\lambda * \alpha_N)(x),\end{aligned}$$

where $h_\lambda(x) = \lambda h(\lambda x)$ and $\alpha_N(dx) = \dfrac{1}{N}[\delta_{x_1} + \ldots \delta_{x_N}]$ is the empirical distribution of the configuration $(x_1, \ldots, x_N)$ on S. We denote by $\hat{\Phi}(\varrho)$ the expected value

$$E^{\mu_\varrho}[\Phi(\omega)] = \hat{\Phi}(\varrho),$$

where μ_ϱ is the Gibbs measure with density ϱ or activity $\lambda = \lambda(\varrho)$. If we define

$$\eta_{N,\lambda}(\underline{x}) = \int_S \hat{\Phi}(\varrho_\lambda(x)) J(x) dx,$$

then we want to prove

Theorem 1.

$$\lim_{\lambda \to \infty} \limsup_{N \to \infty} E^{\bar{f}_N} |\xi_N(\underline{x}) - \eta_{N,\lambda}(\underline{x})| = 0.$$

As in [4] we establish Theorem 1.1 in two steps.

Theorem 4.2.

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^{\bar{f}_N}|\xi_N(\underline{x})-\eta_{N,N_\varepsilon}(x)|=0.$$

Theorem 4.3.

$$\lim_{\substack{\varepsilon\to 0\\ \lambda\to\infty}}\limsup_{N\to\infty} E^{\bar{f}_N}\int_S |\varrho_\lambda(x)-\varrho_{\varepsilon N}(x)|\,dx=0.$$

Let $\hat{f}_N(\underline{x},\underline{v})=\int_S \bar{f}_N(\underline{x}+a,v)\,da$. Then by convexity we have that $\hat{f}_N$ satisfies

$$H_N(\hat{f}_N)\leqq AN,$$

$$I_N(\hat{f}_N)\leqq \frac{C}{N}.$$

In exactly the same way as in [4] the proof of Theorem 4.2 can be reduced to

Theorem 4.4.

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^{\bar{f}_N}\left|\frac{N\varepsilon}{2}\int_{|x|\leqq 1/N\varepsilon}\Phi(\omega_N^x)\,dx-\hat{\Phi}(\varrho_{N\varepsilon}(0))\right|=0.$$

Proof of Theorem 4.4. Both $\int_{|y|\leqq 1/N\varepsilon}\Phi(\omega_y^N)\,dy$ and $\varrho_{N\varepsilon}(0)$ depend only on the configuration $\underline{x}$ in the interval $\left[-\frac{1}{\varepsilon N}-\frac{l}{N},\frac{1}{\varepsilon N}+\frac{l}{N}\right]$ centered around the origin of S. Here the function Φ is assumed to depend only on the configuration in $[-l,l]$. If we project $\hat{f}_N d\mu_N$ onto configurations on this interval and expand the interval by a factor N, we will get a point process $\hat{v}_{N,\varepsilon}$ on the interval $\left[-\frac{1}{\varepsilon}-l,\frac{1}{\varepsilon}+l\right]$. All we have to prove is that

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} E^{\hat{v}_{N,\varepsilon}}\left|\frac{\varepsilon}{2}\int_{-1/\varepsilon}^{1/\varepsilon}\Phi(\omega^x)\,dx-\hat{\Phi}(\varrho_\varepsilon(0))\right|=0,$$

where ω^x is the configuration ω translated by x in space. Since the density of particles is 1 under $\hat{f}_N d\mu_N$, $\hat{v}_{N,\varepsilon}$ is compact and we denote by $\hat{\mathscr{B}}_\varepsilon$ the set of all limit points of $\hat{v}_{N,\varepsilon}$ as $N\to\infty$. We have to prove that

$$\lim_{\varepsilon\to 0}\sup_{v\in\hat{\mathscr{B}}_\varepsilon} E^v\left[\left|\frac{\varepsilon}{2}\int_{-1/\varepsilon}^{1/\varepsilon}\Phi(\omega^x)\,dx-\hat{\Phi}(\varrho_\varepsilon(0))\right|\right]=0.$$

Let $\mu_{n,l}^\omega$ be the Gibbs measure on $[-l,l]$ with exterior boundary condition ω and n particles inside $[-l,l]$. We denote by Γ_l the convex hull of these measures, and by $\Gamma_l^{(1)}$ those for which expected particle density in $[-l,l]$ is at most 1. Then according to Theorem 2.1,

$$\lim_{k\to\infty}\sup_{v\in\Gamma_{l+k}^{(1)}} E^v\left[\left|\frac{1}{2k}\int_{-k}^{k}\Phi(\omega^x)\,dx-\hat{\Phi}(\varrho_{1/k}(0))\right|\right]=0,$$

and we therefore need only establish the lemma

Lemma 4.5. *For every l and ε*

$$\hat{\mathscr{B}}_\varepsilon\subset\Gamma_{(1/\varepsilon)+l}^{(1)}.$$

Proof. Let c_0 be the range of the interaction V, i.e. $V(x)=0$ if $|x|\geqq c_0$. Let us consider the limiting point process on the interval $\left[-\frac{1}{\varepsilon}-l-c_0, \frac{1}{\varepsilon}+l+c_0\right]$, i.e. any distribution $\nu\in\hat{\mathscr{B}}_{\varepsilon'}$, where $1/\varepsilon'+l=1/\varepsilon+l+c_0$. We want to show that $\nu\in\Gamma^{(1)}_{l+(1/\varepsilon)}$.

Consider the function

$$u_N(\underline{x},\underline{v})=\sum_{i=1}^{N} v_i\Phi(\omega_N^0)g(Nx_i),$$

where Φ is any function of the configuration of points in $[-l-c_0, l+c_0]$ which is bounded continuous and differentiable, and g is any positive continuous function with compact support contained in $[-l,l]$. By direct computation we have

$$\begin{aligned}\sum_{i,j} 2V'(N(x_i-x_j))\Phi(\omega_N^0)g(N(x_i))-\sum_{i,j}\frac{1}{N}v_iv_j\frac{\partial}{\partial x_i}(\Phi(\omega_N^0)g(N(x_j)))\\ =u_N(\underline{x},\underline{v})-\frac{1}{N^2}(L_Nu_N)(\underline{x},\underline{v}).\end{aligned}$$

On integrating by parts:

$$\begin{aligned}|\textstyle\int u_N(\underline{x},\underline{v})\hat{f}_N(\underline{x},\underline{v})d\mu_N| &= \left|\int_S da\int u_N(\underline{x}+a,\underline{v})\bar{f}_N(\underline{x},\underline{v})d\mu_N\right|\\ &= \left|\int_S da\int\Phi(\omega_N^a)\sum_{i=1}^{N} g(N(x_i-a))\frac{\partial\bar{f}_N}{\partial v_i}d\mu_N\right|\\ &\leqq \|\Phi\|_\infty\left(\int\sum_i g(Nx_i)\hat{f}_N d\mu_N\right)^{1/2}\\ &\quad\times\left(\int_S da\int\frac{1}{\bar{f}_N}\sum\left(\frac{\partial\bar{f}_N}{\partial v_i}\right)^2 g(N(x_i-a))d\mu_N\right)^{1/2}\\ &\leqq \|\Phi\|_\infty\left(\frac{C}{N}\cdot\int\frac{1}{\bar{f}_N}\sum\left(\frac{\partial\bar{f}_N}{\partial v_i}\right)^2 d\mu_N\right)^{1/2}\leqq\frac{C'}{N}.\end{aligned}$$

If g_N is any probability density with respect to μ_N and $\hat{g}_N$ is its average

$$\int_S g_N(\underline{x}+a,\underline{v})da=\hat{g}_N(\underline{x},\underline{v}),$$

$$\begin{aligned}\textstyle\int u_N(\underline{x},\underline{v})\hat{g}_N d\mu_N &= \int_S\int u_N(\underline{x}+a,\underline{v})g_N(\underline{x},\underline{v})d\mu_N da\\ &\leqq\|\Phi\|_\infty\left(\int\frac{1}{N}\sum|v_i|g_N(\underline{x},\underline{v})d\mu_N\right)\\ &\leqq\|\Phi\|_\infty\left(\int\frac{1}{N}\sum|v_i|^2 g_N(\underline{x},\underline{v})d\mu_N\right)^{1/2}\\ &\leqq\|\Phi\|_\infty\left[\frac{4}{N}\log\int\exp\left[\frac{1}{4}\sum v_i^2\right]d\mu_N+\frac{4}{N}H(g_N)\right]^{1/2}\\ &\leqq C'\|\Phi\|_\infty,\end{aligned}$$

provided $H(g_N) \leqq CN$. If we now use the explicit formula $\hat{f}_N = \frac{1}{T}\int_0^T \hat{f}_N^t dt$ and calculate

$$\int L_N u_N \hat{f}_N d\mu_N = \frac{1}{T}\int_0^T \int u_N \frac{\partial \hat{f}_N^t}{\partial t} dt d\mu_N$$
$$= \frac{1}{T}\left[\int u_N \hat{f}_N^T d\mu_N - \int u_N \hat{f}_N^0 d\mu_N\right],$$

then

$$\left|\int L_N u_N \hat{f}_N d\mu_N\right| \leqq 2C' \|\Phi\|_\infty$$

because $H(\hat{f}_N^T) \leqq H(\hat{f}_N^0) \leqq CN$.

We have therefore established

$$\lim_{N\to\infty} E^{\hat{f}_N}\left[\sum_{i,j} 2V'(N(x_i - x_j))\Phi(\omega_N^0)g(Nx_i) - \frac{1}{N}\sum_{i,j} v_i v_j \frac{\partial}{\partial x_i}(\Phi(\omega_N^0)g(Nx_j))\right] = 0. \quad (4.4)$$

Let us define the following $N \times N$ matrix

$$F_{ij}(\underline{x}) = \frac{1}{N}\frac{\partial}{\partial x_i}(\Phi(\omega_N^0)g(Nx_j)).$$

Because of the scaling $\{F_{ij}(\underline{x})\}$ are uniformly bounded. Then by the entropy estimates of Sect. 3,

$$\int \frac{1}{N}\sum_{i=1}^N (v_i^2 - 1)F_{ii}(\underline{x})\hat{f}_N(\underline{x}, \underline{v})d\mu_N$$
$$\leqq \int \hat{f}_N^*(\underline{x})d\mu_N^*(\underline{x})\left\{\log\int \exp\left[\frac{1}{N}\sum(v_i^2 - 1)F_{ii}(\underline{x})\right]G_N(d\underline{v}) + \int \hat{f}_N(\underline{v}|\underline{x})\log \hat{f}_N(\underline{v}|\underline{x})G_N(d\underline{v})\right\}$$
$$\leqq \int \hat{f}_N^*(\underline{x})d\mu_N^*(\underline{x})\left\{\log\int \exp\left[\frac{1}{N}\sum(v_i^2 - 1)F_{ii}(\underline{x})\right]G_N(d\underline{v})\right\} + \frac{2C}{TN}.$$

By an elementary calculation, essentially because of the law of large numbers we can conclude that the last term goes to zero with N. Therefore

$$\limsup_{N\to\infty} \frac{1}{N}\sum [v_i^2 - 1]F_{ii}(\underline{x})\hat{f}_N(\underline{x}, \underline{v})d\mu_N \leqq 0.$$

Since the same argument works with change of sign we actually have

$$\lim_{N\to\infty}\left|\frac{1}{N}\int \sum (v_i^2 - 1)F_{ii}(\underline{x})\hat{f}_N(\underline{x}, \underline{v})d\mu_N\right| = 0. \quad (4.5)$$

Using an exactly similar argument on the off diagonal elements

$$\frac{1}{N}\int \sum_{i\neq j} v_i v_j F_{ij}(\underline{x})\hat{f}_N(\underline{x},\underline{v})d\mu_N$$
$$\leqq \int \hat{f}_N^*(\underline{x})d\mu_N^* \left\{\frac{1}{\sigma}\log\int \exp\left[\frac{\sigma}{N}\sum_{i\neq j} v_i v_j F_{ij}(\underline{x})\right]G_N(d\underline{v}) + \frac{2C}{\sigma N T}\right\}.$$

We can estimate by direct calculation

$$\int \exp\left[\frac{\sigma}{N}\sum_{i\neq j} v_i v_j F_{ij}\right]G_N(d\underline{v}) = \int \exp\left[\frac{\sigma}{N}\sum_{i\neq j} v_i v_j \bar{F}_{ij}\right]G_N(d\underline{v})$$
$$= \exp\left[\frac{1}{2}\log\det\left|I - \frac{2\sigma}{N}\bar{F}\right|\right],$$

where $\bar{F}$ is the matrix $\bar{F}_{ij}$ for $i\neq j$ and $\bar{F}_{ii}=0$ with $\bar{F}_{ij}=\frac{1}{2}(F_{ij}+F_{ji})$. Since $\bar{F}_{ij}$ is bounded it is easy to obtain the estimate (note $\operatorname{tr}\bar{F}=0$)

$$\log\det\left|I-\frac{2\sigma}{N}\bar{F}\right| \leqq C\frac{\sigma^2}{N^2}\operatorname{Tr}(\bar{F})^2 \leqq C\cdot\sigma^2,$$

provided σ is suitably small. We therefore obtain

$$\limsup_{N\to\infty}\frac{1}{N}\int\sum_{i\neq j} v_i v_j F_{ij}(\underline{x})\hat{f}_N(\underline{x},\underline{v})d\mu_N$$
$$\leqq C\sigma \quad \text{for all small } \sigma>0$$
$$=0.$$

Again since the argument is insensitive to sign we have

$$\limsup_{N\to\infty}\left|\frac{1}{N}\int\sum_{i\neq j} v_i v_j F_{ij}(\underline{x})\hat{f}_N(\underline{x},\underline{v})d\mu_N\right| = 0. \tag{4.6}$$

Combining (4.4) with (4.5) and (4.6) we get

$$\lim_{N\to\infty} E^{\hat{f}_N}\left[\sum_{i,j} 2V'(N(x_i-x_j))\Phi(\omega_N^0)g(Nx_i) - \frac{1}{N}\sum\frac{\partial}{\partial x_j}(\Phi(\omega_N^0)g(Nx_j))\right] = 0. \tag{4.7}$$

Since the function $\Phi(\omega)$ can be used just as well to cut off configurations having too many particles, we get from (4.7)

$$E^{\nu}\left[\sum_{i,j} 2V'(y_i-y_j)\Phi(\omega^0)g(y_i) - \sum\frac{\partial}{\partial y_i}(\Phi(\omega^0)g(y_i))\right] = 0 \tag{4.8}$$

for any ν which is a limit point. But (4.8) is enough to guarantee that $\nu\in\Gamma^{(1)}_{l+1/\varepsilon}$.

To see this we choose the function $\Phi(\omega)$ to be of the form $\psi_1(\omega_{\Lambda_l^c})\psi_2(\omega_{\Lambda_l})$, where ψ_1 and ψ_2 are functions of the configurations in the interior of $\Lambda_l=[-l,l]$ and its exterior Λ_l^c. We can choose $\psi_2(\omega_{\Lambda_l})$ to be of the form

$$\psi_2(\omega_{\Lambda_l}) = \phi(y_1,\dots,y_n) \quad \text{if } |\omega_{\Lambda_l}|=n$$
$$=0 \quad \text{otherwise},$$

where $|\omega_{A_l}|$ is the cardinality of the configuration inside $[-l, l]$ and $y_1, \ldots, y_n$ are the locations of the n particles inside $[-l, l]$. Here $\phi(y_1, \ldots, y_n)$ is a smooth symmetric function of $y_1, \ldots, y_n$. Although such a $\psi_2(\omega_{A_l})$ is not smooth on configuration space it is easily approximated by smooth ones. There is no difficulty because the probability of finding a particle on the boundary is zero under v. If we denote by $v = v_{A_l}(dy_1, \ldots, dy_n|\omega_{A_l^c}, n)$ the conditional distribution of the configuration on $[-l, l]$ given the configuration on $\omega^c_{A_l}$ and the cardinality $|\omega_A|$ of ω_A, then

$$\int \left\{ \sum_{i=1}^{n} \sum_{y'_j \in \omega^c_{A_l}} 2V'(y_i - y'_j)\phi(y_1, \ldots, y_n) g(y_i) - \sum_{i=1}^{n} \frac{\partial}{\partial y_i} (\phi(y_1, \ldots, y_n) g(y_i)) \right\} dv' = 0 .$$

This identifies v' as the canonical Gibbs measure.

Proof of Theorem 4.3. For any configuration $\underline{x} \in S^N$ we have associated the empirical measure

$$\alpha_N(dx) = \frac{1}{N} \sum \delta_{x_i}(dx)$$

and smoothed version

$$\varrho_\lambda(x) = \frac{\lambda}{N} \sum_{i=1}^{N} h(\lambda(x - x_i)) .$$

We have to prove that $\varrho_\lambda(x)$ for λ large but fixed and $\varrho_{\varepsilon N}(x)$ with ε small but fixed are close. This amounts to proving that $\varrho_{\varepsilon N}(x)$ does not have any oscillations. Just as in [4] we calculate the Young measures associated with $\varrho_{\varepsilon N}(\cdot)$ and show that as $N \to \infty$ and $\varepsilon \to 0$, these converge to a degenerate measure for almost every configuration. Corresponding to any density $\varrho(x)$ we define a measure π on $S \times R^+$ by

$$\int F(x, \varrho)\pi(dx, d\varrho) = \int_S F(x, \varrho(x))dx .$$

Clearly

$$\int \varrho\pi(dx, d\varrho) = \int \varrho(x)dx = 1 ,$$
$$\int F(x)\pi(dx, d\varrho) = \int F(x)dx .$$

If we denote by $\mathcal{M}$ the space of probability measures on $S \times R^+$, then through the map $\varrho(\cdot) \to \pi$ we can map the random density

$$\varrho_{\varepsilon N}(x) = (h_{\varepsilon N} * \alpha_N)(x)$$

into $\mathcal{M}$. If we consider the pair α_N and π we get a map of S^N into $M_1(S) \times \mathcal{M}$. We start with our basic $\hat{f}_N d\mu_N$ on S^N and denote by $\hat{Q}_{N,\varepsilon}$ the induced distribution on $M_1(S) \times \mathcal{M}$. In order to establish Theorem 4.3, the main step, after establishing compactness of $\hat{Q}_{N,\varepsilon}$ as $N \to \infty$ and then as $\varepsilon \to 0$, is to show

Theorem 4.6. *Let $\hat{Q}$ be any limit point of $\hat{Q}_{N,\varepsilon}$ as $N \to \infty$ and then as $\varepsilon \to 0$. Then for almost all (α, π) with respect to $\hat{Q}$*

$$\alpha(dx) = \varrho(x)dx ,$$
$$\pi(dx, d\varrho) = dx\delta_{\varrho(x)}(d\varrho)$$

for some function $\varrho(x) \in L_1(S)$.

We will prove Theorem 4.6 by means of the following lemmas.

Lemma 4.7. *Let $\hat{Q}$ be the limit measure as above. Then*

$$\hat{Q}[\alpha : \alpha(dx) = \varrho(x)dx \text{ for some } \varrho \in L_1(S)] = 1,$$
$$E^{\hat{Q}}\left[\int_S \varrho^2(x)dx\right] \leqq C \cdot B,$$

where C is a universal constant and B is the bound on entropy.

Proof. Identical to Lemma 7.7 in [4].

Lemma 4.8. *Let $\hat{Q}_\varepsilon$ be any limit measure of $\hat{Q}_{N,\varepsilon}$ as $N \to \infty$. Then*

(i) $$\hat{Q}_\varepsilon[(\alpha, \pi) : \int \varrho\pi(dx, d\varrho) = 1] = 1,$$

(ii) $$\hat{Q}_\varepsilon[(\alpha, \pi) : \pi(dx, d\varrho) = dx\pi_x(d\varrho)] = 1,$$

(iii) $$\hat{Q}_\varepsilon[(\alpha, \pi) : \alpha(dx) = \varrho(x)dx, \quad \text{with} \quad \varrho(x) = \int \varrho\pi_x(d\varrho)] = 1.$$

Proof. Since this lemma is based only on entropy bounds, like Lemma 4.7 this proof is identical to Lemma 7.8 of [4].

It remains to prove that as $\varepsilon \to 0$ the limit points of $\hat{Q}_\varepsilon$ are supported by degenerate Young measures. Consider a function g which is nonnegative, smooth, symmetric with compact support in $[-\frac{1}{4}, \frac{1}{4}]$ and having $\int g(x)dx = 1$. For every N, δ, and λ we define

$$G_{\delta,N,\lambda}(x) = \int_{-\infty}^{x} g_{\delta,N,\lambda}(y)dy,$$

where

$$g_{\delta,N,\lambda}(x) = N\delta g(N\delta x) - \lambda g(\lambda x).$$

We note that $G_{\delta,N,\lambda}$ has small compact support on R and is therefore well defined as a function on S provided $N\delta \geqq 1$ and $\lambda \geqq 1$. We consider the following test function on $S^N \times R^N$:

$$W_{\delta,N,\lambda}(\underset{\sim}{x}, \underset{\sim}{v}) = \frac{1}{N} \sum_{i=1}^{N} v_i \sum_{j \neq i} G_{\delta,N,\lambda}(x_i - x_j).$$

We then have by calculation

$$\begin{aligned}\frac{1}{N^2} \sum_{i,j} \Bigg\{ 2 \sum_{k \neq i} NV'(N(x_i - x_k)) \sum_{j \neq i} G_{\delta,N,\lambda}(x_i - x_j) \\ - \sum_l v_i v_l \frac{\partial}{\partial x_l} \Big(\sum_{j \neq i} G_{\delta,N,\lambda}(x_i - x_j) \Big) \Bigg\} \\ = -W_{\delta,N,\lambda}(\underset{\sim}{x}, \underset{\sim}{v}) + \frac{1}{N^2} L_N W_{\delta,N,\lambda}.\end{aligned} \tag{4.9}$$

We integrate by parts and use Schwartz' inequality to obtain

$$
\begin{aligned}
|\int W_{\delta,N,\lambda}(\underline{x},\underline{v})\hat{f}_N(x,v)d\mu_N| &= \left|\frac{1}{N}\int \sum v_i \sum_{j\neq i} G_{\delta,N,\lambda}(x_i-x_j)\hat{f}_N(\underline{x},\underline{v})d\mu_N\right| \\
&= \left|\int \frac{1}{N}\sum_{i=1}^N \frac{\partial \hat{f}_N}{\partial v_i}\sum_{j\neq i} G_{\delta,N,\lambda}(x_i-x_j)d\mu_N\right| \\
&\leqq \left(\int \frac{1}{N}\sum_{i=1}^N \left(\sum_{j\neq i} G_{\delta,N,\lambda}(x_i-x_j)\right)^2 \hat{f}_N d\mu_N\right)^{1/2} \\
&\quad \times \left(\int \frac{1}{N}\sum_{i=1}^N \frac{1}{\hat{f}_N}\left(\frac{\partial \hat{f}_N}{\partial v_i}\right)^2 d\mu_N\right)^{1/2} \\
&\leqq \sqrt{\frac{C}{T}}\left(\int \frac{1}{N^3}\sum_{i=1}^N \left(\sum_{j\neq i} G_{\delta,N,\lambda}(x_i-x_j)\right)^2 \hat{f}_N d\mu_N\right)^{1/2} \\
&\leqq \sqrt{\frac{C}{T}}\left(\int \frac{1}{N^2}\sum_{i,j} G^2_{\delta,N,\lambda}(x_i-x_j)\hat{f}_N d\mu_N\right)^{1/2}.
\end{aligned} \tag{4.10}
$$

From the definition of $\hat{f}_N$,

$$
\frac{1}{N^2}\int (L_N W_{\delta,N,\lambda})\hat{f}_N d\mu_N = \frac{1}{N^2 T}\left(\int W_{\delta,N,\lambda}\hat{f}_N^T d\mu_N - \int W_{\delta,N,\lambda} f_N^0 d\mu_N\right),
$$

since $H(\hat{f}_N^T) \leqq H(\hat{f}_N^0) \leqq CN$, by Lemma 3.1 and Schwartz's inequality,

$$
\begin{aligned}
&|\int W_{\delta,N,\lambda}\hat{f}_N^t d\mu_N| \\
&\leqq \left(\int \frac{1}{N}\sum_{i=1}^N v_i^2 \hat{f}_N^t d\mu\right)^{1/2}\left(\int \frac{1}{N}\sum_{i=1}^N \left(\sum_{j\neq i} G_{\delta,N,\lambda}(x_i-x_j)\right)^2 \hat{f}_N^t d\mu_N\right)^{1/2} \\
&\leqq C_1\left(\int \frac{1}{N}\cdot N\cdot N^2\cdot C_2^2\cdot f_N^t d\mu_N\right)^{1/2} \\
&\leqq C_3 N.
\end{aligned} \tag{4.11}
$$

We have used here the fact that $G_{\delta,N,\lambda}$ is bounded by some C_2. Using (4.9), (4.10), and (4.11) it follows that

$$
\begin{aligned}
\lim_{\lambda\to\infty}\lim_{N\to\infty}\int \frac{1}{N^2}\sum_{i=1}^N \Bigg\{ &2\sum_{k\neq i} V'(N(x_i-x_k))\left(\sum_{j\neq i} G_{\delta,N,\lambda}(x_i-x_j)\right. \\
&\left.\left. - \sum_{l=1}^N v_i v_l \frac{\partial}{\partial x_l}\left(\sum_{j\neq i} G_{\delta,N,\lambda}(x_i-x_j)\right)\right)\right\}\hat{f}_N(\underline{x},\underline{v})d\mu_N = 0.
\end{aligned} \tag{4.12}
$$

Let $A_N^{ij}(\underline{x})$ be the symmetric matrix

$$
\begin{aligned}
[A_N(\underline{x})]_{ij} &= \frac{1}{N} g_{\delta,N,\lambda}(x_i-x_j) \quad \text{for} \quad i\neq j \\
&= 0 \qquad\qquad\qquad\qquad \text{for} \quad i=j.
\end{aligned}
$$

We can then use an argument similar to one used in Theorem 4.1 [Eqs. (4.7) and (4.8)] to get

$$\int \frac{1}{N^2} \sum_{i \neq j} v_i v_j g_{\delta,N,\lambda}(x_i - x_j) \hat{f}_N d\mu_N$$
$$\leqq \frac{1}{\sigma} \int \hat{f}_N^*(\underline{x}) d\mu_N^* \left\{ \frac{1}{2} \log \det \left[I - \frac{2\sigma}{N} A_N(\underline{x}) \right] \right\} + \frac{C}{2TN\sigma}.$$

Just as before we conclude that

$$\lim_{N\to\infty} \left| \int \frac{1}{N^2} \sum_{i \neq j} v_i v_j g_{\delta,N,\lambda}(x_i - x_j) \hat{f}_N d\mu_N \right| = 0. \tag{4.13}$$

We treat the remaining term in the same manner:

$$\int \frac{1}{N^2} \sum_{i=1}^{N} (v_i^2 - 1) \sum_{j \neq i} g_{\delta,N,\lambda}(x_i - x_j) \hat{f}_N d\mu_N$$
$$\leqq \frac{1}{\sigma} \int \hat{f}_N^*(\underline{x}) d\mu_N^* \left\{ \log \int \exp \left[\frac{\sigma}{N^2} \sum_{i=1}^{N} (v_i^2 - 1) \right.\right.$$
$$\left.\left. \times \left(\sum_{j \neq i} g_{\delta,N,\lambda}(x_i - x_j) \right) \right] G_N(dv) \right\} + \frac{C}{TN\sigma}.$$

If we denote $\sum_{j \neq i} g_{\delta,N,\lambda}(x_i - x_j)$ by ξ_i then $|\xi_i| \leqq CN^2$. Moreover

$$\int \exp \left[\frac{\sigma \xi_i}{N^2} (v_i^2 - 1) \right] e^{-v_i^2/2} \frac{1}{\sqrt{2\pi}} dv_i = e^{-\frac{\sigma \xi_i}{N^2} - \frac{1}{2} \log \left(1 - \frac{2\xi_i \sigma}{N^2}\right)}.$$

Therefore for small σ,

$$\log \int \exp \left[\frac{\sigma}{N^2} \sum_{i=1}^{N} (v_i^2 - 1) \xi_i \right] G_N(dv) = -\frac{\sigma}{N^2} \sum \left[\xi_i - \frac{1}{2} \log \left(1 - \frac{2\xi_i \sigma}{N^2} \right) \right]$$
$$\leqq -\frac{\sigma}{N^2} \sum \xi_i + \frac{\sigma}{N^2} \sum \xi_i + c_1 \frac{\sigma^2}{2N^4} \sum \xi_i^2$$
$$\leqq c_2 \sigma^2.$$

We have now as before

$$\lim_{N\to\infty} \left| \frac{1}{N^2} \int \sum_{i=1}^{N} (v_i^2 - 1) \sum_{j \neq i} g_{\delta,N,\lambda}(x_i - x_j) \hat{f}_N d\mu_N \right| = 0. \tag{4.14}$$

If we combine (4.12), (4.13), and (4.14) we get

$$\lim_{\lambda\to\infty} \lim_{N\to\infty} \left| \int \frac{1}{N^2} \sum_{i=1}^{N} \left\{ \sum_{j,k \neq i} NV'(N(x_i - x_j)) G_{\delta,N,\lambda}(x_i - x_j) \right.\right.$$
$$\left.\left. - \sum_{j \neq i} g_{\delta,N,\lambda}(x_i - x_j) \right\} \hat{f}_N d\mu_N \right| = 0. \tag{4.15}$$

From this point on the rest of the proof proceeds exactly like in [4]. First we note that with $\delta = 1$ and $\lambda = 1$,

$$\sup_N \left| \int \frac{1}{N^2} \sum_{i=1}^{N} \{ NV'(N(x_i - x_j)) G_{1,N,1}(x_i - x_j) \} \hat{f}_N d\mu_N \right| \leqq C. \tag{4.16}$$

This provides us with the estimate

$$\sup_N \left| \int \frac{1}{N} \sum_{i=1}^{N} \left\{ \sum_{j=1}^{N} \psi(N(x_i - x_j)) \right\}^2 \hat{f}_N d\mu_N \right| \leqq C. \tag{4.17}$$

According to Lemma 7.9 of [4] this will give the required uniform integrability to take the limits as $N \to \infty$ and then $\lambda \to \infty$ and $\delta \to 0$. We remark that once $N \to \infty$, according to Lemma 4.5 we are in the local Gibbs situation and that puts us in the circumstances of [4]. In particular we get Lemma 7.10 of [4] establishing for any possible limit point the bound

$$E^Q \int_S dx \int \varrho^3 \pi_x(d\varrho) \leqq C. \tag{4.18}$$

5. Hydrodynamic Limit

We will prove in this section the main part namely (iv) for Theorem 1.1. Parts (i) and (ii) were essentially established in the previous section. We will prove (iii) in the next section. Of course we need some estimates to prove compactness. These will also be deferred to the next section.

For any function J on S which is smooth by Eq. (1.1) we have

$$\frac{1}{N} \sum_{i=1}^{N} J(x_i(T)) - \frac{1}{N} \sum_{i=1}^{N} J(x_i(0)) = \int_0^T \sum_{i=1}^{N} J'(x_i(t)) v_i(t) dt. \tag{5.1}$$

On the other hand

$$\begin{aligned} d\left(\sum_{i=1}^{N} J'(x_i(t)) v_i(t) \right) = {} & N \sum_{i=1}^{N} J''(x_i(t)) v_i^2(t) dt \\ & - N \sum_{i=1}^{N} \Big(2 \sum_{j \neq i} N V'(N(x_i(t) - x_j(t)))\Big) J'(x_i(t)) dt \\ & - \frac{N^2}{2} \sum_{i=1}^{N} J'(x_i(t)) v_i(t) dt \\ & + N \sum_{i=1}^{N} J'(x_i(t)) dw_i(t), \end{aligned} \tag{5.2}$$

we can rewrite this in the form

$$\begin{aligned} \int_0^T \sum_{i=1}^{N} J'(x_i(t)) v_i(t) dt = {} & \frac{2}{N} \int_0^T \sum_{i=1}^{N} J''(x_i(t)) v_i^2(t) dt \\ & - \frac{2}{N} \int_0^T \sum_{i=1}^{N} \Big(2 \sum_{j \neq i} N V'(N(x_i(t) - x_j(t)))\Big) J'(x_i(t)) ct \\ & + \frac{2}{N} \int_0^T \sum_{i=1}^{N} J'(x_i(t)) dw_i(t) \\ & - \frac{2}{N^2} \left(\sum_{i=1}^{N} J'(x_i(T)) v_i(T) - \sum_{i=1}^{N} J'(x_i(0)) v_i(0) \right). \end{aligned} \tag{5.3}$$

Combining it with (5.1)

$$\begin{aligned}&\frac{1}{N}\sum J(x_i(T))-\frac{1}{N}\sum J(x_i(0))\\&=2\int_0^T F_N(t)dt+\frac{2}{N}\int_0^T\sum_{i=1}^N J'(x_i(t))dw_i(t)\\&\quad-\frac{2}{N^2}\left(\sum_{i=1}^N J'(x_i(T))v_i(T)-\sum_{i=1}^N J'(x_i(0))v_i(0)\right),\end{aligned}\tag{5.4}$$

where

$$F_N(t)=\frac{1}{N}\sum_{i=1}^N\left\{v_i^2(t)J''(x_i(t))-\sum_{j\neq i}2NV'(N(x_i(t)-x_j(t)))J'(x_i(t))\right\}.\tag{5.5}$$

From (i) of Theorem 3.1 it is easy to show that

$$\lim_{N\to\infty}E^{f_N^0}\left(\frac{2}{N^2}\left|\sum_{i=1}^N J'(x_i(T))v_i(T)-\sum_{i=1}^N J'(x_i(0))v_i(0)\right|\right)=0,\tag{5.6}$$

and because $w_1(t),\dots,w_N(t)$ are independent Brownian motions

$$\lim_{N\to\infty}E^{f_N^0}\left|\frac{2}{N}\int_0^T\sum_{i=1}^N J'(x_i(t))dw_i(t)\right|=0.\tag{5.7}$$

The next step is to replace v_i^2 by 1 in (5.5).

Lemma 5.1.

$$\lim_{N\to\infty}E^{f_N^0}\left(\int_0^T\left|\frac{1}{N}\sum J''(x_i(t))(v_i^2(t)-1)\right|dt\right)=0.$$

Proof. By entropy inequality and logarithmic Sobolev inequality,

$$\begin{aligned}&\int\left|\frac{1}{4}\sum J''(x_i)(v_i^2-1)\right|\bar{f}_N d\mu_N\\&\leqq\int f_N^*(\underline{x})d\mu_N^*\log\left\{\int\exp\left[\left|\frac{1}{N}\sum_{i=1}^N(v_i^2-1)J''(x_i)\right|\right]G_N(d\underline{v})\right\}+\frac{C}{2NT}.\end{aligned}$$

By the law of large numbers

$$\int\exp\left[\left|\frac{1}{N}\sum_{i=1}^N(v_i^2-1)J''(x_i)\right|\right]G_N(d\underline{v})\to 1$$

as $N\to\infty$, uniformly over $\underline{x}$. This proves the lemma.

Now the velocities are completely out of the picture and we only have to look at

$$\int_0^T\frac{2}{N}\sum_{i=1}^N\left\{J''(x_i(t))-\sum_{j\neq i}2NV'(N(x_i(t)-x_j(t)))J'(x_i(t))\right\}dt.$$

We replace $J'(x_i(t))$ by $\frac{1}{2}[J'(x_i(t))-J'(x_j(t))]$, which we can, because of the skew symmetry of V'. As in [4], we proceed to replace

$$J'(x_i(t))-J'(x_j(t))\quad\text{by}\quad(x_i(t)-x_j(t))J''(x_i(t))$$

to get

$$\int_0^T \frac{2}{N} \sum_{i=1}^N J''(x_i(t)) \left(1 + \sum_{j=1}^N \psi(N(x_i(t) - x_j(t))\right) dt,$$

where

$$\psi(z) = -zV'(z).$$

Now we can use Theorem 4.1 as in [4] and show that any weak limit will satisfy (iv) of Theorem 1.1.

6. Some Auxiliary Lemmas

Lemma 6.1. *There exists a constant C such that for all N,*

$$\int \frac{1}{N} \sum_{i=1}^N \left| \sum_{j=1}^N \psi(N(x_i - x_j)) \right|^2 \bar{f}_N d\mu_N \leqq C.$$

Proof. This is precisely (4.17).

Lemma 6.2 (Compactness). *For every smooth $J(\cdot)$ on S, and $\varepsilon > 0$,*

$$\lim_{\delta \to 0} \limsup_{N \to \infty} P^{f_N^0} \left[\sup_{\substack{0 \leqq s < t \leqq T \\ |s-t| \leqq \delta}} |\langle J, \xi_N(t) \rangle - \langle J, \xi_N(s) \rangle| \geqq \varepsilon \right] = 0.$$

Proof. According to Eq. (5.1),

$$\begin{aligned} \langle J, \xi_N(t) \rangle - \langle J, \xi_N(s) \rangle &= \frac{1}{N} \sum J(x_i(t)) - \frac{1}{N} \sum J(x_i(s)) \\ &= \int_s^t \sum_{i=1}^N J'(x_i(\sigma)) v_i(\sigma) d\sigma. \end{aligned}$$

We can rewrite this using (5.3) as

$$\begin{aligned} \int_s^t \sum_{i=1}^N J'(x_i(\sigma)) v_i(\sigma) d\sigma &= \frac{2}{N} \int_s^t \sum_{i=1}^N J''(x_i(\sigma)) v_i^2(\sigma) d\sigma \\ &\quad - \frac{2}{N} \int_s^t \sum_{i=1}^N 2 \sum_{j \neq i} N V'(N(x_i(\sigma) - x_j(\sigma))) J'(x_i(\sigma)) d\sigma \\ &\quad + \frac{2}{N} \int_s^t \sum_{i=1}^N J'(x_i(\sigma)) d\omega_i(\sigma) \\ &\quad - \frac{2}{N^2} \left[\sum_{i=1}^N J'(x_i(t)) v_i(t) - \sum_{i=1}^N J'(x_i(s)) v_i(s) \right] \\ &= A_1(s,t) + A_2(s,t) + A_3(s,t) + A_4(s,t), \end{aligned}$$

$$A_1(s,t) = \frac{2}{N} \int_s^t \sum_{i=1}^N J''(x_i(\sigma)) [v_i^2(\sigma) - 1] d\sigma + \frac{2}{N} \int_s^t \sum J''(x_i(\sigma)) d\sigma,$$

$$|A_1(s,t)| \leqq 2 \int_0^T \left| \frac{1}{N} \sum J''(x_i(\sigma)) (v_i^2(\sigma) - 1) \right| d\sigma + 2|t-s| \, \|J''\|_\infty.$$

Since $E^{f_N^0}\left[\int_0^T\left|\frac{1}{N}\sum J''(x_i(\sigma))(v_i^2(\sigma)-1)\right|d\sigma\right]\to 0$ as $N\to\infty$ it follows that

$$\lim_{\delta\downarrow 0}\overline{\lim_{N\to\infty}} P^{f^0}\left[\sup_{\substack{0\leqq s\leqq t\leqq T\\ |t-s|\leqq\delta}}|A_1(s,t)|\geqq\varepsilon\right]=0,$$

$$A_2(s,t)=-\frac{2}{N}\int_s^t\sum_{i=1}^N NV'(N(x_i(\sigma)-x_j(\sigma)))(J'(x_i(\sigma)))d\sigma,$$

and

$$\begin{aligned}|A_2(s,t)|&\leqq\frac{2}{N}\|J''\|_\infty\int_s^t\sum_{i,j}\psi(N(x_i(\sigma)-x_j(\sigma)))d\sigma\\ &\leqq(t-s)^{1/2}\cdot\frac{2}{N}\|J''\|_\infty\left[\int_s^t\Big(\sum_{i,j}\psi(N(x_i(\sigma)-x_j(\sigma)))\Big)^2d\sigma\right]^{1/2}\\ &\leqq(t-s)^{1/2}\,2\|J''\|_\infty\left(\int_0^T\frac{1}{N}\sum_{i=1}^N\Big[\sum_{j\neq i}\psi(N(x_i(\sigma)-x_j(\sigma)))\Big]^2d\sigma\right)^{1/2}.\end{aligned}$$

From Lemma 6.1

$$\lim_{\delta\to 0}\limsup_{N\to\infty}P^{f_N^0}\left[\sup_{\substack{0\leqq s\leqq t\leqq T\\ |t-s|\leqq\delta}}|A_2(s,t)|\geqq\varepsilon\right]=0.$$

From Doob's inequality

$$P^{f_N^0}\left[\sup_{0\leqq t\leqq T}\left|\frac{2}{N}\int_0^t\sum_{i=1}^N J'(x_i(\sigma))d\omega_i(\sigma)\right|\geqq\varepsilon\right]\leqq\frac{2}{\varepsilon^2}\cdot\frac{4}{N^2}\cdot N\cdot\|J'\|_\infty^2.$$

Therefore

$$\lim_{N\to\infty}P^{f_N^0}\left[\sup_{0\leqq s\leqq t\leqq T}|A_3(s,t)|\geqq\varepsilon\right]=0.$$

In order to handle $A_4(s,t)$ it is sufficient to prove

$$\lim_{N\to\infty}P^{f_N^0}\left[\sup_{0\leqq t\leqq T}\frac{1}{N^2}\left|\sum_{i=1}^N J'(x_i(t))v_i(t)\right|\geqq\varepsilon\right]=0. \tag{6.1}$$

Let us denote by $\eta_N(t)=\sum_{i=1}^N J'(x_i(t))v_i(t)$. We can rewrite (5.2) in the form

$$\begin{aligned}d\eta_N(t)=&-\frac{N^2}{2}\eta_N(t)dt\\ &+N\sum_{i=1}^N J''(x_i(t))v_i^2(t)dt\\ &-N\sum_{i=1}^N\Big(2\sum_{j\neq i}NV'(N(x_i(t)-x_j(t)))J'(x_i(t))\Big),dt\\ &+N\sum_{i=1}^N J'(x_i(t))d\omega_i(t).\end{aligned} \tag{6.2}$$

Integrating this we get

$$\begin{aligned}\eta_N(t) = \eta_N(0)\, e^{-N^2 t/2} & \\ + N^2 \int_0^t A_N(s) e^{-(N^2/2)(t-s)} ds & \\ + N^2 \int_0^t B_N(s) e^{-(N^2/2)(t-s)} ds & \\ + N^2 \int_0^t e^{-(N^2/2)(t-s)} dM_N(s)\,, & \qquad (6.3)\end{aligned}$$

where

$$A_N(s) = \frac{1}{N} \sum J''(x_i(s)) v_i^2\,,$$

$$|B_N(s)| \leqq \frac{1}{N} \|J''\|_\infty \sum_{i,j} \psi(N(x_i(s) - x_j(s)))\,,$$

$$M_N(s) = \frac{1}{N} \int_0^s \sum J'(x_i(\sigma)) d\omega_i(\sigma)\,.$$

Since we have a uniform bound on

$$\begin{aligned}E^{f_N^0}[|\eta_N(0)|] &\leqq \log \int \exp[|\sum J'(x_i) v_i|] d\mu_N + CN \\ &\leqq C'N\,,\end{aligned}$$

$$P^{f_N^0}\left[\frac{1}{N^2} \sup_{0 \leqq t \leqq T} |\eta_N(0)\, e^{-N2t/2}| \geqq \delta\right] \to 0 \quad \text{as} \quad N \to \infty\,.$$

In order to establish (6.1) we need only prove

$$P^{f_N^0}\left[\sup_{0 \leqq t < T} \int_0^t e^{-(N^2/2)(s-t)} |A_N(s)| ds \geqq \varepsilon\right] \to 0 \quad \text{as} \quad N \to \infty\,, \qquad (6.4)$$

$$P^{f_N^0}\left[\sup_{0 \leqq t < T} \int_0^t e^{-(N^2/2)(s-t)} |B_N(s)| ds \geqq \varepsilon\right] \to 0 \quad \text{as} \quad N \to \infty\,, \qquad (6.5)$$

and

$$P^{f_N^0}\left[\sup_{0 \leqq t < T} \left|\int_0^t e^{-(N^2/2)(s-t)} dM_N(s)\right| \geqq \varepsilon\right] \to 0 \quad \text{as} \quad N \to \infty\,,$$

$$\begin{aligned}|A_N(s)| &\leqq \left|\frac{1}{N} \sum J''(x_i(s))\right| + \left|\frac{1}{N} \sum J''(x_i(s))(v_i^2(s) - 1)\right| \\ &\leqq \|J''\|_\infty + \left|\frac{1}{N} \sum J''(x_i(s))(v_i^2(s) - 1)\right|. \qquad (6.6)\end{aligned}$$

Clearly,

$$\sup_{0\leq t\leq T}\int_0^T e^{-(N^2/2)(t-s)}ds\to 0 \quad\text{as}\quad N\to\infty,$$

$$\sup_{0\leq t\leq T}\int_0^T e^{-(N^2/2)(t-s)}\left|\frac{1}{N}\sum J''(x_i(s))(v_i^2(s)-1)\right|ds$$
$$\leqq\int_0^T\left|\frac{1}{N}\sum J''(x_i(s))(v_i^2(s)-1)\right|ds,$$

and since $E^{f_N^0}\left[\int_0^T\left|\frac{1}{N}\sum J''(x_i(s))(v_i^2(s)-1)\right|ds\right]\to 0$ as $N\to\infty$ we have (6.4).

We estimate

$$\int_0^t e^{-(N^2/2)(t-s)}|B_N(s)|ds\leqq\left(\int_0^t e^{N^2(t-s)}ds\right)^{1/2}\left(\int_0^t|B_N(s)|^2ds\right)^{1/2}$$
$$\leqq\frac{1}{N}\left(\int_0^T|B_N(s)|^2ds\right)^{1/2}$$

$$\int_0^T|B_N(s)|^2ds\leqq\frac{1}{N}\int_0^T\sum_{i=1}^N\left(\sum_{j=1}^N\psi(N(x_i(s)-x_j(s)))\right)^2ds.$$

Since we have a uniform bound on the expectation of the right-hand side from Lemma 6.1 we have (6.5).

Finally we turn to (6.6). Let us pretend N^2T is an integer. Then

$$\sup_{0\leq t\leq T}\left|e^{-(N^2/2)t}\int_0^t e^{(N^2/2)s}dM_N(s)\right|$$
$$\leqq\sup_{1\leqq k\leqq N^2T}e^{-(N^2/2)(k-1)/N^2}\sup_{0\leqq t\leqq k/N^2}\left|\int_0^t e^{N^2s/2}dM_N(s)\right|.$$

We can estimate

$$P^{f_N^0}\left[\sup_{0\leq t\leq T}\left|e^{-N^2t/2}\int_0^t e^{(N^2/2)s}dM_N(s)\right|\geqq\varepsilon\right]$$
$$\leqq\sum_{k=1}^{N^2T}P^{f_N^0}\left[\sup_{0\leqq t\leqq k/N^2}\left|\int_0^t e^{-(N^2/2)s}dM_N(\varepsilon)\right|\geqq\varepsilon e^{(k-1)/2}\right]$$
$$\leqq\sum_{k=1}^{N^2T}\frac{e^{-(k-1)}}{\varepsilon^2}E^{f_N^0}\left[\int_0^{k/N^2}e^{(N^2/2)s}dM_N(s)\right]^2$$
$$\leqq\sum_{k=1}^{N^2T}\frac{1}{\varepsilon^2}\cdot e^{-(k-1)}e^k\|J'\|_\infty^2\cdot\frac{1}{N^3}$$
$$=\frac{T}{N}\cdot e\cdot\|J'\|_\infty^2\cdot\frac{1}{\varepsilon^2}$$

$\to 0$ as $N\to\infty$ and we are done.

Lemma 6.3. *Let Q be any limit point of the distributions of $\xi_N(\cdot)$ under $P^{f_N^0}$ as $N\to\infty$, where f_N^0 satisfies $H_N(f_N^0)\leqq CN$. Then*

$$E^Q\left(\int_0^T dt\int_S d\theta\frac{1}{q(\theta,t)}\left[\frac{\partial}{\partial\theta}P(q(\theta,t))\right]^2\right)\leqq\frac{C}{2},$$

with the same constant.

Proof. Given two integers n and l we consider a family of functions $u_1(x), \ldots, u_n(x)$, $F_1(x), \ldots, F_l(x)$ which are smooth functions on S. We consider also a family $g_1(y_1, \ldots, y_l), \ldots, g_n(y_1, \ldots, y_l)$ of smooth functions with compact support on R^l. We use these functions to define a family $U_1, \ldots, U_n$ of functions on $S^N \times R^N$ and $G_1, \ldots, G_n$ on S^N,

$$U_r(\underline{x}, \underline{v}) = \sum_{j=1}^{N} v_j u_r(x_j),$$

$$G_r(\underline{x}) = g_r\left(\frac{1}{N}\sum_{j=1}^{N} F_1(x_j), \ldots, \frac{1}{N}\sum_{j=1}^{N} F_l(x_j)\right).$$

By a direct calculation for each r,

$$\frac{1}{N}\sum_{i=1}^{N}\left\{2\sum_{j=1}^{N} NV'(N(x_i - x_j))u_r(x_i) - v_i^2 u_r'(x_i)\right\}$$
$$= -\frac{1}{2}U_r(\underline{x}, \underline{v}) - \frac{1}{N^2}L_N U_r(x, v). \tag{6.7}$$

We want to prove first that

$$\lim_{N\to\infty}\frac{1}{N^2}\int (L_N U_r)(\underline{x}, \underline{v}) \cdot G_r(\underline{x})\bar{f}_N d\mu_N = 0. \tag{6.8}$$

In fact since $G_r(\underline{x})$ is only a function of $\underline{x}$ and L_N is first order in x derivatives,

$$(L_N U_r)G_r = L_N(U_r G_r) - U_r(L_N G_r)$$

and

$$\frac{1}{N^2}\int L_N(U_r G_r)\bar{f}_N d\mu_N = \frac{1}{N^2 T}\left[\int U_r G_r f_N^T d\mu_N - \int U_r G_r f_N^0 d\mu_N\right].$$

Since both f_N^T and f_N^0 satisfy entropy bounds,

$$H_N(f_N) \leqq CN,$$

$$\left|\int U_r G_r f_N d\mu_N\right| \leqq \|G_r\|\,\|U_r\|\sqrt{N}\left(\int \sum v_i^2 f_N d\mu_N\right)^{1/2}$$
$$\leqq C'N,$$

where f_N can be either f_N^T or f_N^0. Therefore

$$\lim_{N\to\infty}\frac{1}{N^2}\int L_N(U_r G_r)\bar{f}_N d\mu_N = 0. \tag{6.9}$$

Moreover

$$\frac{1}{N^2}\int U_r(L_N G_r)\bar{f}_N d\mu_N = \frac{1}{N^2}\int \sum_{i,j} v_i v_j u_r(x_i)\frac{\partial G_r}{\partial x_j}(\underline{x})\bar{f}_N d\mu_N.$$

By the arguments we have used in the proof of Theorem 4.6 we can replace $v_i v_j$ by δ_{ij} and because it is obvious that

$$\lim_{N\to\infty}\frac{1}{N^2}\int \sum_{r=1}^{N} u_r(x_i)\frac{\partial G_r}{\partial x_i}(\underline{x})\bar{f}_N d\mu_N = 0,$$

we conclude that

$$\lim_{N\to\infty} \frac{1}{N^2} \int U_r(L_N G_r)\bar{f}_N d\mu_N = 0. \tag{6.10}$$

(6.8) follows immediately from (6.9) and (6.10).

We now turn to the relation (6.7). Let us use $\bar{f}_N$ to construct a measure $\bar{Q}_N$ on $M_1(S)$ by the map $(x_1, \ldots, x_N) \to \frac{1}{N}(\delta_{x_1} + \ldots \delta_{x_N})$ and let $\bar{Q}$ be any weak limit. According to the results of Sects. 4 and 5,

$$\lim_{N\to\infty} \int \frac{1}{N} \sum_{i=1}^{N} \left\{ 2 \sum_{j=1}^{N} NV'(N(x_i - x_j))(u_r(x_i) - u_r'(x_i)) \right\} G_r(\underline{x})\bar{f}_N d\mu_N$$

along the subsequence giving the weak limit $\bar{Q}$ can be represented as

$$-E^{\bar{Q}}\{\int P(q(x))u_r'(x)G_r(\int F_1(x)q(x)dx, \ldots, \int F_l(x)q(x)dx)\}.$$

If we now use (6.7) and (6.8) and sum over $r = 1, 2, \ldots$, we obtain

$$\begin{aligned}
&\left| E^{\bar{Q}} \left\{ \int P(q(x)) \sum_{r=1}^{n} u_r'(x) G_r(\int F_1(x)q(x)dx, \ldots, \int F_l(x)q(x)dx)dx \right\} \right| \\
&\leqq \limsup_{N\to\infty} \left| \frac{1}{2} \int \sum_{r=1}^{n} U_r(\underline{x}, \underline{v}) G_r(\underline{x}) \bar{f}_N d\mu_N \right| \\
&= \limsup_{N\to\infty} \left| \frac{1}{2} \int \sum_{r=1}^{n} \sum_{i=1}^{N} u_r(x_i) G_r(\underline{x}) v_i \bar{f}_N d\mu_N \right| \\
&= \limsup_{N\to\infty} \left| \frac{1}{2} \int \sum_{r=1}^{n} \sum_{i=1}^{N} u_r(x_i) G_r(\underline{x}) \frac{\partial \bar{f}_N}{\partial v_i} d\mu_N \right| \\
&\leqq \limsup_{N\to\infty} \frac{1}{2} \left(\int \frac{1}{\bar{f}_N} \sum_{i=1}^{N} \left(\frac{\partial \bar{f}_N}{\partial v_i} \right)^2 d\mu_N \right)^{1/2} \\
&\quad \times \left(\int \sum_{i=1}^{N} \left(\sum_{r=1}^{n} u_r(x_i) G_r(\underline{x}) \right)^2 \bar{f}_N d\mu_N \right)^{1/2} \\
&\leqq \limsup_{N\to\infty} \frac{1}{2} \left(\frac{2C}{NT} \right)^{1/2} N^{1/2} \left(\int \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{r=1}^{n} u_r(x_i) G_r(\underline{x}) \right)^2 \bar{f}_N d\mu_N \right)^{1/2} \\
&= \frac{1}{2} \left(\frac{2C}{T} \right)^{1/2} \left(E^{\bar{Q}} \int q(x)dx \left(\sum_{r=1}^{n} u_r(x) G_r(\int F_1(x)q(x)dx, \ldots, \int F_l((x)q(x)dx) \right) \right)^{1/2}.
\end{aligned} \tag{6.11}$$

Denoting by ω the typical point in $M_1(S)$ we can rewrite (6.11) as

$$\left| E^{\bar{Q}} \int P(q(x)) \frac{\partial \hat{G}}{\partial x}(x, \omega) dx \right| \leqq \left(\frac{C}{2T} \right)^{1/2} (E^{\bar{Q}} \int \hat{G}^2(x, \omega) q(x) dx)^{1/2} \tag{6.12}$$

for test functions $\hat{G}$ from a suitable class. First we have it for $\hat{G}$ of the form

$$\sum u_r(x) G_r(\langle F_1, q \rangle, \ldots, \langle F_l, q \rangle).$$

Then we can obtain the inequality for $\hat{G}$ of the form

$$\sum u_r(x)\hat{G}_r(\omega).$$

The next step is to use test functions of general form $\hat{G}(x,\omega)$ but with reasonable bounds on $\partial\hat{G}/\partial x$. Then by standard regularization techniques one can prove

$$E^{\bar{Q}}\int \frac{1}{q(x)}\left(\frac{\partial P(q(x))}{\partial x}\right)^2 dx \leqq \frac{C}{2T}. \tag{6.13}$$

It is easy to see that if Q is a weak limit on $C[[0,T]; M_1(S)]$ then $\bar{Q}$ is just the average marginal distribution over the time interval $[0,T]$. Therefore

$$\begin{aligned} E^Q \int_0^T \int \frac{1}{q(x,t)}\left(\frac{\partial}{\partial x} P(q(x,t))\right)^2 dx dt &\leqq \frac{C}{2T}\cdot T \\ &= \frac{C}{2}. \end{aligned} \tag{6.14}$$

Lemma 6.4. *There is at most one weak solution of the equation*

$$\frac{\partial \varrho(t,x)}{\partial t} = 2(P(\varrho(t,x)))_{xx}$$

with initial condition $\varrho(0,x)=\varrho_0(x)$ *among the class of nonnegative solutions satisfying*

$$\int_0^T \int \varrho^3(t,x)\,dt dx < \infty$$

$$\int_0^T \int \left(\frac{\partial P(\varrho(t,x))}{\partial x}\right)^2 \frac{1}{\varrho(t,x)}\,dt dx < \infty.$$

Proof. This has been carried out in [4].

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